Counterexamples to the extendibility of positive maps

Mizanur Rahaman

June 30, 2022
Ecole Normale Supérieure de Lyon

Joint work with
Giulio Chiribella, Ken Davidson and Vern Paulsen

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$$\widetilde{f}(\mathcal{C}) \subseteq [0, \infty).$$

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then f can be extended to a positive linear functional on $M_n(\mathbb{C})$.

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Replace the range of the linear function

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can we extend this to $\widetilde{\Phi}$ such that

$$\widetilde{\Phi}(PSD(n, \mathbb{C})) \subseteq PSD(m, \mathbb{C})?$$

Arveson Extension Theorem (1969)

Let $\mathcal{S} \subseteq M_n(\mathbb{C})$ be an operator system and H be a finite-dimensional Hilbert space and let

$$\Phi : \mathcal{S} \to B(H)$$

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Question: What if we have a positive map, instead of completely positive map? Can we extend?
Ans: NO!

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Ans: NO! Arveson gave a counterexample! The norm seems to be an obstruction.

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$$\mathcal{A}_n := \left\{ \begin{pmatrix} aI_n & B \\ C & dI_n \end{pmatrix} : a, d \in \mathbb{C},\ B, C \in M_n(\mathbb{C}) \right\} \subseteq M_2(M_n(\mathbb{C})),$$

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$$\Phi_n : \mathcal{A}_n \to M_{2n}(\mathbb{C})$$

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$$\Phi_n \begin{pmatrix} aI_n & B \\ C & dI_n \end{pmatrix} = \begin{pmatrix} aI_n & \frac{1}{4}B^t \\ \frac{1}{4}C^t & dI_n \end{pmatrix},$$

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where X^t denotes the transpose of $X \in M_n(\mathbb{C})$. Then Φ_n is a unital, positive map with $\|\Phi_n\| = 1$, that does not admit a positive extension to $M_{2n}(\mathbb{C})$ for $n > 16$.

Another counterexample

$$\mathcal{T}_n := \left\{ \begin{pmatrix} A & bI_n \\ cI_n & dI_n \end{pmatrix} : A \in M_n(\mathbb{C}), b, c, d \in \mathbb{C} \right\}$$

and a map $\Gamma_n : \mathcal{T}_n \to \mathcal{T}_n$ by

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The map Γ_n is a positive isometry on $\mathcal{T}_n$ which has no positive extension to the full matrix algebra $M_{2n}(\mathbb{C})$.

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If $\mathcal{S} \subseteq M_n(\mathbb{C})$ and given $B(H)$ for finite dimensional H, we define two cones

$$\mathcal{C}_1 = \{x \in \mathcal{S} \otimes B(H) | x = \sum_{i=1}^{r} a_i \otimes b_i, a_i \in \mathcal{S}^+, \ b_i \in B(H)^+\}.$$

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If $\mathcal{C}_1 = \mathcal{C}_2$, then every positive map $\Phi : \mathcal{S} \to B(H)$ can be extended to a positive map on $M_n(\mathbb{C})$.
Conversely, if $\mathcal{C}_1 \subsetneq \mathcal{C}_2$, then there exists a positive map $\Phi : \mathcal{S} \to B(H)$ that can not be extended positively to $M_n(\mathbb{C})$.

THANK YOU!

Consider $E = \mathbb{R}^2$, $\mathcal{C} = \{(x, y) : y > 0\} \cup \{(x, 0) : x > 0\}$. Let $F \subseteq E$ is the x-axis.

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Check that it can not be extended to positive functional on E.