Linear growth of quantum circuit complexity

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Based on work with:

   

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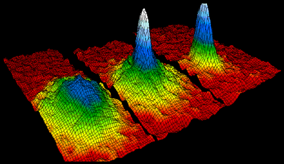
Quantum complexity

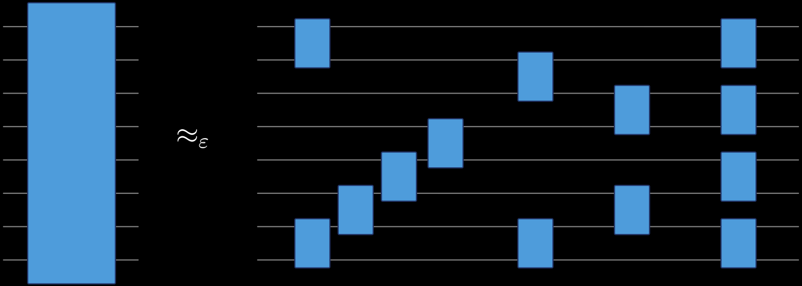
Quantum (circuit) complexity is a well-established concept in quantum information theory.

- Which operations are hard and which are easy?
- Classical analogue is one of the most pervasive objects in CS.
- Definition of topological phases of matter.
- In the AdS/CFT correspondence.



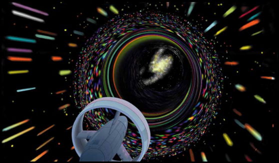
Quantum circuit complexity

How many 2-local gates are necessary to implement a unitary (or a state)?



Denote the number of gates in a minimal decomposition by $\mathcal{C}_\varepsilon(U)$.
Also, exact circuit complexity $\mathcal{C}(U) := \mathcal{C}_0(U)$.

Wormhole growth



AdS

Ψ

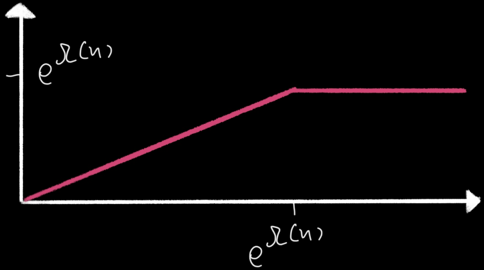
CFT

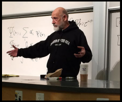
$$\text{Complexity} = \text{Volume?}$$

Susskind, Fort. Phys. Standford, Susskind, Phys. Rev. D Hartman, Maldacena, JHEP

Complexity growth: A universal phenomenon?

How does $\mathcal{C}_\varepsilon(\exp(itH))$ grow for interacting system H.



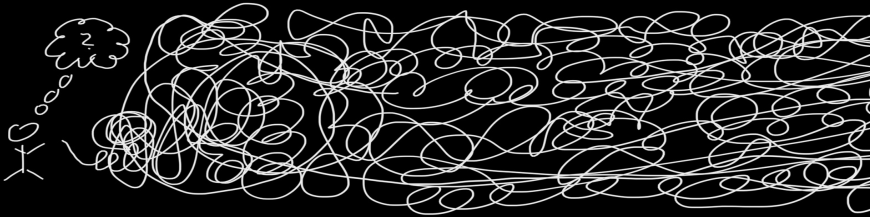


Brown, Susskind, PRA

(Also recurrence after very long times: Michał's talk!)

Oszmaniec, Horodecki, Hunter-Jones, preprint

Notoriety of circuit complexity



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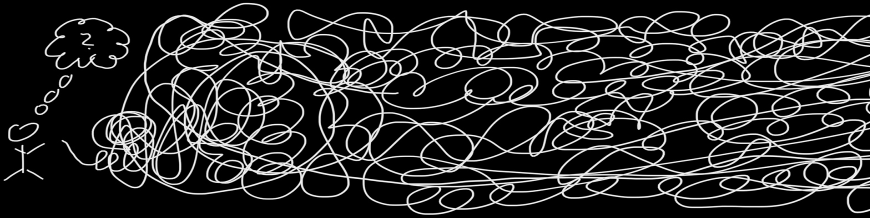


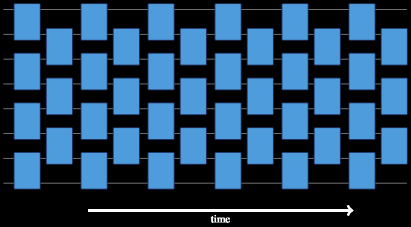
- Almost no superpolynomial complexity lower bound for an explicit unitary.
- Ruling out "accidental short-cuts".

Jia, Wolf, preprint

How to make progress?

- Superpolynomial complexity for particular Hamiltonians from conjectures in computational complexity theory. Aaronson, Susskind and Brandão, Bohdanowicz

- Random quantum circuits on n qubits: Draw gates iid from the Haar measure on $\mathrm{SU}(4)$ or from fixed gate set G.



Brandão, Chemissany, Hunter-Jones, Kueng, Preskill, PRX Q

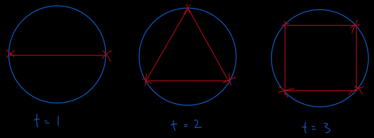
Counting arguments for complexity

Most unitaries are very complex:

- Uniform measure of balls $B_\varepsilon(U)$ is doubly exponentially small.
- Set M_R: quantum circuits with R gates from a gate set G have at most $|G|^R$ unitaries.
- $\implies$ Exponentially many gates required to cover full unitary group.

Partial derandomization: Unitary designs

- Unitary t-designs are pseudorandom distribuitions: indistiguishable from Haar for degree t polynomials.



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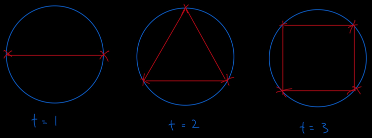

- Defined via t-fold twirl:

$$M(\nu, t) := \int U^{\otimes t} \otimes \overline{U}^{\otimes t} \mathrm{d}\nu(U) = \int U^{\otimes t} \otimes \overline{U}^{\otimes t} \mathrm{d}\mu_H(U).$$

- $\int U^{\otimes t} \otimes \overline{U}^{\otimes t} \mathrm{d}\mu_H(U) =$ Projector onto permutations.

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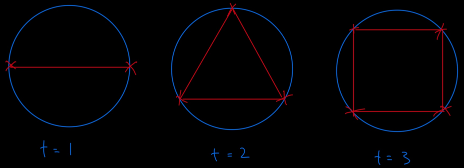

- Defined via t-fold twirl:

$$M(\nu, t) := \int U^{\otimes t} \otimes \overline{U}^{\otimes t} \mathrm{d}\nu(U) \approx \int U^{\otimes t} \otimes \overline{U}^{\otimes t} \mathrm{d}\mu_H(U).$$

Circuit lower bounds for unitary designs

- For U drawn from a unitary t-designs:

$$C_\varepsilon(U) \geq \Omega(t).$$

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- Draw $|\phi\rangle$ from t-design. From union bound:

$$\mathrm{P}\left[\max_{|\psi\rangle \in M_R} |\langle\psi|\phi\rangle| \geq (1-\delta)\right] \leq \sum_{|\psi\rangle \in M_R} \mathrm{P}[|\langle\psi|\phi\rangle| \geq (1-\delta)]$$

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- From Markov's inequality:

$$\mathrm{P}[|\langle\psi|\phi\rangle| \geq (1-\delta)] = \mathrm{P}[|\langle\psi|\phi\rangle|^{2t} \geq (1-\delta)^{2t}] \leq \frac{\mathbb{E}|\langle\psi|\phi\rangle|^{2t}}{(1-\delta)^{2t}}$$

Brandão, Chemissany, Hunter-Jones, Kueng, Preskill, PRX Q

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Brandão, Chemissany, Hunter-Jones, Kueng, Preskill, PRX Q

Circuit lower bounds for unitary designs

- For U drawn from a unitary t-designs:

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- Combined we find:

$$\mathrm{P}\left[\max_{|\psi\rangle\in M_R} |\langle\psi|\phi\rangle| \geq (1-\delta)\right] \leq |G|^R (1-\delta)^{-2t} \binom{2^n+t-1}{t}^{-1}$$

- Scales roughly like $(2^n/t)^{-t} e^{O(R)}$.

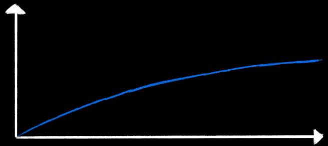
Brandão, Chemissany, Hunter-Jones, Kueng, Preskill, PRX Q

Sublinear scaling of complexity

- Random circuits form unitary t-designs in depth $T = O(nt^{11})$.
- For circuit complexity:

$$C_\varepsilon(U) \geq \Omega(T^{1/11}),$$

for $t \leq 2^{n/2}$.



Brandão, Harrow, Horodecki, Comm. Math. Phys.

Limitations

What can we hope for with current techniques?

- Current techniques might be improved.
- Spectral gap method of Brandão, Harrow and Horodecki:

$$\lambda_2(M(\nu, t)) := ||M(\nu, t) - M(\mu_H, t)||_\infty = 1 - \frac{\Delta(H_n)}{n}.$$

  - Many-body techniques to bound spectral gap of frustration-free H_n.
  - Combine with exponential bound from convergence result.
  - Translate to better notion of approximate design via norm inequality.

Dissecting the BHH exponent

$$10.41 \quad \approx \quad \underbrace{1}_{1.} + \left(\underbrace{2}_{2.} + \underbrace{0.5}_{3.} \right) \times \left(\underbrace{\log_2 5 + \log_2(e)}_{4.} \right).$$

1. From spectral gap to better notions of approximate designs.
2. Approximate orthogonality of permutations in the regime $t^2 \leq 2^n$.
3. Artefact of the proof technique.
4. Exponential bound on the spectral gap of the moment operator:

$$\lambda_2(M(\nu, t)) \leq 1 - \frac{1}{n(5e)^n}.$$

New bounds on the design depth

Theorem (Improved bounds for unitary designs)

Random quantum circuits for unitary t-designs in depth

$$T = O(nt^{5+o(1)})$$

for $t \leq 2^{n/2}$. *JH, preprint*

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$$5 + o(1) \quad = \quad \underbrace{1}_{1.} + \left(\underbrace{2}_{2.} + \underbrace{1.5 \frac{1}{\sqrt{\log_2(t)}}}_{3.} \right) \times \underbrace{\log_2(4)}_{4.},$$

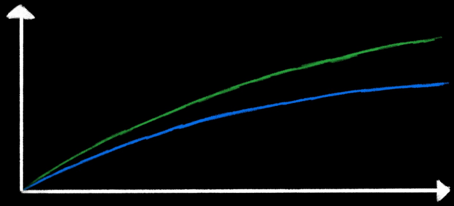
Improved bound on spectral gap:

$$\lambda_2(M(\nu, t)) \leq 1 - 120000^{-1} n^{-5} 2^{-2n}.$$

Improved growth of circuit complexity

For the quantum circuit complexity:

$$C_\varepsilon(U) \geq \Omega(T^{1/(5+o(1))}),$$



Cliffords

Many faces:

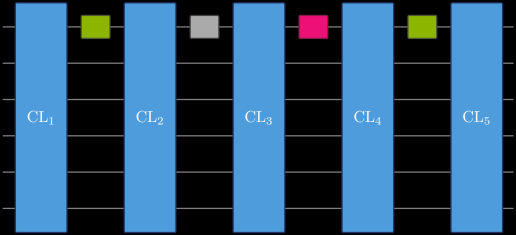
- Generated by Phase gate, Hadamard and controlled NOT: $\mathrm{Cl}(2^n) := \langle\{S, H, \mathrm{CZ}\}\rangle$.
- Normalizer of the Pauli group: $\mathrm{Cl}(2^n) := \{U \in U(2^n), U\mathcal{P}U^\dagger \subseteq \mathcal{P}\}$.
- Symplectic group on vector space over finite field.

Many properties:

- Polynomially bounded complexity.
- Unitary 2-design.

Clifford interleaved walks

- Convergence via path coupling technique on unitary group.
- Near optimal convergence of auxiliary walk:



- Approximate uniform measure on Clifford group by local random walk.

JH, Montealegre-Mora, Heinrich, Eisert, Gross, Roth

Dimension counting and exact circuit complexity

- $\dim(SU(2^n)) = 4^n - 1$.
- Define sets

$$\mathcal{U}(R) := \{U, U = U_R \dots U_1, U_i \text{ 2-local gates}\}.$$

- $\dim \mathcal{U}(R) \leq \dim SU(4) * R = 15R$.
- Not enough parameters to generate a set of positive measure.

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- $\dim \mathcal{U}(R) \leq \dim SU(4) * R = 15R$.
- Not enough parameters to generate a set of positive measure.
- Partial derandomization: Lower bounds on

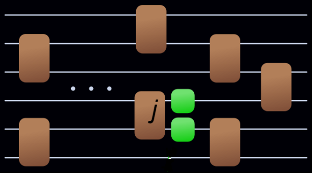
$$\dim \mathcal{U}_{\mathrm{bw}}(T) := \dim\{U, U = U_{nT} \dots U_1, \text{ in brickwork arrangement}\}.$$

Tools

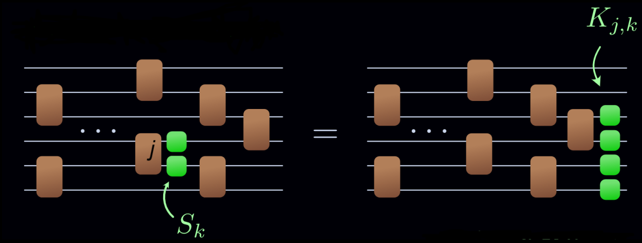
$\mathcal{C}(e^{\mathrm{i}H})$

$\dim \mathcal{U}_{\mathrm{bw}}(T)$

- Toolkit from topology and (semi)algebraic geometry.
- Reduced to combinatorial problem: Inserting Pauli operators into circuits:



Many independent matrices



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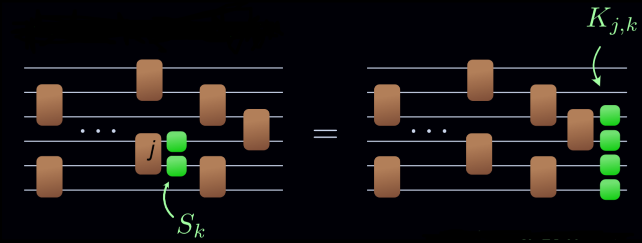

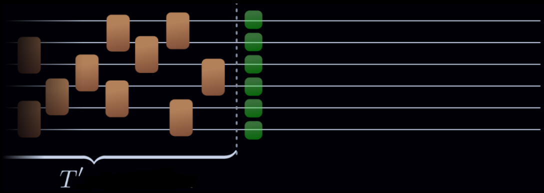

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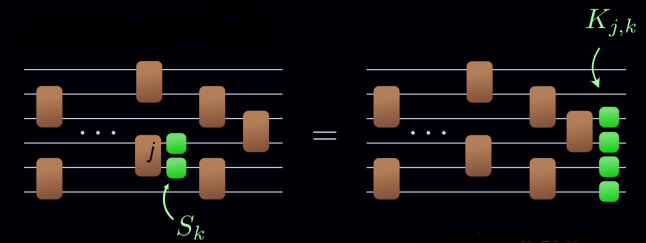
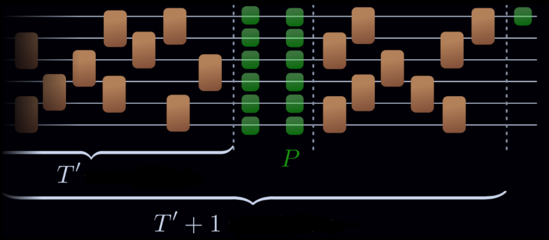




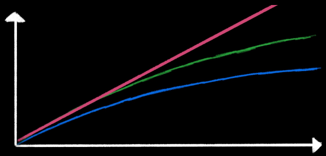
Linear growth of exact circuit complexity

Theorem (Brown-Susskind for exact circuit complexity)

U = *random quantum circuit in brickwork architecture.*

$$\mathcal{C}(U) \geq \frac{\#\text{ of gates}}{9n^2} - \frac{n}{3}\,.$$

with unit probability, until the number of gates grows to $\#\text{ of gates}/n^2 \geq 4^n - 1$.



JH, Faist, Kothakonda, Eisert, Yunger Halpern, Nature Physics, Li, preprint

Future science

- Perfect incompressibility of polynomial quantum circuits.
- Quantum circuit lower bounds for output distributions.
- Brown-Susskind for error robust circuit complexity.