Quantum information & quantum gravity

1. The black hole information paradox
2. Holographic quantum gravity
3. Entanglement entropy in holography
4. Tensor network toy models
5. Reconstruction, error correction & locality
6. Black hole information paradox in holography

1 The black hole information paradox

Classically:

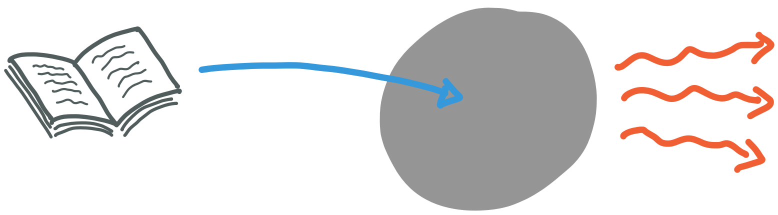
- nothing escapes a black hole
- black hole has no degrees of freedom (no hair) beyond mass, charge, angular momentum

So: if something falls into a black hole it never comes out and the information is lost to outside observers

A semiclassical computation shows:

- black holes radiate (Hawking radiation)
- this radiation is purely thermal, containing no information about the black hole
- after some finite time, the black hole evaporates.

Black hole information paradox (BHIP)



If we collect all Hawking radiation, can we reconstruct the book after evaporation?
It appears we can not... contradicting unitarity of the universe! → a sharper version of the paradox: AMPS (Almheiri et al.)

Solutions

- accept information loss
- quantum gravity corrections at the final stage of evaporation, information comes out at the end
- Hawking radiation stops at some point
- no smooth horizon

Solution needs a theory of quantum gravity
at scale $l_P \approx 1.6 \times 10^{-35}$ m
en strength $G_N \approx 6.7 \times 10^{-11}\ m^3 kg^{-1} s^{-2}$

Way beyond experimentally accessible scales, but relevant near black hole singularity

The Page curve

Consider R radiation system
B black hole interior

Assume unitarity (so the universe is a pure state)

$$|\psi_{BR}\rangle = \sum_i \sqrt{\lambda_i}\, |e_i\rangle \otimes |f_i\rangle$$ (Schmidt decomposition)

(entanglement) entropy of the radiation

$$H(R)_\psi = -\sum \lambda_i \log \lambda_i$$

Hawking radiation = thermal ≈ maximally mixed

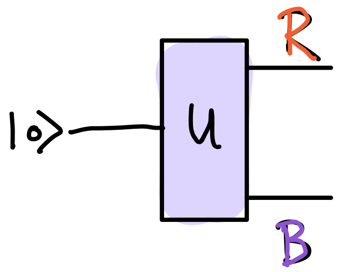
$$H(R)_\psi = \log(d_R)$$

On the other hand:

$$H(R)_\psi = H(B)_\psi \leq \log(d_B)$$

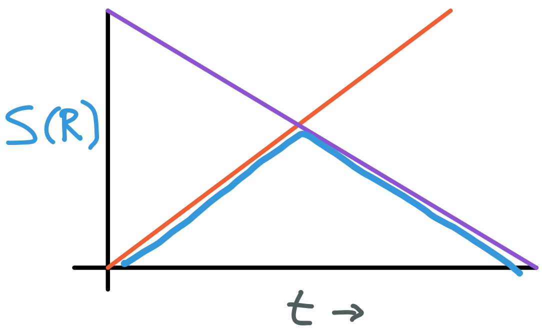
which goes to zero at the end!

Proposal: model black hole evolution by random unitary this makes some sense: black holes equilibrate very quickly, so the corresponding quantum evolution should be very 'chaotic' and random unitaries (or better: random Hamiltonians) provide good intuition.



with high probability for large d_R, d_B

$$H(R) \approx \min(\log d_R, \log d_B)$$



Black hole entropy

Hawking radiation $\longrightarrow$ thermodynamic entropy

String theory $\longrightarrow$ counting microstates

$$S_B = \frac{\text{Area}}{4 G_N}$$

in $c=1$, $k_B=1$, $\hbar=1$ units

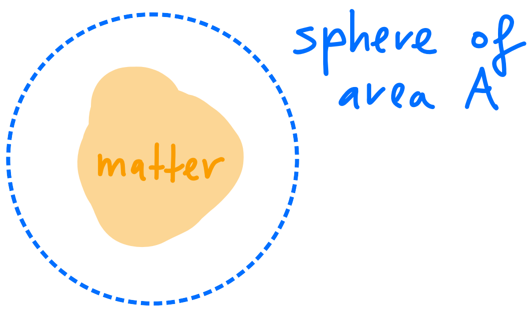
for a black hole of solar mass

$S_{BH} \sim 10^{78}$

$t_{evap} \sim 10^{66}$ years

② Holographic quantum gravity

$S_B = \frac{\text{Area}}{4 G_N}$ is surprising! (Susskind)



Suppose matter is not a black hole & contained in a sphere of area A. Collapse in additional matter. Second law implies
no entropy decrease)

$$\text{entropy} \leq \frac{A}{4 G_N}$$

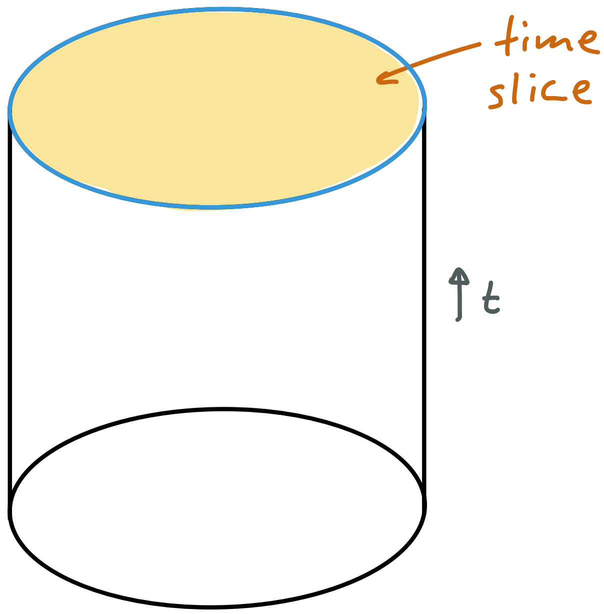
Would expect volume scaling...

Suggests that 'secretly' quantum gravity theory has one dimension fewer: holography

AdS/CFT

(Maldacena, ..)

provides a concrete realization of this idea in certain string theories.



quantum gravity on bulk AdS_{d+1} space

$\updownarrow$

nongravitational conformal field theory (CFT_d) on the boundary

conjectured equality of partion functions

$$Z_{AdS}[\phi_0] = Z_{CFT}[\phi_0]$$

$\shortparallel$ path integral with boundary conditions $\phi = \phi_0$

$\shortparallel$ path integral with ϕ_0 as sources for operators

weak gravity $\longleftrightarrow$ strongly interacting CFT

G_N small

central charge ~ # degrees of freedom large

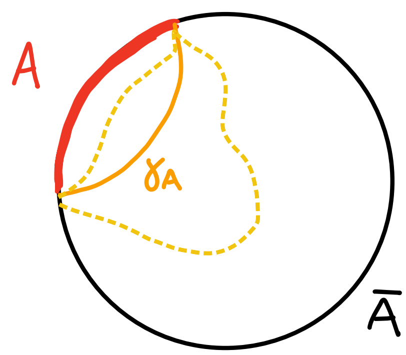
For small G_N

$$Z_{AdS} = \int d\phi \, e^{-S_{grav}(\phi)} \approx e^{-I(\phi)}$$

saddle point = gravity geometry

"Dictionary" relating concepts on both sides

③ Entanglement entropy in holography



Entropy $H(\rho_A)$?

Claim: computed as

$$H(\rho_A) = \frac{\text{Area}(\gamma_A)}{4 G_N}$$

bulk surface with $\partial\gamma_A = \partial A$ of minimal area

Ryu Takayanagi formula

Here we assume time-independence

Comments:

① both $H(\rho_A)$, Area(γ_A) are infinite
→ need to regulate, diverge in the same way

② sanity check: AdS_3/CFT_2
$H(\rho_A) \sim \log L$ for interval of length L
γ_A is a geodesic in hyperbolic disk → length $\log L$

The replica trick

Basic fact: $\mathrm{tr}[\rho^k] = \mathrm{tr}[\rho^{\otimes k}\, \tau_k]$

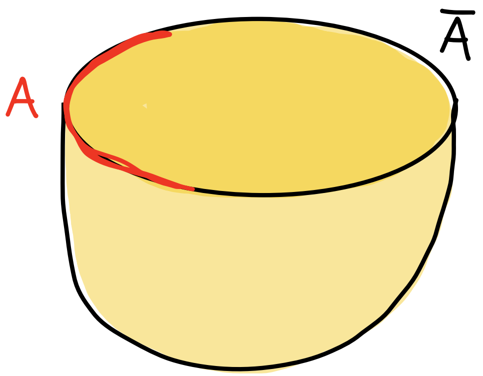
τ_k permutes cyclically

$\tau_k |i_1 \dots i_k\rangle = |i_2 \dots i_k\, i_1\rangle$

Want to compute Rényi entropy

$$H_k(A) = \frac{1}{1-k} \log\left(\mathrm{tr}\,\rho_A^k\right) \qquad \lim_{k\to 1} H_k(A) = H(A)$$

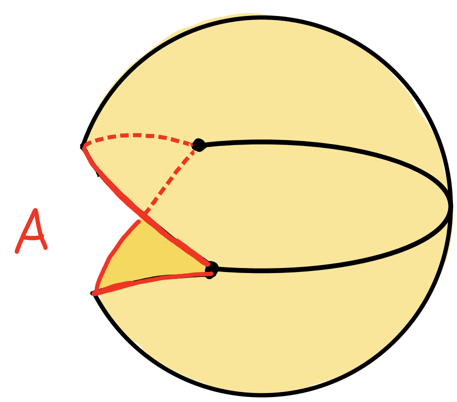
Boundary state is prepared by path integral



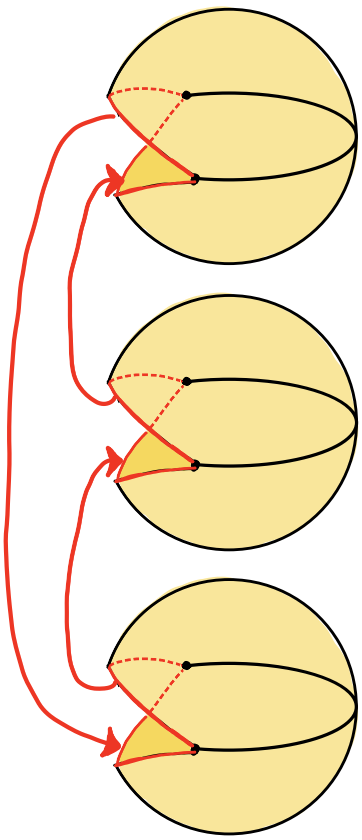
$= |\psi\rangle$

Reduced density matrix

$\rho_A = \mathrm{tr}_{\bar{A}}\, |\psi\rangle\langle\psi|$

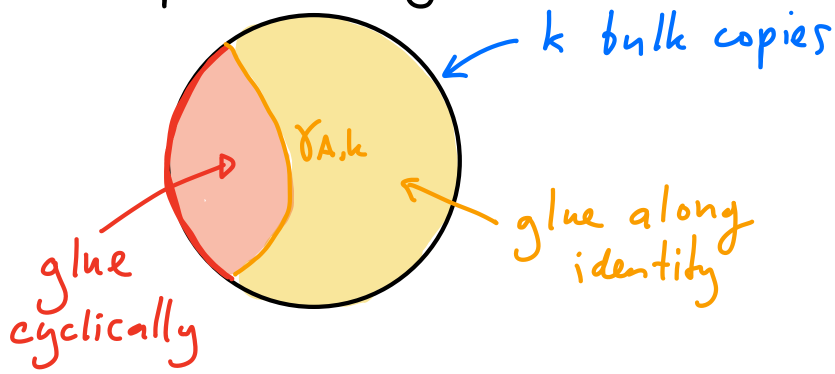


$\mathrm{tr}[\rho_A^k]$: take k copies and glue cyclically



AdS/CFT: (Maldacena-Lewkowycz)

this is computed by a corresponding bulk path integral!



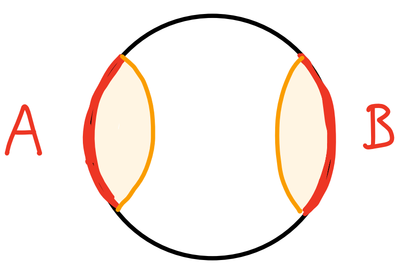
small G_N : $\mathrm{tr}[\rho_A^k] \approx e^{-I_{grav}}$

glued geometry

the surface $\gamma_{A,k}$ depends on k!

Analyzing this action & taking a limit $k \to 1$ gives RT-formula.

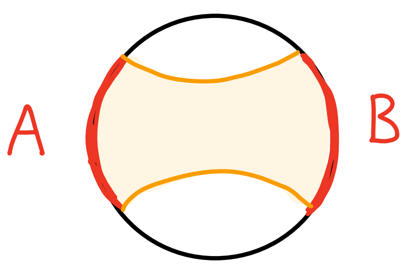
Interesting example of RT formula:



$\gamma_{AB} = \gamma_A \cup \gamma_B$

"disconnected"

$I(A:B) = H(A) + H(B) - H(AB)$
≈ 0



$\gamma_{AB} \neq \gamma_A \cup \gamma_B$

"connected"

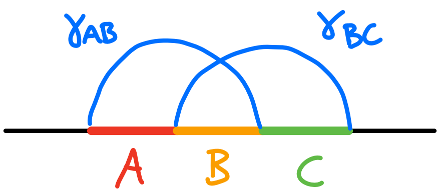
$I(A:B) = H(A) + H(B) - H(AB)$
> 0

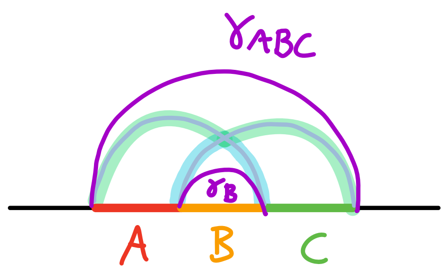
This is one of the motivations for the slogan

"Boundary entanglement corresponds to the connectedness of the bulk space"

Entropy inequalities

Can easily prove strong subadditivity for holographic states (Headrick)





$$H(AB) + H(BC) \geq H(ABC) + H(B)$$

$\geqslant \gamma_B$

$\geqslant \gamma_{ABC}$

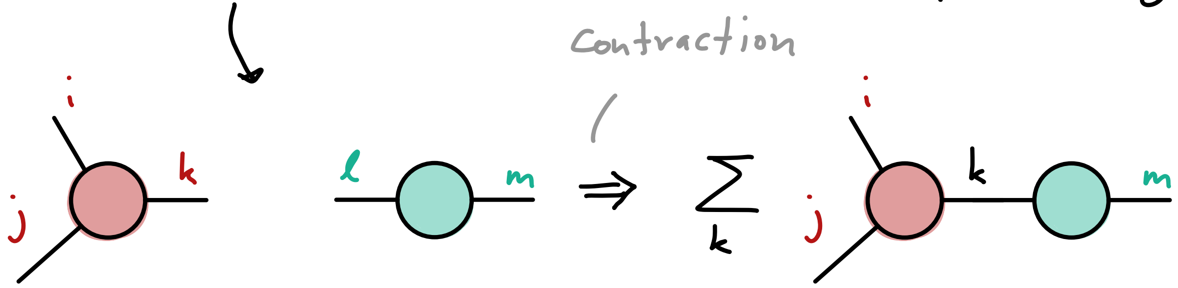
In fact, there are entropy inequalities beyond SSA which are valid for holographic states, but not in general. There is a complete finite list of such inequalities for any number of parties. (Bao et al.)

④ Tensor network toy models

Interesting quantum information structure in holography, but complicated quantum gravity computations. Can we find simple finite dimensional models with similar behaviour?

(Pastawski et al., Swingle)

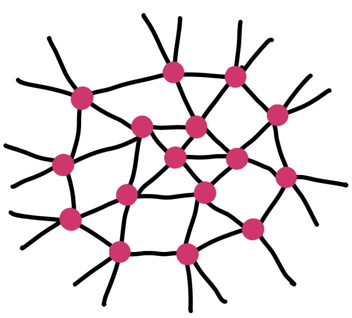
Tensor networks as toy models for holography



A_{ijk} $\qquad$ $B_{\ell m}$ $\qquad$ $\sum_k A_{ijk} B_{km}$

$\sum_{i,j,k=1}^{D} A_{ijk} |ijk\rangle$ $\qquad$ D = bond dimension

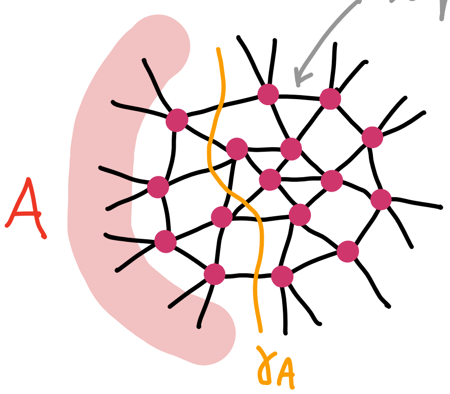
Tensor network = tensors contracted along some network



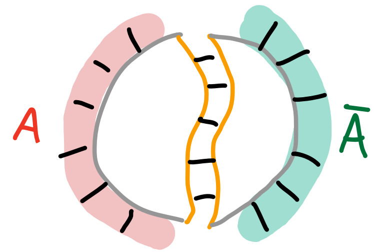
← state lives on "dangling edges"

Idea: tensors in the middle represent bulk, dangling edges are boundary theory

Examples: HaPPY code, MERA, dimer models suppose each edge has bond dimension D



Any cut in the network separating A from its boundary complement gives an entropy upper bound



$$\operatorname{rank}(\rho_A) \leq D^{|\gamma_A|}$$

so $H(\rho_A) \leq \log(D)\,|\gamma_A|$

In particular, if γ_A is a minimal cut for A

$$H(\rho_A) \leq \log(D)\,|\gamma_A|$$

Claim: for independent uniformly random tensors

$$H(\rho_A) \approx \log(D)\,|\gamma_A| \qquad \text{(Hayden et al.)}$$

Toy model for RT-formula!

Sketch proof:

two main ingredients

1. $\operatorname{tr}[\rho^2] = \operatorname{tr}[\rho^{\otimes 2} \mathcal{F}] \qquad \mathcal{F}|ij\rangle = |ji\rangle$

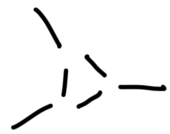
2. $\mathbb{E}\,|\psi\rangle\langle\psi| = \frac{I}{d} \qquad \mathbb{E}(|\psi\rangle\langle\psi|)^{\otimes 2} = \frac{I + \mathcal{F}}{d(d-1)}$

   for uniformly (Haar) random $|\psi\rangle$ of dimension d

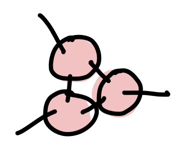
   (follows from unitary invariance)

PEPS (projected entangled pairs) construction of the state: graph $G = (V, E)$

- start with maximally entangled pairs on each edge: $\bigotimes_{e \in E} |\varphi_e\rangle \quad |\varphi_e\rangle = \frac{1}{\sqrt{D}} \sum |ii\rangle$



- project onto tensor $D^{\frac{1}{2}d(v)} |\psi_v\rangle$ at each vertex v, where $d(v)$ is the degree of v, and $|\psi_v\rangle$ is a Haar random tensor of dimension $D^{d(v)}$



$$|\Psi\rangle = \left(I \otimes \bigotimes_{v \in V} D^{\frac{1}{2}d(v)} \langle\psi_v| \right) \bigotimes_{e \in E} |\varphi_e\rangle$$

$$\rho = |\Psi\rangle\langle\Psi|$$
$$= \mathrm{tr}\Big[\big(I \otimes \bigotimes_{v \in V} D^{d(v)} |\psi_v\rangle\langle\psi_v|\big) \bigotimes_{e \in E} |\varphi_e\rangle\langle\varphi_e| \Big]$$

$$-\log(\mathrm{tr}[\rho_A^2]) = H_2(\rho_A) \leq H(\rho_A) \leq \log(D)|\gamma_A|$$

Jensen inequality:

$$-\mathbb{E}\log(\mathrm{tr}[\rho_A^2]) \geq -\log(\mathbb{E}\,\mathrm{tr}[\rho_A^2])$$

So we need to compute:

$$\mathbb{E}\,\mathrm{tr}[\rho_A^2] \overset{1}{=} \mathbb{E}\,\mathrm{tr}[F_A\, \rho_A^{\otimes 2}]$$

$$= \mathbb{E}\,\mathrm{tr}[F_A \otimes I_{\bar{A}}\, \rho^{\otimes 2}]$$

$$= \mathbb{E}\,\mathrm{tr}\Big[F_A \otimes I_{\bar{A}} \otimes \bigotimes_{v\in V} D^{2d(v)} |\psi_v\rangle\langle\psi_v|^{\otimes 2} \Big(\bigotimes_{e\in E} |\varphi_e\rangle\langle\varphi_e|^{\otimes 2}\Big)\Big]$$

$$\overset{2}{\approx} \mathrm{tr}\Big[F_A \otimes I_{\bar{A}} \bigotimes_{v\in V} (I_v + F_v)\Big(\bigotimes_{e\in E} |\varphi_e\rangle\langle\varphi_e|^{\otimes 2}\Big)\Big]$$

$$= \sum_{\Gamma \subset V} \mathrm{tr}\Big[F_{A\cup\Gamma} \otimes I_{\bar{A}\cup\bar{\Gamma}} \Big(\bigotimes_{e\in E} |\varphi_e\rangle\langle\varphi_e|^{\otimes 2}\Big)\Big]$$

For each edge $e=(xy)$:

$x,y \in A\cup\Gamma$: $\mathrm{tr}[(|\varphi_e\rangle\langle\varphi_e|)^2] = 1$

$x,y \in \bar{A}\cup\bar{\Gamma}$: $\mathrm{tr}[\,|\varphi_e\rangle\langle\varphi_e|]^2 = 1$

$x \in A\cup\Gamma,\ y \in \bar{A}\cup\bar{\Gamma}$: $\mathrm{tr}[(\mathrm{tr}_y|\varphi_e\rangle\langle\varphi_e|)^2] = \frac{1}{D}$

$$= \sum_{\gamma \text{ cut for } A} D^{-|\gamma|} = D^{-|\gamma_A|}(1 + O(D^{-1}))$$

if γ_A is a minimal cut (assume unique for convenience)

This can be used to show that for large D whp

$$H(\rho_A) = \log(D)\,|\gamma_A| - o(1)$$

Comment:

1. $$\mathbb{E}\, \mathrm{tr}[\rho_A^2] \approx \sum_{\gamma} D^{-|\gamma|} = \sum_{\{S_v = \pm 1\}} e^{-\log(D) \sum_{e=(xy)} \frac{1}{2} - \frac{1}{2} S_x S_y}$$

   is a partition function for a classical spin model with domain walls

2. the dominant configuration is analogous to the saddle point glueing in the gravitational replica trick.

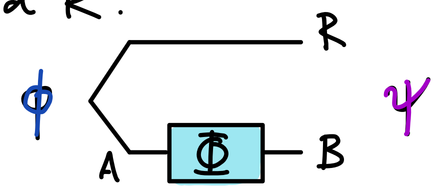
3. similar computation for $H_k(\rho_A)$, find ~ independent of k (flat spectrum). Can make the model more realistic by replacing link states $|\phi_e\rangle$ by non-max entangled states (Cheng et al.)

5 Reconstruction, error correction & locality

Back to the black hole information paradox: does the book radiate out of the black hole?

Recovery: System A, channel $\Phi_{A\to B}$
then we can recover A if there exists a channel $\Psi_{B\to A}$ s.t. $\Psi_{B\to A} \circ \Phi_{A\to B} = \mathcal{I}_A$

We can 'recover' A if for a maximally entangled state ϕ_{AR} the state $\psi_{BR} = \Phi_{A\to B} \otimes I_R(\phi_{AR})$ is maximally entangled between B and R.

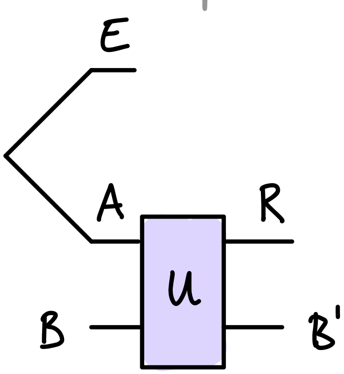


Decoupling criterion:

If ψ_{BRE} is a purification of ψ_{BR}, then

$$\psi_{BR} \text{ max entangled} \leftrightarrow \psi_{RE} = \frac{I_R}{d_R} \otimes \psi_E$$

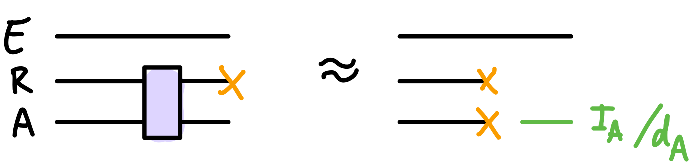
We can recover $\Phi_{A\to B}$ if no information leaks to the complementary channel



Alice's diary falls into the black hole. How much radiation should we collect to reconstruct it?

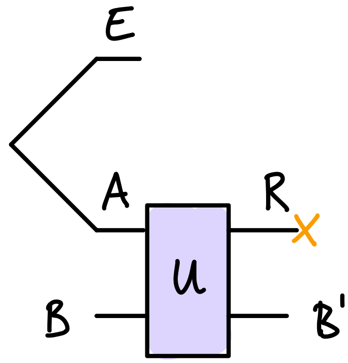
Decoupling theorem

ρ_{ARE} any state, random unitary U_{BR}



$$\mathbb{E}_U \left\| \mathrm{tr}_R\left[U_{AR}\,\rho_{ARE}\,U_{AR}^\dagger\right] - \rho_E \otimes \frac{I_A}{d_A} \right\|_1^2 \le \frac{d_A d_E}{d_R}\, 2^{-H_2(\rho)}$$

Pure initial state: can recover if



$\approx$

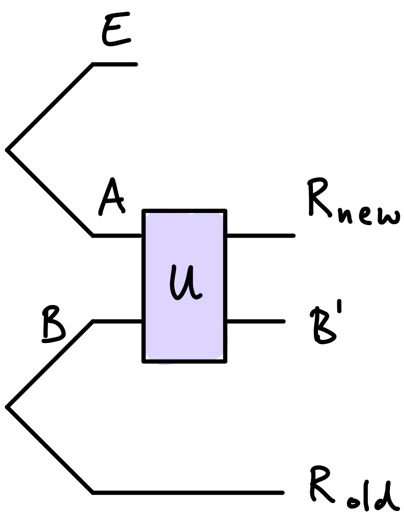
E — $\frac{I_E}{d_E}$

B' — $\frac{I_B}{d_B}$

need $d_R \gtrsim d_A d_{B'} \cdot \frac{1}{\varepsilon^2}$ for error ε

$\longrightarrow$ If A is n qubits, then information comes out $n + 2\log(\frac{1}{\varepsilon})$ qubits after Page time

Old black hole: already collected a lot of radiation

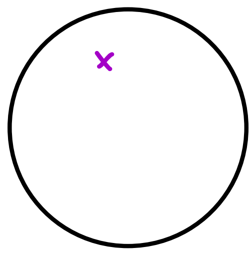


Decoupling: information comes out if R_{new} is $n + \log(\frac{1}{\varepsilon})$ qubits

$d_{R_{old}} - d_B$

"old black holes are information mirrors"

Recovery in holography



act with operator ϕ in the bulk

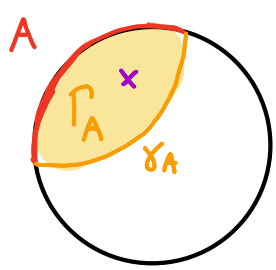
$\downarrow$

there should be a corresponding boundary operator O_ϕ

Where does O_ϕ act? What is the meaning of locality on both sides of the duality?

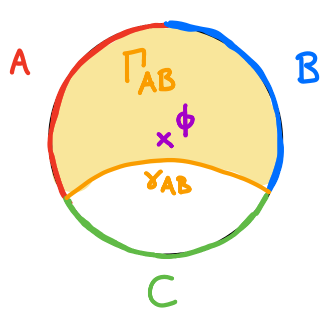
Subregion duality

Let Γ_A be the region between A and minimal surface γ_A, then any ϕ acting in Γ_A can be reconstructed as an operator O_ϕ on A



Γ_A is the entanglement wedge

strange features:



can reconstruct ϕ as O_{AB}, O_{BC} or O_{AC}

$\longrightarrow$ no common support... $AB \cap BC \cap AC = \emptyset$

Should be interpreted as an error correcting code

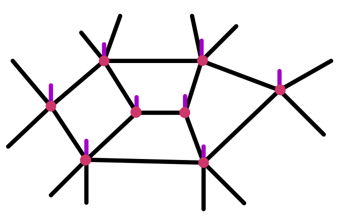
a subset of bulk states (eg a classical geometry with low energy excitations) = code subspace encoded in a set of boundary states.

Information encoded in Γ_A is protected against erasure of $\bar{A}$

Note that $\phi \to O_\phi$ not unique since we restrict to a subspace

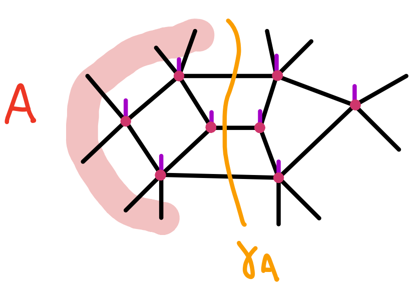
Tensor network model for QEC

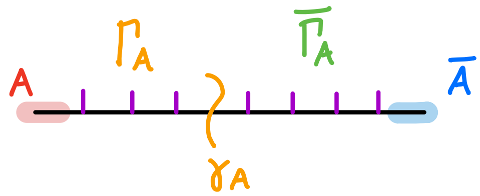
add dangling bulk legs



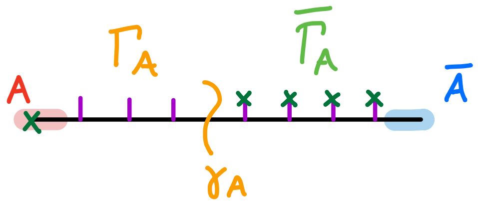
this defines an encoding map from bulk to boundary

Let us take bulk legs of dimension $d \ll D$ = bond dimension





Decoupling criterion for recovery:



same calculations as before

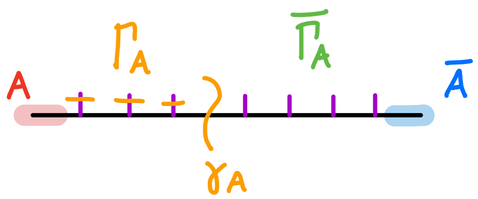
$$\rho_{\Gamma_A \bar{A}} \approx \frac{I_{\Gamma_A}}{d_{\Gamma_A}} \otimes \rho_{\bar{A}}$$

$\Longrightarrow$ can recover Γ_A from A

Quantum RT-formula

If one has a mixed bulk state, how to compute entropies?

Consider RTN, max mixed states at bulk sites



also need to cut bulk edges

$\rightarrow$ exactly the same computation

$$H(\rho_A) \approx \log(D)|\gamma_A| + \log(d)|\Gamma_A|$$

In holography: quantum RT-formula

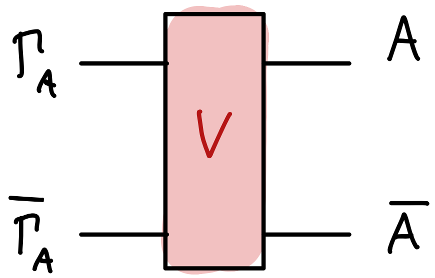
$$H(\rho_A) = \frac{\mathrm{Area}(\gamma_A)}{4G_N} + H_{bulk}(\Gamma_A)$$

bulk entropy in entanglement wedge

(Harlow et al.)

In fact quantum RT-formula is equivalent to subregion duality!

Basic version of this:



V is encoding isometry

for $\rho_{\Gamma_A \bar{\Gamma}_A}$ let $\rho_{A\bar{A}} = V \rho_{\Gamma_A \bar{\Gamma}_A} V^\dagger$

Quantum RT $\quad H(\rho_A) = c + H(\rho_{\Gamma_A})$

for all ρ $\qquad$ for some constant c

$$\Updownarrow$$

Recovery $\quad \exists$ channel Φ_A such that

$$\Phi_A(\rho_A) = \rho_{\Gamma_A} \quad \forall \rho$$

⑥ Black hole information paradox in holography

AdS/CFT strongly suggests unitarity

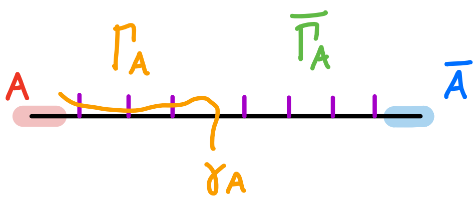
More generalizations of RT formula

- time-dependent version: HRT formula (Hubeny et al.)

  (will not discuss here! also need to consider different choices of spatial slice)

- versions with large bulk entropy:

  consider again RTN, but now with bulk dimension d not small compared to D



again: exactly the same computation (interpreting the bulk dangling edges as boundary edges)

when computing appropriate γ_A need to minimize

$$\log(D)\,|\gamma_A| + \log(d)\,|\Gamma_A|$$

This suggests in holography

$$H(A) = \min_{\gamma_A} \left\{ \frac{\mathrm{Area}(\gamma_A)}{4G_N} + H_{\mathrm{bulk}}(\Gamma_A) \right\}$$

quantum extremal surface (QES) formula

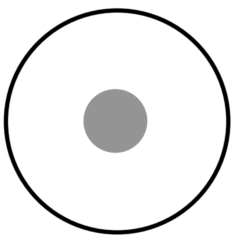
one should really consider time dependent version ...

Note: evidence for QES formula is more limited than for regular RT-formula

QES formula & islands

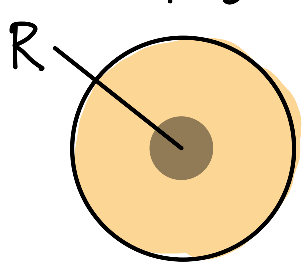
(Penington, ...)

Consider a set-up in AdS/CFT with a radiating black hole. We collect the radiation at the boundary, and we consider the full boundary (without the radiation!)



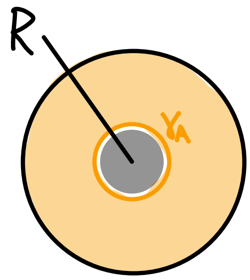
Two potential surfaces:

1 Empty surface:



$$\underbrace{\frac{\mathrm{Area}(\emptyset)}{4G_N}}_{0} + \underbrace{H_{\mathrm{bulk}}(\Gamma_A)}_{\log(d_R)}$$

② Surface ~ at horizon

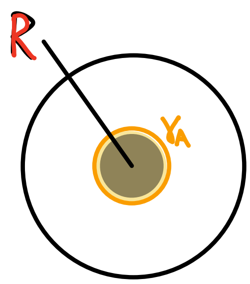


$$\frac{\text{Area}(\gamma_A)}{4G_N} + \underbrace{H_{bulk}(\Gamma_A)}_{=0} = \frac{\text{Area (BH)}}{4G_N}$$

QES gives: $H = \min\left\{ \frac{\text{Area (BH)}}{4G_N}, \log(d_R) \right)$

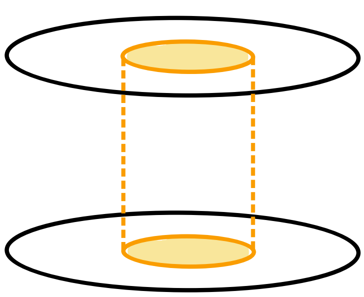
$\longrightarrow$ Page curve!

Alternative: consider the radiation system R
then we can have "islands"



by subregion duality: can reconstruct black hole interior from R!

In this case, the saddle in the replica trick should look like:



"replica wormholes"

$\hookrightarrow$ very explicit in JT gravity, a 1+1-dimensional model which is very similar to random tensor model

Comment: the information processing tasks in holography are one-shot tasks.

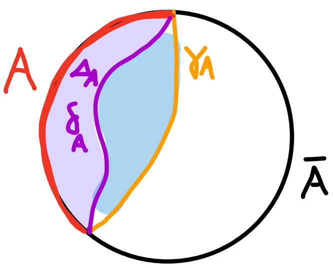
small G_N ~ many copy asymptotics in QI

(Akers-Penington)

In the presence of large bulk entropy, one has to be careful and sometimes use smooth min/max entropies (e.g. superposition of BH & no BH)

e.g. can reconstruct Γ_A on A if for any surface δ_A contained in Γ_A, with associated interior region Δ_A we have

$$H^{\varepsilon}_{max}(\Gamma_A \setminus \Delta_A \mid \Delta_A) < \frac{\text{Area}(\delta_A) - \text{Area}(\gamma_A)}{4 G_N}$$



References

this talk is partly based on previous lectures by Michael Walter, you can find slides at his home page.

Black hole information paradox

Harlow - Jerusalem lectures on black holes & quantum information

Hayden, Preskill - Black holes as mirrors

Quantum information in AdS/CFT

Harlow - TASI lectures on the emergence of the bulk in AdS/CFT

Chen, Czech, Wang - Quantum information in holographic duality

Islands & recent progress on the BHIP

Almheiri, Hartman, Maldacena, Shaghoulian, Tajdini - The entropy of Hawking radiation

these mostly are reviews/lecture notes, containing references to original work. If you are interested in a specific topic, feel free to email me, then I am happy to suggest more detailed sources!

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