A relativistic discrete spacetime formulation of 3+1 QED

Nathanaël Eon, Giuseppe Di Molfetta, Giusppe Magnifico, Pablo Arrighi

arXiv:2205.03148

Particle physics on a chessboard?

Nathanaël Eon, Giuseppe Di Molfetta, Giusppe Magnifico, Pablo Arrighi

arXiv:2205.03148

What we want.

Fundamentally discrete space and time

Quantum electrodynamics (QED) from first principles

... *(perspective)* standard model of particles on a chessboard

What we want. Why?

- Fundamentally discrete space and time
- Quantum electrodynamics (QED) from first principles
- ... *(perspective)* standard model of particles on a chessboard

Describe QED as a quantum circuit on a chessboard

- Explainability
- Efficient quantum simulation

What we want. Why? How?

- Fundamentally discrete space and time
- Quantum electrodynamics (QED) from first principles
- ... *(perspective)* standard model of particles on a chessboard

Describe QED as a quantum circuit on a chessboard

- Explainability
- Efficient quantum simulation

Three challenges:

- 'Relativistic' discretization ($\Delta_t = \Delta_x$)
- Multi-particle
- Three spatial dimensions

Specifications

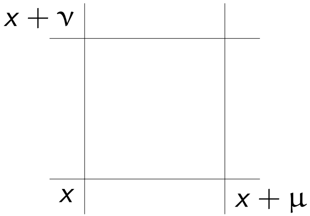
A symmetry: gauge invariance

Fundamental particles: specific commutation relations

Many particles

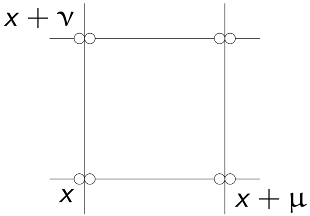
Electromagnetic contribution

Some conventions



Some conventions

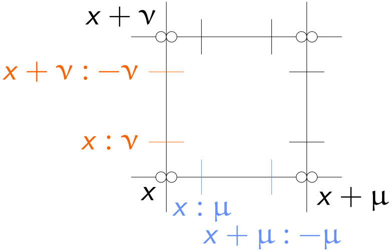
The matter particles, *fermions*, at x



Some conventions

The matter particles, *fermions*, at x

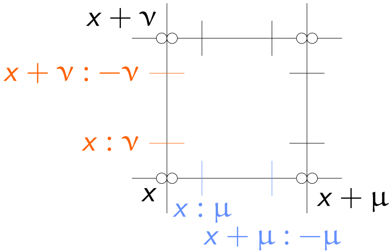
The interaction particles, *boson/gauge field*, at $x : \eta$

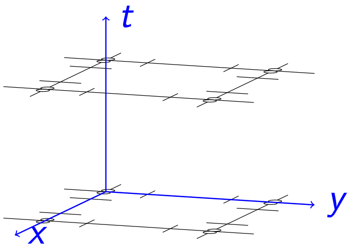


Some conventions

The matter particles, *fermions*, at x

The interaction particles, *boson/gauge field*, at $x : \eta$





Framework

Two fermions per site

Framework

Two fermions per site

Gauge invariance (of the evolution T):

$$T \circ g_\varphi = g_\varphi \circ T$$

Framework

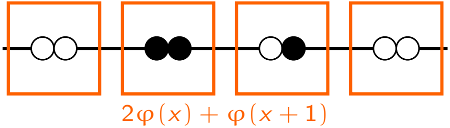
Two fermions per site

Gauge invariance (of the evolution T):

$$T \circ g_\varphi = g_\varphi \circ T$$

Gauge transformations:

$$g_{x,\varphi} : |l\rangle^x \mapsto e^{il\varphi(x)} |l\rangle^x$$
$$|l\rangle^y \mapsto |l\rangle^y \qquad \text{if } x \neq y.$$



Framework

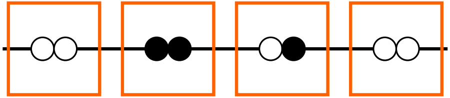
Two fermions per site

Gauge invariance (of the evolution T):

$$T \circ g_\varphi = g_\varphi \circ T$$

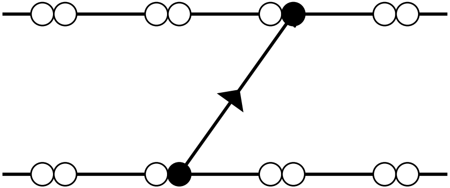
Gauge transformations:

$$\begin{aligned} g_{x,\varphi} : |l\rangle^x &\mapsto e^{il\varphi(x)}\,|l\rangle^x \\ |l\rangle^y &\mapsto |l\rangle^y \qquad \text{if } x \neq y. \end{aligned}$$



$2\varphi(x) + \varphi(x+1)$

Transport



Framework and non gauge invariance

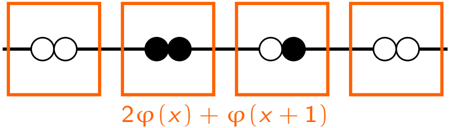
Two fermions per site

Gauge invariance (of the evolution T):

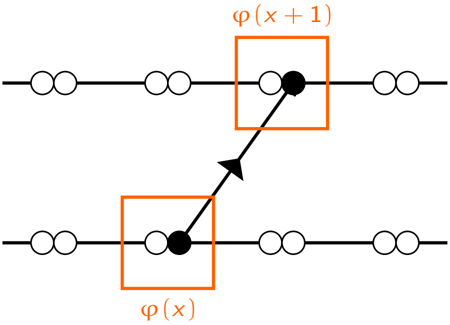
$$T \circ g_\varphi = g_\varphi \circ T$$

Gauge transformations:

$$g_{x,\varphi} : |l\rangle^x \mapsto e^{il\varphi(x)} |l\rangle^x$$
$$|l\rangle^y \mapsto |l\rangle^y \qquad \text{if } x \neq y.$$



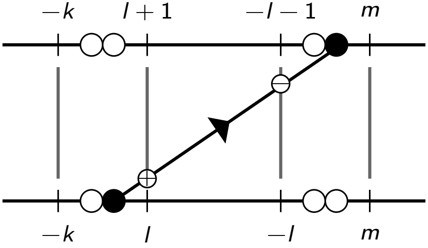
Transport



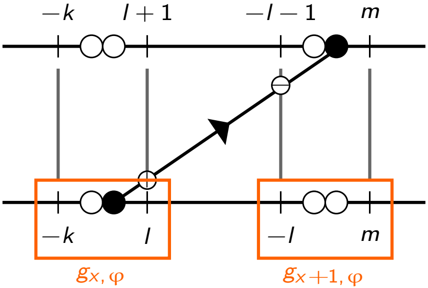
$$T \circ g_\varphi \longrightarrow \varphi(x)$$
$$\neq g_\varphi \circ T \longrightarrow \varphi(x+1)$$

Gauge invariance using a gauge field (bouncer)

Gauge invariance using a gauge field (bouncer)

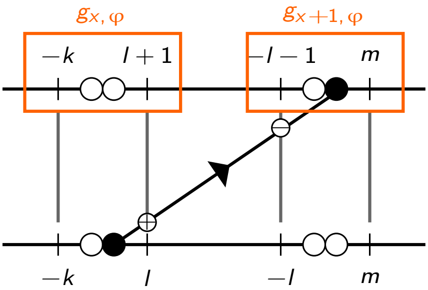


Gauge invariance using a gauge field (bouncer)



$$g_{\varphi} \longrightarrow (l - k + 1)\,\varphi(x) + (m - l)\,\varphi(x+1)$$

Gauge invariance using a gauge field (bouncer)



$g_\varphi \longrightarrow (l+1-k)\varphi(x) + (m-l-1+1)\varphi(x+1)$

$$=$$

$g_\varphi \longrightarrow (l-k+1)\varphi(x) + (m-l)\varphi(x+1)$

1. A required symmetry—Gauge invariance
2. Particle specifications—fermionic anti-commutation
3. Working with many particles—second quantization
4. Adding electromagnetic contribution—full QED
5. Conclusion and perspectives

Operators and their commutations

Specifications:

- $a^\dagger_{x,j}$: creates a fermion of type j at position x
- $V^\dagger_{x:\mu}$: raises the gauge field at position x in direction μ

Commutation relations

$$\{a_{x,j}, a^\dagger_{y,k}\} = \delta_{x,y}\delta_{j,k}$$
$$[V_{x:\mu}, V_{y:\nu}] = 0$$
$$[V_{x:\mu}, a_{y,j}] = 0$$

Operators and their commutations

Specifications:

- $a^\dagger_{x,j}$: creates a fermion of type j at position x
- $V^\dagger_{x:\mu}$: raises the gauge field at position x in direction μ

Commutation relations

$$\{a_{x,j}, a^\dagger_{y,k}\} = \delta_{x,y}\delta_{j,k}$$
$$[V_{x:\mu}, V_{y:\nu}] = 0$$
$$[V_{x:\mu}, a_{y,j}] = 0$$

Working with commuting qubits: how to implement the anti-commutation?
Answer: using the Jordan-Wigner transform

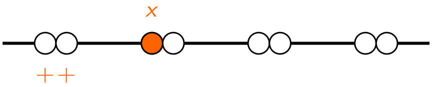
The Jordan-Wigner transform

$$a^\dagger_{x,j} = |1\rangle^{x,j}\langle 0| \prod_{y \prec (x,j)} Z_y$$

$|00 \;\; 00 \;\; 00 \;\; 00\rangle$

The Jordan-Wigner transform

$$a^\dagger_{x,j} = |1\rangle^{x,j} \langle 0| \prod_{y \prec (x,j)} Z_y$$



$|00\ \ 00\ \ 00\ \ 00\rangle$

$|00\ \ 10\ \ 00\ \ 00\rangle$

The Jordan-Wigner transform

$$a^\dagger_{x,j} = |1\rangle^{x,j}\langle 0| \prod_{y \prec (x,j)} Z_y$$

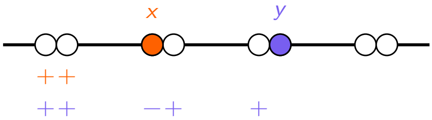
$|00\ \ 00\ \ 00\ \ 00\rangle$

$++$ $\qquad |00\ \ 10\ \ 00\ \ 00\rangle$

$++\quad -+\quad +\qquad - \, |00\ \ 10\ \ 01\ \ 00\rangle$

The Jordan-Wigner transform

$$a^\dagger_{x,j} = |1\rangle^{x,j}\langle 0| \prod_{y \prec (x,j)} Z_y$$



$|00\ \ 00\ \ 00\ \ 00\rangle$

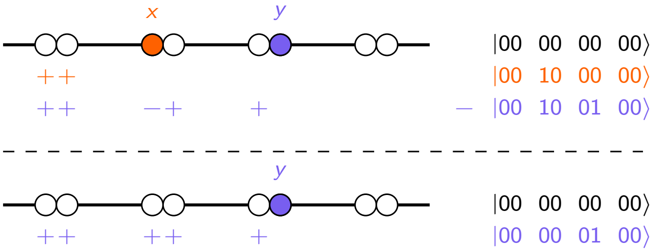
$|00\ \ 10\ \ 00\ \ 00\rangle$

$-\ |00\ \ 10\ \ 01\ \ 00\rangle$

$|00\ \ 00\ \ 00\ \ 00\rangle$

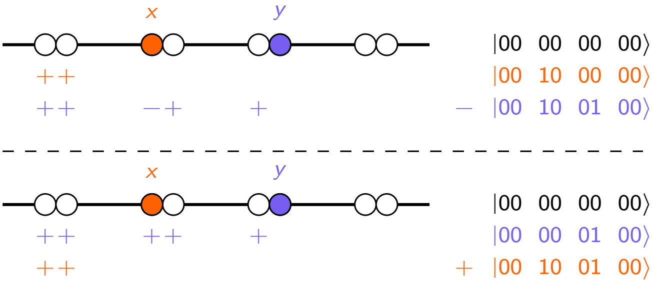
The Jordan-Wigner transform

$$a^{\dagger}_{x,j} = |1\rangle^{x,j}\langle 0| \prod_{y \prec (x,j)} Z_y$$



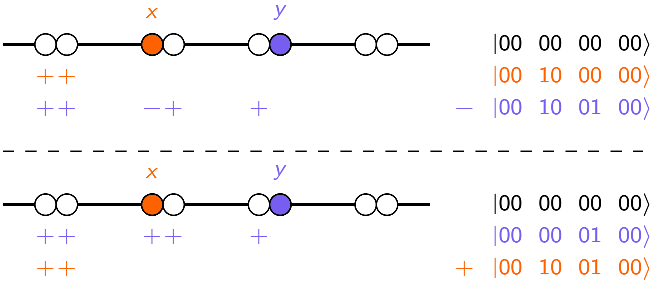
The Jordan-Wigner transform

$$a^{\dagger}_{x,j} = |1\rangle^{x,j}\langle 0| \prod_{y \prec (x,j)} Z_y$$



The Jordan-Wigner transform

$$a^\dagger_{x,j} = |1\rangle^{x,j} \langle 0| \prod_{y \prec (x,j)} Z_y$$



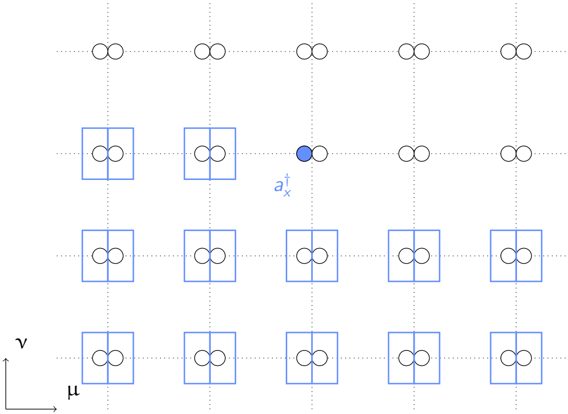
$$a^\dagger_y a^\dagger_x = -a^\dagger_x a^\dagger_y$$

The Jordan-Wigner transform

$$a^\dagger_{x,j} = |1\rangle^{x,j} \langle 0| \prod_{y \prec (x,j)} Z_y$$

In 2D: fix an order

2D



Locality of the Jordan-Wigner transform?

In short:

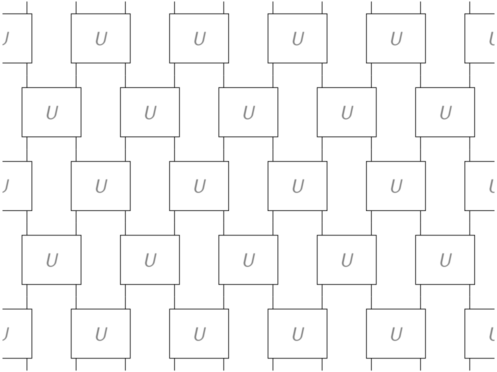
- 1D: Yes
- 2D (and higher): No.

Solution

- Use the gauge field

1. A required symmetry—Gauge invariance
2. Particle specifications—fermionic anti-commutation
3. **Working with many particles—second quantization**
4. Adding electromagnetic contribution—full QED
5. Conclusion and perspectives

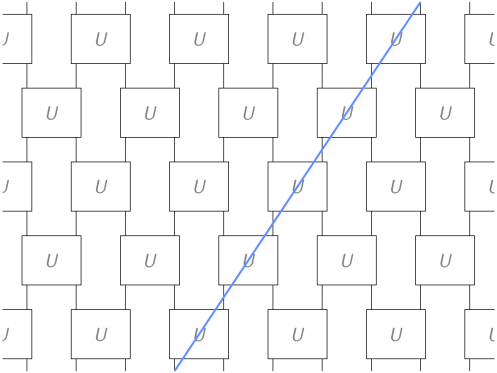
Single-particle QW



QW unitary:

$$U = \begin{pmatrix} U_{00} & -e^{i\theta} U_{10}^{*} \\ U_{10} & e^{i\theta} U_{00}^{*} \end{pmatrix} \begin{matrix} |01\rangle \\ |10\rangle \end{matrix}$$

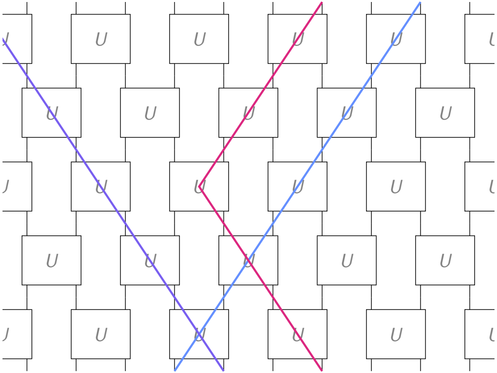
Single-particle QW



QW unitary:

$$U = \begin{pmatrix} U_{00} & -e^{i\theta} U_{10}^* \\ U_{10} & e^{i\theta} U_{00}^* \end{pmatrix} \begin{matrix} |01\rangle \\ |10\rangle \end{matrix}$$

Multi-particle QW



QW unitary:

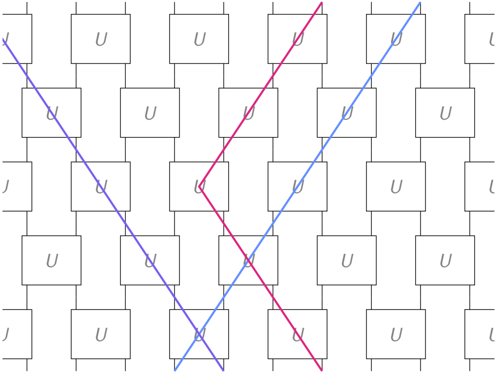
$$U = \begin{pmatrix} U_{00} & -e^{i\theta} U_{10}^* \\ U_{10} & e^{i\theta} U_{00}^* \end{pmatrix} \begin{matrix} |01\rangle \\ |10\rangle \end{matrix}$$

QCA unitary:

$$\mathbf{U} = 1 \oplus U \oplus e^{i\phi}$$

$$= \left(\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline 0 & U_{00} & -e^{i\theta} U_{10}^* & 0 \\ 0 & U_{10} & e^{i\theta} U_{00}^* & 0 \\ \hline 0 & 0 & 0 & e^{i\phi} \end{array}\right) \begin{matrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{matrix}$$

Multi-particle QW



QW unitary:

$$U = \begin{pmatrix} U_{00} & -e^{i\theta} U_{10}^* \\ U_{10} & e^{i\theta} U_{00}^* \end{pmatrix} \begin{matrix} |01\rangle \\ |10\rangle \end{matrix}$$

QCA unitary:

$$\mathbf{U} = 1 \oplus U \oplus e^{i\phi}$$

$$= \left(\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline 0 & U_{00} & -e^{i\theta} U_{10}^* & 0 \\ 0 & U_{10} & e^{i\theta} U_{00}^* & 0 \\ \hline 0 & 0 & 0 & e^{i\phi} \end{array}\right) \begin{matrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{matrix}$$

Who is ϕ?

Multi-particle QW: computing ϕ

$$\mathbf{U} = \left(\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline 0 & U_{00} & -e^{i\theta} U_{10}^* & 0 \\ 0 & U_{10} & e^{i\theta} U_{00}^* & 0 \\ \hline 0 & 0 & 0 & e^{i\phi} \end{array}\right)$$

$\phi \neq \theta \implies$ intrication. e.g. $U = Id$

$$\mathbf{U} = \left(\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & e^{i\phi} \end{array}\right) = C[P_\phi]$$

Multi-particle QW: computing ϕ

$$\mathbf{U} = \left(\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline 0 & U_{00} & -e^{i\theta} U_{10}^* & 0 \\ 0 & U_{10} & e^{i\theta} U_{00}^* & 0 \\ \hline 0 & 0 & 0 & e^{i\phi} \end{array}\right)$$

$\phi \neq \theta \Longrightarrow$ intrication. e.g. $U = Id$

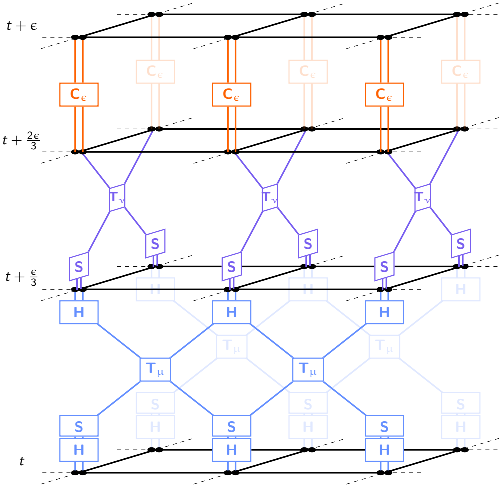
$$\mathbf{U} = \left(\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & e^{i\phi} \end{array}\right) = C[P_\phi]$$

No interaction $\Longrightarrow \phi = \theta$

$$\mathbf{U} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & U_{00} & U_{01} & 0 \\ 0 & U_{10} & U_{11} & 0 \\ 0 & 0 & 0 & U_{00} U_{11} - U_{01} U_{10} \end{pmatrix}$$

Free multi-particle "Dirac" QW

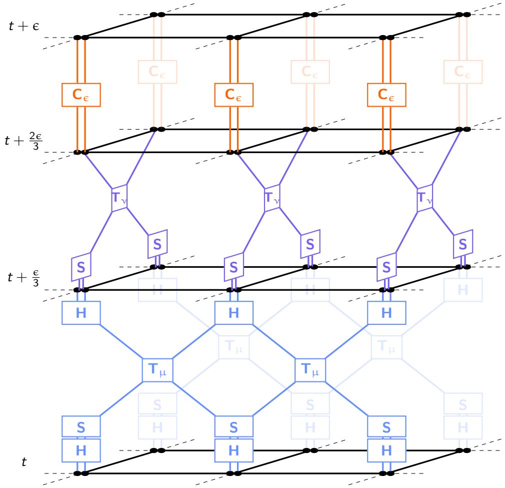
$$\mathbf{D_F} = \mathbf{C}_\epsilon \, [\mathbf{T}_\nu \mathbf{S}] \, [\mathbf{H}\mathbf{T}_\mu \mathbf{S}\mathbf{H}]$$



Free multi-particle "Dirac" QW

$$\mathbf{D_F} = \mathbf{C}_\epsilon \, [\mathbf{T}_\nu \mathbf{S}] \, [\mathbf{H}\mathbf{T}_\mu \mathbf{S}\mathbf{H}]$$

Gauge invariance insured by transport acting on gauge field.



Free multi-particle "Dirac" QW

$$\mathbf{D_F} = \mathbf{C}_\epsilon \, [\mathbf{T}_\nu \mathbf{S}] \, [\mathbf{H} \mathbf{T}_\mu \mathbf{S} \mathbf{H}]$$

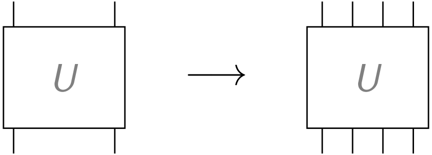
$$\mathrm{QW} = \left[\bigotimes_x C_\epsilon \right] \left[\bigotimes_{(x,1),(x+\nu,0)} T_\nu \bigotimes_x S \right] \left[\left(\bigotimes_x H_\mu \right) \left(\bigotimes_{(x,1),(x+\mu,0)} T_\mu \bigotimes_x S \right) \left(\bigotimes_x H_\mu^\dagger \right) \right]$$

Going three dimensional

2D Dirac Eq.: two amplitudes (spin up and down)

3D Dirac Eq.: four amplitudes

$2 \longrightarrow 4$ qubits:

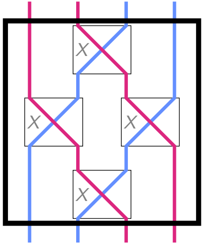


U is a 16×16 unitary. Divide up in 2-qubit gates.
For the swap:

$$S = (1 \oplus X \oplus 1)(X \oplus X)(1 \oplus X \oplus 1)$$

$$\longrightarrow \mathbf{S} = (I \otimes \mathbf{X} \otimes I)(\mathbf{X} \otimes \mathbf{X})(I \otimes \mathbf{X} \otimes I)$$

S

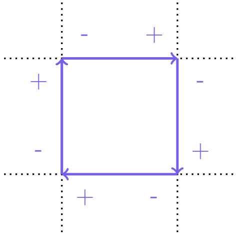


Electromagnetic terms

Magnetic term: gauge field dynamics

$$P\left|\widetilde{n}\right\rangle = \left|\widetilde{n+1}\right\rangle$$

$$\mathbf{D_M} = e^{i\pi k'(P+P^\dagger)}$$



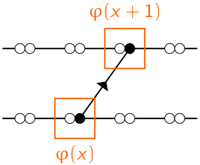
Electric term: interaction

$$E\left|l\right\rangle = l\left|l\right\rangle$$

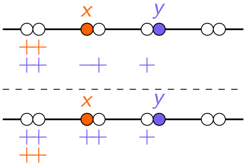
$$\mathbf{D_E} = e^{i\pi k E^2}$$

Conclusion

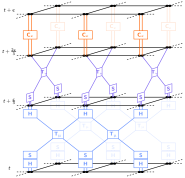
Gauge invariance



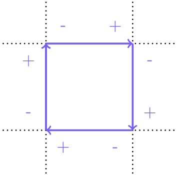
Jordan-Wigner



Multi-particle



Electromagnetic



Perspectives

Experiment

- Implementation on quantum devices / experiments

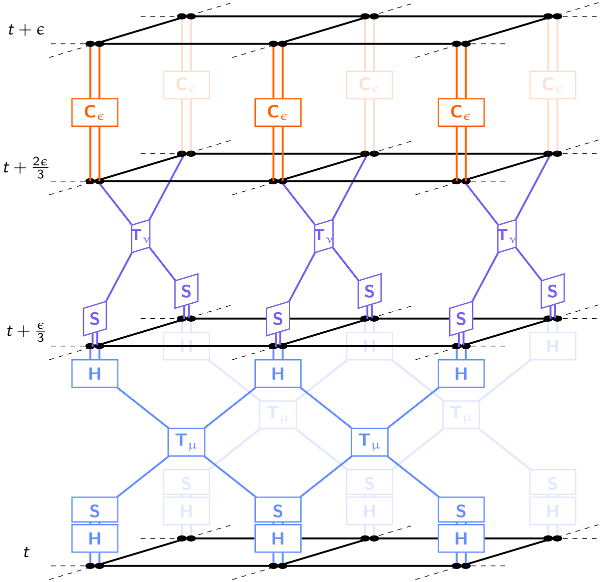
Extensions

- QCD (non-abelian gauge theory)
- Different types of lattice / graph

Theory

- Make the CA ‘plastic’ (proper continuous limit to QED)
- Link between fermions and interacting-hardcore-bosons ?

Thank you



$t+\epsilon$
$t+\frac{2\epsilon}{3}$
$t+\frac{\epsilon}{3}$
t
C_ϵ
T_ν
S
H
T_μ

Appendices

Locality

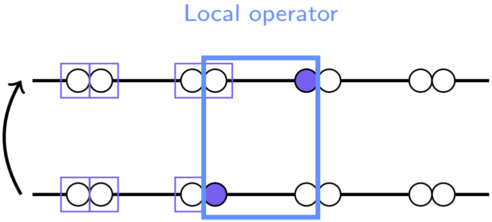
Is it local?

Is the map $a_x^\dagger \mapsto a_{x+1}^\dagger$ local ?

$$a_{x+1}^\dagger a_x = |1\rangle^{x+1} \langle 0| \prod_{x \prec y \prec x+1} Z_y |0\rangle^x \langle 1|$$

Answer

1D: yes



Locality

Is it local?

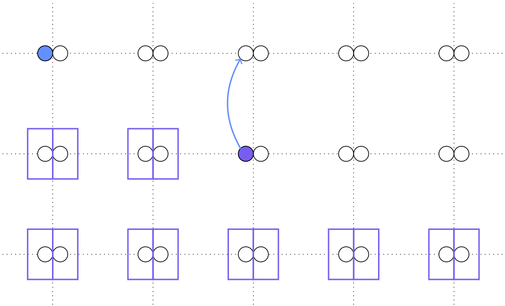
Is the map $a_x^\dagger \mapsto a_{x+1}^\dagger$ local ?

$$a_{x+1}^\dagger a_x = |1\rangle^{x+1} \langle 0| \prod_{x \prec y \prec x+1} Z_y \, |0\rangle^x \langle 1|$$

Answer

1D: yes

2D or higher: no



Locality

Is it local?

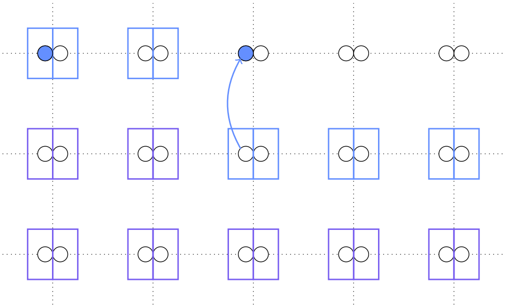
Is the map $a_x^\dagger \mapsto a_{x+1}^\dagger$ local ?

$$a_{x+1}^\dagger a_x = |1\rangle^{x+1} \langle 0| \prod_{x \prec y \prec x+1} Z_y \, |0\rangle^x \langle 1|$$

Answer

1D: yes

2D or higher: no



Locality

Is it local?

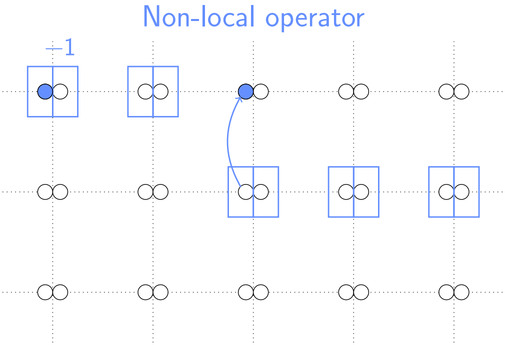
Is the map $a_x^\dagger \mapsto a_{x+1}^\dagger$ local ?

$$a_{x+1}^\dagger a_x = |1\rangle^{x+1} \langle 0| \prod_{x \prec y \prec x+1} Z_y \, |0\rangle^x \langle 1|$$

Answer

1D: yes

2D or higher: no



Non locality due to the JW order

Gauge field to the rescue

Operators

$$a^\dagger_{x,j} = |1\rangle^{x,j} \langle 0| \prod_{y \prec (x,j)} Z_y$$

Transport

$a^\dagger_{x+\eta,j}$ $\qquad a_{x,k}$

Gauge field to the rescue

Operators

$$a^\dagger_{x,j} = |1\rangle^{x,j}\langle 0| \prod_{y \prec (x,j)} Z_y$$

$$V_{x:\eta} = s_{x:\eta} s^\dagger_{x+\eta:-\eta}$$

Transport

$$a^\dagger_{x+\eta,j} V^\dagger_{x:\eta} a_{x,k} = a^\dagger_{x+\eta,j}(s_{x+\eta:-\eta} s^\dagger_{x:\eta}) a_{x,k}$$

Gauge field to the rescue

Parity of gauge field is considered as a fermion: included in the JW order

Operators

$$a^\dagger_{x,j} = |1\rangle^{x,j} \langle 0| \prod_{y \prec (x,j)} Z_y$$

$$V_{x:\eta} = s_{x:\eta} s^\dagger_{x+\eta:-\eta}$$

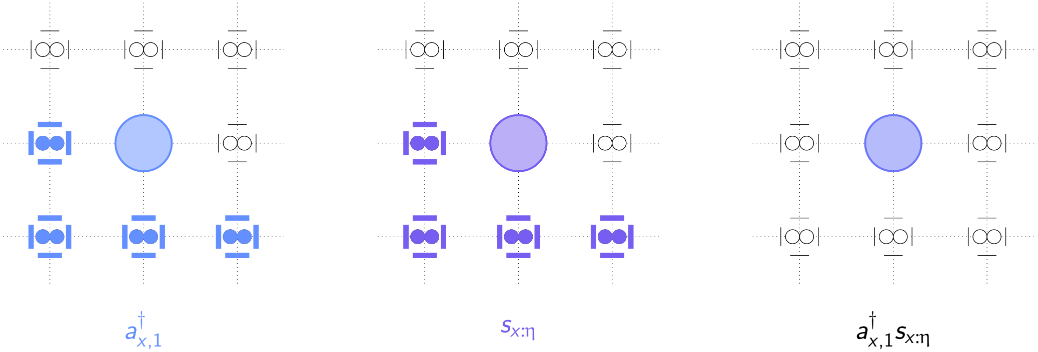
$$s_{x:\eta} = r_{x:\eta} \prod_{y \prec x:\eta} Z_y$$

$$Z |l\rangle = (-1)^l |l\rangle$$

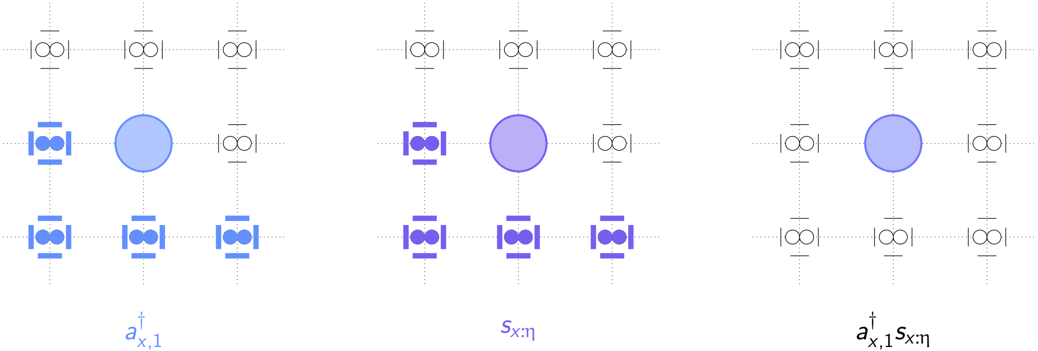
Transport

$$\begin{aligned} a^\dagger_{x+\eta,j} V^\dagger_{x:\eta} a_{x,k} &= a^\dagger_{x+\eta,j} (s_{x+\eta:-\eta} s^\dagger_{x:\eta}) a_{x,k} \\ &= (a^\dagger_{x+\eta,j} s_{x+\eta:-\eta})(s^\dagger_{x:\eta} a_{x,k}) \end{aligned}$$

Locality in higher dimensions!



Locality in higher dimensions!



$$a^\dagger_{x+\eta,j} V^\dagger_{x:\eta} a_{x,k} = (a^\dagger_{x+\eta,j} s_{x+\eta:-\eta})(s^\dagger_{x:\eta} a_{x,k})$$

Electric contribution (interaction)

Simplest contribution using the electric operator

$$E\,|l\rangle = l\,|l\rangle$$

that is **anisotropic**

$$E^2_{x:\mu} = E^2_{x+\mu:-\mu}$$

and **unitary**

$$\mathbf{D_E} = e^{\frac{i}{2}\epsilon^2 g_E^2 E^2}$$

Electric contribution (interaction)

Simplest contribution using the electric operator

$$E\,|l\rangle = l\,|l\rangle$$

that is **anisotropic**

$$E^2_{x:\mu} = E^2_{x+\mu:-\mu}$$

and **unitary**

$$\mathbf{D_E} = e^{\frac{i}{2}\epsilon^2 g_E^2 E^2}$$

Matches the Trotterization of the electric Hamiltonian:

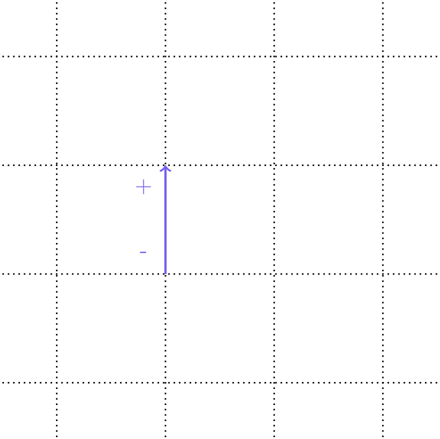
$$\mathcal{H}_E^{2d} = \frac{g_E^2}{2}\Delta_x \sum_x \left(E^2_{x:\mu} + E^2_{x:\nu}\right)$$

$$\prod_{x:\eta} \mathbf{D_E} = e^{i\Delta_t \mathcal{H}_E}$$

Magnetic contribution (gauge dynamics)

Gauge invariant operator on gauge field

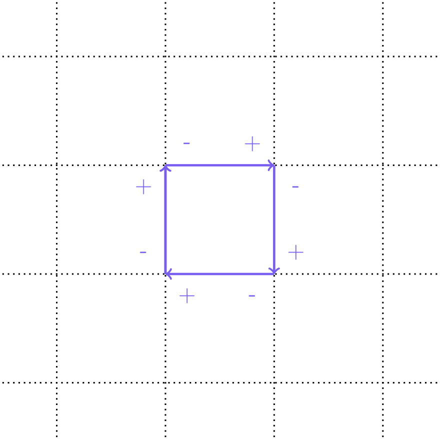
$$V^{\dagger}_{x:\nu}$$



Magnetic contribution (gauge dynamics)

Gauge invariant operator on gauge field

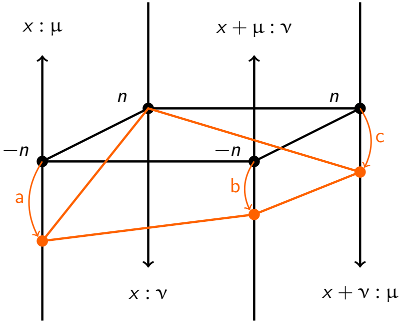
$$P_{x,\mu,\nu} = V^{\dagger}_{x+\mu:-\mu} V^{\dagger}_{x+\nu+\mu:-\nu} V^{\dagger}_{x+\nu:\mu} V^{\dagger}_{x:\nu}$$
$$= V_{x:\mu} V_{x+\mu:\nu} V^{\dagger}_{x+\nu:\mu} V^{\dagger}_{x:\nu}$$

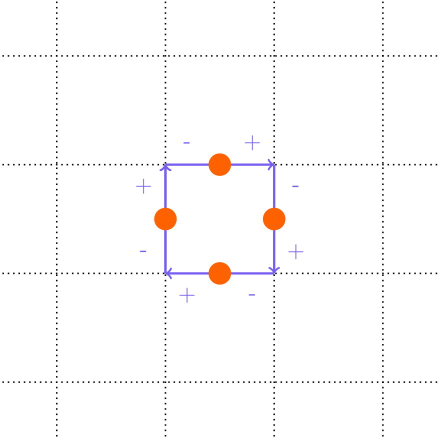


Magnetic contribution (gauge dynamics)

Reformulation of gauge state

$$\left|\widetilde{abcn}\right\rangle = V^a V^b V^{\dagger c} \left|-n, -n, n, n\right\rangle$$

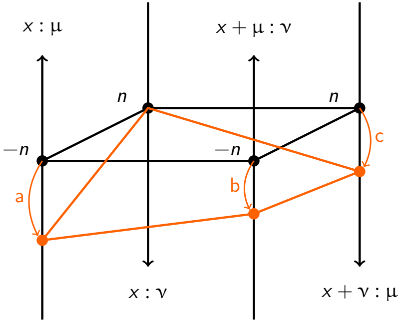




Magnetic contribution (gauge dynamics)

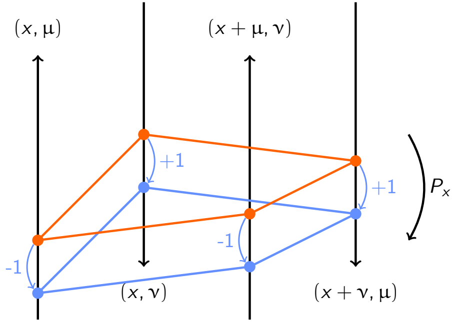
Reformulation of gauge state

$$\left|\widetilde{abcn}\right\rangle = V^a V^b V^{\dagger c} \left|-n, -n, n, n\right\rangle$$



Gauge invariant 'plaquette' operator

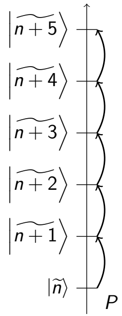
$$P\left|\widetilde{abcn}\right\rangle = \left|ab\widetilde{c, n+1}\right\rangle$$



Magnetic contribution (quantum walk)

$$P\left|\widetilde{n}\right\rangle = \left|\widetilde{n+1}\right\rangle$$

$$P^{\dagger}\left|\widetilde{n+1}\right\rangle = \left|\widetilde{n}\right\rangle$$



Magnetic contribution (quantum walk)

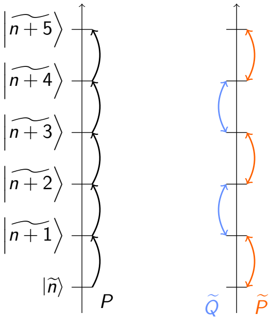
$$P\,|\widetilde{n}\rangle = \left|\widetilde{n+1}\right\rangle$$

$$P^\dagger \left|\widetilde{n+1}\right\rangle = |\widetilde{n}\rangle$$

$$\widetilde{P} = \sum_{n\in 2\mathbb{Z}} \left|\widetilde{n}\right\rangle\!\left\langle\widetilde{n+1}\right| + \left|\widetilde{n+1}\right\rangle\!\left\langle\widetilde{n}\right|$$

$$\widetilde{Q} = \sum_{n\in 2\mathbb{Z}+1} \left|\widetilde{n}\right\rangle\!\left\langle\widetilde{n+1}\right| + \left|\widetilde{n+1}\right\rangle\!\left\langle\widetilde{n}\right|$$

$$P + P^\dagger = \widetilde{P} + \widetilde{Q}$$



Magnetic contribution (quantum walk)

$$P\,|\widetilde{n}\rangle = \left|\widetilde{n+1}\right\rangle$$
$$P^\dagger \left|\widetilde{n+1}\right\rangle = |\widetilde{n}\rangle$$

$$\widetilde{P} = \sum_{n\in 2\mathbb{Z}} \left|\widetilde{n}\right\rangle\!\left\langle\widetilde{n+1}\right| + \left|\widetilde{n+1}\right\rangle\!\left\langle\widetilde{n}\right|$$
$$\widetilde{Q} = \sum_{n\in 2\mathbb{Z}+1} \left|\widetilde{n}\right\rangle\!\left\langle\widetilde{n+1}\right| + \left|\widetilde{n+1}\right\rangle\!\left\langle\widetilde{n}\right|$$

$$P + P^\dagger = \widetilde{P} + \widetilde{Q}$$

Taking the exponential

$$\mathbf{D}_{\widetilde{\mathbf{P}}} = exp\left(i\epsilon^2 \frac{g_m^2}{2}\widetilde{P}\right)$$

one obtains the dynamics (n even)

$$\mathbf{D}_{\widetilde{\mathbf{P}}}\,|\widetilde{n}\rangle = c\,|\widetilde{n}\rangle + is\left|\widetilde{n+1}\right\rangle$$
$$\mathbf{D}_{\widetilde{\mathbf{P}}}\left|\widetilde{n+1}\right\rangle = is\,|\widetilde{n}\rangle + c\left|\widetilde{n+1}\right\rangle$$

reminiscent of a ‘QW’

$$\begin{pmatrix} c & is \\ is & c \end{pmatrix}$$

Magnetic contribution (quantum walk)