A geometrical description of the universal $1 \to 2$ asymmetric quantum cloning region

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Universal Quantum Cloning

Theorem (No-cloning theorem)

$\nexists\, T : \mathcal{M}_d \xrightarrow{CPTP} \mathcal{M}_d \otimes \mathcal{M}_d$, $\forall \rho$ *pure*,

$$T(\rho) = \rho \otimes \rho.$$

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Theorem (No-broadcasting theorem)

$\nexists\, T : \mathcal{M}_d \xrightarrow{CPTP} \mathcal{M}_d \otimes \mathcal{M}_d,\ \forall \rho \text{ pure},$

$$T_1(\rho) = \rho \quad \text{and} \quad T_2(\rho) = \rho.$$

Approximate Universal Quantum Cloning

$$T : \mathcal{M}_d \xrightarrow{\text{CPTP}} \mathcal{M}_d \otimes \mathcal{M}_d, \; \forall \rho \text{ pure}, \qquad \begin{array}{l} T_1(\rho) = \rho \\ T_2(\rho) = \rho \end{array}$$

Approximate Universal Quantum Cloning

$$T : \mathcal{M}_d \xrightarrow{\text{CPTP}} \mathcal{M}_d \otimes \mathcal{M}_d, \ \forall \rho \text{ pure}, \quad \begin{aligned} T_1(\rho) &= p \cdot \rho + (1 - p)/d \cdot I \\ T_2(\rho) &= p \cdot \rho + (1 - p)/d \cdot I \end{aligned}$$

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Approximate Universal Quantum Cloning

Definition (Approximate universal quantum cloning)

$$\textbf{Isotropic} \quad \left\{ p \in [0,1]^2 \,\middle|\, T : \mathcal{M}_d \xrightarrow{\text{CPTP}} \mathcal{M}_d \otimes \mathcal{M}_d,\ \forall \rho \text{ pure},\ \begin{array}{l} T_1(\rho) = p_1 \cdot \rho + (1 - p_1)/d \cdot I \\ T_2(\rho) = p_2 \cdot \rho + (1 - p_2)/d \cdot I \end{array} \right\}$$

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Worst $\sup_{T \text{ CPTP}} \sum_{i=1}^{2} \alpha_i \cdot \inf_{\rho \text{ pure}} F\big(\rho, T_i(\rho)\big)$

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Worst $\sup_{T \text{ CPTP}} \sum_{i=1}^{2} \alpha_i \cdot \inf_{\rho \text{ pure}} F\big(\rho, T_i(\rho)\big)$

Average $\sup_{T \text{ CPTP}} \sum_{i=1}^{2} \alpha_i \cdot \mathbb{E}_{\rho \text{ pure}} F\big(\rho, T_i(\rho)\big)$

Approximate Universal Quantum Cloning

Theorem

- ***Worst*** *and* ***Average*** *are equal.*
- *Optimal channels for* ***Worst****/****Average*** *are as in* ***Isotropic****.*

Choi Matrix

Definition
The Choi matrix C_T of a linear map $T : \mathcal{M}_d \to \mathcal{M}_{d'}$ is defined by

$$C_T = (\mathrm{id}_d \otimes T)\Big(\sum_{ij=1}^{d} |ii\rangle\langle jj|\Big).$$

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Theorem

$$T \text{ CPTP}$$

$$\Longleftrightarrow$$

$$C_T \geq 0 \quad \text{and} \quad \mathrm{Tr}_{d'} C_T = I$$

Optimal Cloning Map

Isotropic $$\left\{ p \in [0,1]^2 \,\middle|\, T : \mathcal{M}_d \xrightarrow{\text{CPTP}} \mathcal{M}_d \otimes \mathcal{M}_d,\ \forall \rho \text{ pure}, \begin{array}{l} T_1(\rho) = p_1 \cdot \rho + (1 - p_1)/d \cdot I \\ T_2(\rho) = p_2 \cdot \rho + (1 - p_2)/d \cdot I \end{array} \right\}$$

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Theorem
*The Choi matrix C_T of an optimal channel for **Worst**/**Average** is of the form*

$$C_T = \sum_{\sigma \in \mathfrak{S}_3} c_\sigma \cdot R^\Gamma(\sigma),$$

where $R(\sigma)(v_1 \otimes v_2 \otimes v_3) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes v_{\sigma^{-1}(3)}$.

Quantum Cloning Region

Isotropic $\left\{ p \in [0,1]^2 \,\middle|\, T : \mathcal{M}_d \xrightarrow{\text{CPTP}} \mathcal{M}_d \otimes \mathcal{M}_d,\ \forall \rho \text{ pure},\ \begin{matrix} T_1(\rho) = p_1 \cdot \rho + (1 - p_1)/d \cdot I \\ T_2(\rho) = p_2 \cdot \rho + (1 - p_2)/d \cdot I \end{matrix} \right\}$

Theorem
Isotropic *is the union of a family of ellipses indexed by* $\lambda \in [0, d]$, *given by*

$$\frac{t^2}{a^2} + \frac{(s-c)^2}{b^2} \leq \lambda^2 \quad \text{and} \quad \begin{cases} s = p_1 + p_2 \\ t = p_1 - p_2 \end{cases}$$

with $a = \frac{1}{\sqrt{d^2-1}}$, $b = \frac{1}{d^2-1}$ *and* $c = \frac{\lambda d - 2}{d^2-1}$.

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Quantum Cloning Region (qubits)

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