From Pauli matrices to quantum Itô formula

Yan Pautrat

Institut Fourier
UMR 5582
BP 74, 38402 Saint-Martin d'Hères Cedex
France

Abstract

This paper answers important questions raised by the recent description, by Attal, of a robust and explicit method to approximate basic objects of quantum stochastic calculus on bosonic Fock space by analogues on the state space of quantum spin chains. The existence of that method justifies a detailed investigation of discrete-time quantum stochastic calculus. Here we fully define and study that theory and obtain in particular a discrete-time quantum Itô formula, which one can see as summarizing the commutation relations of Pauli matrices.

An apparent flaw in that approximation method is the difference in the quantum Itô formulas, discrete and continuous, which suggests that the discrete quantum stochastic calculus differs fundamentally from the continuous one and is therefore not a suitable object to approximate subtle phenomena. We show that flaw is only apparent by proving that the continuous-time quantum Itô formula is actually a consequence of its discrete-time counterpart. [1]

Introduction

From an early stage in the development of the theory of quantum stochastic calculus on bosonic Fock space, simpler discrete-time versions based on toy Fock spaces have been considered as a source of inspiration, but only by formal analogy; for example, such ideas undermine the presentation of the

[1] **Keywords**: quantum probability, quantum stochastic integrals, Fock spaces, toy Fock space, quantum Itô formula
AMS classification:81S25, 60H05

field in [Mey]. Yet it was not believed the analogy could be upgraded to a useful tool.

The recent paper [At3] by Attal showed such beliefs wrong. That paper describes a completely explicit realization of toy Fock space $T\Phi$ as a subspace of the usual Fock space $\Phi = \Gamma_{sym}(L^2(\mathbb{R}_+))$ and similarly fundamental noises on $T\Phi$ are expressed in terms of increments of quantum noises on Φ. These realizations depend on some scale; the interesting property here is that, when that scale goes to zero, these objects approximate their continuous-time counterparts. The simplicity of the method is surprising, but it should be remarked that its discovery relies heavily on the picturesque abstract Itô calculus description of Fock space (see [Mey] or [At2]). Discrete-time objects are naturally simpler than continuous-time ones; here the simplification is a major one since, as should be clear from the exposition in this paper, it reduces many problems to finite-dimensional ones. There is therefore reasonable hope that continous-time problems can be answered via the approximation scheme.

The goal of this paper is to pave the road for systematic application of this program. The first step is a rigorous treatment of discrete-time quantum stochastic calculus. The state space for that theory is an infinite-dimensional toy Fock space; such a space naturally appears as state space of a chain of two-levels atoms, and in particular, our fundamental noises $a^+, a^-, a^\circ, a^\times$ are just linear combinations of the usual Pauli matrices. The most natural definitions of integrals $\sum h_i a_i^\varepsilon$, $\varepsilon = +, \circ, -, \times$, turn out to be discrete-time transcriptions of the objects of the Attal-Lindsay theory of quantum stochastic integration (see [A-L]), which extends the earlier versions (developed successively in [H-P], [B-L], [A-M]). This means both that an important role is played by discrete-time abstract Itô calculus and that our integrals enjoy many properties.

Another issue which we address here is the lack, in Attal's paper, of a relation between a quantum stochastic integral on Fock space and the integral representation of its discrete-time approximation. Such a representation exists under fairly general assumptions, as was proved by the author in [Pa1]. What's more, these representations are explicit and expressed in terms of the discrete-time abstract Itô calculus. Here we relate the discrete-time Itô calculus to its continuous-time counterpart; this allows us to express the integral representation of the approximation of a quantum stochastic integral, in terms of the original integrands.

Such calculations in turn allow us to answer a question which is a probable reason why toy Fock approximation of Fock space quantum stochastic calculus was not believed to be a relevant object. Let us describe that question more precisely: on toy Fock space, there is, as we prove it, a quantum

Itô formula describing the composition of two integrals

$$\sum_{i\in\mathbb{N}} h_i^\varepsilon a_i^\varepsilon \sum_{i\in\mathbb{N}} k_i^\eta a_i^\eta = \sum_{i\in\mathbb{N}} h_i^\varepsilon (\sum_{j<i} k_j^\eta a_j^\eta) a_i^\varepsilon + \sum_{i\in\mathbb{N}} (\sum_{j<i} h_j^\varepsilon a_j^\varepsilon) k_i^\eta a_i^\eta + \sum_{i\in\mathbb{N}} h_i^\varepsilon k_i^\eta a_i^{\varepsilon.\eta},$$

where $a^{\varepsilon\cdot\eta}$ is actually $a^\varepsilon a^\eta$, so that it is given by the following table:

$\Rsh$	$-$	$\circ$	$+$	$\times$
$-$	0	a^-	$a^\times - a^\circ$	a^-
$\circ$	0	a°	a^+	a°
$+$	a°	0	0	a^+
$\times$	a^-	a°	a^+	$a^\times$

which, we recall, is a consequence of the Pauli matrices commutation relations. We call that table the *discrete time* Itô table. On the other hand, it is known that on the Fock space, and under some analytical conditions, (see [At1],[A-L], [A-M]) there is also an Itô formula, of similar form

$$\int_0^\infty H_t^\varepsilon \, da_t^\varepsilon \int_0^\infty K_t^\eta \, da_t^\eta = \int_0^\infty H_t^\varepsilon (\int_0^t K_s^\eta \, da_s^\eta) da_t^\varepsilon + \int_0^\infty (\int_0^t H_s^\varepsilon \, da_s^\varepsilon) K_t^\eta \, da_t^\eta + \int_0^\infty H_t^\varepsilon K_t^\eta \, da_t^{\varepsilon.\eta}$$

for the continuous time integrals, but here $da^{\varepsilon\cdot\eta}$ is given by

$\Rsh$	$-$	$\circ$	$+$	$\times$
$-$	0	da^-	$da^\times$	0
$\circ$	0	da°	da^+	0
$+$	0	0	0	0
$\times$	0	0	0	0

which we call the *continuous time* Itô table.

From the slight difference in the two Itô tables it may seem that there exist fundamental differences between discrete and continuous stochastic calculus. Both in order to relieve the approximation scheme from this apparent defect and to show its efficiency we will actually reprove the continuous-time Itô formula from the discrete-time one and the approximation results – which, we must note, do not depend on any composition formula.

This paper is organized as follows: in section one we develop a full theory of quantum stochastic calculus on toy Fock space and recall the statement of the theorem of representation as discrete quantum stochastic integrals which we use in the sequel. In section two we recall Attal's approximation method, describe the relation between discrete and continuous-time abstract Itô calculus and then compute the integral representations of approximations of continuous-time quantum stochastic integrals; we will see that the form of the projections is not as trivial as one would expect. In section three we recover the Itô formula with the associated continuous time Itô table from our approximation scheme and the commutation relations for Pauli matrices.

For notational simplicity, this paper only describes the case of simple toy Fock space. The case of higher multiplicity Fock spaces is a simple consequence and is described in full detail in the author's thesis [Pa2], which the interested reader can consult for a general exposition of the applications of the approximation method.

1 Stochastic calculus on toy Fock space

1.1 Definitions

Since our main goal is to reproduce as closely as possible the structure of Fock space continuous calculus, we define many objects by analogy with the continuous-time case. We therefore refer the reader to expositions of the theory of quantum stochastic calculus using abstract Itô calculus, *e.g.* [A-M] or [At2].

A basic property of the Fock space $\Phi = \Gamma_{sym}(L^2(\mathbb{R}_+))$ is its *Guichardet interpretation* that makes explicit a unitary equivalence with the space $L^2(\mathcal{P})$ of square-integrable functions over the set $\mathcal{P}_{\mathbb{R}_+}$ of finite subsets of $\mathbb{R}_+$ (see section 2, [Gui], or the above references). Similarly, the toy Fock space $\mathrm{T}\Phi$ is most naturally defined as the antisymmetric Fock space over $l^2(\mathbb{N})$. Nevertheless, we define it at once by its "Guichardet form". We denote by $\mathcal{P}$ the set of finite subsets of $\mathbb{N}$, that is, elements of $\mathcal{P}$ are either or the form $\{i_1, \ldots, i_n\}$ or the empty set $\emptyset$. The toy Fock space $\mathrm{T}\Phi$ is then defined as the space $l^2(\mathcal{P})$ of square-integrable functions on the set $\mathcal{P}$ of finite subsets of $\mathbb{N}$, that is, $\mathrm{T}\Phi$ is the space of all maps $f : \mathcal{P} \mapsto \mathbb{C}$, such that

$$\sum_{A \in \mathcal{P}} |f(A)|^2 < +\infty,$$

When $\mathrm{T}\Phi$ is seen as $l^2(\mathcal{P})$, a natural basis arises, that of the indicators $\mathbb{1}_A$ of elements A of $\mathcal{P}$; we will denote by X_A these vectors, and by Ω the vector $X_\emptyset$, called the *vacuum vector*. Every vector $f \in \mathrm{T}\Phi$ thus admits an orthogonal decomposition of the form

$$f = \sum_{A \in \mathcal{P}} f(A) X_A.$$

The toy Fock space has an important property of tensor product decomposition: for any partition $\cup N_j$ of $\mathbb{N}$, denote by $\mathrm{T}\Phi_{N_j}$ the space $l^2(\mathcal{P}_{N_j})$

where $\mathcal{P}_{N_j}$ is the set of finite subsets of N_j; then $\mathrm{T\Phi}_{N_j}$ can be identified with a subspace of TΦ and one has the explicit isomorphism

$$\mathrm{T\Phi} = \bigotimes_j \mathrm{T\Phi}_{N_j},$$

where we identify any X_A with $\bigotimes_j X_{A\cap N_j}$. We will mainly consider cases where the N_j's are of the form $\{0,\ldots,i-1\}$ or $\{i,\ldots\}$; we therefore introduce the notations

$$\mathrm{T\Phi}_i = \mathrm{T\Phi}_{\{0,\ldots,i-1\}}, \quad \mathrm{T\Phi}_{[i} = \mathrm{T\Phi}_{\{i,\ldots\}}.$$

A particular family of elements of TΦ will be useful in the sequel: it is the family of *exponential vectors*. To every u in $l^2(\mathbb{N})$ we associate a function on $\mathcal{P}$ by

$$e(u)(A) = \prod_{i\in A} u(i) \text{ for } A \in \mathcal{P},$$

and it can be seen to define a vector in TΦ by the inequality

$$n! \sum_{|A|=n} \left|\prod_{i\in A} |u(i)|\right|^2 \le \left(\sum_{i\ge 0} |u(i)|^2\right)^n,$$

but this yields only $\|e(u)\|^2 \le e^{\|u\|^2}$ and no (simple) formula for $< e(u), e(v) >$. The family of exponential vectors is total but contrarily to the case of exponentials of the Fock space $\Gamma_{sym}(L^2(\mathbb{R}_+))$, a family of exponentials of distinct functions is not necessarily linearly independent: consider for example the case of $u = (0,\ldots)$, $v = (1,0,\ldots)$ and $w = (2,0,\ldots)$, for which $e(u) - 2e(v) + e(w) = 0$.

Note that the tensor decomposition of an exponential vector is simple: any $e(u)$ can be decomposed for example as $e(u_i) \otimes e(u_{[i})$ where u_i is the restriction of u to $\{0,\ldots,i-1\}$ and $u_{[i}$ is the restriction of u to $\{i, i+1\ldots\}$.

Fundamental operators on TΦ One defines for all $i \in \mathbb{N}$ three operators by their action on the basis $\{X_A\}$:

$$a_i^+ X_A = \begin{cases} X_{A\cup\{i\}} & \text{if } i \notin A \\ 0 & \text{otherwise,} \end{cases}$$

$$a_i^- X_A = \begin{cases} X_{A\setminus\{i\}} & \text{if } i \in A \\ 0 & \text{otherwise,} \end{cases} \tag{1.1}$$

$$a_i^\circ X_A = \begin{cases} X_A & \text{if } i \in A \\ 0 & \text{otherwise,} \end{cases}$$

These operators are closable, of bounded closures (with norm 1), and we will keep the same notations for their closures, which we call operators of *creation* (a^+), *annihilation* (a^-) and *conservation* (a°). Besides, one should remark that they are of the form $\mathrm{Id} \otimes a_i^\varepsilon \otimes \mathrm{Id}$ in $T\Phi_i \otimes T\Phi_{\{i\}} \otimes T\Phi_{[i+1}$. For notational simplicity we define for all i the operator $a_i^\times$ to be the identity operator.

Relations with Pauli matrices A more physical description of our framework would start with the following: quantum mechanically speaking, a particle with, for example, two energy levels should be described by the complex vector space of dimension two: $\mathbb{C}^2$. The customary description for the most important operators of position and momentum, which we denote for a few lines by Q and P respectively, is

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad P = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

and Q, P satisfy the commutation relation

$$QP - PQ = 2i\,\mathrm{Id}.$$

The *-algebra generated by Q and P is the whole of the algebra of complex 2×2 matrices; that algebra is also linearly generated by Id, Q, P and QP. If we denote for consistency Q, P, $-iQP$ by σ_x, σ_y, σ_z respectively, we obtain the famous Pauli matrices. We therefore have a basis Id, σ_x, σ_y, σ_z with particular algebraic relations.

Now if we denote by Ω, X the canonical basis $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, there is a more natural basis for the vector space of 2×2 matrices, that is, Id, a^+, a^-, a° with

$$\begin{array}{lll} a^+\Omega = X & a^-\Omega = 0 & a^\circ\Omega = 0 \\ a^+X = 0 & a^-X = \Omega & a^\circ X = X, \end{array} \tag{1.2}$$

and we have the relations

$$a^+ = \frac{1}{2}(\sigma_x - i\sigma_y) \quad a^- = \frac{1}{2}(\sigma_x + i\sigma_y) \quad a^\circ = \frac{1}{2}(\mathrm{Id} - \sigma_z), \tag{1.3}$$

so that the commutation relations for a^+, a^-, a° are straightforward consequences of those for Pauli matrices. One may wonder why we did not choose the fourth operator of the canonical basis of linear operators on $\mathbb{C}^2$ instead of the identity; it is only in order to stay as close as can be to the continuous-time case.

Now our toy Fock space is simply the state space for a spin chain, that is, an infinity of distinguishable particles with two energy levels; the operators a_i^ε we consider are the natural ampliations of the above operators a^ε, $\varepsilon = +, -, \circ$. More precisely, for any i, a_i^ε is $\mathrm{Id} \otimes a^\varepsilon \otimes \mathrm{Id}$ in the decomposition $T\Phi = T\Phi_i \otimes T\Phi_{\{i\}} \otimes T\Phi_{(i}$. The relation of the objects we consider with the physically more customary Pauli matrices is therefore clear.

1.2 Abstract Itô calculus on TΦ

The main difference between Itô calculus on Toy Fock space and on regular Fock space is that predictability should replace adaptability for a simpler transcription of the classical results. Therefore we define the (everywhere defined) *predictable* projection and gradient at time $i \in \mathbb{N}$, of a vector f in $T\Phi$ by

$$p_i f(M) = \mathbb{1}_{M<i}\, f(M)$$
$$d_i f(M) = \mathbb{1}_{M<i}\, f(M \cup \{i\})$$

where $\mathbb{1}_{M<i}$ is the indicator of the event denoted by $M < i$ which is "$j < i$ for all j in M" (note that $\emptyset < i$ for all i).

The above operators are called *predictable* because for any $f \in T\Phi$, both $(p_i f)_{i\geq 0}$ and $(d_i f)_{i\geq 0}$ are *predictable processes*, that is, are sequences of vectors such that the i-th vector belongs to $T\Phi_i$. In contrast with the continuous time case, there is no definition problem for the d_i's as individual operators. We will write, to simplify notations,

$$d_A = d_{i_1} \dots d_{i_n} \text{ if } A = \{i_1 < \cdots < i_n\},$$

and

$$d_\emptyset = \mathrm{Id}.$$

The other essential tool for quantum Itô calculus is the abstract Itô integral:

Definition 1.1 *A predictable process of vectors $(f_i)_{i\geq 0}$ is said to be Itô-integrable if*

$$\sum \|f_i\|^2 < +\infty.$$

One then defines its Itô integral as the sum of mutually orthogonal terms

$$\sum_i f_i\, X_i.$$

It can be alternatively described as the vector $I(f)$ such that

$$I(f)(M) = f_{\vee M}(M - \vee M) \text{ and } I(f)(\emptyset) = 0,$$

where $\vee M$ denotes the largest element in the n-uple M.

Let us stress the fact that in $f_i\, X_i$ the product is just a tensor product in $T\Phi_i \otimes T\Phi_{[i}$ thanks to the previsiblity of the process: f_i belongs to $T\Phi_i$, X_i belongs to $T\Phi_{[i}$. The condition for the alternative definition to actually define a square-integrable function of A is easily seen to be the above Itô-integrability condition.

Substituting the equality $d_i f = \sum_{A<i} f(A+i) X_A$ in the chaotic decomposition of a vector f yields the following results:

Proposition 1.2 *Any $f \in T\Phi$ admits a unique decomposition of the form*

$$f \;=\; f(\emptyset)\Omega + \sum_{i\in\mathbb{N}} d_i f\, X_i$$

and one has the associated isometry formula:

$$\|f\|^2 = |f(\emptyset)|^2 + \sum_{i\in\mathbb{N}} \|d_i f\|^2.$$

This decomposition is called the predictable representation *of f.*

The isometry formula polarizes to the following adjoint relation:

$$\langle \sum_{i\in\mathbb{N}} f_i X_i, g\rangle = \sum_{i\in\mathbb{N}} \langle f_i, d_i g\rangle$$

for all $g \in T\Phi$ and all Itô-integrable process $(f_i)_{i\geq 0}$ of vectors of $T\Phi$.

1.3 Quantum stochastic integration on $T\Phi$

First of all we have to define predictability of an operator on $T\Phi$; the following definition is a natural extension of the classical predictability, in the sense that it is satisfied by a "quantized" i-predictable random variable (for the relation between quantum and classical stochastic calculus see [At2]).

Definition 1.3 *An operator is i-predictable if it is of the form $h \otimes Id$ in* $T\Phi_i \otimes T\Phi_{[i}$.

It is clear from this definition that an i-predictable operator is bounded and therefore can be extended to an everywhere defined operator of the above form. We will therefore always assume predictable operators to be everywhere defined and bounded.

The following lemma can be deduced from the former definition and links our definition with a more algebraic approach which would be a transcription of Attal and Lindsay's definition in [A-L]:

Lemma 1.4 *A bounded operator h on $\mathrm{T}\Phi$ is i-predictable if and only if it satisfies the following conditions:*

- *Its domain $Dom\, h$ is stable by p_i and by all operators d_j, $j \geq i$.*
- *The following equalities hold on $Dom\, h$:*

$$\begin{aligned} hp_i &= p_i h \text{ and} \\ hd_j &= d_j h \text{ for all } j \geq i. \end{aligned}$$

Proof.
It is clear that an i-predictable operator satisfies the above properties. Conversely, one can prove that the commutation relations in the statement of the lemma are equivalent to the relation

$$hf(M) = (hp_i d_{M\cap\{i,\ldots\}})f(M \cap \{0,\ldots,i-1\}) \tag{1.4}$$

for all f in $Dom\, h$, all M in P. From this relation one can then show that, for any vector of the form $f \otimes g$ in $\mathrm{T}\Phi_i \otimes \mathrm{T}\Phi_{[i}$, one has $f \in Dom\, h$ and $h(f \otimes g) = (hf) \otimes g$. The boundedness of h implies that it is of the form $h \otimes \mathrm{Id}$.

□

We will now define quantum stochastic integrals in discrete time; first remark that we wish these integrals to give analogues of predictable representations for operators. This means that we want integrals to be formally of the form $\sum_i h_i\, a_i$, where a_i denotes an "elementary action at time i". What's more, we wish to be able to consider the classical case where h_i, a_i are multiplication operators, and yet the composition $h_i a_i$ should involve no probabilistic interpretation, so that the operators h_i and a_i should be tensor-product independent and the composition $h_i a_i$ be a tensor decomposition in $\mathrm{T}\Phi_i \otimes \mathrm{T}\Phi_{\{i\}}$. We have remarked already that a_i^+, a_i^-, a_i°, Id, is a basis for $\mathrm{T}\Phi_{\{i\}}$; for all these reasons we will consider integrals as series of the form $\sum_i h_i^\varepsilon a_i^\varepsilon$ where every h_i^ε is i-predictable.

We call *predictable process* a process $(h_i)_{i\in\mathbb{N}}$ of operators, such that every h_i is i-predictable.

Definition 1.5 *Let $(h_i^\varepsilon)_{i\in\mathbb{N}}$ be a predictable process. For any ε in $\{+,-,\circ,\times\}$, we define the integral of $(h_i^\varepsilon)_{i\in\mathbb{N}}$ with respect to a^ε, as the operator series $\sum_{i\in\mathbb{N}} h_i^\varepsilon a_i^\varepsilon$ where this series means that*

- *its domain Dom* $\sum_i h_i^\varepsilon a_i^\varepsilon$ *is the set of all* $f \in \mathrm{T}\Phi$ *such that*

$$\begin{cases} \text{for all } M \in \mathcal{P}, \sum_{i\in\mathbb{N}} |h_i^\varepsilon a_i^\varepsilon f(M)| < +\infty \\ M \mapsto \sum_{i\in\mathbb{N}} h_i^\varepsilon a_i^\varepsilon f(M) \text{ is square-integrable.} \end{cases}$$

- *the vector* $\sum_i h_i^\varepsilon a_i^\varepsilon f$ *is defined by*

$$(\sum_{i\in\mathbb{N}} h_i^\varepsilon a_i^\varepsilon f)(M) = \sum_{i\in\mathbb{N}} (h_i^\varepsilon a_i^\varepsilon f(M))$$

  for all M *in* $\mathcal{P}$.

For some of the results to come we will need more restrictive summability assumptions; we therefore define *restricted integrals*:

Definition 1.6 *Let* $(h_i^\varepsilon)_{i\in\mathbb{N}}$ *be a predictable process. For any* ε *in* $\{+,-,\circ,\times\}$, *we define the restricted integral* $\sum_{i\in\mathbb{N}}^R h_i^\varepsilon a_i^\varepsilon$ *of* $(h_i^\varepsilon)_{i\in\mathbb{N}}$ *with respect to* a^ε, *as the restriction of the integral* $\sum_{i\in\mathbb{N}} h_i^\varepsilon a_i^\varepsilon$ *to the set of vectors* f *in Dom* $\sum_i h_i^\varepsilon a_i^\varepsilon$ *which are such that*

$$M \mapsto \sum_{i\in\mathbb{N}} |h_i^\varepsilon a_i^\varepsilon f(M)|$$

is a square-integrable function on $\mathcal{P}$.

We have mentioned already the relation between the above described, natural definitions of integrals and discrete-time analogues of Attal and Lindsay's algebraic definitions of quantum stochastic integrals. One can see from the definitions of operators a^ε, $\varepsilon = +,\circ,-$, that

- the quantity $a_i^+ f(M)$ is null if $i \notin M$ and if $i \in M$ then $a_i^+ f(M) = f(M \setminus \{i\})$. Therefore we have for all M in $\mathcal{P}$,

$$\sum_i h_i a_i^+ f(M) = \sum_{i\in M} h_i f(M \setminus \{i\}),$$

  and the action of $\sum_i h_i a_i^+$ gives a "discrete Skorohod integral" of $h_i f$.

- the adapted gradient d_i equals $p_i a_i^-$, so that for all M in$\mathcal{P}$,

$$\sum_i h_i a_i^- f(M) = \sum_{i\notin M} h_i f(M \cup \{i\}),$$

- the above two remarks and the equality $a_i^\circ = a_i^+ a_i^-$ imply that, for all M in $\mathcal{P}$,

$$\sum_i h_i a_i^\circ f(M) = \sum_{i\in M} h_i f(M)$$

and in each of these equalities it is equivalent for one expression or for the other to define a summable series (in the case where $\varepsilon = -$) and to define an element of $l^2(\mathcal{P})$. It is therefore clear that our integrals (including the case $\varepsilon = \times$) are exactly transcriptions of Attal and Lindsay's integrals as defined in [A-L]; in a similar way, the restricted integrals we defined are analogues of their restricted integrals.

In particular these integrals handle just like Attal and Lindsay's, except that, thanks to the discrete-time framework, the integrands h_i are bounded so that one of the domain conditions disappears; yet it is, of all conditions, the one least intrinsic to the integral. We take advantage of these analogies to state a few properties of these integrals and refer the reader to the proofs in [A-L] instead of reproducing rather tedious computations. Of the properties we state here, the first is a discrete-time Hudson-Parthasarathy formula for the action of an integral on the exponential domain; the second is an alternative characterization of restricted integrals, in Attal-Meyer form. The third is the famous *Itô formula* which gives the integral representation for the composition of two stochastic integrals.

Proposition 1.7 (Hudson-Parthasarathy formulas) *Let $(h_i)_{i\geq 0}$ be a predictable process, let $\varepsilon \in \{+, \circ, -, \times\}$ and assume an exponential vector $e(u)$ to be in the domain of the restricted integral $\sum_i^R h_i^\varepsilon a_i^\varepsilon$. Then for all v in $l^2(\mathbb{N})$ one has*

$$\begin{aligned}
\langle e(u), \sum_{i\in\mathbb{N}} h_i^+ a_i^+\, e(v)\rangle &= \sum_{i\in\mathbb{N}} \overline{u(i)}\ \langle e(u\mathbb{1}_{\neq i}), h_i^+ e(v\mathbb{1}_{\neq i})\rangle && \text{if } \varepsilon = +\\
\langle e(u), \sum_{i\in\mathbb{N}} h_i^- a_i^-\, e(v)\rangle &= \sum_{i\in\mathbb{N}} v(i)\ \langle e(u\mathbb{1}_{\neq i}), h_i^- e(v\mathbb{1}_{\neq i})\rangle && \text{if } \varepsilon = -\\
\langle e(u), \sum_{i\in\mathbb{N}} h_i^\circ a_i^\circ\, e(v)\rangle &= \sum_{i\in\mathbb{N}} \overline{u(i)}v(i)\langle e(u\mathbb{1}_{\neq i}), h_i^\circ e(v\mathbb{1}_{\neq i})\rangle && \text{if } \varepsilon = \circ\\
\langle e(u), \sum_{i\in\mathbb{N}} h_i^\times a_i^\times\, e(v)\rangle &= \sum_{i\in\mathbb{N}}\langle e(u), h_i^\times e(v)\rangle && \text{if } \varepsilon = \times
\end{aligned}$$

and every one of the above series is summable. Here $u\mathbb{1}_{\neq i}$ (respectively $v\mathbb{1}_{\neq i}$) represents the sequence which is equal to u (respectively to v), except for the i-th term, which is null.

The following characterization for restricted integrals can be a most useful tool, especially as it very nicely summarizes the domain conditions for an integral to be defined.

Proposition 1.8 (Attal-Meyer characterization) *Let $(h_i^\varepsilon)_{i\in\mathbb{N}}$ be four predictable processes, $\varepsilon = +, \circ, -, \times$; the operator $\sum\sum_{i\in\mathbb{N}}^R h_i^\varepsilon a_i^\varepsilon$ is the maximal* *(in the sense of domains) operator h satisfying*

$$hf = \sum_{i\in\mathbb{N}} h_i d_i f\, X_i + \sum_{i\in\mathbb{N}} h_i^+ p_i f\;\, X_i + \sum_{i\in\mathbb{N}} h_i^- d_i f + \sum_{i\in\mathbb{N}} h_i^\circ d_i f\;\, X_i + \sum_{i\in\mathbb{N}} h_i^\times p_i f,$$

where $h_i = \sum_{j<i} h_j^\varepsilon a_j^\varepsilon$ and where these equalities mean that

- *a vector f is in the domain of h if and only if $h_i d_i f$ is Itô-integrable and the other series are Itô-integrable or summable in norm (depending on ε),*
- *equality holds.*

This characterization turns out to be very useful in many proofs; for example the proof of the following proposition, which seems to reduce to a simple permutation of two summations, is made quite painful because of domain considerations. Proposition 1.8 summarizes very nicely these problems so that the proof becomes a (even then tedious) play with commutation relations between integrals and operators p_i, d_i. Notice that in contrast with the continuous-time Attal-Meyer definition (see section two), the above is not an implicit definition via a kind of integral equation; thanks to the discrete-time summation, an integral stopped at time i is readily defined as a finite sum of operators.

The next theorem expresses the composition of two quantum stochastic integrals in integral form. Note that, in the following proposition, the considered integrals are restricted ones.

Theorem 1.9 (Itô formula) *Let ε and η be two elements of $\{+, \circ, -, \times\}$ and $(h_i^\varepsilon)_{i\in\mathbb{N}}$ and $(k_i^\eta)_{i\in\mathbb{N}}$ be two predictable operator processes on $\mathrm{T}\Phi$. Then the operator*

$$\sum_{i\in\mathbb{N}}^R h_i a_i^\varepsilon \sum_{i\in\mathbb{N}}^R k_i a_i^\eta - \sum_{i\in\mathbb{N}}^R h_i^\varepsilon\, k_i\, a_i^\varepsilon - \sum_{i\in\mathbb{N}}^R h_i k_i^\eta a_i^\eta - \sum_{i\in\mathbb{N}}^R h_i^\varepsilon k_i^\eta a_i^{\varepsilon\cdot\eta}, \tag{1.5}$$

is a restriction of the zero process; the symbol $a^{\varepsilon\cdot\eta}$ is given by the following table

$\Rsh$	$-$	$\circ$	$+$	$\times$
$-$	0	a^-	$a^\times - a^\circ$	a^-
$\circ$	0	a°	a^+	a°
$+$	a°	0	0	a^+
$\times$	a^-	a°	a^+	$a^\times$

(1.6)

so that $a^{\varepsilon\cdot\eta}$ is simply $a^\varepsilon a^\eta$.

The comparison between this theorem and its continuous-time analogue will be the subject of section three.

1.4 Integral representations of operators

Here we simply recall results from [Pa1]. In that paper we characterized operators on $\mathrm{T}\Phi$ which can be represented as quantum stochastic integrals, and obtained explicit formulas for the integrands. For our purposes here, the most useful result is the following:

Theorem 1.10 *Let h be an operator on $\mathrm{T}\Phi$ such that all vectors X_A belong to $Dom\, h \cap Dom\, h^*$. Then the integral operator with integrands*

$$\begin{cases} h_i^+ \; p_i &= d_i h p_i \\ h_i^- \; p_i &= p_i h a_i^+ p_i \\ h_i^\circ \; p_i &= d_i h a_i^+ p_i - p_i h p_i \end{cases} \tag{1.7}$$

and $\lambda = \langle \Omega, h\Omega \rangle$ is such that

$$h - (\lambda + \sum_{i \geq 0} h_i^+ a_i^+ + \sum_{i \geq 0} h_i^- a_i^- + \sum_{i \geq 0} h_i^\circ a_i^\circ)$$

is a restriction of the zero process and the set $\{X_A, A \in \mathcal{P}\}$ is in its domain.

This theorem is not quite enough if we want to consider predictable processes of operators, that is, sequences $(h_i)_i$ of operators such that h_i is i-predictable, and represent such a process by

$$h_i = \sum_{\varepsilon=+,\circ,-,\times} \sum_{j<i} h_j^\varepsilon a_j^\varepsilon.$$

Note that the presence of an integral with respect to $a^\times$ is unavoidable if we want the h_j^ε 's to be independent of i. Minor adaptations of the above result allow us to give the following description of representations of processes; moreover, the boundedness of predictable operators simplifies the analytical problems:

Corollary 1.11 *Let $(h_j)_{j\in\mathbb{N}}$ be a predictable process of operators on $\mathrm{T}\Phi$. Then for every j the operator h_j is equal to*

$$\lambda + \sum_{i<j} h_i^+ a_i^+ + \sum_{i<j} h_i^- a_i^\circ + \sum_{i<j} h_i^\circ a_i^\circ + \sum_{i<j} h_i^\times a_i^\times$$

where

$$\begin{cases} h_i^+ p_i = & d_i h_{i+1} p_i \\ h_i^- p_i = & p_i h_{i+1} a_i^+ p_i \\ h_i^\circ p_i = & d_i h_{i+1} a_i^+ p_i - p_i h_{i+1} p_i \\ h_i^\times p_i = & p_i (h_{i+1} - h_i) p_i \end{cases} \tag{1.8}$$

and $\lambda = \langle \Omega, h_0 \Omega \rangle$.

2 Approximations of continuous-time integrals

2.1 A reminder on quantum stochastic calculus

We shall here recall briefly some necessary definitions and results from quantum stochastic calculus on regular Fock space. Again we refer the reader to [Mey] or [At2] for details on the general framework and on the definition given here of quantum stochastic integrals.

First of all, thanks to Guichardet's interpretation we can simply define the Fock space as $\Phi = L^2(\mathcal{P})$, *i.e.* the set of functions on the set $\mathcal{P}$ of finite subsets of $\mathbb{R}_+$. More precisely, we equip $\mathcal{P}$ with a measured space structure using the fact that it is the union of the set of n-tuples and of the empty set. On n-tuples we simply consider n-th dimensional Borel sets and Lebesgue measure; the empty set is defined to be an atom of mass one. The canonical variable will be denoted by σ, and the infinitesimal volume element by $d\sigma$. Note that we denote the sets of finite subsets of $\mathbb{N}$ or $\mathbb{R}_+$ by the same symbol $\mathcal{P}$: the context should always prevent confusion.

The elements of Φ can be seen as the functions, defined on all increasing simplices $\Sigma_n = \{t_1 < \cdots < t_n\}$ of $\mathbb{R}_+$, such that

$$\sum_n \int_{\Sigma_n} |f(t_1, \cdots, t_n)|^2 \, dt_1 \cdots dt_n < +\infty. \tag{2.1}$$

It is clear from this *chaotic representation* that Φ is isomorphic to the chaos space of any normal martingale (see [Mey] or [At2] and the references therein) *e.g.* the Brownian motion, the compensated Poisson process, the Azéma martingales, etc. We shall label as Φ_t the analogous set of functions defined on simplices of $[0, t]$; Φ_t will be canonically included in Φ.

A particular set of elements in Φ is relevant, the *exponential domain*: an exponential over a function $u \in L^2(\mathbb{R}_+)$ is defined by

$$\mathcal{E}(u)(\sigma) = \prod_{s \in \sigma} u(s). \tag{2.2}$$

It is an element of Φ, as one can see that $\|\mathcal{E}(u)\|^2 = e^{\|u\|^2}$. Besides, the exponential domain $\mathcal{E}(L^2(\mathbb{R}_+))$ is total in Φ and exponentials are easy to handle so that it is a domain of choice for many proofs.

Abstract Itô calculus Let us consider for all t the element χ_t of Φ defined as follows:

$$\chi_t(\sigma) = \begin{cases} \mathbb{1}_{s<t} \text{ if } \sigma = \{s\} \\ 0 \text{ otherwise.} \end{cases}$$

The isomorphism from Φ to any chaos space sends χ_t to the Brownian motion at time t when onto the chaos space of Brownian motion, to the Poisson process at time t when onto the chaos space of Poisson process, etc. One can define an integral of *adapted processes* $(f_t)_{t\geq 0}$ of elements of Φ (that is, such that $f_t \in \Phi_t$ for all t), with respect to the curve $(\chi_t)_{t\geq 0}$, denoted

$$I(f) = \int f_t \, d\chi_t$$

and satisfying

$$\|I(f)\|^2 = \int_{\mathbb{R}_+} \|f_t\|^2 dt, \tag{2.3}$$

as soon as the latter real-valued integral is finite; the complete construction uses the isometry property (2.3) for step processes. This integral is called the *abstract Itô integral.*

There is an alternate construction for this integral:

$$I(f)(\sigma) = f_{\vee\sigma}(\sigma-) \tag{2.4}$$

where $\vee\sigma$ is the largest element in σ and $\sigma- = \sigma \setminus \{\vee\sigma\}$. The natural conditions for this to be well defined can be seen to be the same as above, namely the square-integrability of the process $(\|f_t\|)_{t\geq 0}$.

Let us define the two fundamental operators of abstract Itô calculus on Φ:

- the *adapted projection* P_t is defined for all t, as the orthogonal projection onto Φ_t. Explicitly, for any $f \in \Phi$,

$$P_t f(\sigma) = \mathbb{1}_{\sigma<t} f(\sigma); \tag{2.5}$$

- the *adapted gradient* is defined by

$$D_t f(\sigma) = \mathbb{1}_{\sigma<t} f(\sigma \cup \{t\}). \tag{2.6}$$

As in the discrete-time case, "$\sigma < t$" means "$s < t$ for all $s \in \sigma$".

Substituting (2.6) in (2.4) yields immediately the following analogues to Proposition 1.2:

$$f = f(\emptyset) + \int D_t f \, d\chi_t \tag{2.7}$$

and

$$\|f\|^2 = |f(\emptyset)|^2 + \int \|D_t f\|^2 dt. \tag{2.8}$$

Now notice that we have not been precise in our definition of the operators D_t; actually it is quite ill-defined in the sense that an individual D_t is not a well-defined operator. All one can say is, thanks to formula (2.8), that for any f, $D_t f$ is defined for almost all t.

Quantum stochastic integrals We shall here define integrals

$$\int_0^\infty H_s da_s^\varepsilon$$

with respect to the three quantum noises da^+, da°, da^- and to time dt, which we denote by $da^\times$ for simplicity.

The heuristics of the Attal-Meyer quantum stochastic calculus, which we present in a simplified way, derives from the fact that the noises, which will turn out to be differentials of continuous-time fundamental operators, should act just like the fundamental operators of toy Fock space (compare with (1.1)):

- any da_t^ε acts only on $\Phi_{[t,t+dt]}$, which from (2.7) can be seen as "generated" by Ω and $d\chi_t$ and
- the da_t^ε are given by the following table:

$$\begin{array}{llll} da_t^+\Omega = d\chi_t & da_t^-\Omega = 0 & da_t^\circ\Omega = 0 & da_t^\times\Omega = dt\,\Omega \\ da_t^+ d\chi_t = 0 & da_t^- d\chi_t = dt\,\Omega & da_t^\circ d\chi_t = d\chi_t & da_t^\times d\chi_t = 0. \end{array}$$

These heuristics allow us to define integrals $\int H_s^\varepsilon da_s^\varepsilon$ for adapted processes $(H_s^\varepsilon)_{s\geq 0}$, that is, processes of operators such that for almost all s, all $f \in Dom\ H_s$,

- the vectors $P_s f$ and $D_u f$ belong to $Dom\, H_s^\varepsilon$ for almost all $u \geq s$
- the equalities $H_s^\varepsilon P_s f = P_s H_s^\varepsilon f$ and $H_s^\varepsilon D_u f = D_u H_s^\varepsilon f$ hold for almost all $u \geq s$.

In that case, a formal computation (see [A-M] or [Mey]) leads us to give the following definition: the integral $\sum_{\varepsilon=+,\circ,-} \int_0^\infty H_s^\varepsilon da_s^\varepsilon$ is defined as the only operator H which satisfies the following equality:

$$Hf = \int_0^\infty H_s D_s f d\chi_s + \int_0^\infty H_s^+ P_s f d\chi_s + \int_0^\infty H_s^- D_s f ds + \int_0^\infty H_s^\circ D_s f d\chi_s \tag{2.9}$$

with $H_s = P_sHP_s$. That is, f is in the common domain of the integrals if and only if the right-hand-side is well defined and equality holds. One can define an integral $\int_a^b H_s da_s^\varepsilon$ as the integral of the process equal to H_s for $s \in [a,b]$ and zero otherwise; one then notices that (2.9) holds equivalently with $H_s = \int_0^s H_r^\varepsilon\, da_r^\varepsilon$. We also define the integral of an adapted process $(H_s^\times)_{s\geq 0}$ as the strong integral $\int H_s^\times\, ds$. The operators a_t^ε are then defined as the integrals $\int_0^t da_s^\varepsilon$ in the above sense.

We give here as a corollary the formulas of Hudson and Parthasarathy, which were the cornerstone of the first theory of quantum stochastic integration on Fock space (described in [H-P]).

Hudson-Parthasarathy formula Let us consider a quantum stochastic integral H defined on the exponential domain (see (2.2)). Then the following equality holds for all $u, v \in L^2(\mathbb{R}_+)$, almost all $t \in \mathbb{R}_+$:

$$\langle \mathcal{E}(u)\,, H\mathcal{E}(v)\rangle = \int_0^\infty \phi(s)\langle \mathcal{E}(u)\,, H_s^\varepsilon \mathcal{E}(v)\rangle ds \tag{2.10}$$

where

$$\phi(s) = \begin{cases} \overline{u}(s) & \text{if } \varepsilon = + \\ v(s) & \text{if } \varepsilon = - \\ \overline{u}(s)v(s) & \text{if } \varepsilon = \circ \\ 1 & \text{if } \varepsilon = \times. \end{cases}$$

2.2 Explicit formulas for the projections of integrals

In this section we use the embedding of toy Fock space in regular Fock space defined by Attal in [At3] and the formulas (1.7) to obtain the projection of an integral operator in discrete time. Let us first recall the definitions of [At3]: we denote by $\mathfrak{P}$ the filter of partitions of $\mathbb{R}_+$ partially ordered by inclusion. A generic element of $\mathfrak{P}$ is denoted by $\mathcal{S} = \{0 = t_0 < t_1 < \cdots\}$, and its mesh size by $|\mathcal{S}|$. For any $\mathcal{S}$ in $\mathfrak{P}$ we define the following objects on Φ:

$$\begin{aligned} a_i^- &= \mathrm{Id} \otimes \frac{a^-_{t_{i+1}} - a^-_{t_i}}{\sqrt{t_{i+1} - t_i}} P^{(1)} \otimes \mathrm{Id} \\ a_i^+ &= \mathrm{Id} \otimes P_i^{(1)} \frac{a^+_{t_{i+1}} - a^+_{t_i}}{\sqrt{t_{i+1} - t_i}} \otimes \mathrm{Id} \\ a_i^\circ &= \mathrm{Id} \otimes P^{(1)}(a^\circ_{t_{i+1}} - a^\circ_{t_i})P^{(1)} \otimes \mathrm{Id}, \end{aligned}$$

where $P^{(1)}$ represents the projection on the chaos of order 1, and where the tensor decomposition is meant in $\Phi_{t_i} \otimes \Phi_{[t_i,t_{i+1}]} \otimes \Phi_{[t_{i+1}}$. The space $\mathrm{T}\Phi(\mathcal{S}) \subset \Phi$

is defined as the closed subspace spanned by the vectors $X_A = \prod_{i\in A} X_i$ for $A \in \mathcal{P}$; it is isomorphic to the toy Fock space, and this isomorphism sends restrictions of the above operators a_i^ε to the operators defined in the previous section. We denote by $\mathbb{E}_{\mathcal{S}}$ the projection on the subspace $\mathrm{T}\Phi(\mathcal{S})$.

The main relations for our computations are given in the following lemma:

Lemma 2.1 *Let us fix a given partition $\mathcal{S} = \{0 = t_0 < t_1 < \cdots < t_n < \cdots\}$. One has for all $f \in \Phi$,*

$$p_i \mathbb{E}_{\mathcal{S}} f = \mathbb{E}_{\mathcal{S}} P_{t_i} f,$$

$$d_i \mathbb{E}_{\mathcal{S}} f = \frac{1}{\sqrt{t_{i+1}-t_i}}\, \mathbb{E}_{\mathcal{S}} \int_{t_i}^{t_{i+1}} P_{t_i} D_t f dt,$$

and

$$\mathbb{E}_{\mathcal{S}} \int_0^\infty f_t d\chi_t = \sum_{i\geq 0} \frac{1}{\sqrt{t_{i+1}-t_i}}\, \mathbb{E}_{\mathcal{S}} \int_{t_i}^{t_{i+1}} P_{t_i} f_t\, dt\; X_i.$$

Proof.
The third equality is a consequence of the predictable representation property on toy Fock space and of the second equality; the first is straightforward. We therefore prove only the second one.

Since $d_i = p_i a_i^-$, one has:

$$d_i \mathbb{E}_{\mathcal{S}} f = p_i a_i^- \mathbb{E}_{\mathcal{S}} f = \frac{1}{\sqrt{t_{i+1}-t_i}} \mathbb{E}_{\mathcal{S}} P_{t_i}((a_{t_{i+1}}^- - a_{t_i}^-) f),$$

but

$$(a_{t_{i+1}}^- - a_{t_i}^-) f = \int_{t_i}^\infty (a_{t\wedge t_{i+1}}^- - a_{t_i}^-) D_t f\, d\chi_t + \int_{t_i}^{t_{i+1}} D_t f dt$$

by the Attal-Meyer formulas, so

$$\begin{aligned} d_i \mathbb{E}_{\mathcal{S}} f &= \frac{1}{\sqrt{t_{i+1}-t_i}} p_i \mathbb{E}_{\mathcal{S}} (\int_{t_i}^{t_{i+1}} D_t f dt) \\ &= \frac{1}{\sqrt{t_{i+1}-t_i}} \mathbb{E}_{\mathcal{S}} (\int_{t_i}^{t_{i+1}} P_{t_i} D_t f dt). \end{aligned}$$

□

As a corollary, we obtain the following straightforward lemma:

Lemma 2.2 *Let u belong to $L^2(\mathbb{R}_+)$. The projection $\mathbb{E}_{\mathcal{S}}\mathcal{E}(u)$ is again an exponential vector in $\mathrm{T}\Phi$ over the function $\widetilde{u}(i) = \frac{1}{\sqrt{t_{i+1}-t_i}} \int_{t_i}^{t_{i+1}} u(s) ds$. When seen as a vector of Φ, it is not an exponential vector if $u \neq 0$, but one has for all $t_i \leq t < t_{i+1}$,*

$$D_t e(\widetilde{u}) = \frac{\widetilde{u}(i)}{\sqrt{t_{i+1}-t_i}} e(\widetilde{u}_i).$$

Now, if we want to apply Theorem 1.10 to approximations $\mathbb{E}_{\mathcal{S}} H \mathbb{E}_{\mathcal{S}}$ of integrals H, we need to make some assumptions on these integrals.

(HD) The integrals $\int_0^\infty H_s^\varepsilon \, da_s^\varepsilon$ and $\int_0^\infty (H_s^\varepsilon)^* da^{\varepsilon'}$ are well-defined on $\mathcal{E}(L^2(\mathbb{R}_+))$ and all its images by any $\mathbb{E}_{\mathcal{S}}$.

Here ε' is defined by $+' = - \quad -' = + \quad \circ' = \circ$.

The case of $\varepsilon = \times$ will be treated later. The assumption **(HD)** implies in particular, if we denote by H the integral $\int_0^\infty H_s \, da_s^\varepsilon$, that H^* equals $\int_0^\infty (H_s^\varepsilon)^* da^{\varepsilon'}$ on $\mathcal{E}(L^2(\mathbb{R}_+))$ and all its projections by $\mathbb{E}_{\mathcal{S}}$. It also implies that the projections $\mathbb{E}_{\mathcal{S}} H \mathbb{E}_{\mathcal{S}}$ and $\mathbb{E}_{\mathcal{S}} H^* \mathbb{E}_{\mathcal{S}}$ are defined on all finite linear combinations of X_A's. Indeed, by Lemma 2.2

$$\mathbb{E}_{\mathcal{S}} \mathcal{E}(1) = \Omega$$

$$\mathbb{E}_{\mathcal{S}} \mathcal{E}(\mathbb{1}_{[t_i,t_{i+1}]}) = \sqrt{t_{i+1}-t_i}\, X_i + \Omega$$

$$\mathbb{E}_{\mathcal{S}} \mathcal{E}(\mathbb{1}_{[t_i,t_{i+1}] \cup [t_j,t_{j+1}]}) = \sqrt{t_{i+1}-t_i}\sqrt{t_{j+1}-t_j}\, X_{i,j} + \sqrt{t_{i+1}-t_i}\, X_i + \sqrt{t_{j+1}-t_j}\, X_j + \Omega$$

and so on. We therefore apply Theorem 1.10 to obtain the following:

Proposition 2.3 *Let $H = \int H_t^\varepsilon da_t^\varepsilon$ be a quantum stochastic integral on Φ that satisfies the assumptions* (**HD**). *Then $\mathbb{E}_{\mathcal{S}} H \mathbb{E}_{\mathcal{S}}$ has a representation as a discrete quantum stochastic integral which holds at least on vectors $\{X_A, A \in \mathcal{P}\}$ and the integrands h_i^+, h_i^-, h_i° are given by:*

- *for $\varepsilon = +$,*

$$\begin{aligned} h_i^+ &= \frac{1}{\sqrt{t_{i+1}-t_i}} \mathbb{E}_{\mathcal{S}} \int_{t_i}^{t_{i+1}} P_{t_i} H_t^+ dt \\ h_i^- &= 0 \\ h_i^\circ &= \frac{1}{t_{i+1}-t_i} \mathbb{E}_{\mathcal{S}} \int_{t_i}^{t_{i+1}} P_{t_i} H_t^+ (a_t^+ - a_{t_i}^+) dt \end{aligned}$$

- *for $\varepsilon = -$,*

$$\begin{aligned} h_i^+ &= 0 \\ h_i^- &= \frac{1}{\sqrt{t_{i+1}-t_i}} \mathbb{E}_{\mathcal{S}} \int_{t_i}^{t_{i+1}} P_{t_i} H_t^- dt \\ h_i^\circ &= \frac{1}{t_{i+1}-t_i} \mathbb{E}_{\mathcal{S}} \int_{t_i}^{t_{i+1}} P_{t_i} (a_t^- - a_{t_i}^-) H_t^- dt \end{aligned}$$

- *for* $\varepsilon = \circ$,

$$\begin{aligned} h_i^+ &= 0 \\ h_i^- &= 0 \\ h_i^\circ &= \frac{1}{t_{i+1}-t_i}\,\mathbb{E}_{\mathcal{S}}\int_{t_i}^{t_{i+1}} P_{t_i} H_t^\circ dt \end{aligned}$$

where the equalities are over $\mathrm{T}\Phi_i$ *(considering, for the right-hand-side,* $\mathrm{T}\Phi_i$ *as a subspace of* Φ_{t_i}*) and where all operator integrals are in the strong sense.*

In the case where $\varepsilon = \circ$*, the integral representation has the exponential domain of* $\mathrm{T}\Phi$ *in its restricted domain.*

Remarks:

- note that for any $s < t$, $(a_t^+ - a_s^+)P_s$ is bounded with norm $\sqrt{t-s}$ since, on Φ_s, $(a_t^+ - a_s^+)$ is simply multiplication by $\chi_t - \chi_s$. As a consequence, $P_s(a_t^- - a_s^-)$ is also bounded on the exponential domain.

- It is rather disappointing that the discrete-time integrals are not defined on a larger space, for example on the exponential domain of $\mathrm{T}\Phi$ in the case where $\varepsilon = +$ or $-$. Let us discuss this phenomenon: notice first that, in any case,

$$h_i^+ a_i^+ + h_i^- a_i^- + h_i^\circ a_i^\circ = \mathbb{E}_{\mathcal{S}}\int_{t_i}^{t_{i+1}} H_s^\varepsilon\, da_s^\varepsilon \mathbb{E}_{\mathcal{S}}.$$

Denote $H_i = \int_{t_i}^{t_{i+1}} H_s^\varepsilon\, a_s^\varepsilon$. For all $f \in Dom\, h$, all $A \in \mathcal{P}$, one has

$$\sum_i |h_i^\varepsilon a_i^\varepsilon f(A)| < +\infty \quad \text{and} \quad \sum_i |h_i^\circ a_i^\circ f(A)| < +\infty, \tag{2.11}$$

so that the problems will arise from square-summability over A in $\mathcal{P}$. To prove the above claim (2.11), notice first that it is straightforward when $\varepsilon = +$ (the sum is finite) and continue with $\varepsilon = -$. The sum $\sum_i |\mathbb{E}_{\mathcal{S}} H_i \mathbb{E}_{\mathcal{S}} f(A)|$ is smaller than

$$\sum_i \frac{1}{\sqrt{(t_{i_1+1}-t_{i_1})\dots(t_{i_n+1}-t_{i_n})}}\int_{t_{i_1}}^{t_{i_1+1}}\dots\int_{t_{i_n}}^{t_{i_n+1}} |H_i \mathbb{E}_{\mathcal{S}} f(s_1,\dots,s_n)|\, ds_1\dots ds_n \tag{2.12}$$

if $A = \{i_1, \ldots, i_n\}$. This is smaller than

$$(\int_{t_{i_n}}^{t_{i_n+1}} \|H_s D_s \mathbb{E}_{\mathcal{S}} f\|^2 \, ds)^{1/2} + \sum_i \int_{t_i}^{t_{i+1}} \left\|H_s^- D_s \mathbb{E}_{\mathcal{S}} f\right\|$$

which is finite. Besides, $\mathbb{E}_{\mathcal{S}} H_i \mathbb{E}_{\mathcal{S}}$ is $h_i^- a_i^- + h_i^\circ a_i^\circ$ and the condition $\sum_i |h_i^\circ a_i^\circ f(A)|$ is straightforward since only a finite number of terms in the sum are nonzero and this implies the condition on $\sum_i h_i^- a_i^-$.

Then one has $\sum_{A\in\mathcal{P}} |\sum_i \mathbb{E}_{\mathcal{S}} H_i \mathbb{E}_{\mathcal{S}} f(A)|^2 < +\infty$ since it is dominated by $\|\sum_i H_i \mathbb{E}_{\mathcal{S}} f\|^2$. Therefore one deduces that

$$\sum_{A\in\mathcal{P}} |\sum_i (h_i^\varepsilon a_i^\varepsilon + h_i^\circ a_i^\circ) f(A)|^2 < +\infty, \tag{2.13}$$

but there is no reason why one should have

$$\sum_{A\in\mathcal{P}} |\sum_i h_i^+ a_i^+ f(A)|^2 < +\infty \quad \text{and} \quad \sum_{A\in\mathcal{P}} |\sum_i h_i^\circ a_i^\circ f(A)|^2 < +\infty;$$

choose for example $\varepsilon = +$ and $f = \mathcal{E}(u)$ to see in detail what happens. For $t_i \leq t < t_{i+1}$ one has

$$P_t e(\widetilde{u}) = e(\widetilde{u}_i) + \widetilde{u}(i) e(\widetilde{u}_i) \frac{\chi_t - \chi_{t_i}}{t_{i+1} - t_i} \tag{2.14}$$

which implies (2.13) by the Attal-Meyer formulation but this does not imply that

$$\int_0^\infty \left\|H_t^+ e(\widetilde{u}_i)\right\|^2 dt + \int_0^\infty \left\|H_t^+ \widetilde{u}(i) e(\widetilde{u}_i) \frac{\chi_t - \chi_{t_i}}{t_{i+1} - t_i}\right\|^2 dt < +\infty.$$

One can on the other hand obtain the definiteness of the integral $\sum_i h_i^\circ a_i^\circ$ on the exponential domain from the fact that for $t_i \leq t < t_{i+1}$,

$$\frac{\widetilde{u}(i)}{\sqrt{t_{i+1} - t_i}} H_t^\circ e(\widetilde{u}_i) = H_t^\circ D_t e(\widetilde{u}).$$

The equality (2.14) is also the reason for the surprising presence of a a° integral in the projection of an integral with respect to a^ε when ε is $+$ or $-$. We prove now that that parasite integral vanishes in the limit; yet, since we do not know if that integral is well-defined beyond the linear span of $\{X_A, A \in \mathcal{P}\}$, we have to prove that result on a subdomain of $\mathcal{E}(L^2(\mathbb{R}_+))$ such that its projections $\mathbb{E}_{\mathcal{S}}$ belong to that linear span. The set of exponential vectors of square-integrable functions with compact support is such a subdomain.

Lemma 2.4 *Let $H = \int_0^\infty H_s\, da_s^\varepsilon$ be an integral that satisfies the assumptions* **(HD)** *with $\varepsilon = +$ or $-$. Then the parasite a° integral which arises in the projection $\mathbb{E}_\mathcal{S} H \mathbb{E}_\mathcal{S}$ vanishes, in the sense that the net*

$$\big(\mathbb{E}_\mathcal{S} \sum_{i\in\mathbb{N}} h_i^\circ a_i^\circ \mathbb{E}_\mathcal{S}\big)_{\mathcal{S}\in\mathfrak{P}}$$

tends to zero in the w^*-topology on the set of exponentials of functions with compact support.

Proof.
Take for example $\varepsilon = +$; then one can see from the discrete Hudson--Parthasarathy equation (Proposition 1.7) that

$$\begin{aligned}
\langle \mathcal{E}(u), \mathbb{E}_\mathcal{S} \sum_i h_i^\circ a_i^\circ \mathbb{E}_\mathcal{S} \mathcal{E}(v)\rangle &= \sum_i \overline{\widetilde{u}}(i)\widetilde{v}(i)\langle e(\widetilde{u}\mathbb{1}_{\neq i}), h_i^\circ e(\widetilde{v}\mathbb{1}_{\neq i})\rangle \\
&= \sum_i \frac{\overline{\widetilde{u}}(i)}{\sqrt{t_{i+1}-t_i}} \int_{t_i}^{t_{i+1}} \langle e(\widetilde{u}_i), H_t^+ P_t(e(\widetilde{v}_{i+1}) - e(\widetilde{v}_i))\rangle\, dt \\
&= \sum_i \int_{t_i}^{t_{i+1}} \langle D_t \mathbb{E}_\mathcal{S}\mathcal{E}(u), H_t^+ P_t(e(\widetilde{v}_{i+1}) - e(\widetilde{v}_i))\rangle.
\end{aligned}$$

Now, by the assumptions **(HD)** we know that

$$\int_0^\infty \big\|(H_t^-)^* D_t \mathbb{E}_\mathcal{S}\mathcal{E}(u)\big\|\, dt < +\infty$$

whereas $\|e(\widetilde{v}_{i+1}) - e(\widetilde{v}_i)\|$ is of order $\widetilde{v}(i)$, which is smaller than

$$\sqrt{\int_{t_i}^{t_{i+1}} \|v(t)\|^2 dt}$$

and converges to zero uniformly in i.

□

Proof of Proposition 2.3
Let us prove for example the case $\varepsilon = +$. Let us consider the action of h_i^+, h_i^-, h_i° on a vector X_A with $A < i$. One has by (1.7) and Lemma 2.1,

$$\begin{aligned}
h_i^+ X_A &= d_i \mathbb{E}_\mathcal{S} H X_A \\
&= \frac{1}{\sqrt{t_{i+1}-t_i}} \mathbb{E}_\mathcal{S} \int_{t_i}^{t_{i+1}} P_{t_i} D_t H X_A\, dt
\end{aligned}$$

and the Attal-Meyer equations yield $D_t H X_A = H_t^+ X_A$, and we obtain the expression of h_i^+. The value of h_i^- is easily computed:

$$\begin{aligned} h_i^- X_A &= p_i \mathbb{E}_{\mathcal{S}} H a_i^+ X_A \\ &= \mathbb{E}_{\mathcal{S}} P_{t_i} X_{A\cup\{i\}} \\ &= 0, \end{aligned}$$

because, again by straightforward application of the Attal-Meyer formula,

$$HX_{A\cup\{i\}} = \int_{t_i}^{t_{i+1}} H_s \frac{X_A}{\sqrt{t_{i+1}-t_i}}\, d\chi_s + \int_{t_i}^{t_{i+1}} H_s^+ X_A \frac{\chi_s - \chi_{t_i}}{\sqrt{t_{i+1}-t_i}}\, d\chi_s.$$

Finally,

$$\begin{aligned} h_i^\circ X_A &= d_i \mathbb{E}_{\mathcal{S}} H a_i^+ X_A - p_i \mathbb{E}_{\mathcal{S}} H X_A \\ &= \frac{1}{\sqrt{t_{i+1}-t_i}} \mathbb{E}_{\mathcal{S}} \int_{t_i}^{t_{i+1}} P_{t_i} D_t H X_{A\cup\{i\}}\, dt - \mathbb{E}_{\mathcal{S}} P_{t_i} H X_A. \end{aligned}$$

From the above computation,

$$P_{t_i} D_t H X_{A\cup\{i\}} = \frac{1}{\sqrt{t_{i+1}-t_i}} P_{t_i} H_t X_A + \frac{1}{\sqrt{t_{i+1}-t_i}} P_{t_i} H_t^+ (X_A(\chi_t - \chi_{t_i}));$$

now we have

$$P_{t_i} H_t X_A - P_{t_i} H X_A = 0$$

for $t \geq t_i$ and

$$H_t^+(X_A(\chi_t - \chi_{t_i})) = H_t^+(a_t^+ - a_{t_i}^+) X_A.$$

The proof is complete. The other two cases are treated exactly in the same way. The definiteness of the restricted integral on the exponential domain when $\varepsilon = \circ$ was proved in the discussion following the statement of Proposition 2.3.

□

Suppose now that we want to project an integral $H = \int H_s^\times \, da_s^\times$; it is possible to compute an integral representation for $\mathbb{E}_{\mathcal{S}} H \mathbb{E}_{\mathcal{S}}$, but the coefficients do not express simply in terms of $(H_t^\times)_t$. The same is true if we compute the integral representation of the process $(\mathbb{E}_{\mathcal{S}} H_{t_i} \mathbb{E}_{\mathcal{S}})_i$. The reason is the following: the representation of H as $\int_0^\infty H_s^\times \, ds$ is not unique; if we compute projections according to our schemes, be it projections of the process or projections of the operator only, we compute an unique representation, that is, we try to compute a representation with more information than the original one, so that one can not relate explicitly the coefficients of the projection to the process $(H_s^\times)_{s\in\mathbb{R}_+}$. This is why it will be more convenient to take a completely different approach to project integrals $\int_0^\infty H_s^\times \, da_s^\times$. We only state the following proposition, which shows an alternative way to project integrals with respect to time and is proven immediately by an Attal-Meyer argument:

Proposition 2.5 *Let $H = \int_0^{+\infty} H_s^\times da_s^\times$ be an integral in Φ which satisfies the assumptions* **(HD)**. *Then $\mathbb{E}_\mathcal{S} H \mathbb{E}_\mathcal{S}$ is equal on the exponential domain to the restricted integral $\sum_{i\geq 1}^R h_i'^\times a_i^\times$, where*

$$h_i'^\times = \mathbb{E}_\mathcal{S} \int_{t_{i-1}}^{t_i} H_t^\times dt \, \mathbb{E}_\mathcal{S}.$$

We should emphasize here the fact that the representation given above is not a contradiction to the formulas (1.8): it is just another consequence of the fact that in discrete-time also, the representation of *one* operator h as $h = \sum_{\varepsilon=+,\circ,-,\times} \sum_i h_i^\varepsilon a_i^\varepsilon$ is not unique.

3 Convergence of the Itô table

3.1 A proof of the Itô formula by approximation

In this subsection we want to prove that the Itô formula for continuous-time quantum stochastic integrals is a limit of the one for discrete-time integrals; to achieve this we actually reprove the quantum Itô formula for regular semi-martingales as defined in [At1], using nothing but our approximation scheme and the Itô formula on toy Fock space. Throughout this section we will make the following assumptions on the operator integrals

$$H = \int_0^\infty H_t^\varepsilon da_t^\varepsilon :$$

(HS)
1. the integrands H_t^ε are bounded operators such that $t \mapsto \|H_t^\varepsilon\|$ is:
   - square-integrable if $\varepsilon = +$ or $-$,
   - integrable if $\varepsilon = \times$,
   - essentially bounded if $\varepsilon = \circ$
2. H is a bounded operator on Φ.

Notice that these assumptions are those made for regular semimartingale processes as defined in [At1].

Let us fix the notations to be used in the rest of the paper: we will consider two integrals

$$H = \int_0^\infty H_s^\varepsilon \, da_s^\varepsilon \quad \text{et} \quad K = \int_0^\infty K_s^\eta \, da_s^\eta$$

which satisfy the assumptions **(HS)**; ε and η can take the values $+, -, \circ$ or$\times$. The projections $\mathbb{E}_{\mathcal{S}} H \mathbb{E}_{\mathcal{S}}$, $\mathbb{E}_{\mathcal{S}} K \mathbb{E}_{\mathcal{S}}$ will be denoted by h, k respectively. In the case where ε or η is different from $\times$, the processes $(h_i^\varepsilon)_{i\in\mathbb{N}}$, $(k_i^\eta)_{i\in\mathbb{N}}$ and $(h_i^\circ)_{i\in\mathbb{N}}$, $(k_i^\circ)_{i\in\mathbb{N}}$ are as defined in Proposition 2.3; we will discuss again later the case of $\varepsilon = \times$. If ε is $+$ or $-$ then we have seen that h is equal to

$$h = \sum_i h_i^\varepsilon a_i^\varepsilon + \sum_i h_i^\circ a_i^\circ;$$

we then denote by $\widetilde{h}^\circ$ the integral $\sum_i h_i^\circ a_i^\circ$ which we will usually call the "parasite" term. As before, h_j will denote the integral stopped at time j

$$h_j = \sum_{i<j} h_i^\varepsilon a_i^\varepsilon + \sum_{i<j} h_i^\circ a_i^\circ;$$

and $\widetilde{h}_j^\circ$ the integral

$$\widetilde{h}_j^\circ = \sum_{i<j} h_i^\circ a_i^\circ.$$

The proof will be done in four steps. In the first step, we will discuss the validity of obtained integrals. We show that they are valid in the restricted sense on the whole of Fock space, when $\varepsilon = +, \circ$ or $-$. This will allow us to apply the discrete Itô formula freely. To simplify further proofs, we give an alternative description of the projection where $\varepsilon = \times$ integrals. In the second step, we show that the unwanted a° integrals that appear when projecting integrals with respect to a^+ or a^- vanish, as well as the terms they create when two projections are composed. In the third step we prove that, asymptotically, one can compute the composition of two projections using the *continuous* Itô table. Last, we show that the remaining discrete-time integrals obtained after composition do converge to the continuous-time integrals we are looking for.

Validity of the discrete integrals

First let us assume that $\varepsilon \neq \times$. With such assumptions, it is straightforward from general stochastic integration theory on Φ that $H = \int_0^\infty H_t^\varepsilon da_t^\varepsilon$ is the strong limit of the $\int_0^T H_t^\varepsilon da_t^\varepsilon$ as T goes to infinity, with uniform norm estimates. As a consequence, H is the strong sum of all $\int_{t_i}^{t_{i+1}} H_t^\varepsilon da_t^\varepsilon$ and $\mathbb{E}_\mathcal{S} H \mathbb{E}_\mathcal{S}$ is the strong sum of all

$$\mathbb{E}_\mathcal{S} \int_{t_i}^{t_{i+1}} H_t^\varepsilon dt \, \mathbb{E}_\mathcal{S} = h_i^+ a_i^+ + h_i^- a_i^- + h_i^\circ a_i^\circ.$$

The following lemma, which will allow us to use full power of the Attal-Meyer formulation, proves that the obtained integral representations are valid in the restricted sense on the whole of $\mathrm{T}\Phi$.

Lemma 3.1 *Let ε equal $+, -$ or $\circ$ and the integral $\int_0^\infty H_s^\varepsilon da_s^\varepsilon$ satisfy the assumptions* **(HS)**. *Then the integral $\sum_i h_i^\varepsilon a_i^\varepsilon$ associated to $\mathbb{E}_\mathcal{S} H \mathbb{E}_\mathcal{S}$ has the whole of $\mathrm{T}\Phi$ for restricted domain.*

Proof.
By the Attal-Meyer characterization of restricted integrals it is enough to prove that for all f in Φ, the vector $\mathbb{E}_\mathcal{S} f$ of $\mathrm{T}\Phi$ is such that $\sum_i \|h_i d_i \mathbb{E}_\mathcal{S} f\|^2$ and $\sum_i \left\|h_i^+ p_i \mathbb{E}_\mathcal{S} f\right\|^2$, $\sum_i \left\|h_i^- d_i \mathbb{E}_\mathcal{S} f\right\|$ or $\sum_i \|h_i^\circ d_i \mathbb{E}_\mathcal{S} f\|^2$, depending on ε, are finite. This is obtained from the fact that $\|h_i\|$ and $\|h_i^\circ\|$ are uniformly bounded and that

$$\left\|h_i^\pm\right\|^2 \leq \int_{t_i}^{t_{i+1}} \|H_s\|^2 ds$$

is square-integrable.

□

Now let us consider the case when $\varepsilon = \times$. The representation of $\mathbb{E}_\mathcal{S} H \mathbb{E}_\mathcal{S}$ given by Proposition 2.5 would eventually lead us to compare $\int_{t_{i-1}}^{t_i}$ integrals to $\int_{t_i}^{t_{i+1}}$ integrals. To avoid this problem we give the following alternative description:

Proposition 3.2 *Let $H = \int_0^\infty H_s^\times ds$ be an operator satisfying* **(HS)**. *Then $\mathbb{E}_\mathcal{S} H \mathbb{E}_\mathcal{S}$ is the strong limit on $\mathrm{T}\Phi$ of the series $\sum_i h_i^\times a_i^\times$, where*

$$h_i^\times p_i = \mathbb{E}_\mathcal{S} \int_{t_i}^{t_{i+1}} P_{t_i} H_t^\times dt \, \mathbb{E}_\mathcal{S} p_i.$$

Proof.
The series $\sum_i h_i^\times a_i^\times$ has clearly $\mathrm{T}\Phi$ as restricted domain by the Attal-Meyer characterization. We are going to prove that, for all u in $L^2(\mathbb{R}_+)$,

$$(\sum_i h_i^\times a_i^\times - \sum_i h_i'^\times) \mathbb{E}_\mathcal{S} \mathcal{E}(u)$$

converges to zero in norm. Since the considered operators are norm-bounded, this will prove strong convergence of TΦ. The above quantity is

$$\sum_{i\geq 0} h_i^\times e(\widetilde{u}_i)e(\widetilde{u}_{[i}) - \sum_{i\geq 0} h'^\times_{i+1} e(\widetilde{u}_{i+1})e(\widetilde{u}_{[i+1})$$
$$= \sum_i \Big(\mathbb{E}_\mathcal{S}\int_{t_i}^{t_{i+1}} P_{t_i}H_t^\times e(\widetilde{u}_i)\,dt \otimes e(\widetilde{u}_{[i}) - \mathbb{E}_\mathcal{S}\int_{t_i}^{t_{i+1}} H_t^\times e(\widetilde{u}_{i+1})\,dt \otimes e(\widetilde{u}_{[i+1})\Big),$$

and replacing $e(\widetilde{u}_i)$, $e(\widetilde{u}_{i+1})$ in the two integrals by $P_t e(\widetilde{u})$ creates an error term which smaller than

$$\sum_i |\widetilde{u}(i)| \int_{t_i}^{t_{i+1}} \left\|H_s^\times\right\| ds\; \|e(\widetilde{u})\| + \sum_i |\widetilde{u}(i+1)| \int_{t_i}^{t_{i+1}} \left\|H_s^\times\right\| ds\; \|e(\widetilde{u})\|\,,$$

so that it converges to zero. A similar estimate holds for the substitution of $e(\widetilde{u}_{[i})$, $e(\widetilde{u}_{[i+1})$ by $(\Omega + \chi_{t_{i+1}} - \chi_t/\sqrt{t_{i+1}-t_i}\,\widetilde{u}(i)) \otimes e(\widetilde{u}_{[i+1})$. What we end up with, after these substitutions, is

$$\sum_i \mathbb{E}_\mathcal{S}\int_{t_i}^{t_{i+1}} (P_{t_i} - \mathrm{Id})H_t^\times P_t e(\widetilde{u})\,dt.$$

This can be rewritten as

$$\mathbb{E}_\mathcal{S}\int_0^\infty (P_{t_i} - \mathrm{Id})H_t^\times P_t e(\widetilde{u})\,dt$$

where the i is actually a $i(t)$. Now Lebesgue's theorem applies and shows that the above integral tends to zero with the mesh size of the partition.

□

Remark: notice that a projection $\mathbb{E}_\mathcal{S}\int H_s^\times da_s^\times \mathbb{E}_\mathcal{S}$ will, in our scheme, be composed only with bounded operators. Besides, we will from now on only be interested in weak convergences so that one can always consider adjoint relations. Because of that, we will systematically replace the projection $\mathbb{E}_\mathcal{S} H \mathbb{E}_\mathcal{S}$ of an integral $H = \int H_s^\times da_s^\times$ by the series described in Proposition 3.2.

The vanishing of parasite terms

In this paragraph we show that the unwanted a° from the projection, as well as the terms they induce after composition, vanish in the limit.

Proposition 3.3 *Let $\varepsilon, \eta \in \{+,-,\circ,\times\}$ and let H, K be two operator integrals satisfying the assumptions* **(HS)**. *Then the net*

$$\big(\mathbb{E}_{\mathcal{S}} H \mathbb{E}_{\mathcal{S}}\, \mathbb{E}_{\mathcal{S}} K \mathbb{E}_{\mathcal{S}} - \mathbb{E}_{\mathcal{S}} \langle e(\widetilde{u})\,, \sum_i h_i^\varepsilon a_i^\varepsilon \, \sum_i k_i^\eta a_i^\eta \, \mathbb{E}_{\mathcal{S}}\big)_{\mathcal{S}\in\mathfrak{P}}$$

tends to zero on the exponential domain in the w^*-topology.

Proof.
If both ε and η are $\circ$, there is nothing to do but recall that $\mathbb{E}_{\mathcal{S}} H \mathbb{E}_{\mathcal{S}}$ and $\mathbb{E}_{\mathcal{S}} K \mathbb{E}_{\mathcal{S}}$ converge strongly on Φ and are uniformly bounded in norm. If one of ε, η is $\times$, the corresponding projection can be immediately replaced by the integral with respect to $a^\times$ thanks to the emphasized consequence of Proposition 3.2.

To work out the other cases notice that

$$hk \;=\; ((h-\widetilde{h}^\circ)+\widetilde{h}^\circ)((k-\widetilde{k}^\circ)+\widetilde{k}^\circ) \text{ if } \varepsilon \text{ and } \eta \text{ are both } + \text{ or } -,$$

$$hk \;=\; ((h-\widetilde{h}^\circ)+\widetilde{h}^\circ)k \text{ if for example } \varepsilon \text{ only is } + \text{ or } -\,.$$

Besides, $h - \widetilde{h}^\circ = \sum_i h_i^\varepsilon a_i^\varepsilon$, so that one only has to show that $\widetilde{h}^\circ k$, hk° and $h^\circ \widetilde{k}^\circ$ tend to zero in our weak sense. Thanks to adjointness properties the proof reduces to proving that

$$\widetilde{h}^\circ k \text{ converges to zero for } \varepsilon = -,+ \text{ and any } \eta \tag{3.1}$$

and

$$\widetilde{h}^\circ \widetilde{k}^\circ \text{ converges to zero when } \varepsilon \text{ and } \eta \text{ are both } + \text{ or } - \tag{3.2}$$

where convergence is meant weakly on the exponential domain.

We first prove (3.1); for this let us state the simple estimate

$$\|h_i^\circ e(\widetilde{u}_i)\| \le \frac{1}{\sqrt{t_{i+1}-t_i}} \int_{t_i}^{t_{i+1}} \|H_t^\varepsilon\|\, dt \;\, \|e(\widetilde{u}_i)\|\,, \tag{3.3}$$

obtained by using the first remark after Proposition 2.3. This estimate implies

$$\left\|\widetilde{h}_i^\circ e(\widetilde{u}_i)\right\| \le \|u\| \sqrt{\int_0^\infty \|H_t^\varepsilon\|^2 dt} \quad \exp\|u\|^2/2 \tag{3.4}$$

Observe that

$$\langle e(\widetilde{u})\,, \widetilde{h}^\circ k e(\widetilde{v})\rangle = \langle \widetilde{h^\circ}^* e(\widetilde{u}), k e(\widetilde{v})\rangle$$

and that, since $k = \mathbb{E}_{\mathcal{S}} K \mathbb{E}_{\mathcal{S}}$ is bounded and $(\widetilde{h}^\circ)^* e(\widetilde{u})$ is uniformly bounded by (3.4), one can replace $e(\widetilde{v})$ by anything which tends to it in norm with the mesh size $|\mathcal{S}|$. One can approximate $K\mathcal{E}(v)$ by a linear combination of exponential vectors; let us suppose for simplicity that $K\mathcal{E}(v)$ is approximated by a single vector $\mathcal{E}(w)$. Then, since

$$\begin{aligned}\|ke(\widetilde{v})-e(\widetilde{w})\| &= \|\mathbb{E}_{\mathcal{S}}\mathcal{E}(w)-\mathbb{E}_{\mathcal{S}}K\mathbb{E}_{\mathcal{S}}\mathcal{E}(v)\|\\ &\leq \|\mathbb{E}_{\mathcal{S}}\mathcal{E}(w)-\mathbb{E}_{\mathcal{S}}K\mathcal{E}(v)\|+\|\mathbb{E}_{\mathcal{S}}K\mathcal{E}(v)-\mathbb{E}_{\mathcal{S}}K\mathbb{E}_{\mathcal{S}}\mathcal{E}(v)\|\\ &\leq \|K\mathcal{E}(v)-\mathcal{E}(w)\|+\|K\|\,\|\mathcal{E}(v)-\mathbb{E}_{\mathcal{S}}\mathcal{E}(v)\|\,,\end{aligned}$$

one can replace $ke(\widetilde{v})$ by $e(\widetilde{w})$. Now our assumption reduces to showing that $\langle e(\widetilde{u})\,,\widetilde{h}^{\circ}e(\widetilde{w})\rangle$ tends to zero with $|\mathcal{S}|$. It is equal to

$$\begin{aligned}&\sum_i \overline{\widetilde{u}(i)}\widetilde{v}(i)\langle e(\widetilde{u}_i)\,,h_i^{\circ}e(\widetilde{w}_i)\rangle\langle e(\widetilde{u}_{[i+1}),e(\widetilde{v}_{[i+1})\rangle\\ &= \sum_i \frac{\overline{\widetilde{u}(i)}\widetilde{v}(i)}{t_{i+1}-t_i}\int_{t_i}^{t_{i+1}}\langle e(\widetilde{u}_i)\,,P_{t_i}(a_t^- - a_{t_i}^-)H_t^- e(\widetilde{w}_i)\rangle dt\,\langle e(\widetilde{u}_{[i+1}),e(\widetilde{v}_{[i+1})\rangle\end{aligned}$$

if for example $\varepsilon=-$ (the case $\varepsilon=+$ is proved by dual computations).By an estimate similar to (3.4) we have

$$\left|\langle e(\widetilde{u})\,,\widetilde{h}^{\circ}e(\widetilde{w})\rangle\right| \leq \|u\|\,\|w\|\exp\|u\|\exp\|w\|\sqrt{\sup_i\int_{t_i}^{t_{i+1}}\left\|H_t^-\right\|^2 dt}$$

and the last term converges to zero as the mesh size of the partition goes to zero.

To prove (3.2) let us write

$$\widetilde{h}^{\circ}\widetilde{k}^{\circ}=\sum_i h_i^{\circ}\widetilde{k}_i^{\circ}a_i^{\circ}+\sum_i \widetilde{h}_i^{\circ}k_i^{\circ}a_i^{\circ}+\sum_i h_i^{\circ}k_i^{\circ}a_i^{\circ}$$

using the discrete time Itô formula. That implies

$$\langle e(\widetilde{u})\,,\widetilde{h}^{\circ}\widetilde{k}^{\circ}e(\widetilde{v})\rangle=\sum_i \overline{\widetilde{u}}(i)\widetilde{v}(i)\langle e(\widetilde{u}_i)\,,(h_i^{\circ}\widetilde{k}_i^{\circ}+\widetilde{h}_i^{\circ}k_i^{\circ}+h_i^{\circ}k_i^{\circ})e(\widetilde{v}_i)\rangle$$

up to uniformly bounded factors $\langle e(\widetilde{u}_{[i+1}),e(\widetilde{v}_{[i+1})\rangle$. For the sake of lisibility, we will forget them and all their avatars from now on. Using estimates as (3.3) and the fact that $\widetilde{h}_i^{\circ}$, $\widetilde{k}_i^{\circ}$ are bounded with norms $\leq\|H\|\,,\ \|K\|$, one has a majoration of $\left|\langle e(\widetilde{u})\,,\widetilde{h}^{\circ}\widetilde{k}^{\circ}e(\widetilde{v})\rangle\right|$ by three terms of the kind

$$\sum_i |\widetilde{u}(i)\widetilde{v}(i)|\sqrt{\int_{t_i}^{t_{i+1}}\|K_t\|^2 dt}$$

and since the series $\sum\widetilde{u}(i)\widetilde{v}(i)$ is convergent, this tends to zero with the mesh size of the partition.

□

We now move on to the third step of our proof, contained in the following proposition.

Proposition 3.4 *With the assumptions of Proposition 3.3, the net*

$$\Big(\mathbb{E}_{\mathcal{S}}H\mathbb{E}_{\mathcal{S}}\,\mathbb{E}_{\mathcal{S}}K\mathbb{E}_{\mathcal{S}} - \mathbb{E}_{\mathcal{S}}\big(\sum_i h_i^\varepsilon k_i a_i^\varepsilon + \sum_i h_i k_i^\eta a_i^\eta + \sum_i h_i^\varepsilon k_i^\eta a_i^{\varepsilon.\eta}\big)\mathbb{E}_{\mathcal{S}}\Big)_{\mathcal{S}\in\mathfrak{P}}$$

where $\varepsilon.\eta$ is computed using formally the continuous *Itô table, tends to zero on the exponential domain in the w*-topology.*

Proof.
All that is left to prove is that

$$\text{for } (\varepsilon,\eta)=(+,-),\ \text{one has } \sum_i h_i^+ k_i^- a_i^\circ \underset{|\mathcal{S}|\to 0}{\to} 0 \tag{3.5}$$

and

$$\text{for } (\varepsilon,\eta)=(-,+),\ \text{one has } \sum_i h_i^- k_i^+ a_i^\circ \underset{|\mathcal{S}|\to 0}{\to} 0. \tag{3.6}$$

plus the convergence to zero in all cases involving an integral with respect to $\times$. The proofs of (3.5), (3.6) are the same; let us prove for example (3.5):

$$|\langle e(\widetilde{u})\,,\sum_i h_i^- k_i^+ a_i^\circ e(\widetilde{v})\rangle| \leq \sum_i |\widetilde{u}(i)|\,|\widetilde{v}(i)|\,\left\|h_i^{-*}e(\widetilde{u}_i)\right\|\,\left\|k_i^+e(\widetilde{v}_i)\right\|$$

up to a constant factor, and since

$$\left\|h_i^{-*}e(\widetilde{u}_i)\right\| \leq \sqrt{\int_{t_i}^{t_{i+1}}\left\|H_t^{-*}\right\|^2 dt}\ \ \|e(\widetilde{u}_i)\|$$

with a similar estimate for $\left\|k_i^+e(\widetilde{v}_i)\right\|$,one concludes as before.

Now in the case where ε, for example, is $\times$, one writes the usual equalities

$$\begin{aligned}\langle e(\widetilde{u})\,,a^\eta(h^\times k^\eta)e(\widetilde{v})\rangle &= \sum_{i\in\mathbb{N}}\langle e(\widetilde{u}_i)\,,h_i^\times k_i^\eta e(\widetilde{v}_i)\rangle\langle e(\widetilde{u}_{[i]}),e(\widetilde{v}_{[i]})\rangle\\ &= \sum_{i\in\mathbb{N}}\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}}\langle e(\widetilde{u}_i)\,,H_t^\times K_s^\eta e(\widetilde{v}_i)\rangle\langle e(\widetilde{u}_{[i]}),e(\widetilde{v}_{[i]})\rangle\,dt\,ds\end{aligned}$$

(keep in mind that $\times.\eta=\eta$ in all cases), and this is dominated by the following quantities:

- $\sum_i \int_{t_i}^{t_{i+1}} \left\|H_t^\times\right\| dt \ \int_{t_i}^{t_{i+1}} \left\|K_t^\times\right\| dt$ if $\eta = \times$,
- $\sum_i \sqrt{t_{i+1}-t_i} \ \int_{t_i}^{t_{i+1}} \left\|H_t^\times\right\| dt \sqrt{\int_{t_i}^{t_{i+1}} \|K_t^\eta\|^2 dt}$ if $\eta = +, -$,
- $\sum_i (t_{i+1}-t_i) \ \int_{t_i}^{t_{i+1}} \left\|H_t^\times\right\| dt \ \|K^\circ\|_\infty$ if $\eta = \circ$

where all majorations are up to constant factors. In all three cases the summed term in the majorant is a summable one multiplied by a vanishing one.

□

We now apply these results to prove the final result:

Theorem 3.5 (Itô formula in continous time) *Let* $H = \int_0^\infty H_s^\varepsilon \, da_s^\varepsilon$ *and* $K = \int_0^\infty K_s^\eta \, da_s^\eta$ *be two continuous-time integrals satisfying the assumptions* **(HS)**. *Then the following equality holds on* Φ*:*

$$\int_0^\infty H_t^\varepsilon da_t^\varepsilon \int_0^\infty K_t^\eta da_t^\eta = \int_0^\infty H_t^\varepsilon K_t da_t^\varepsilon + \int_0^\infty H_t K_t^\eta da_t^\eta + \int_0^\infty H_t^\varepsilon K_t^\eta da_t^{\varepsilon.\eta}$$

where $a^{\varepsilon.\eta}$ *is computed using the* continuous *Itô table.*

Remark : let us repeat that this reproves the Itô formula on regular Fock space knowing nothing but its counterpart on the toy Fock space.

Proof.
We will prove that for any $u, v \in L^2(\mathbb{R}_+)$ one has

$$\begin{aligned}\langle \mathcal{E}(u)\,, \int_0^\infty H_t^\varepsilon da_t^\varepsilon \int_0^\infty K_t^\eta da_t^\eta \, \mathcal{E}(v)\rangle \quad &= \\ &\langle \mathcal{E}(u)\,, (\int_0^\infty H_t^\varepsilon K_t da_t^\varepsilon + \int_0^\infty H_t K_t^\eta da_t^\eta + \int_0^\infty H_t^\varepsilon K_t^\eta da_t^{\varepsilon.\eta})\, \mathcal{E}(v)\rangle.\end{aligned}$$

By Proposition 3.4, it suffices to show that

$$\begin{aligned}\langle e(\widetilde{u})\,, \sum_i h_i^\varepsilon k_i a_i^\varepsilon \, e(\widetilde{v})\rangle \quad &\to \quad \langle \mathcal{E}(u)\,, \int_0^\infty H_s^\varepsilon K_s \, da_s^\varepsilon \, \mathcal{E}(v)\rangle \\ \langle e(\widetilde{u})\,, \sum_i h_i k_i^\eta a_i^\eta \, e(\widetilde{v})\rangle \quad &\to \quad \langle \mathcal{E}(u)\,, \int_0^\infty H_s K_s^\eta \, da_s^\eta \, \mathcal{E}(v)\rangle \\ \langle e(\widetilde{u})\,, \sum_i h_i^\varepsilon k_i^\eta a_i^{\varepsilon.\eta} \, e(\widetilde{v})\rangle \quad &\to \quad \langle \mathcal{E}(u)\,, \int_0^\infty H^\varepsilon K^\eta \, da_s^{\varepsilon.\eta} \, \mathcal{E}(v)\rangle\end{aligned}$$

but one can see that the previous propositions apply to the integrals $\int H^\varepsilon_s K_s da^\varepsilon_s$, $\int H_s K^\eta_s da^\eta_s$, $\int H^\varepsilon_s K^\eta_s da^{\varepsilon.\eta}_s$ so that one also has that

$$\begin{aligned}
\langle e(\widetilde{u}) \, , \sum_i (H^\varepsilon K)^\varepsilon_i a^\varepsilon_i \, e(\widetilde{v})\rangle &\;\to\; \langle \mathcal{E}(u) \, , \int_0^\infty H^\varepsilon_s K_s da^\varepsilon_s \, \mathcal{E}(v)\rangle \\
\langle e(\widetilde{u}) \, , \sum_i (H K^\eta)^\eta_i a^\eta_i \, e(\widetilde{v})\rangle &\;\to\; \langle \mathcal{E}(u) \, , \int_0^\infty H_s K^\eta_s da^\eta_s \, \mathcal{E}(v)\rangle \\
\langle e(\widetilde{u}) \, , \sum_i (H^\varepsilon K^\eta)^{\varepsilon.\eta}_i \, e(\widetilde{v})\rangle &\;\to\; \langle \mathcal{E}(u) \, , \int_0^\infty H^\varepsilon_s K^\eta_s da^{\varepsilon.\eta}_s \, \mathcal{E}(v)\rangle
\end{aligned}$$

where $(HK^\eta)^\eta_i$, $(H^\varepsilon K)^\varepsilon_i$ are the integrands associated by Proposition 2.3 (or by Proposition 3.2) to the integral $\int H_s K^\eta_s da^\eta_s$, etc. It suffices then to prove that

$$\langle e(\widetilde{u}) \, , \sum_i (h^\varepsilon_i k_i - (H^\varepsilon K)^\varepsilon_i) a^\varepsilon_i \, e(\widetilde{v})\rangle \;\to\; 0 \tag{3.7}$$

$$\langle e(\widetilde{u}) \, , \sum_i (h_i k^\eta_i - (H K^\eta)^\eta_i) a^\eta_i \, e(\widetilde{v})\rangle \;\to\; 0 \tag{3.8}$$

$$\langle e(\widetilde{u}) \, , \sum_i (h^\varepsilon_i k^\eta_i - (H^\varepsilon K^\eta)^{\varepsilon.\eta}_i) a^{\varepsilon.\eta}_i \, e(\widetilde{v})\rangle \;\to\; 0 \tag{3.9}$$

The convergences (3.7) and (3.8) derive one from another by adjointness. Let us prove the different cases one by one.

Proof of (3.7)
First consider (3.7) in the case $\varepsilon = -$ or $+$: let us take for example $\varepsilon = +$.

$$\langle e(\widetilde{u}) \, , a^\varepsilon (h^\varepsilon k - (H^\varepsilon K)^\varepsilon_i) e(\widetilde{v})\rangle = \sum_i \overline{\widetilde{u}(i)} \langle e(\widetilde{u}_i) \, , (h^+_i k_i - (H^+ K)^+_i) e(\widetilde{v}_i)\rangle .$$

The above quantities are equal to

$$\begin{aligned}
&\sum_i \overline{\widetilde{u}(i)} \langle e(\widetilde{u}_i) \, , \frac{1}{\sqrt{t_{i+1} - t_i}} \int_{t_i}^{t_{i+1}} P_{t_i} (H^+_t k_i - H^+_t K_t) e(\widetilde{v}_i) dt\rangle \\
= \;\; &\sum_i \frac{\overline{\widetilde{u}(i)}}{\sqrt{t_{i+1} - t_i}} \int_{t_i}^{t_{i+1}} \langle H^{+*}_t e(\widetilde{u}_i), (k_i - K_t) e(\widetilde{v}_i)\rangle dt
\end{aligned}$$

So the norm of the left-hand-side is smaller than

$$\begin{aligned}
&\sum_i \frac{|\widetilde{u}(i)|}{\sqrt{t_{i+1}-t_i}} \int_{t_i}^{t_{i+1}} \left\| H_t^{+*} \right\| \left\| (k_i - K_t) e(\widetilde{v}_i) \right\| dt \\
&\leq \sum_i |\widetilde{u}(i)| \sqrt{\int_{t_i}^{t_{i+1}} \| H_t \|^2 \| (k_i - K_t) e(\widetilde{v}_i) \|^2 dt} \\
&\leq \| \widetilde{u} \|_{l^2}^2 \sqrt{\int_0^\infty \left\| H_t^+ \right\|^2 \| (k_i - K_t) e(\widetilde{v}_i) \|^2 dt} \qquad (3.10)
\end{aligned}$$

by repeated use of the Cauchy-Schwarz formula and convenient erasing of constant terms. The index i in the last line is actually a $i(t)$.

But, since $k_i = \mathbb{E}_{\mathcal{S}} K_{t_i} \mathbb{E}_{\mathcal{S}}$,

$$\| (k_i - K_t) e(\widetilde{v}_i) \| \leq \| K_{t_i} e(\widetilde{v}_i) \| + \| K_t e(\widetilde{v}_i) \| .$$

If $\eta = +, \circ, -$, then (K_t) is an operator martingale, so, since $t_i \leq t$ with $e(\widetilde{v}_i) \in \Phi_{t_i}$,

$$\begin{aligned}
\| (k_i - K_t) e(\widetilde{v}_i) \| &\leq \| \mathbb{E}_{\mathcal{S}} K_{t_i} e(\widetilde{v}_i) \| + \| K_t e(\widetilde{v}_i) \| \\
&\leq 2 \| K e(\widetilde{v}_i) \| \\
&\leq 2 \| K \| \, \| e(\widetilde{v}) \| .
\end{aligned}$$

A majoration of the same kind is immediately obtained in the case $\eta = \times$ since $\| K_t \| \leq \int \| K_s^\times \| \, ds$. One can then apply Lebesgue's dominated convergence theorem to the integral in (3.10). Besides,

$$\| (k_i - K_t) e(\widetilde{v}_i) \| \leq \| (\mathbb{E}_{\mathcal{S}} K_{t_i} - K_{t_i}) e(\widetilde{v}) \| + \| (K_{t_i} - K_t) P_{t_i} \mathbb{E}_{\mathcal{S}} \mathcal{E}(v_{t_i}) \|$$

and both terms on the right-hand-side tend to zero a.e.

We now consider (3.7) in the case $\varepsilon = \circ$: consider the quantity

$$\sum_i \overline{\widetilde{u}(i)} \widetilde{v}(i) \langle e(\widetilde{u}_i) \, , (h_i^\circ k_i - (H^\circ K)_i^\circ) e(\widetilde{v}_i) \rangle$$

where we forget once again the last factor. It is equal to

$$\begin{aligned}
&\sum_i \frac{\overline{\widetilde{u}(i)} \widetilde{v}(i)}{t_{i+1}-t_i} \int_{t_i}^{t_{i+1}} \langle e(\widetilde{u}_i) \, , H_t^\circ (k_i - K_t) e(\widetilde{v}_i) \rangle dt \\
&= \int_0^\infty \frac{\overline{\widetilde{u}(i)} \widetilde{v}(i)}{t_{i+1}-t_i} \langle e(\widetilde{u}_i) \, , H_t^\circ (k_i - K_t) e(\widetilde{v}_i) \rangle dt.
\end{aligned}$$

The bracket is uniformly bounded, and $t \mapsto \frac{\widetilde{u}(i)}{\sqrt{t_{i+1}-t_i}}, \frac{\widetilde{v}(i)}{\sqrt{t_{i+1}-t_i}}$ where once again i is actually a $i(t)$, tend to u,v in $L^2(\mathbb{R}_+)$ by the martingale convergence theorem. One can therefore consider, instead of the above, the quantity

$$\int_0^\infty \overline{u(t)} v(t) \langle e(\widetilde{u}_i)\,, H_t^\circ (k_i - K_t) e(\widetilde{v}_i)\rangle dt$$

so that one can now apply Lebesgue's theorem in the same way as in the previous case.

Now turn to (3.7) in the case $\varepsilon = \times$: we have

$$\begin{aligned} &\sum_i \langle e(\widetilde{u}_i)\,, (h_i^\times k_i - (H^\times K)_i^\times) e(\widetilde{v}_i)\rangle \\ &= \sum_i \int_{t_{i-1}}^{t_i} \langle H_t^{\times *} e(\widetilde{u}_i), (\mathbb{E}_{\mathcal{S}} K_{t_i} - K_t) e(\widetilde{v}_i)\rangle dt, \end{aligned}$$

and we conclude as before.

Proof of (3.9)
Let us now prove (3.9). The following lemma will be most useful:

Lemma 3.6 *One has the following estimates:*

- *if* $\eta = +$, *then* $k_i^\eta a_i^\eta e(\widetilde{v}_{i+1}) = k_i^\eta e(\widetilde{v}\mathbb{1}_{\neq i}) X_i$, *and*

$$\|k_i^\eta a_i^\eta e(\widetilde{v}_{i+1})\| \leq \sqrt{\int_{t_i}^{t_{i+1}} \|K_t^\eta\|^2 dt}\ \ \|\mathcal{E}(v)\|\,.$$

- *if* $\eta = -$, *then* $k_i^\eta a_i^\eta e(\widetilde{v}_{i+1}) = \widetilde{v}(i)\, k_i^\eta e(\widetilde{v}\mathbb{1}_{\neq i})$ *and*

$$\|k_i^\eta a_i^\eta e(\widetilde{v}_{i+1})\| \leq \sqrt{\int_{t_i}^{t_{i+1}} \|K_t^\eta\|^2 dt}\ \ |\widetilde{v}(i)|\, \|\mathcal{E}(v)\|\,.$$

- *if* $\eta = \circ$, *then* $k_i^\eta a_i^\eta e(\widetilde{v}_{i+1}) = \widetilde{v}(i)\, k_i^\eta e(\widetilde{v}\mathbb{1}_{\neq i}) X_i$ *and*

$$\|k_i^\eta a_i^\eta e(\widetilde{v}_{i+1})\| \leq \sup \|K_s^\circ\|\ |\widetilde{v}(i)|\, \|\mathcal{E}(v)\|\,.$$

- *if* $\eta = \times$, *then* $k_i^\times a_i^\times e(\widetilde{v}_{i+1}) = \left(k_i^\times e(\widetilde{v}_i)\right)(\Omega + \widetilde{v}(i) X_i)$ *and*

$$\|k_i^\eta a_i^\eta e(\widetilde{v}_{i+1})\| \leq \int_{t_i}^{t_{i+1}} \left\|K_t^\times\right\| dt\ (1 + |\widetilde{u}(i)|).$$

First let us treat the case where one of ε or η is $\times$; we take for example $\varepsilon = \times$. In this case we have to show that

$$\langle e(\widetilde{u})\,,\sum_i h_i^\times k_i^\eta a_i^\eta e(\widetilde{v})\rangle$$

vanishes when the mesh size of the partition tends to zero. But that quantity is smaller in norm than

$$\sum_i \left\|(h_i^\times)^* e(\widetilde{u})\right\| \, \|k_i^\eta a_i^\eta e(\widetilde{v})\|$$

which in turn is smaller than a constant times

$$\sum_i \int_{t_i}^{t_{i+1}} \left\|H_t^{\times *}\right\| dt \; \|k_i^\eta a_i^\eta e(\widetilde{v}_{i+1})\| \,. \tag{3.11}$$

From Lemma 3.6, we obtain that whatever η, (3.11) is a sum of terms of the form $\int_{t_i}^{t_{i+1}}\left\|H_t^{\times *}\right\|$ times a term that vanishes uniformly in i with the mesh size of the partition.

There are four non trivial cases left: (ε,η) equal to $(-,+)$, $(\circ,\circ)$, $(\circ,+)$ and $(-,\circ)$. The last two cases have similar proofs; let us prove them first. We therefore consider (3.9) in the case $(\varepsilon,\eta) = (-,\circ)$ or $(\circ,+)$: take for example $(\varepsilon,\eta) = (\circ,+)$. What we want to prove is that

$$\langle e(\widetilde{u})\,,\sum_i (h_i^\circ k_i^- - (H^\circ H^+)_i^+) a_i^+ e(\widetilde{v})\rangle \underset{|\mathcal{S}|\to 0}{\longrightarrow} 0.$$

This quantity is equal to

$$\begin{aligned}
&\sum_i \overline{\widetilde{u}(i)} \langle e(\widetilde{u}_i)\,, (h_i^\circ k_i^+ - \frac{1}{\sqrt{t_{i+1}-t_i}} \int_{t_i}^{t_{i+1}} P_{t_i} H_t^\circ K_t^+ dt) e(\widetilde{v}_i)\rangle \\
&\quad = \sum_i \overline{\widetilde{u}(i)} \langle e(\widetilde{u}_i)\,, \frac{1}{\sqrt{t_{i+1}-t_i}} \int_{t_i}^{t_{i+1}} H_t^\circ (\frac{1}{\sqrt{t_{i+1}-t_i}}\, k_i^+ - K_t^+) dt \; e(\widetilde{v}_i)\rangle
\end{aligned}$$

hence the norm of the left-hand-side is, up to a factor term depending only on u, v, smaller than:

$$\begin{aligned}
&\sum_i \frac{|\widetilde{u}(i)|}{\sqrt{t_{i+1}-t_i}} \int_{t_i}^{t_{i+1}} \|H_t^\circ\| \, \|(\frac{1}{\sqrt{t_{i+1}-t_i}}\, k_i^+ - K_t^+) e(\widetilde{v}_i)\| dt \\
&\quad \le \; \sup \|H_t^\circ\| \sum_i |\widetilde{u}(i)| \sqrt{\int_{t_i}^{t_{i+1}} \|(\frac{1}{\sqrt{t_{i+1}-t_i}}\, k_i^+ - K_t^+) e(\widetilde{v}_i)\|^2 dt} \,.
\end{aligned}$$

Since $e(\widetilde{v}_{i+1}) - e(\widetilde{v}_i) = \widetilde{v}(i)e(\widetilde{v}_i)X_i$, substituting $e(\widetilde{v}_i)$ with $e(\widetilde{v}_{i+1})$ creates an error term which is smaller than

$$\Big(\sum_i \int_{t_i}^{t_{i+1}} \Big\| (\frac{1}{t_{i+1}-t_i} \int_{t_i}^{t_{i+1}} K_s^+ ds - K_t^+)\widetilde{v}(i)e(\widetilde{v}_i)X_i) \Big\|^2 dt\Big)^{1/2}$$
$$\leq \Big(2\sum_i \int_{t_i}^{t_{i+1}} (\frac{1}{(t_{i+1}-t_i)^2} \Big(\int_{t_i}^{t_{i+1}} \|K_s\|\, ds\Big)^2 + \|K_t\|^2)\, |\widetilde{v}(i)|\, dt\Big)^{1/2}$$

up to a constant factor; but that is smaller by the Cauchy-Schwarz inequality than

$$\Big(\sum_i \int_{t_i}^{t_{i+1}} (\frac{1}{t_{i+1}-t_i} \int_{t_i}^{t_{i+1}} \|K_s\|^2 ds + \|K_t\|^2)\, |\widetilde{v}(i)|^2\, dt\Big)^{1/2}$$
$$= \Big(\sum_i |\widetilde{v}(i)|^2 \,(\int_{t_i}^{t_{i+1}} \|K_s\|^2 ds + \int_{t_i}^{t_{i+1}} \|K_t\|^2 dt)\Big)^{1/2}$$

up to constant factors again. This tends to zero with the mesh size of the partition.

Using the adaptation of operators, one sees easily that, once $e(\widetilde{v}_{i+1})$ has been substituted with $e(\widetilde{v}_i)$, it can be in turn substituted with $e(\widetilde{v})$; the usual majorations allow one to substitute it then with $\mathcal{E}(v)$. The convergence to zero of

$$\sum_i \int_{t_i}^{t_{i+1}} \| \frac{1}{t_{i+1}-t_i} \int_{t_i}^{t_{i+1}} K_s^+ \mathcal{E}(v) ds - K_t^+ \mathcal{E}(v)\|^2 dt$$

is then a simple consequence of the L^2 martingale convergence theorem.

Now consider (3.9) in the case $(\varepsilon, \eta) = (-, +)$: what we must show vanishes is

$$\sum_i \langle e(\widetilde{u}_i)\,,(h_i^- k_i^+ - (H^- K^+)_i^\times) e(\widetilde{v}_i)\rangle.$$

We show that

$$\sum_i \langle e(\widetilde{u}_i)\,,\Big(h_i^- k_i^+ - \mathbb{E}_{\mathcal{S}} \int_{t_i}^{t_{i+1}} H_t^- K_t^+ dt\Big) e(\widetilde{v}_i)\rangle$$

vanishes. Its norm is easily shown to be smaller than

$$\sum_i \int_{t_i}^{t_{i+1}} \|H_t^-\|\, \|(\frac{1}{\sqrt{t_{i+1}-t_i}}\, k_i^+ - K_t^+) e(\widetilde{v}_i)\| dt$$
$$\leq \ (\int \|H^-\|^2)^{1/2} \sqrt{\sum_i \int_{t_i}^{t_{i+1}} \|(\frac{1}{\sqrt{t_{i+1}-t_i}}\, k_i^+ - K_t^+) e(\widetilde{v}_i)\|^2 dt}$$

and one concludes using Proposition 3.2.

Last, consider (3.9) in the case $(\varepsilon, \eta) = (\circ, \circ)$:we prove that

$$\langle e(\widetilde{u})\,, \sum_i (h_i^\circ k_i^\circ - (H^\circ K^\circ)_i^\circ) a_i^\circ e(\widetilde{v})\rangle \underset{|\mathcal{S}|\to 0}{\longrightarrow} 0.$$

This is equal to

$$\sum_i \overline{\widetilde{u}(i)}\widetilde{v}(i)\langle e(\widetilde{u}_i)\,, (h_i^\circ k_i^\circ - (H^\circ K^\circ)_i^\circ) e(\widetilde{v}_i)\rangle$$

up to the usual last factor in the sum. The above line is equal to:

$$\begin{aligned} &\sum_i \overline{\widetilde{u}(i)}\widetilde{v}(i)\langle e(\widetilde{u}_i)\,, \frac{1}{t_{i+1}-t_i}\int_{t_i}^{t_{i+1}} (H_t^\circ k_i^\circ - H_t^\circ K_t^\circ) e(\widetilde{v}_i) dt\rangle \\ =\ &\int_0^\infty \frac{\overline{\widetilde{u}(i)}\widetilde{v}(i)}{t_{i+1}-t_i}\langle H_t^{\circ *} e(\widetilde{u}_i), \frac{1}{t_{i+1}-t_i}\int_{t_i}^{t_{i+1}} (K_s^\circ - K_t^\circ) e(\widetilde{v}_i) ds\rangle dt \end{aligned}$$

As in the proof of 3.1 we can replace $\frac{\overline{\widetilde{u}(i)}\widetilde{v}(i)}{t_{i+1}-t_i}$ by $\overline{u(t)}v(t)$. The norm of the integrated function is then smaller than

$$|u(t)v(t)|\ 2\sup_s \|H_s^{\circ *}\| \sup_s \|K_s^\circ\| \, \|\mathcal{E}(u)\| \, \|\mathcal{E}(v)\|$$

which is integrable. By Lebesgue's theorem, the considered quantity tends to zero with the mesh size of the partition.

□

3.2 A remark on the classical Itô formula

It is well known that the classical Itô formula for quantum stochastic integrals with respect to any normal martingale is a consequence of the quantum Itô formula. Indeed, any normal martingale, that is, any martingale M with square bracket $[M]_t = t$, can be identified with a multiplication operator on Fock space. That operator has a quantum stochastic integral representation (see [At2]), so that its angle bracket can be obtained from the Itô formula.

Therefore, we have proved that, once the normality of the martingale is known, the value of the angle bracket is deduced from the quantum stochastic integral representation of the multiplication operator and the commutation relations for Pauli matrices. There is nothing very deep here since the integral representation of the multiplication operator itself is derived from the structure equation of the martingale (see for example [At2]), but it may help and shed some light on the general computations we have made.

The Brownian motion $(B_t)_{t\geq 0}$ can be identified to the operator process $(a_t^+ + a_t^-)_{t\geq 0}$. If we consider a partition $\mathcal{S}$ with constant steps δ, then the approximation of the multiplication operator by B_t will be $\sum_{i|t_i\leq t}\sqrt{\delta}(a_i^+ + a_i^-)$ plus some terms which we have shown can be, in the limit when δ goes to zero, neglected. Besides, $a_i^+ + a_i^-$ is σ_x so that

$$\left(\sqrt{\delta}(a_i^+ + a_i^-)\right)^2 \delta\sigma_x^2 = \delta I.$$

The operator δI is the approximation of the deterministic process $(t)_{t\geq 0}$. This, as we have shown, implies that $d\langle B\rangle_t = dt$.

Another example is the compensated Poisson process $(N_t - t)_{t\geq 0} = (X_t)_{t\geq 0}$. It can be identified with the operator process $(a_t^+ + a_t^- + a_t^\circ)_{t\geq 0}$. If we consider again a partition $\mathcal{S}$ with constant step δ, $a_t^+ + a_t^- + a_t^\circ$ is projected to $\sum_{i|t_i\leq t}(\sqrt{\delta}a_i^+ + \sqrt{\delta}a_i^- + a_i^\times)$ plus asymptotically negligible terms. Since $\sqrt{\delta}a_i^+ + \sqrt{\delta}a_i^- + a_i^\times = \sqrt{\delta}\sigma_x - \frac{1}{2}\sigma_z + \frac{1}{2}I$, we obtain

$$\begin{aligned}\left(\sqrt{\delta}a_i^+ + \sqrt{\delta}a_i^- + a_i^\times\right)^2 &= \delta\sigma_x^2 + \frac{1}{4}\sigma_z^2 + \frac{1}{4}I \\ &\quad - \frac{1}{2}\sqrt{\delta}(\sigma_x\sigma_z + \sigma_z\sigma_x) + \sqrt{\delta}\sigma_x - \frac{1}{2}\sigma_z \\ &= \left(\sqrt{\delta}a_i^+ + \sqrt{\delta}a_i^- + a_i^\times\right) + \delta I\end{aligned}$$

because $\sigma_x\sigma_z + \sigma_z\sigma_x = 0$ and $\sigma_x^2 = \sigma_z^2 = I$. That, as we have shown, implies that $d\langle X\rangle_t = X_t + t$.

Much more interesting remarks can be made on the relation between random walks and normal martingales from the viewpoint of our approximation scheme, in particular in higher dimensional cases. In the paper [A-P], in collaboration with Stéphane Attal, we show that limits of random walks can be completely determined by the order of magnitude, with respect to the time scale, of the coefficients of their (discrete-time) structure equations.

A Proof of Lemma 3.1

References

[At1] S.ATTAL: An algebra of non commutative bounded semimartingales, square and angle quantum brackets, *Journal of functional analysis 124* (1994), Academic Press, pp. 292-332.

[At2] S.ATTAL: Classical and quantum stochastic calculus, *Quantum probability and related topics X* (1998), World Scientific, pp.1-52.

[At3] S.Attal: Approximating the Fock space with the toy Fock space, *Sminaire de Probabilits XXXVI*, 477–491, Springer Verlag.

[A-L] S.Attal and J.M.Lindsay: Quantum stochastic calculus with maximal operator domain, *Ann. Probab.* 32 (2004), no. 1A, 488–529.

[A-M] S.Attal and P.A.Meyer: Interprétations probabilistes et extension des intégrales stochastiques non commutatives, *Séminaire de probabilités XXVII* (1993), Springer Verlag.

[A-P] S.Attal and Y.Pautrat: Some remarks on (n+1)-level atom chains and n-dimensional noises, to appear in *Annales de l'Institut Henri Poincaré*.

[B-L] V.P.Belavkin and J.M.Lindsay: The kernel of a Fock space operator II, *Quantum Probability and Related Topics VIII* (1993), World scientific.

[Gui] A.Guichardet: Symmetric Hilbert spaces and related topics, Lecture Notes in Mathematics 261 (1970), Springer Verlag.

[H-P] R.L.Hudson and K.R.Parthasarathy: Quantum Itô's formula and stochastic evolutions, *Communications in Mathematical Physics* 93 (1984), no. 3.

[Mey] P.A.Meyer: Quantum probability for probabilists, Lecture Notes in Mathematics 1538 (1993), Springer Verlag.

[Pa1] Y.Pautrat: Kernel and integral representations of operators on infinite dimensional toy Fock space, to appear in *Séminaire de Probabilités.*

[Pa2] Y.Pautrat: Des matrices de Pauli aux bruits quantiques, *Thèse de doctorat de l'Université Joseph Fourier*, available on `http://math-doc.ujf-grenoble.fr`.