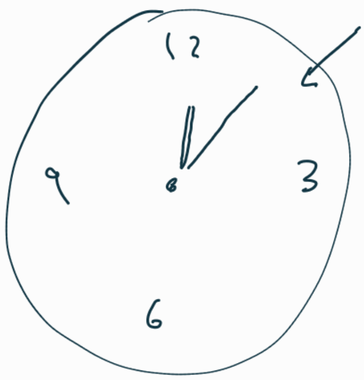
Lyon 28/June/2022

Clock tutorial by M. Woods

(see slides at end of notes)

- v. active area of research. Will aim to cover some of the most relevant results in the field.
- Q. clock emittes "ticks" at regular intervals.
- Explain that we will not discuss stopwatches.

- "ticks" represent changes in a classical register. [show clock slide]



clockface or "register"

(dim = $N_T + 1$)

"clockwork"

(dim = d)

"Quantum".

• clocks go "tick-tick tick" and the register changes between distinguishable states w/ each "tick".

• Reg. classical: $\{|k\rangle_R\}_{k=0}^{N_T}$ → associate a particular n° of ticks

- definite register state:

$\rho_c(t) \otimes |j\rangle\langle j|_R$ ← clock has ticked j times. @ time t.

- We want ensembles of different reg. states

$$\sum_{j=0}^{N_T} P_j(t)\, \rho^0_{j,c}(t) \otimes |j\rangle\langle j|_R$$ ← this describes our

- Dynamics:

$$\mathcal{M}^t_{CR\to CR}\big(\underbrace{\rho^0_c \otimes |0\rangle\langle 0|_R}_{\text{initial state of clockwork.}}\big) = \sum_{j=0}^{N_T} P_j(t)\, \rho_{j,c}(t) \otimes |j\rangle\langle j|_R$$

- Point out no zeno effect.

Axiomatic Definition of clock

- Useful to extend register states to integers via

$|j\rangle_R := |j \bmod N_T+1\rangle_R$ for $j \in \mathbb{Z} \setminus \{0, 1, \ldots, N_T+1\}$.

- Now introduce 4 axioms which restrict the dynamics to something we can think of as a clock.

1) The self-timing condition: For all $t_1, t_2 \geq 0$:

$$\mathcal{M}^{t_1+t_2}_{CR\to CR} = \mathcal{M}^{t_2}_{CR\to CR} \circ \mathcal{M}^{t_1}_{CR\to CR}.$$

(Markovian condition).

- Necessary for clock to be autonomous (no external timing)

2) The zeroth order condition:

$$\mathcal{M}^0_{CR\to CR} = I_{CR}$$

$$\lim_{t\to 0} \| M^t_{CR\to CR} - I_{CR} \| = 0$$

3) The leading order condition:

$$P(l,t|k) := \mathrm{tr}\left[M^t_{CR\to CR}(\rho^0_C \otimes |k\rangle\langle k|_R) \, |l\rangle\langle l|_R \right]$$

$\tilde{p}^{(n)}_l(t)$

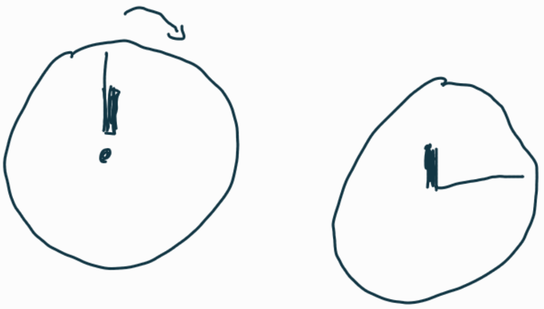
$$\lim_{t\to 0^+} \frac{\sum_{\substack{l=0 \\ l\notin\{k,k+1\}}}^{N_T} P(l,t|k)}{P(k+1,t|k)} = 0 \quad \forall\, t\geq 0,\ k\in\mathbb{N}.$$

4) Register translation symmetry condition

$$\mathrm{tr}_R\left[M^t_{CR\to CR}(\rho^0_C \otimes |k\rangle\langle k|_R) \cdot |k+l\rangle\langle k+l|_R \right] =$$

$$\mathrm{tr}_R\left[M^t_{CR\to CR}(\rho^0_C \otimes |k'\rangle\langle k'|_R) \, |k'+l\rangle\langle k'+l|_R \right], \quad \forall\, t\in\mathbb{R},\ k,k',l \in\mathbb{Z}$$

10 ticks



10 ticks

— 15 min

Structure thm [Explicit clock representation]. The pair $(\rho^0_C;\ M^t_{CR\to CR},\ t\geq 0)$ form a clock (i.e. satisfy axioms 1) to 4)) if and only if $\exists$ a Hermitian operator H, and 2 sets of linear operators $(L_j)_{j=1}^{N_L}$, $(J_j)_{j=1}^{N_L}$ acting on $\mathcal{H}_C$

s.t. $\forall\, t\geq 0$ and $N_T\in\mathbb{N}$:

$M^t \quad\quad t\,\delta_{CR}(\cdot)$ Jump ops

$$\mathcal{M}_{CR\to CR}^{(\cdot)} = e$$

↙ noise

$$\mathcal{L}_{CR}(\cdot) = -i\,[\tilde{H}, (\cdot)] + \sum_{j=1}^{N_L} \tilde{L}_j(\cdot)\tilde{L}_j^{\dagger} - \frac{1}{2}\{\tilde{L}_j^{\dagger}\tilde{L}_j, (\cdot)\}$$

$$+ \sum_{j=1}^{N_L} \tilde{J}_j(\cdot)\tilde{J}_j^{\dagger} - \frac{1}{2}\{\tilde{J}_j^{\dagger}\tilde{J}_j, (\cdot)\}$$

↙ Jump ops ticks

$$\tilde{H} := H_c \otimes \mathbb{1}_R, \quad \tilde{L}_j = L_j \otimes \mathbb{1}_R, \quad \tilde{J}_j = J_j \otimes O_R$$

$$O_R := |1\rangle\langle 0|_R + |2\rangle\langle 1|_R + |3\rangle\langle 2|_R + \dots + |N_T\rangle\langle N_T - 1|_R + |0\rangle\langle N_T|.$$

If $|0\rangle\langle N_T|$ → 0 get cut-off reg. clock.

~ Special form of Lindblad master (also known as G.K.S.L master eq.)

{- G. Lindblad
{- V. Gorini, A. Kossakowski, and E. Sudarshan.

Clock accuracy

- Want to measure how accurate the clock is.
- Introduce "delay function of kth tick" $P^{(k)}_{tick}(t)$:

* $P^{(k)}_{tick}(t)\cdot \delta t$ = probability of ticking $k-1$ times in interval $[0, t)$ followed by ticking once in time interval $[t, t+\delta t]$.

- Maths : $$\rho_{CR}^{(k-1)}(t) = \langle k-1|_R \left(\mathcal{M}^t_{CR\to CR}\left(\rho^0_{CR} \otimes |0\rangle\langle 0|_R\right)\right) |k-1\rangle_R$$

$$p^{(\kappa)}_{\text{tick }\kappa}(t) := \lim_{\delta t \to 0} \frac{\operatorname{tr}\left[|\kappa\rangle\langle\kappa|_R \left(\mathcal{M}^{\delta t}_{CR \to CR}\left(\rho^{(\kappa-1)}_{CR}(t)\right) - \rho^{(\kappa-1)}_{CR}(t) \right) \right]}{\delta t}$$

- A usefull measure of accuracy for the κth tick is R_κ:

$$R_\kappa := \left(\frac{\mu_\kappa}{\sigma_\kappa}\right)^2 \; ; \quad \mu_\kappa = \int_0^\infty dt \cdot p^{(\kappa)}_{\text{tick }\kappa}(t) \cdot t$$

$$\sigma_\kappa = \sqrt{\int_0^\infty dt \; p^{(\kappa)}_{\text{tick }\kappa}(t) \; (t-\mu_\kappa)^2}$$

- Note: under $\mathcal{M}^{t}_{CR\to CR} \mapsto \mathcal{M}^{at}_{CR\to CR}$, we find $\begin{cases} \mu_\kappa \mapsto a\mu_\kappa \\ \sigma_\kappa \mapsto a\sigma_\kappa \end{cases}$

$\therefore \; R_\kappa \mapsto R_\kappa$

Mapping $t \to at$ corresponds to a re-scaling of time. We want our maps to be invariant under said re-scaling.

Reset clocks

[Slide: structure theorem]

- There are clocks where clockwork is reset to its initial state ρ^0_c after every time the clock ticks:

$$\sum_{j=1}^{N_L} \mathcal{I}_j \, \rho_c \, \mathcal{I}_j^+ \propto \rho^0_c \qquad \forall \, \rho_c \in S(\mathcal{H}_c)$$

w/ ρ^0_c = initial state of the clockwork.

- Note that $\underbrace{p^{(\kappa)}(t_\kappa | t_{\kappa-1}) \, \delta t}_{\text{prob. of } \kappa^{\text{th}} \text{ tick occurring during interval } [t_\kappa, t_\kappa+\delta t] \text{ given that } (\kappa-1)^{\text{th}} \text{ tick occurred @ time } t_{\kappa-1}} = \underbrace{p^{(1)}_{\text{tick }1}(t_\kappa - t_{\kappa-1}) \, \delta t}_{}$ → Delay function 1st tick shifted in time

• consequences are:

Let $P_{\text{ticks}}(t_1, t_2)$ be prob. of 1^{st}, 2^{nd} ticks at times t_1, t_2

$$P_{\text{ticks}}(t_1, t_2) = P^{(1)}(t_2|t_1) \cdot P^{(1)}_{\text{ticks}}(t_1) = P^{(1)}_{\text{ticks}}(t_2 - t_1) \cdot P^{(1)}_{\text{ticks}}(t_1).$$

$$P^{(2)}_{\text{tick}}(t_2) = \int_0^{t_2} dt_1\, P_{\text{ticks}}(t_1, t_2) = \int_0^{t_2} dt_1\, P^{(1)}_{\text{ticks}}(t_2 - t_1)\, P^{(1)}_{\text{ticks}}(t_1).$$

half way 1h (15 min break)

∴ by induction:

$$P^{(k)}_{\text{ticks}}(t_k) = \int_0^{t_k} dt_{k-1} \cdot \int_0^{t_{k-1}} dt_{k-2} \cdots \int_0^{t_3} dt_2 \int_0^{t_2} dt_1$$

$$P^{(1)}_{\text{ticks}}(t_k - t_{k-1}) \cdot P^{(1)}_{\text{ticks}}(t_{k-1} - t_{k-2}) \cdots P^{(1)}_{\text{ticks}}(t_3 - t_2) \cdot P^{(1)}_{\text{ticks}}(t_1).$$

$$=: \left(P^{(1)}_{\text{ticks}}\right)^{*k}(t_k)$$

now calculate mean and standard dev.

$$\left.\begin{aligned} \mu_k &= k \cdot \mu_1 \\ \sigma_k &= \sqrt{k} \cdot \sigma_1 \end{aligned}\right\} \therefore R_k = \left(\frac{k\mu_1}{\sqrt{k}\,\sigma_1}\right)^2 = k \cdot R_1 \quad (\text{for reset clocks}).$$

classical clocks

Definition: a clock is classical if $\exists\ \{|c_\ell\rangle\}_\ell$

s.t. $\operatorname{tr}_R\left[\mathcal{M}^t_{CR\to CR}(\rho^0_c \otimes |k\rangle\langle k|_R)\right] = \sum_m P_{m,k}(t)\, |c_m\rangle\langle c_m|_c \quad \forall t \geq 0,\ \forall k \in \mathbb{Z}.$

can be viewed as the clock-work suffering decoherence.

⟹ structure lemma: All classical clocks satisfy:

1) $$\mathcal{M}^t_{CR\to CR}\left(\sum_{l,m} P_{l,m}\, |c_l\rangle\langle c_l|_c \otimes |m\rangle\langle m|_R\right) = \sum_{l,m} P'_{l,m}(t)\, |c_l\rangle\langle c_l| \otimes |m\rangle\langle m|_R.$$

2) $$\underbrace{\operatorname{Diag}\left\{\mathcal{M}^t_{CR\to CR}(\rho_c \otimes |0\rangle\langle 0|)\right\}}_{=:\, V_{CR}(t)} = e^{tM} \cdot \underbrace{\operatorname{Diag}\left\{\rho_c \otimes |0\rangle\langle 0|\right\}}_{=:\, V_{CR}(0)}$$

$$\mathcal{M} = \tilde{\mathcal{N}} + \tilde{\mathcal{T}}$$

Tick generator

No tick generator

$$\tilde{\mathcal{N}} = \mathcal{N} \oplus 1_R \,, \quad \tilde{\mathcal{T}} = \mathcal{T} \oplus 0_R$$

$$[\mathcal{N}]_{mn} = -\delta_{m,n} \langle c_m | \sum_j (L_j^\dagger L_j + J_j^\dagger J_j) | c_m \rangle + \sum_j | \langle c_m | L_j | c_n \rangle |^2$$

$$[\mathcal{T}]_{mn} = \sum_j |\langle c_m | J_j | c_n \rangle|^2$$

<u>Example</u>: Ladder clock

[see slides]

· observe: { 1) it's reset
2) it classical

∴ only need to calculate delay function of 1st tick.

$$\mathrm{Diag}_l\left(\rho^{(0)}_{CR}(t)\right) = \mathrm{Diag}_l\left(|0\rangle\langle 0|_R \left[\mathcal{M}^t_{CR\to CR}\left(\rho^0_C \otimes |0\rangle\langle 0|_R\right)\right] |0\rangle\langle 0|_R \right)$$

$$= \left[e^{t\tilde{\mathcal{N}}} \cdot \mathrm{Diag}(\rho^{(0)}_{CR}) \right]_l$$

$$P^{(1)}_{\text{ticks}}(t) = \lim_{\delta t \to 0} \frac{\mathrm{tr}\left[|1\rangle\langle 1|_R \left(\mathcal{M}^{\delta t}_{CR\to CR}\left(\rho^{(0)}_{CR}(t)\right) - \rho^{(0)}_{CR}(t) \right) \right]}{\delta t}$$

Since clock channel is classical can plug "top line" into "bottom line".

$$= \sum_{l=1}^{d} \left[\mathcal{T} e^{t\tilde{\mathcal{N}}} \mathrm{Diag}(\rho^{(0)}_{CR}) \right]_l = e^{-t} \cdot \underbrace{\frac{t^{d-1}}{(d-1)!}}$$

Poisson distribution of d-1 events in interval [0, t]

- Question for audience: what are the "events"?

[Ans: each event is one "hop" up one step of the ladder.
In each infinitesimal time step we either stay on current step or hop to next step.]

$\therefore \ \mu_1 = d \ ; \ \sigma_1 = \sqrt{d} \implies R_K = K \cdot d.$

• 2 results which characterize the accuracy of classical clocks:

<u>Lemma</u>: For every c. clock w/ accuracy R and dim d
$\exists$ a c. reset clock of dim d and accuracy $R' \geq R$.

- Explain why important.

<u>Thm</u>:

All c. clocks of dim d satisfy

$$R_K \leq K d \quad , \quad K, d \in \mathbb{N}^+.$$

$\therefore$ we see that the simple ladder clock is optimal.

Notation proof: write $R(d+1)$ in terms of $R(d)$.
conclude $R(d+1) = R(d) + 1$.

<u>Q. Clocks</u>

[See slide for Quasi-ideal clock].

• Point out that it is a reset clock.

$\therefore$ only need to calculate accuracy of 1st tick.

$$P^{(1)}_{tick}(t) = \mathrm{tr}\left[\hat{V} \cdot e^{tG} \rho_c^0 \, e^{tG^\dagger}\right]$$

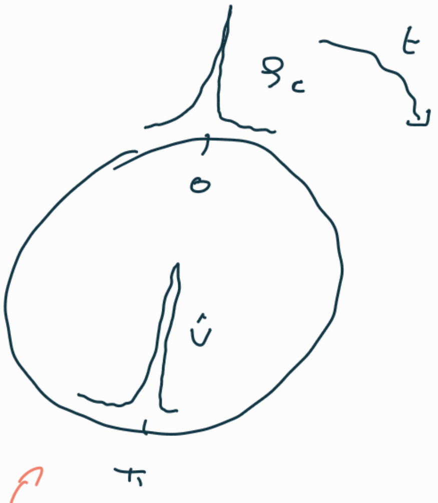
$$\text{w/} \quad \begin{cases} \hat{V} := \sum_{j=0}^{d-1} V_j \, |t_j\rangle\langle t_j| \geq 0 \\ \\ G := -iH - V \end{cases}$$

- For intuition, consider $d = \infty$ case.

choose $H = P$, $\hat{V} = \int_0^{2\pi} \dots$

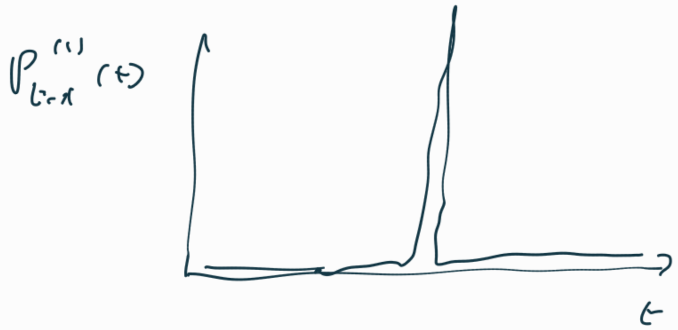
$$\propto \delta(x-\pi)$$

$$\rho_c = |\psi\rangle\langle\psi| , \quad \langle x|\psi\rangle = \delta(x) .$$



explain

wave-function: $\langle x| e^{t(-iH - U(\hat{x}))} |\psi\rangle = e^{-\int_{x-t}^{x} V(x')dx'} \langle x-t|\psi\rangle$



$$\sigma_1 = 0 \quad \therefore \; R_1 = \infty .$$

- Want to mimic in finite dim:

· Def: $V_j = V_0(j/d)$; $V_0 : \mathbb{R} \to \mathbb{R}$, smooth , 2π periodic.

$$\langle t_k | \psi(t_0)\rangle := \psi(t_0, t_k) ;$$

$$\psi(t_0, t) = A\, e^{-\frac{\pi}{\sigma^2}(t-t_0)^2} \cdot e^{i\pi(t-t_0)}$$

Thm: $\langle t_k | e^{t(-iH+V)} |\psi(t_0)\rangle =$

$$= e^{-\int_{k-dt/T_0}^{k} V_0(x)dx} \cdot \psi\left(t_0 - \frac{dt}{T_0} ; k\right) + \varepsilon_v(t,d)$$

$$|\varepsilon_v(t,d)| \le t \cdot \mathrm{poly}(d)\, e^{-\frac{\pi}{4}\left(\frac{d}{\sigma}\right)^2} \cdot \frac{1}{\left(1+\frac{b}{\ln d}\right)^2}$$

$$b \geq \sup_{k \in \mathbb{N}^+} \left(\max_{x \in [0, 2\pi]} \left| V_0^{(k-1)}(x) \right| \right)^{1/k}$$

$\forall\, k \in S_d(t_0 + t d / T_0)$

Explain intuition: [- α - V_0.

Explain proof.

<u>Thm</u>: $\exists$ a sequence of Q. clocks, w/ dim $= 1, 2, 3, \ldots$

s.t. $R_k \sim k d^2$ as $d \to \infty$

<u>Thm</u>: All d-dimensional Q. clocks satisfy

$R_1 \leq c_0 \cdot d^2$

[see slide for low dim]

- other directions: Find the environment and study thermodynamics.

- Physical Realization of Q. clock.

- Connection w/ Lyapunov eq.

Tutorial Overview

- What is a (quantum) clock?
  - Channel with classical register
  - Clock Axioms
  - Structure theorem
  - Examples
  - Clock accuracy
  - Reset clocks
  - Special case of classical clocks
- The ladder clock
- What is the most accurate classical clock?
  - Optimality of reset clocks
  - Upper bounds on classical clocks

References (randomised order):

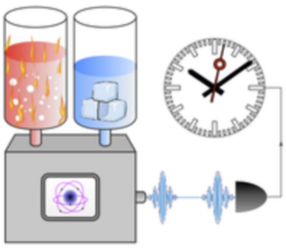
- *The thermodynamics of clocks*, G. J. Milburn [Contemporary Physics Vol 61, (2020)]
- *Quantum clocks are more precise than classical ones*, Mischa P. Woods, Ralph Silva, Gilles Pütz, Sandra Stupar, and Renato Renner [PRX Quantum (2022)]
- *Autonomous Ticking Clocks from Axiomatic Principles*, Mischa P. Woods [Quantum 5, 381 (2021)]
- *Autonomous Quantum Machines & Finite-Sized Clocks*, Mischa P. Woods, Ralph Silva, Jonathan Openheim, [Annales Henri Poincaré (2019)]
- Y. Yang, L. Baumgartner, R. Silva, and R. Renner, *Accuracy enhancing protocols for quantum clocks*, (2019), [arXiv:1905.09707]
- Y. Yang and R. Renner, *Ultimate limit on time signal generation*, (2020), [arXiv:2004.07857]
- *Autonomous quantum clocks: does thermodynamics limit our ability to measure time?* Paul Erker, Mark T. Mitchison, Ralph Silva, Mischa P. Woods, Nicolas Brunner, and Marcus Huber [Phys. Rev. X 7, 031022 (2017)]
- S. Rankovic, Y. Liang, and R. Renner. Quantum clocks and their

- The Quasi-ideal quantum clocks
    - Delay function of 1st tick
    - Analogy with x and p
    - Quasi-ideal control theorem
    - Accuracy and optimality
- Quantum advantage in low dimensions
    - How to compute it
- Other interesting thigs we did not discuss due to time constraints

synchronization - the Alternate Ticks Game. arXiv:1506.01373, (2015).

- *Autonomous Temporal Probability Concentration: Clockworks and the Second Law of Thermodynamics*, Emanuel Schwarzhans, Maximilian P. E. Lock, Paul Erker, Nicolai Friis, Marcus Huber, [Phys. Rev. X 11, 011046 (2021)]
- *Measuring the thermodynamic cost of timekeeping*, A. N. Pearson, Y. Guryanova, P. Erker, E. A. Laird, G. A. D. Briggs, M. Huber, N. Ares, [Phys. Rev. X 11, 021029 (2021)]

Clock Examples:



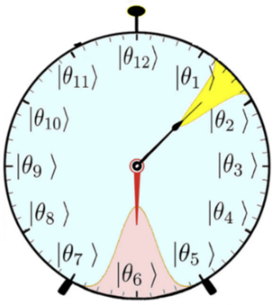
1) Thermodynamic "heat-engine" clock

[P. Erker et. al. PRX (2017)]

$$L_1 = \sqrt{\gamma_h}\sigma_h, \qquad L_2 = \sqrt{\gamma_h e^{-\beta_h E_h}}\sigma_h^\dagger,$$
$$L_3 = \sqrt{\gamma_c}\sigma_c, \qquad L_4 = \sqrt{\gamma_c e^{-\beta_c E_c}}\sigma_c^\dagger,$$
$$J_1 = \sqrt{\Gamma}\,|0\rangle\langle d-1|_w\,, \qquad J_2 = J_3 = J_4 = 0,$$

2) Ladder clock [S. Stupar et.al, arXiv:1806.08812 (2018)]

$$L_j = |c_{j+1}\rangle\langle c_j|\,, \quad J_j = 0, \quad j = 0,1,2,...,d-1$$
$$L_d = 0, \qquad J_d = |c_1\rangle\langle c_d|\,,$$



3) Quasi-Ideal clock [M. Woods et al, PRX Quantum (2019)]

$$L_j = 0, \quad J_j = \sqrt{2V_j}\,|\psi_C\rangle\langle t_j|\,,$$
$$j = 0,1,2,...,d$$

[A. Peres AJP (1980)]

[M. Woods et al, Annales Henri Poincaré (2018)]

Ladder Clock

- $N_L = d$ with

$$L_j = |c_{j+1}\rangle\langle c_j|\,, \quad J_j = 0, \qquad j = 1,2,3,\dots,d-1$$
$$L_d = 0, \qquad J_d = |c_1\rangle\langle c_d|\,,$$
$$H = 0,$$
$$\rho_C^0 = |c_1\rangle\langle c_1|$$

- No tick and tick generators:

$$\mathcal{N} = \begin{bmatrix} -1 & 0 & \dots & & & \dots & 0 \\ 1 & -1 & 0 & \dots & & & \vdots \\ 0 & 1 & -1 & 0 & \dots & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & 0 & 1 & -1 & 0 \\ 0 & \dots & & \dots & 0 & 1 & -1 \end{bmatrix}, \qquad [\mathcal{T}]_{i,j} = \begin{cases} 1 & \text{if } i = 1, j = d \\ 0 & \text{otherwise} \end{cases}$$

Low dimensional accuracy of quasi-ideal clocks