ETH zürich

Converging outer approximations of classical network correlations

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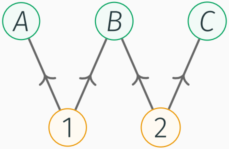
arXiv:2202.04103

Introduction

Causal compatibility

— Causal structure = observed and unobserved nodes + theory for unobserved nodes + connectivity

e.g., classical bilocal network



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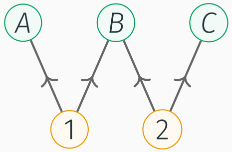
e.g., classical bilocal network 

- Causal compatibility = compatibility of a distribution with a given causal structure

Causal compatibility: applications

- Bell's theorem

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- non-locality without inputs

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- quantifying freedom of choice in Bell's theorem

  Chaves et al, arXiv:2105.05721
- multipartite quantum or post-quantum entanglement

  Coiteux-Roy, Wolfe, Renou, arXiv:2105.09381

Causal compatibility: computational aspects

— Bell-like scenarios: one unobserved node → convex problem → easy 🍰

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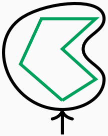
- More general networks: several independent unobserved nodes → nonconvex → hard 🤯

Causal compatibility: approximations

— Inner approximations: based on human or computational exploration of the search space



(lower bounds for maximization problems,

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set of distributions compatible with network

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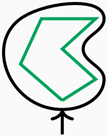

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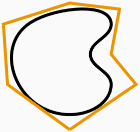
— Outer approximations:

- nontight: entropic or algebraic inequalities
- powerful and asymptotically tight method: inflation



(upper bounds for maximization problems, certify infeasibility)

Objectives of the talk

- Define causal compatibility formally
- Construct outer approximations that are apparently converging

Causal compatibility in classical networks, a.k.a. network locality

Tensor notation

	Standard	Tensor	Components
Outcome distribution	$p(\cdot, \cdot, \cdot)$		$p(a, b, c) =$ 

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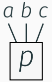
Outcome distribution	$p(\cdot,\cdot,\cdot)$		$p(a,b,c) =$ 
Strategy with two inputs	$p_A(\cdot\|\cdot,\cdot)$		$p_A(a\|\alpha,\beta) =$ 

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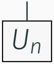
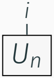
...$\{1,\ldots,n\}$			 $= \frac{1}{n}$

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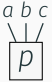
Outcome distribution	$p(\cdot,\cdot,\cdot)$		$p(a,b,c) =$ 
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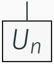
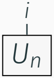
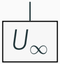
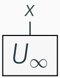
…$\{1,\ldots,n\}$			 $= \frac{1}{n}$
…$[0,1]$			 $= 1$

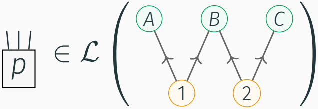
Tensor notation

Special case: deterministic strategies  such that

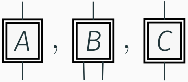
$$= \delta(a - f(\alpha, \beta))$$

for some function $f(\cdot, \cdot)$

Define



iff there exist



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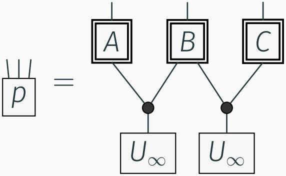
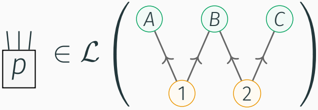


Causal compatibility in classical networks

Define



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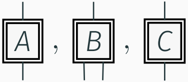

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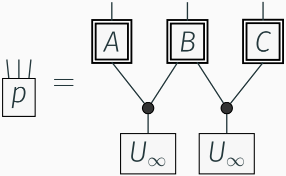
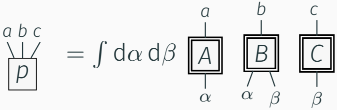


i.e., for all a, b, c :



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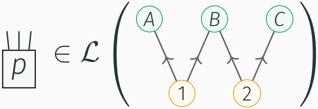
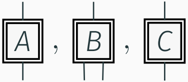
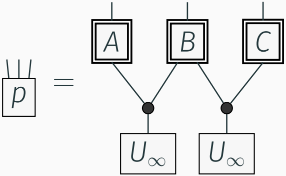
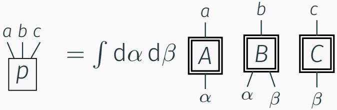


iff there exist  such that



(w.l.o.g)

i.e., for all a, b, c : 

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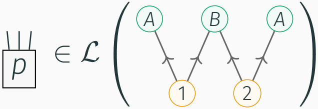
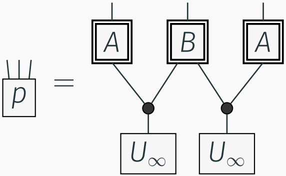
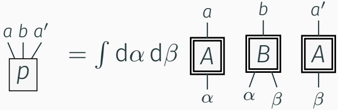


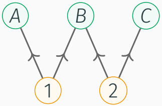
iff there exist ,  such that

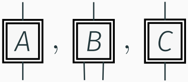


i.e., for all a, b, a' : 

Constructing converging outer approximations

Step 1: Convexification

$\in \mathcal{L}\Big($ 

$\Big)$ iff there exist 

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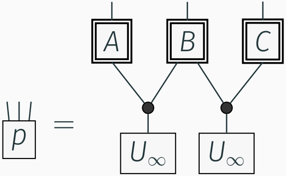
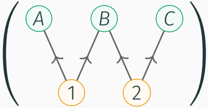
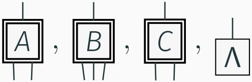


Step 1: Convexification

$p \in \mathcal{I}_{SR}$  iff there exist 

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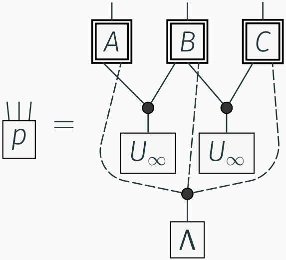

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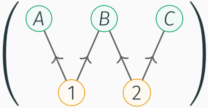

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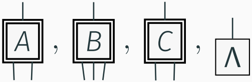
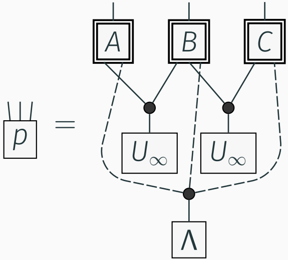


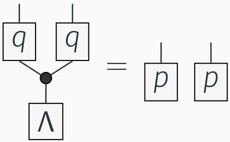
such that



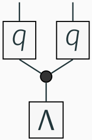
Convex but too permissive. Idea: forbid the agent to use Λ?

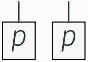
Main lemma

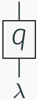
If

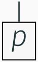


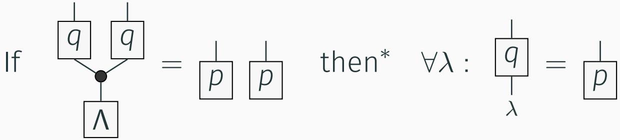
Main lemma

If 

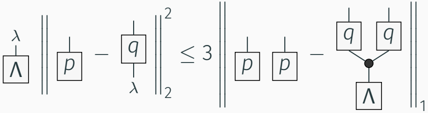
= 

then* $\forall \lambda$: 

= 



*Robust version: $\int \mathrm{d}\lambda$



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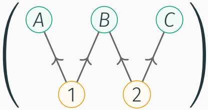

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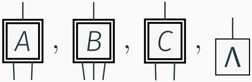
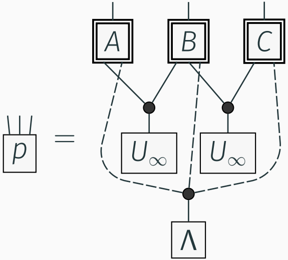


such that



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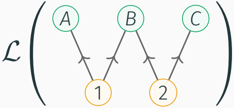
$\in$


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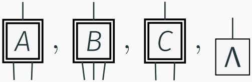

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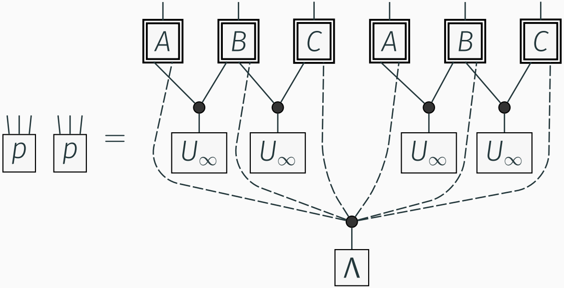

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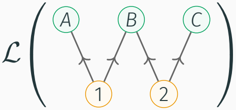

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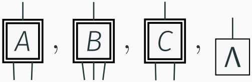

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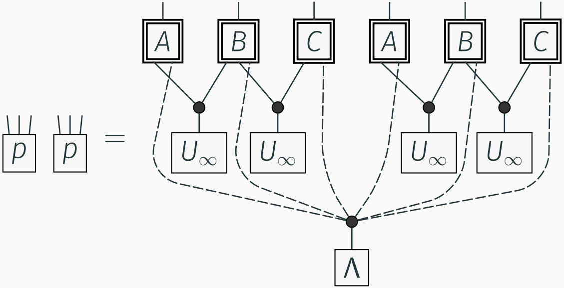
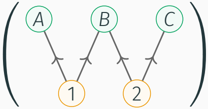
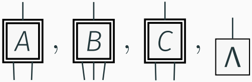


Equivalent to the original problem thanks to the main lemma

Step 2: Restricting the source output cardinality

$\in \mathcal{I}^{(n)}_{\text{restr}}$ 

iff there exist 

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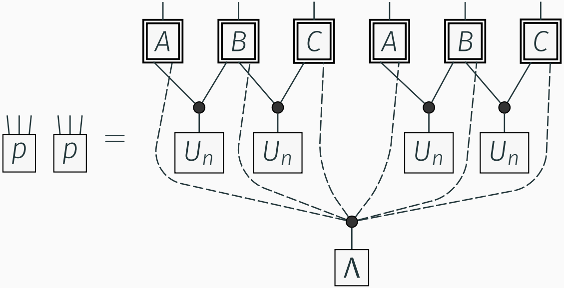

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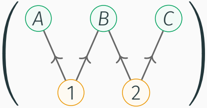
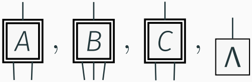
 $\in \mathcal{I}^{(n)}_{\text{restr}}$ 

iff there exist 

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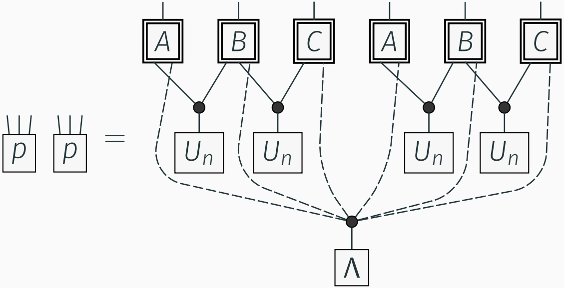

Inner approximation $\mathcal{I}^{(n)}_{\text{restr}} \subseteq \mathcal{L}$, but also $\mathcal{I}^{(n)}_{\text{restr}} \xrightarrow{n\to\infty} \mathcal{L}$

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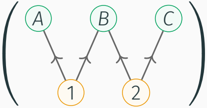
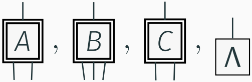
$\in \mathcal{I}^{(n)}_{\text{restr}}$ 

iff there exist 

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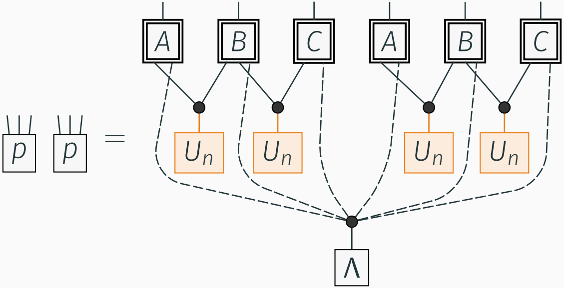
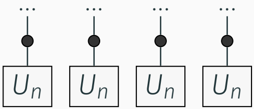
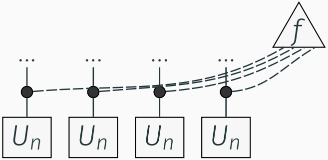


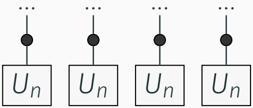
Want outer approximations → let the agents use Λ a little bit?

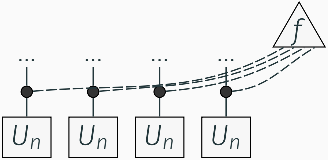
Postselection for additional correlations

Idea: replace 

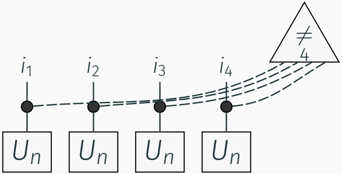
with 

Postselection for additional correlations

Idea: replace 

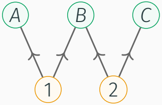
with 

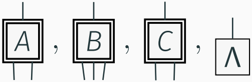
For instance, if $n \geq 4$,



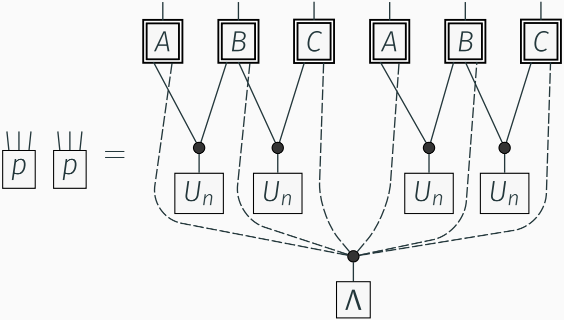
$$= \begin{cases} \dfrac{1}{n(n-1)(n-2)(n-3)} & \text{if } i_x \neq i_y \\ 0 \text{ else} \end{cases}$$

Step 3: Adding postselection

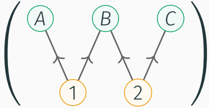
$\in \mathcal{I}^{(n)}_{\text{restr}}$ (

) iff there exist 

such that



Step 3: Adding postselection

$\in \mathcal{I}_f^{(n)}$


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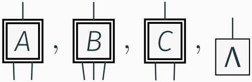

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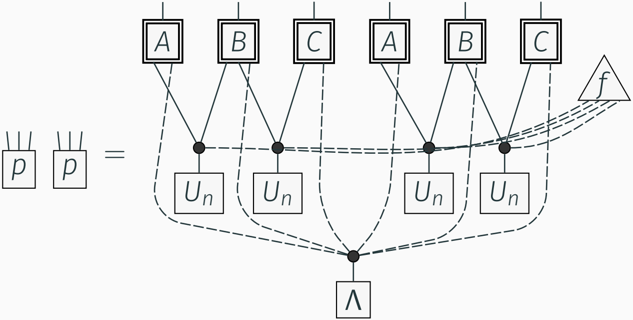

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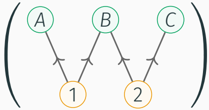

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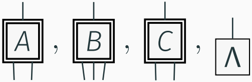

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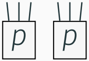
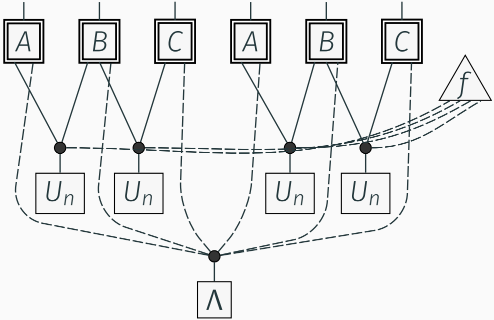
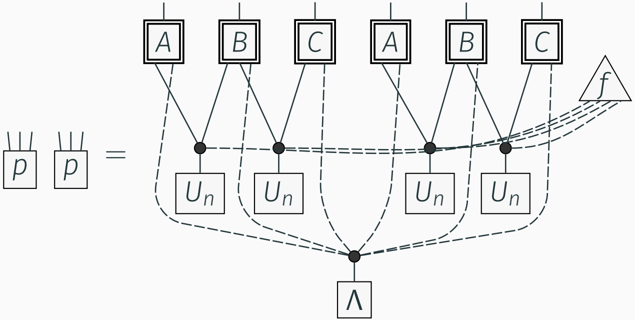


=


$\mathcal{I}_f^{(n)}$ and $\mathcal{L}$ are incomparable for general f. But if $f \to$ "trivial" and $n \to \infty$ then $\mathcal{I}_f^{(n)} \to \mathcal{L}$

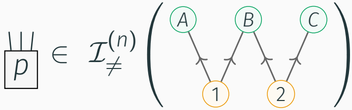
Step 3: Adding postselection

such that



How to choose f? The agents need some asymmetry to use "different parts" of Λ

Step 4: Fixing the postselection



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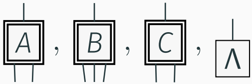
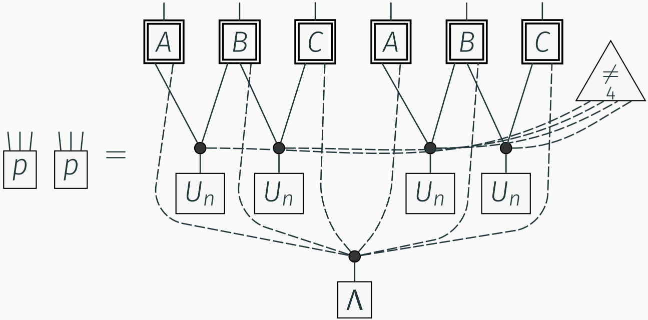
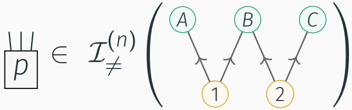


such that



Answer: provide the agents with distinct values from the U_n sources

Step 4: Fixing the postselection



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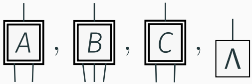

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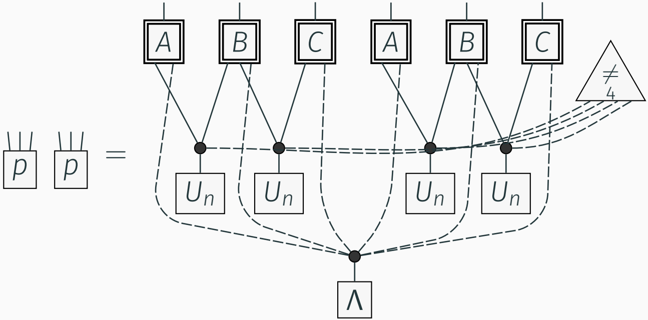

Outer approximation: $\mathcal{L} \subseteq \mathcal{I}_{\neq}^{(n)}$ (left as an exercise).

Furthermore, $n \to \infty$ implies “ $\neq$ ” $\to$ “trivial”, so $\mathcal{I}_{\neq}^{(n)} \xrightarrow{n\to\infty} \mathcal{L}$

Results

— Definition of $\mathcal{I}_{\neq}^{(n)}(\mathcal{N})$ for arbitrary network $\mathcal{N}$

VG, arXiv:2202.04103

Results

- Definition of $\mathcal{I}_{\neq}^{(n)}(\mathcal{N})$ for arbitrary network $\mathcal{N}$
- Proofs of inclusions: $\mathcal{L}(\mathcal{N}) \subseteq \mathcal{I}_{\neq}^{(n)}(\mathcal{N})$ and $\mathcal{I}_{\neq}^{(n+1)}(\mathcal{N}) \subseteq \mathcal{I}_{\neq}^{(n)}(\mathcal{N})$

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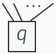
- Convergence: if  $\in \mathcal{I}_{\neq}^{(n)}(\mathcal{N})$, then

inf  $\in \mathcal{L}(\mathcal{N})$ $\Big\|$  $-$  $\Big\|_1 \leq \frac{c(\mathcal{N})}{\sqrt{n}} + \mathcal{O}\left(\frac{1}{n\sqrt{n}}\right)$

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Results

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- Convergence: if $\boxed{p} \in \mathcal{I}_{\neq}^{(n)}(\mathcal{N})$, then

$$\inf_{\boxed{q} \in \mathcal{L}(\mathcal{N})} \left\| \boxed{p} - \boxed{q} \right\|_1 \leq \frac{c(\mathcal{N})}{\sqrt{n}} + \mathcal{O}\left(\frac{1}{n\sqrt{n}}\right)$$

- Drawback: the linear program testing $\boxed{p} \in \mathcal{I}_{\neq}^{(n)}(\mathcal{N})$ takes $e^{\mathrm{poly}(n)}$ time/memory

VG, arXiv:2202.04103

Inflation correspondence

Inflation correspondence

- Two equivalent formulations: $\mathcal{I}_{\neq}^{(n)}(\mathcal{N}) = \mathcal{I}_{\text{std}}^{(n)}(\mathcal{N})$

Navascuès, Wolfe, arXiv:1707.06476

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	Standard inflation	Postselected inflation
Is an outer approximation:		
Converges:		

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- The present proof of convergence extends the original proof to same-strategy networks

Navascués, Wolfe, arXiv:1707.06476

Outlook

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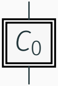
Gisin, arXiv:1708.05556

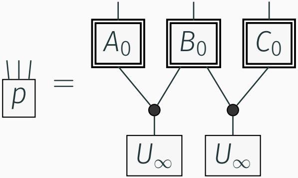
- We still lack convergent outer approximations for quantum causal compatibility, although some progress has been made recently

Ligthart, Gachechiladze, Gross, arXiv:2110.14659

Appendix

Solution: proof that $\mathcal{L} \subseteq \mathcal{I}_{\neq}^{(n)}$ for $n = 4$

Let  $\in \mathcal{L}$ with prob. tensors , ,  such that



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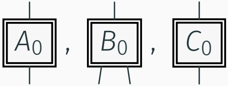
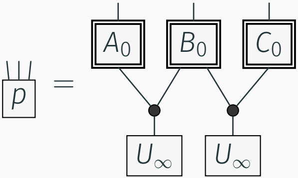
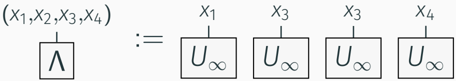
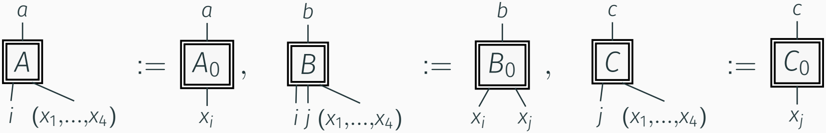
$\in \mathcal{L}$ with prob. tensors 

such that



We let





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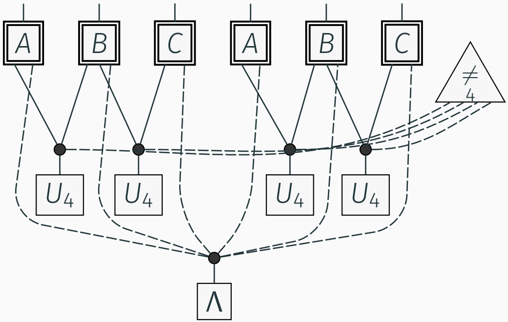

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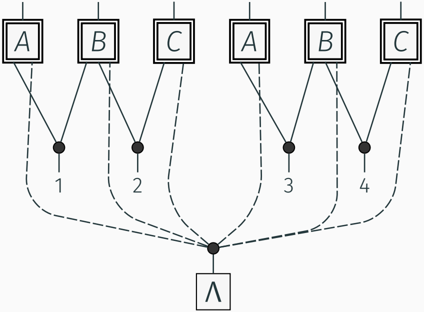

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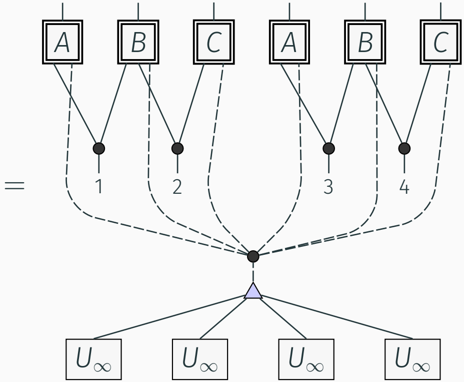

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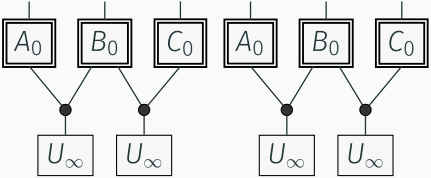
$=$ 

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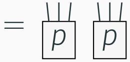
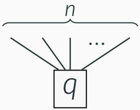


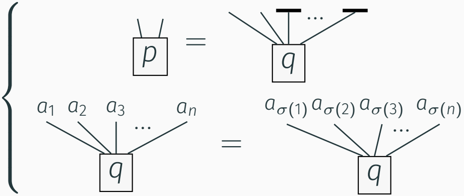
Inflation correspondence: the case of de Finetti

For any distribution 

,

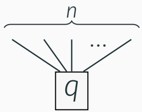
∃ 

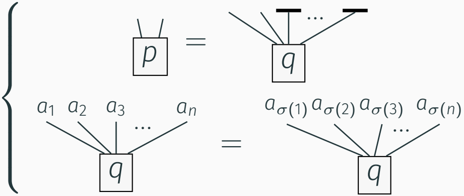
such that



Inflation correspondence: the case of de Finetti

For any distribution ,

$\exists$ 

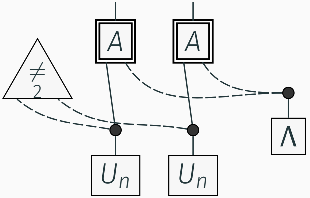
such that 

if and only if

$\exists$ 

, 

such that 

= 

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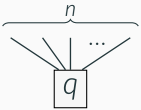
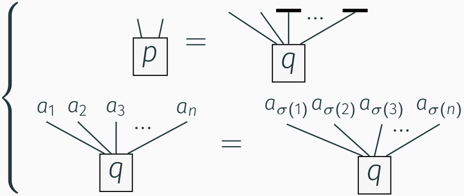
∃ 

such that 

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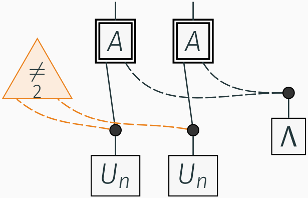
= 

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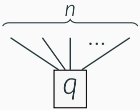
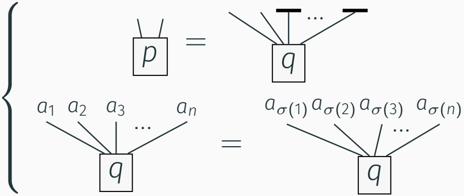
∃ 

such that 

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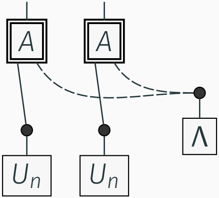
$\overset{2/n}{\approx}$ 

De Finetti equivalence

For any distribution , given $q(\{A_i = \cdot\}_{i=1}^{n})$ such that

 $= q(A_1 = \cdot, A_2 = \cdot)$

and $\forall \{a_i\}_{i=1}^{n}, \forall \sigma \in S_n : \; q(\{A_{\sigma(i)} = a_i\}_{i=1}^{n}) = q(\{A_i = a_i\}_{i=1}^{n})$,

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define first $q =: \sum_\lambda$  $q^{(\lambda)}$ where the $q^{(\lambda)}$ are deterministic distributions,

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$q^{(\lambda)}$ where the $q^{(\lambda)}$ are deterministic distributions, and then

$$:= q^{(\lambda)}(A_i = a)$$

De Finetti equivalence

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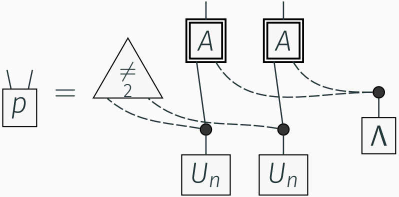

,

define q such that

$q(\{A_i = a_i\}_{i=1}^{n}) = \sum_{\lambda}$  $\frac{1}{n!} \sum_{\sigma \in S_n} \prod_{i=1}^{n}$