

Converging outer approximations of classical network correlations

Victor Gitton

June 29th, 2022 - QIFQT Workshop, ENS Lyon

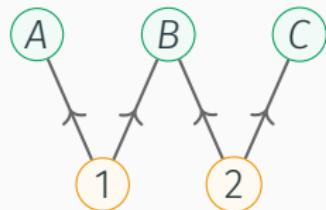
Quantum Information Theory group, Institute for Theoretical Physics, ETH Zürich
arXiv:2202.04103

Introduction

Causal compatibility

- **Causal structure** = observed and unobserved nodes + theory for unobserved nodes + connectivity

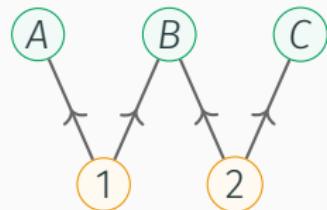
e.g., classical bilocal network



Causal compatibility

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e.g., classical bilocal network



- **Causal compatibility** = compatibility of a distribution with a given causal structure

Causal compatibility: applications

- Bell's theorem

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- non-locality without inputs

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- quantifying freedom of choice in Bell's theorem

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- multipartite quantum or post-quantum entanglement

Coiteux-Roy, Wolfe, Renou, arXiv:2105.09381

Causal compatibility: computational aspects

- Bell-like scenarios: **one** unobserved node \rightarrow convex problem \rightarrow easy 

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- More general networks: **several independent** unobserved nodes \rightarrow nonconvex \rightarrow hard 

Causal compatibility: approximations

- Inner approximations: based on human or computational exploration of the search space



(lower bounds for maximization problems,
certify feasibility)

set of distributions compatible with network

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- Outer approximations:
 - nontight: entropic or algebraic inequalities
 - powerful and asymptotically tight method: **inflation**



(upper bounds for maximization problems,
certify infeasibility)

Objectives of the talk

- Define causal compatibility formally
- Construct outer approximations that are apparently converging

Causal compatibility in classical
networks, a.k.a. network locality

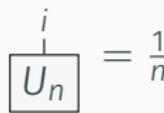
Tensor notation

	Standard	Tensor	Components
Outcome distribution	$p(\cdot, \cdot, \cdot)$		$p(a, b, c) =$ 

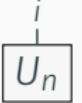
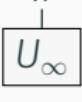
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Strategy with two inputs	$p_A(\cdot \cdot, \cdot)$		$p_A(a \alpha, \beta) =$ 

Tensor notation

	Standard	Tensor	Components
Outcome distribution	$p(\cdot, \cdot, \cdot)$	 A box labeled 'p' with three lines extending from the top.	$p(a, b, c) =$  A box labeled 'p' with three lines labeled 'a', 'b', and 'c' extending from the top.
Strategy with two inputs	$p_A(\cdot \cdot, \cdot)$	 A box labeled 'A' with two lines extending from the top.	$p_A(a \alpha, \beta) =$  A box labeled 'A' with two lines labeled 'a' and 'b' extending from the top.
Uniform distribution on...	$\dots \{1, \dots, n\}$	 A box labeled 'U_n' with one line extending from the top.	 A box labeled 'U_n' with one line labeled 'i' extending from the top, followed by an equals sign and the fraction $\frac{1}{n}$.

Tensor notation

	Standard	Tensor	Components
Outcome distribution	$p(\cdot, \cdot, \cdot)$		$p(a, b, c) =$ 
Strategy with two inputs	$p_A(\cdot \cdot, \cdot)$		$p_A(a \alpha, \beta) =$ 
Uniform distribution on...			
$\dots \{1, \dots, n\}$			 $= \frac{1}{n}$
$\dots [0, 1]$			 $= 1$

Tensor notation

Special case: **deterministic** strategies  such that

$$\begin{array}{c} a \\ \square A \\ \alpha \quad \beta \end{array} = \delta(a - f(\alpha, \beta))$$

for some function $f(\cdot, \cdot)$

Causal compatibility in classical networks

Define

$$\boxed{p} \in \mathcal{L} \left(\begin{array}{c} \textcircled{A} \\ \textcircled{B} \\ \textcircled{C} \\ \textcircled{1} \\ \textcircled{2} \end{array} \right)$$

iff there exist \boxed{A} , \boxed{B} , \boxed{C} such that

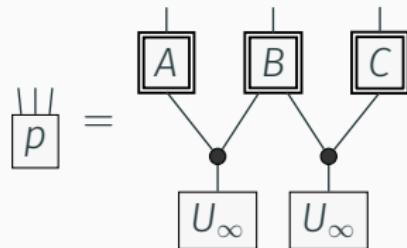
$$\boxed{p} = \boxed{A} \quad \boxed{B} \quad \boxed{C}$$
$$= \boxed{A} \quad \boxed{B} \quad \boxed{C}$$
$$\boxed{U_{\infty}} \quad \boxed{U_{\infty}}$$

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$$p \in \mathcal{L} \left(\begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \\ \text{1} \\ \text{2} \end{array} \right)$$

iff there exist A, B, C such that



i.e., for all a, b, c :

$$\begin{array}{c} a \ b \ c \\ p \end{array} = \int d\alpha d\beta \begin{array}{c} a \\ \text{---} \\ \text{---} \\ \alpha \end{array} \begin{array}{c} b \\ \text{---} \\ \text{---} \\ \alpha \ \beta \end{array} \begin{array}{c} c \\ \text{---} \\ \text{---} \\ \beta \end{array}$$

Causal compatibility in classical networks

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iff there exist A, B, C such that

$$p = \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \end{array} \quad \begin{array}{c} \text{U}_\infty \\ \text{U}_\infty \end{array} \quad (\text{w.l.o.g})$$

i.e., for all a, b, c :

$$\begin{array}{c} a \ b \ c \\ p \end{array} = \int d\alpha d\beta \begin{array}{c} a \\ \text{A} \\ \alpha \end{array} \begin{array}{c} b \\ \text{B} \\ \alpha \ \beta \end{array} \begin{array}{c} c \\ \text{C} \\ \beta \end{array}$$

Causal compatibility: identical strategies

Define

$$p \in \mathcal{L} \left(\begin{array}{c} A \\ \diagup \quad \diagdown \\ B \quad C \\ \diagup \quad \diagdown \\ 1 \quad 2 \end{array} \right)$$

iff there exist A , B such that

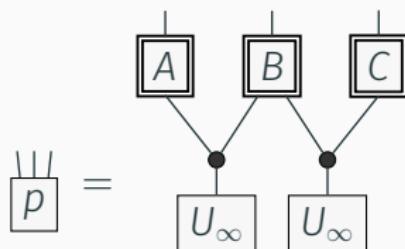
$$p = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ p \end{array} = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ A \\ \text{---} \\ \text{---} \\ \text{---} \\ B \\ \text{---} \\ \text{---} \\ \text{---} \\ A \\ \text{---} \\ \text{---} \\ \text{---} \\ U_\infty \\ \text{---} \\ \text{---} \\ \text{---} \\ U_\infty \end{array}$$

i.e., for all a, b, a' :  $= \int d\alpha d\beta$   

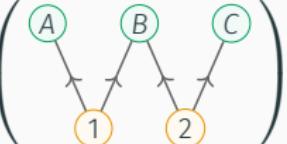
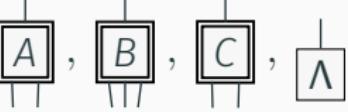
Constructing converging outer approximations

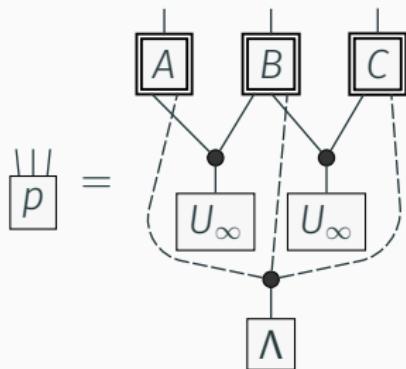
Step 1: Convexification

$\boxed{p} \in \mathcal{L} \left(\begin{array}{c} (A) \\ \swarrow \quad \searrow \\ (1) & (2) \\ \swarrow \quad \searrow \\ (B) & (C) \end{array} \right)$ iff there exist \boxed{A} , \boxed{B} , \boxed{C}
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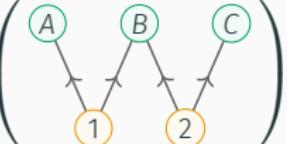


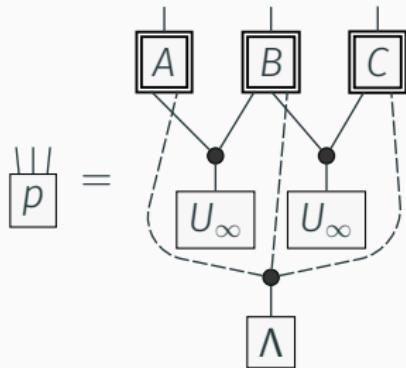
Step 1: Convexification

$p \in \mathcal{I}_{SR}$  iff there exist  such that



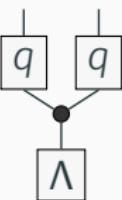
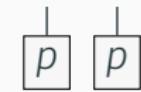
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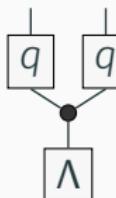
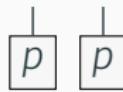
Convex but **too permissive**. Idea: forbid the agent to use Λ ?

Main lemma

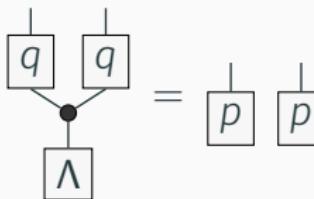
If  = 

The diagram shows a binary tree with a root node labeled Λ . The left child of the root is a node labeled q , and the right child is another node labeled q . Both the q nodes have vertical lines extending upwards from them. To the right of the tree is an equals sign, followed by a diagram showing two separate boxes, each labeled p , with vertical lines extending upwards from them.

Main lemma

If  =  then* $\forall \lambda : \boxed{q} = \boxed{p}$

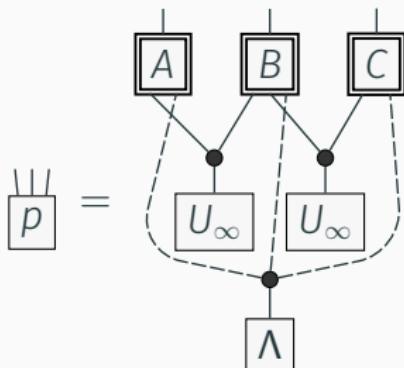
Main lemma

If  =  then* $\forall \lambda$:  = 

*Robust version: $\int d\lambda \left\| \begin{array}{c} \lambda \\ \Lambda \end{array} \right\|_2^2 \leq 3 \left\| \begin{array}{c} p \\ p \end{array} - \begin{array}{c} q \\ \lambda \end{array} \right\|_2^2 \leq 3 \left\| \begin{array}{c} p \\ p \end{array} - \begin{array}{c} q \\ q \\ \bullet \\ \Lambda \end{array} \right\|_1$

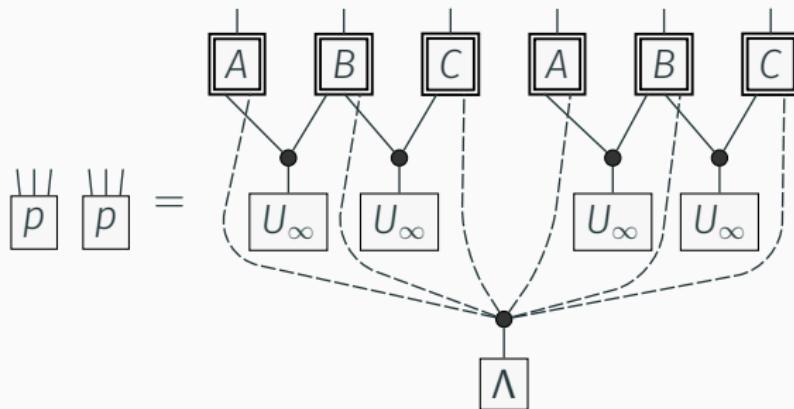
Step 1: Convexification

$p \in \mathcal{I}_{\text{SR}} \left(\begin{array}{c} (A) \\ \diagdown \quad \diagup \\ (1) \quad (2) \\ \diagup \quad \diagdown \\ (B) \quad (C) \end{array} \right)$ iff there exist $\boxed{A}, \boxed{B}, \boxed{C}, \boxed{\Lambda}$
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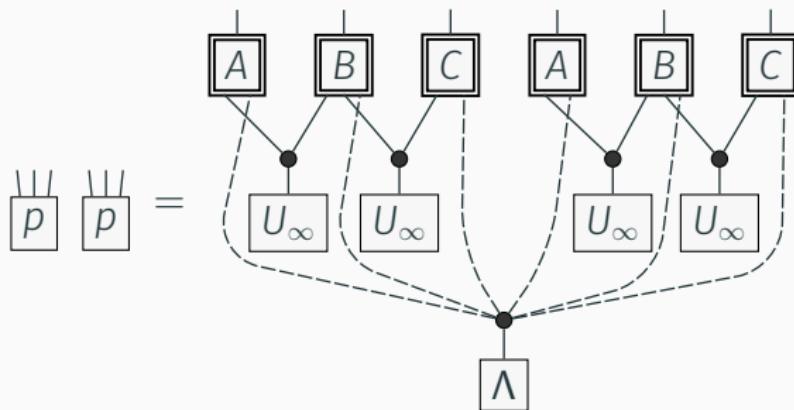
Step 1: Convexification

$p \in \mathcal{L}\left(\begin{array}{c} A \\ \diagdown \quad \diagup \\ 1 \quad 2 \\ \diagup \quad \diagdown \\ B \quad C \end{array}\right)$ iff there exist A, B, C, Λ
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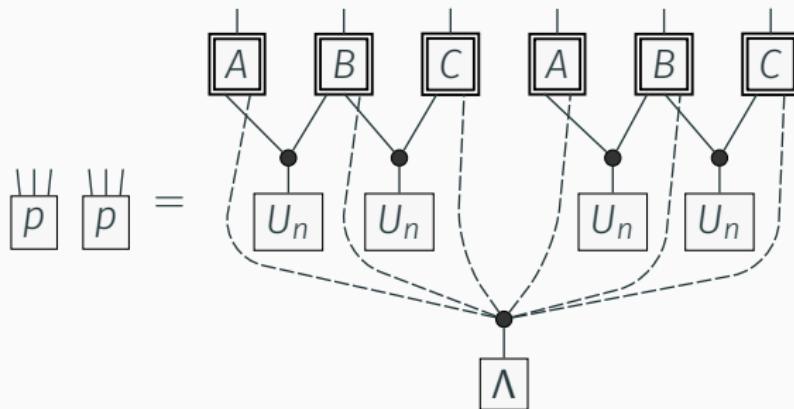
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Equivalent to the original problem thanks to the main lemma

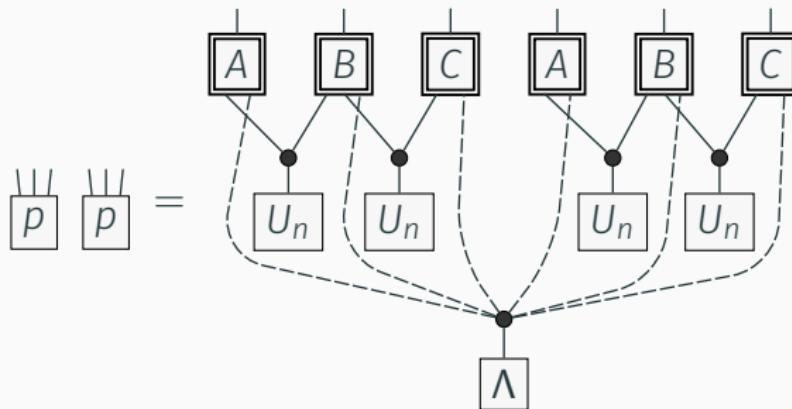
Step 2: Restricting the source output cardinality

$p \in \mathcal{I}_{\text{restr}}^{(n)} \left(\begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \end{array} \middle| \begin{array}{c} 1 \\ 2 \end{array} \right)$ iff there exist $\boxed{A}, \boxed{B}, \boxed{C}, \boxed{\Lambda}$
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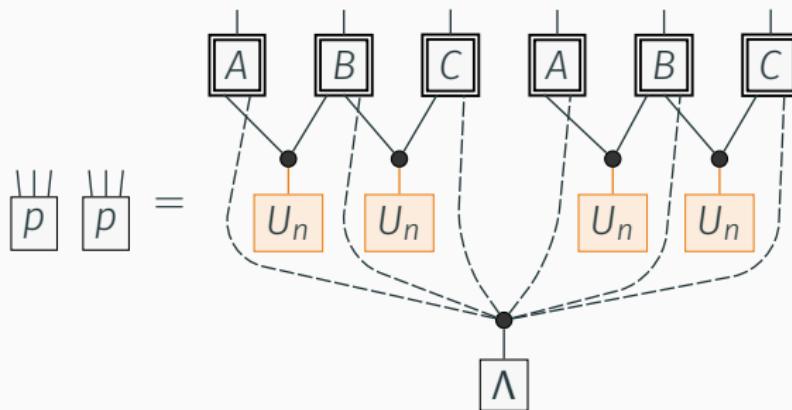
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Inner approximation $\mathcal{I}_{\text{restr}}^{(n)} \subseteq \mathcal{L}$, but also $\mathcal{I}_{\text{restr}}^{(n)} \xrightarrow{n \rightarrow \infty} \mathcal{L}$

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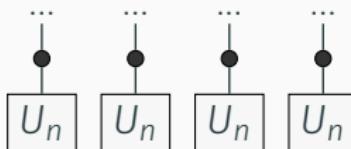
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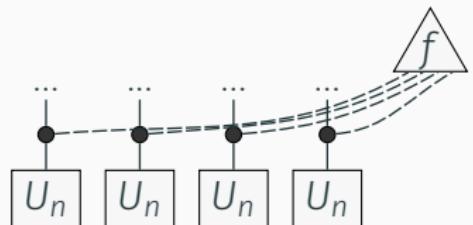
Want outer approximations \rightarrow let the agents use Λ a little bit?

Postselection for additional correlations

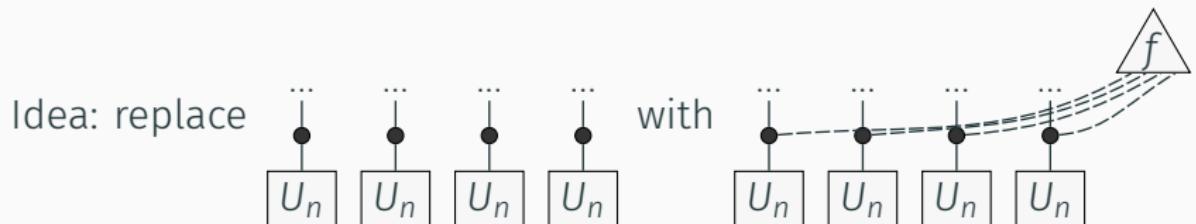
Idea: replace



with



Postselection for additional correlations



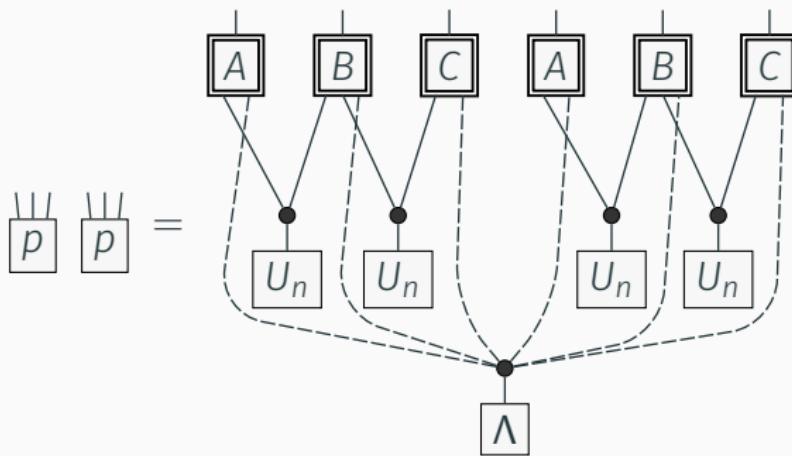
For instance, if $n \geq 4$,

The diagram shows a sequence of four boxes, each labeled U_n , with four indices i_1, i_2, i_3, i_4 above them. A dashed line connects the dots above the boxes. Above the dashed line, a triangle contains the text $\neq 4$. To the right of the dashed line, an equals sign is followed by a piecewise function:

$$= \begin{cases} \frac{1}{n(n-1)(n-2)(n-3)} & \text{if } i_x \neq i_y \\ 0 & \text{else} \end{cases}$$

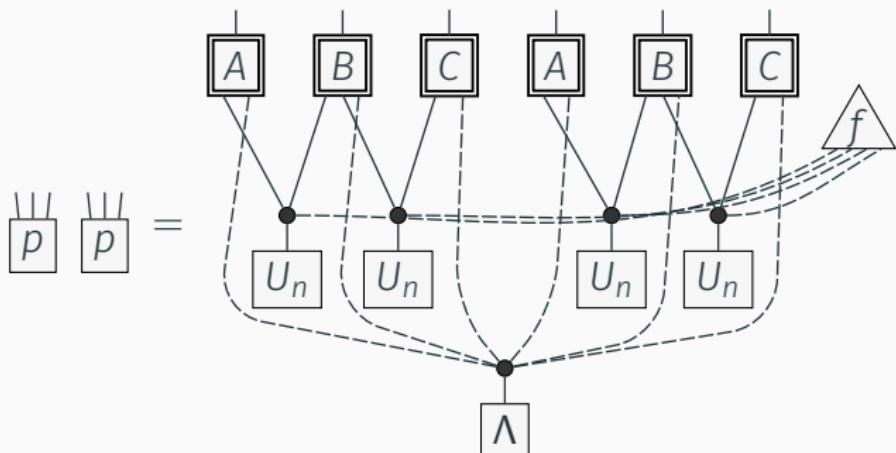
Step 3: Adding postselection

$\boxed{p} \in \mathcal{I}_{\text{restr}}^{(n)} \left(\begin{array}{c} (A) \\ \diagdown \quad \diagup \\ (1) \quad (2) \\ \diagup \quad \diagdown \\ (B) \quad (C) \end{array} \right)$ iff there exist $\boxed{A}, \boxed{B}, \boxed{C}, \boxed{\Lambda}$
such that



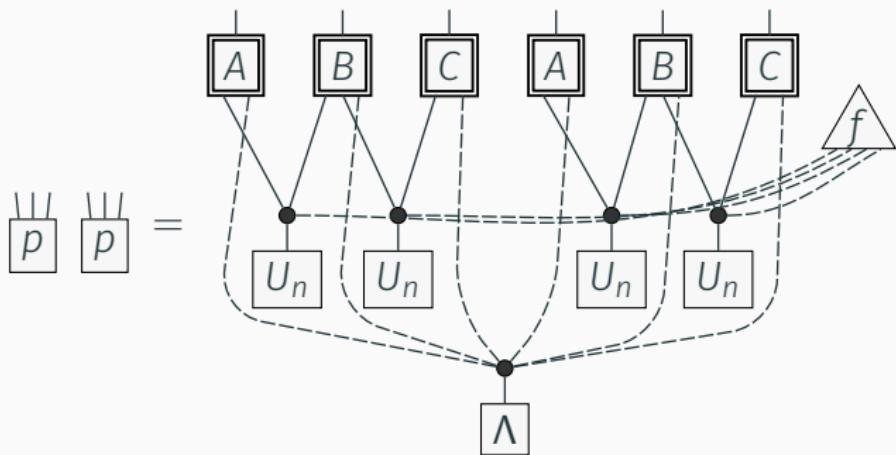
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$\boxed{p} \in \mathcal{I}_f^{(n)} \left(\begin{array}{c} (A) \\ \diagdown \quad \diagup \\ (1) \quad (2) \\ \diagup \quad \diagdown \end{array} \quad \begin{array}{c} (B) \\ \diagdown \quad \diagup \\ (1) \quad (2) \\ \diagup \quad \diagdown \end{array} \quad \begin{array}{c} (C) \\ \diagdown \quad \diagup \\ (1) \quad (2) \\ \diagup \quad \diagdown \end{array} \end{array} \right)$ iff there exist $\boxed{A}, \boxed{B}, \boxed{C}, \boxed{\Lambda}$
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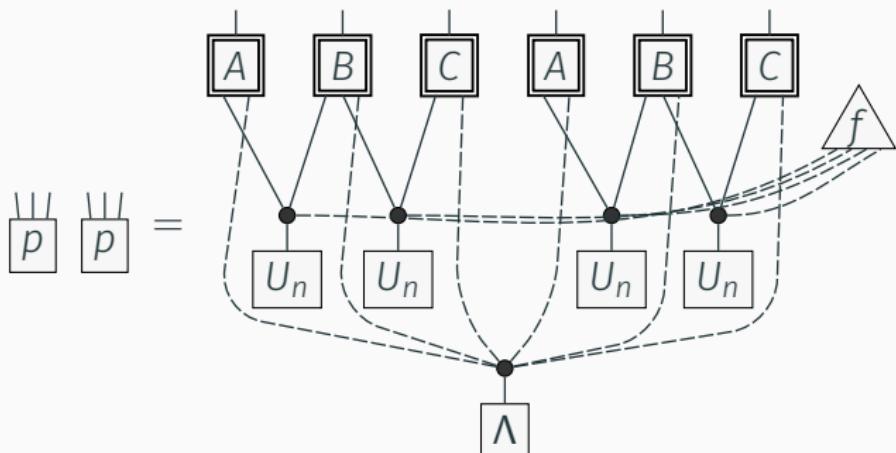
$p \in \mathcal{I}_f^{(n)} \left(\begin{array}{c} (A) \\ \diagdown \quad \diagup \\ (1) \quad (2) \\ \diagup \quad \diagdown \end{array} \right)$ iff there exist $\boxed{A}, \boxed{B}, \boxed{C}, \boxed{\Lambda}$
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$\mathcal{I}_f^{(n)}$ and \mathcal{L} are **incomparable** for general f . But if $f \rightarrow$ “trivial” and $n \rightarrow \infty$ then $\mathcal{I}_f^{(n)} \rightarrow \mathcal{L}$

Step 3: Adding postselection

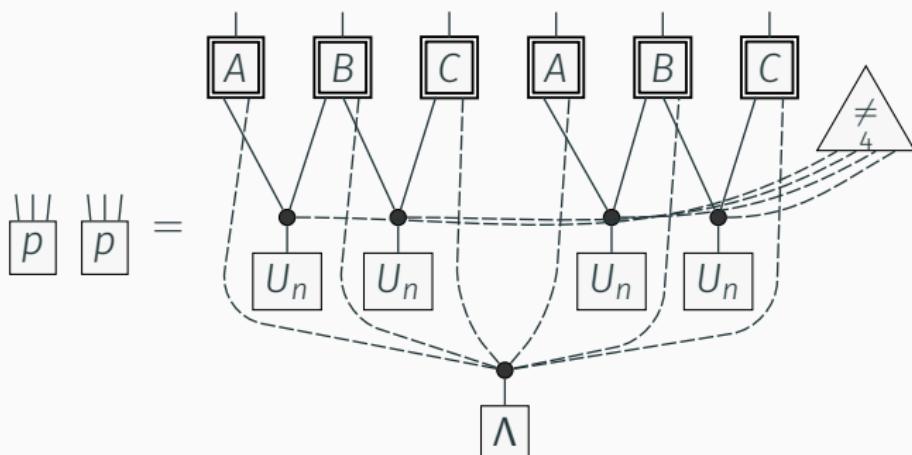
$p \in \mathcal{I}_f^{(n)} \left(\begin{array}{c} (A) \\ \diagdown \quad \diagup \\ (1) \quad (2) \\ \diagup \quad \diagdown \end{array} \right)$ iff there exist A, B, C, Λ
such that



How to choose f ? The agents need some **asymmetry** to use “different parts” of Λ

Step 4: Fixing the postselection

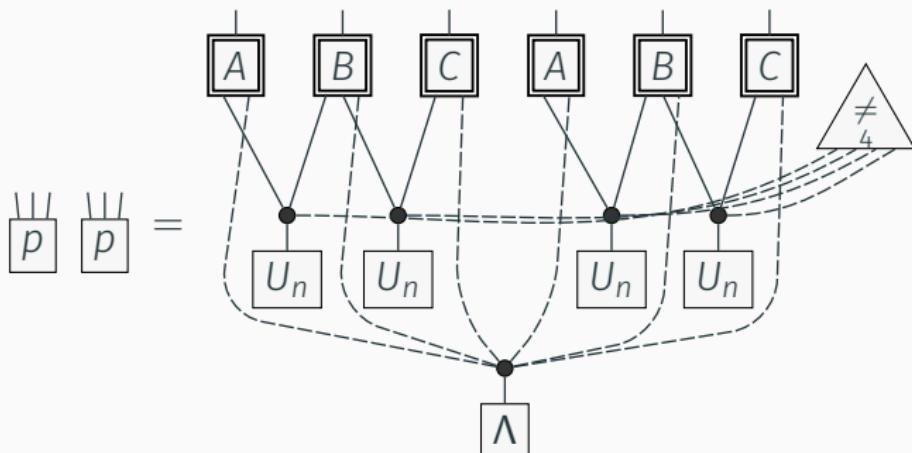
$p \in \mathcal{I}_{\neq}^{(n)} \left(\begin{array}{c} (A) \\ \diagdown \quad \diagup \\ (1) \quad (2) \\ \diagup \quad \diagdown \end{array} \right)$ iff there exist A, B, C, Λ
such that



Answer: provide the agents with **distinct** values from the U_n sources

Step 4: Fixing the postselection

$p \in \mathcal{I}_{\neq}^{(n)} \left(\begin{array}{c} (A) \\ \diagdown \quad \diagup \\ 1 \quad 2 \\ (B) \quad (C) \end{array} \right)$ iff there exist A, B, C, Λ
 such that



Outer approximation: $\mathcal{L} \subseteq \mathcal{I}_{\neq}^{(n)}$ (left as an exercise).

Furthermore, $n \rightarrow \infty$ implies " \neq " \rightarrow "trivial", so $\mathcal{I}_{\neq}^{(n)} \xrightarrow{n \rightarrow \infty} \mathcal{L}$

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- Definition of $\mathcal{I}_{\neq}^{(n)}(\mathcal{N})$ for arbitrary network \mathcal{N}

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Convergence: if $\begin{array}{c} \text{...} \\ \text{p} \\ \text{...} \end{array} \in \mathcal{I}_{\neq}^{(n)}(\mathcal{N})$, then

Results

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- Proofs of inclusions: $\mathcal{L}(\mathcal{N}) \subseteq \mathcal{I}_{\neq}^{(n)}(\mathcal{N})$ and $\mathcal{I}_{\neq}^{(n+1)}(\mathcal{N}) \subseteq \mathcal{I}_{\neq}^{(n)}(\mathcal{N})$
- **Convergence:** if  $\in \mathcal{I}_{\neq}^{(n)}(\mathcal{N})$, then

$$\inf_{\substack{\text{---} \\ q \\ \in \mathcal{L}(\mathcal{N})}} \left\| \begin{array}{c} \text{---} \\ p \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ q \\ \text{---} \end{array} \right\|_1 \leq \frac{c(\mathcal{N})}{\sqrt{n}} + \mathcal{O}\left(\frac{1}{n\sqrt{n}}\right)$$

- Drawback: the linear program testing  $\in \mathcal{I}_{\neq}^{(n)}(\mathcal{N})$ takes $e^{\text{poly}(n)}$ time/memory

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Navascuès, Wolfe, arXiv:1707.06476

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Inflation correspondence

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	Standard inflation	Postselected inflation
Is an outer approximation:		
Converges:		

- The present proof of convergence extends the original proof to same-strategy networks

Outlook

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- Further developments to obtain new tools and solve open problems in classical causal compatibility? E.g., still lack noise-robust proofs of “genuine” multipartite nonlocality

Gisin, arXiv:1708.05556

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Gisin, arXiv:1708.05556

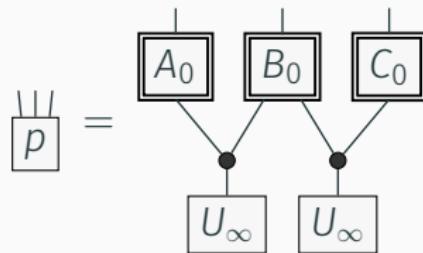
- We still lack convergent outer approximations for **quantum** causal compatibility, although some progress has been made recently

Lighhart, Gachechiladze, Gross, arXiv:2110.14659

Appendix

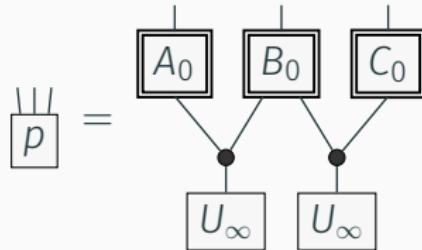
Solution: proof that $\mathcal{L} \subseteq \mathcal{I}_\neq^{(n)}$ for $n = 4$

Let $\begin{array}{c} \text{|||} \\ p \end{array} \in \mathcal{L}$ with prob. tensors $\begin{array}{c} | \\ A_0 \end{array}$, $\begin{array}{c} | \\ B_0 \end{array}$, $\begin{array}{c} | \\ C_0 \end{array}$ such that



Solution: proof that $\mathcal{L} \subseteq \mathcal{I}_\neq^{(n)}$ for $n = 4$

Let $\begin{array}{c} \sqcup \sqcup \\ p \end{array} \in \mathcal{L}$ with prob. tensors A_0, B_0, C_0 such that



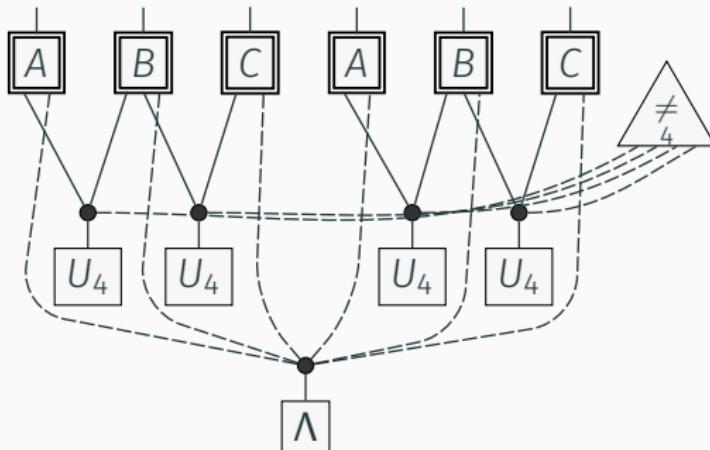
We let

$$\begin{array}{c} (x_1, x_2, x_3, x_4) \\ \Lambda \end{array} := \begin{array}{cccc} x_1 & x_3 & x_3 & x_4 \\ U_\infty & U_\infty & U_\infty & U_\infty \end{array}$$

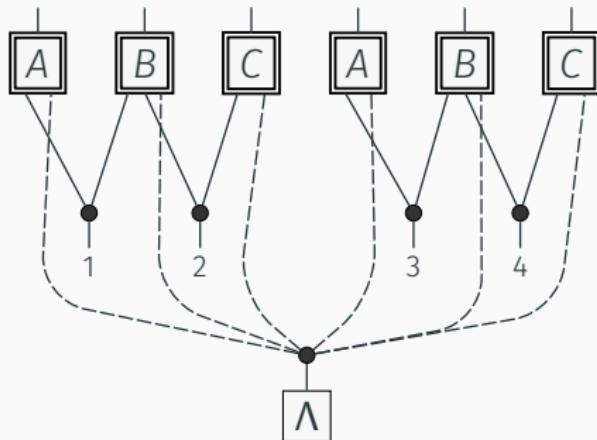
$$\begin{array}{c} a \\ A \end{array} := \begin{array}{c} a \\ A_0 \end{array}, \quad \begin{array}{c} b \\ B \end{array} := \begin{array}{c} b \\ B_0 \end{array}, \quad \begin{array}{c} c \\ C \end{array} := \begin{array}{c} c \\ C_0 \end{array}$$

where $i (x_1, \dots, x_4)$ is mapped to x_i , and $i j (x_1, \dots, x_4)$ is mapped to $x_i x_j$.

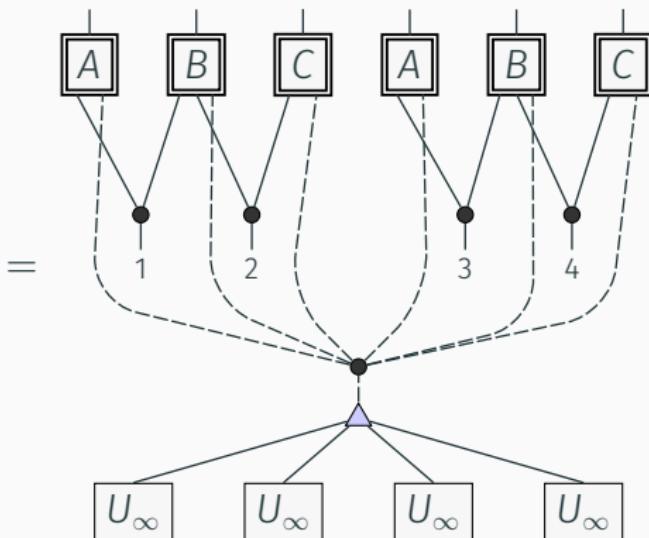
Solution: proof that $\mathcal{L} \subseteq \mathcal{I}_{\neq}^{(n)}$ for $n = 4$



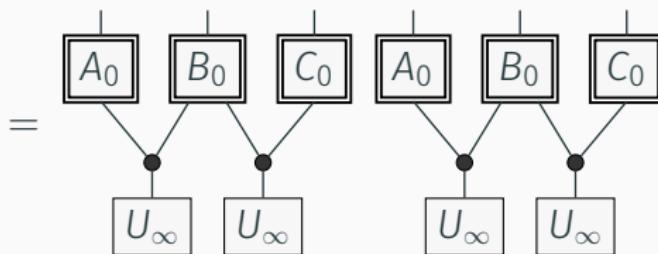
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$$= \begin{array}{c} \diagup \diagdown \\ p \end{array} \quad \begin{array}{c} \diagup \diagdown \\ p \end{array}$$

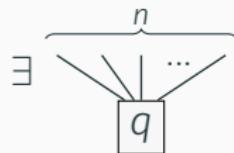
Inflation correspondence: the case of de Finetti

For any distribution p ,

$$\exists \begin{array}{c} n \\ \hline \dots \\ \swarrow \quad \searrow \\ q \end{array} \text{ such that } \left\{ \begin{array}{l} \begin{array}{c} \dots \\ \swarrow \quad \searrow \\ p \end{array} = \begin{array}{c} \dots \\ \swarrow \quad \searrow \\ q \end{array} \\ \begin{array}{c} a_1 \quad a_2 \quad a_3 \quad \dots \quad a_n \\ \swarrow \quad \searrow \\ q \end{array} = \begin{array}{c} a_{\sigma(1)} \quad a_{\sigma(2)} \quad a_{\sigma(3)} \quad \dots \quad a_{\sigma(n)} \\ \swarrow \quad \searrow \\ q \end{array} \end{array} \right.$$

Inflation correspondence: the case of de Finetti

For any distribution \boxed{p} ,



such that

$$\left\{ \begin{array}{l} \boxed{p} = \boxed{q} \\ a_1 a_2 a_3 \dots a_n = a_{\sigma(1)} a_{\sigma(2)} a_{\sigma(3)} \dots a_{\sigma(n)} \end{array} \right.$$

Diagram illustrating the inflation correspondence. The first equation shows a distribution p (box with 2 lines) equal to a distribution q (box with n lines). The second equation shows a sequence of outcomes $a_1, a_2, a_3, \dots, a_n$ equal to a sequence of outcomes $a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}, \dots, a_{\sigma(n)}$, where σ is a permutation of $\{1, 2, 3, \dots, n\}$.

if and only if



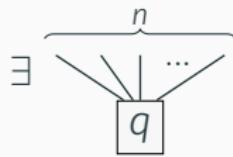
such that

$$\boxed{p} = \begin{array}{c} \neq \\ 2 \end{array} \quad \boxed{A} \quad \boxed{A} \quad \boxed{\Lambda}$$

Diagram illustrating the inflation correspondence. The distribution p (box with 2 lines) is shown to be equal to a triangular symbol with \neq and 2, which is then connected by dashed lines to two boxes labeled A and a box labeled Λ . Below the A boxes are two boxes labeled U_n .

Inflation correspondence: the case of de Finetti

For any distribution \boxed{p} ,

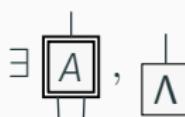


such that

$$\left\{ \begin{array}{l} \boxed{p} = \boxed{q} \\ a_1 a_2 a_3 \dots a_n = a_{\sigma(1)} a_{\sigma(2)} a_{\sigma(3)} \dots a_{\sigma(n)} \end{array} \right.$$

Diagram illustrating the inflation correspondence. The first equation shows a distribution p (box with two lines) equal to a distribution q (box with n lines). The second equation shows a sequence of outcomes $a_1, a_2, a_3, \dots, a_n$ equal to a sequence of outcomes $a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}, \dots, a_{\sigma(n)}$, where σ is a permutation of $\{1, 2, 3, \dots, n\}$.

if and only if



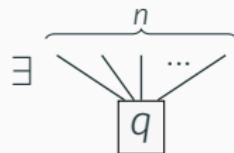
such that

$$\boxed{p} = \boxed{A} \neq \boxed{\Lambda}$$

Diagram illustrating the inflation correspondence. The distribution p (box with two lines) is shown to be equal to a distribution A (box with two lines) but not equal to a parameter Λ (box with one line). The inequality symbol \neq is enclosed in an orange triangle. The distribution A is shown branching into two outcomes U_n and U_n , which then lead to a parameter Λ .

Inflation correspondence: the case of de Finetti

For any distribution \boxed{p} ,

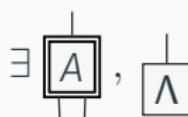


such that

$$\left\{ \begin{array}{l} \boxed{p} = \boxed{q} \\ a_1 a_2 a_3 \dots a_n = a_{\sigma(1)} a_{\sigma(2)} a_{\sigma(3)} \dots a_{\sigma(n)} \end{array} \right.$$

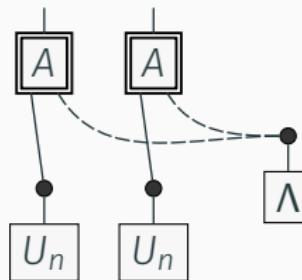
Diagram illustrating the inflation correspondence. The first equation shows a box p with n lines branching out, equal to a box q with n lines branching out. The second equation shows a sequence of outcomes $a_1, a_2, a_3, \dots, a_n$ equal to a sequence of outcomes $a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}, \dots, a_{\sigma(n)}$, where σ is a permutation of $\{1, 2, 3, \dots, n\}$.

if and only if



such that

$$\boxed{p} \underset{2/n}{\approx}$$



De Finetti equivalence

For any distribution $\begin{array}{c} \backslash \\ p \end{array}$, given $q(\{A_i = \cdot\}_{i=1}^n)$ such that

$$\begin{array}{c} \backslash \\ p \end{array} = q(A_1 = \cdot, A_2 = \cdot)$$

and $\forall \{a_i\}_{i=1}^n, \forall \sigma \in S_n : q(\{A_{\sigma(i)} = a_i\}_{i=1}^n) = q(\{A_i = a_i\}_{i=1}^n)$,

De Finetti equivalence

For any distribution \boxed{p} , given $q(\{A_i = \cdot\}_{i=1}^n)$ such that

$$\boxed{p} = q(A_1 = \cdot, A_2 = \cdot)$$

and $\forall \{a_i\}_{i=1}^n, \forall \sigma \in S_n : q(\{A_{\sigma(i)} = a_i\}_{i=1}^n) = q(\{A_i = a_i\}_{i=1}^n)$,

define first $q =: \sum_{\lambda} \boxed{\Lambda}^{\lambda} q^{(\lambda)}$ where the $q^{(\lambda)}$ are deterministic distributions,

De Finetti equivalence

For any distribution $\begin{array}{c} \backslash \\ p \end{array}$, given $q(\{A_i = \cdot\}_{i=1}^n)$ such that

$$\begin{array}{c} \backslash \\ p \end{array} = q(A_1 = \cdot, A_2 = \cdot)$$

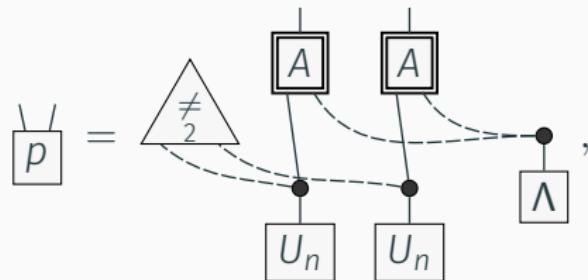
and $\forall \{a_i\}_{i=1}^n, \forall \sigma \in S_n : q(\{A_{\sigma(i)} = a_i\}_{i=1}^n) = q(\{A_i = a_i\}_{i=1}^n)$,

define first $q =: \sum_{\lambda} \begin{array}{c} \lambda \\ \Lambda \end{array} q^{(\lambda)}$ where the $q^{(\lambda)}$ are deterministic distributions, and then

$$\begin{array}{c} a \\ \downarrow \\ A \\ \uparrow \\ i \quad \lambda \end{array} := q^{(\lambda)}(A_i = a)$$

De Finetti equivalence

For any distribution $\begin{smallmatrix} \text{ } \\ \text{ } \\ \text{ } \end{smallmatrix} p$, given $\begin{smallmatrix} \text{ } \\ \text{ } \\ \text{ } \end{smallmatrix} A$ and $\begin{smallmatrix} \text{ } \\ \text{ } \\ \text{ } \end{smallmatrix} \Lambda$ such that



define q such that

$$q(\{A_i = a_i\}_{i=1}^n) = \sum_{\lambda} \begin{smallmatrix} \text{ } \\ \text{ } \\ \text{ } \end{smallmatrix} \lambda \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{i=1}^n \begin{smallmatrix} \text{ } \\ \text{ } \\ \text{ } \end{smallmatrix} \begin{smallmatrix} a_{\sigma(i)} \\ \text{ } \\ \text{ } \end{smallmatrix} A \begin{smallmatrix} \text{ } \\ i \\ \text{ } \\ \text{ } \\ \lambda \end{smallmatrix}$$