Non-Markovian quantum evolution

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Motivations

- To better understand quantum dynamics
- To provide more efficient ways to control quantum systems
- Modern Quantum Technologies
- To control (protect) quantum entanglement
- ....

Notation

n-level quantum system; $\quad n < \infty$

$$\mathcal{H} = \mathbb{C}^n \ ; \qquad \mathcal{B}(\mathcal{H}) = M_n(\mathbb{C})$$

quantum states $\longrightarrow$ density operators in $M_n(\mathbb{C})$

$$\rho \geq 0 \ , \qquad \mathrm{Tr}\, \rho = 1$$

$M_n(\mathbb{C}) \ \longrightarrow \ \mathbb{C}^*$–algebra

Dynamical map

$$\Lambda(t) \;:\; \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H}) \;;\quad t \geq 0$$

$$\Lambda(0) = \mathsf{id}$$

$\Lambda(t)$ completely positive and trace preserving

$$\rho_0 \quad \longrightarrow \quad \rho(t) := \Lambda(t)\rho_0$$

Example: Unitary dynamics

von-Neumann

$$\frac{d\rho}{dt} = -i[H, \rho]$$

$$\Lambda(t)\rho := U(t)\rho U^\dagger(t)$$

$$U(t) = e^{-iHt}$$

$$\Lambda(t_1)\,\Lambda(t_2) \;=\; \Lambda(t_1 + t_2)$$

Markovian semigroup

Master Equation

$$\frac{d}{dt}\rho(t) = \mathcal{L}\,\rho(t)$$

Gorini, Kossakowski, Sudarshan & Lindblad

$$\mathcal{L}\rho \;=\; -i[H,\rho] + \sum_\alpha \left(V_\alpha \rho V_\alpha^\dagger - \frac{1}{2}\{V_\alpha^\dagger V_\alpha, \rho\} \right)$$

$$\Lambda(t) = e^{\mathcal{L}t} \quad \Longrightarrow \quad \Lambda(t+s) = \Lambda(t)\,\Lambda(s)$$

Reduced dynamics

$$\mathcal{H} \otimes \mathcal{H}_R$$

$$\Lambda(t)\rho := \mathrm{Tr}_R \left[e^{-iHt}(\rho \otimes \omega_R) e^{iHt} \right]$$

One obtains Markovian semigroup $\Lambda(t) = e^{tL}$ only under suitable (Markovian) approximation

- weak coupling limit
- singular coupling limit

Genuine $\Lambda(t)$ is NOT of this form!

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How to describe general quantum evolution?

$$\rho \quad \longrightarrow \quad \Lambda(t)\rho$$

General quantum evolution – non-local approach

Nakajima–Zwanzig

$$\frac{d}{dt}\rho(t) = \int_0^t \mathcal{K}(t-u)\rho(u)\,du$$

$\mathcal{K}(t)$ – memory kernel

Conditions for $\mathcal{K}(t)$ are not known

General quantum evolution – non-local approach

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General quantum evolution – local approach

$$\dot{\Lambda}(t) = \dot{\Lambda}(t)\Lambda^{-1}(t)\Lambda(t) =: \mathcal{L}(t)\,\Lambda(t)$$

$$\frac{d}{dt}\rho(t) = \mathcal{L}(t)\,\rho(t)\ ; \quad \rho(0) = \rho_0$$

$$\rho(t) = \Lambda(t)\rho_0\ ; \quad \Lambda(t) = \mathrm{T}\,\exp\left[\int_0^t \mathcal{L}(\tau)d\tau\right]$$

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TC

$$\frac{d}{dt}\,\rho(t) \;=\; \int_0^t \mathcal{K}(t-u)\rho(u)\,du\;; \quad \rho(0)=\rho_0$$

TCL

$$\frac{d}{dt}\,\rho(t) = \mathcal{L}(t)\,\rho(t)\;; \quad \rho(0)=\rho_0$$

Classical problem

$$\rho \quad \longrightarrow \quad (p_1, \ldots, p_n)$$

Classical stochastic dynamics

$$p(0) \quad \longrightarrow \quad p(t) = \Lambda(t)\, p(0)$$

$$p_i(t) = \sum_j \Lambda_{ij}(t)\, p_j(0)$$

$$\Lambda_{ij}(t) \geq 0 \ , \quad \sum_i \Lambda_{ij}(t) = 1$$

Classical problem — semigroup

$$\frac{d}{dt}\, p_i(t) = \sum_j L_{ij}\, p_j(t)$$

Kolmogorov conditions

$$L_{ij} \geq 0 \ , \ \ (i \neq j) \ , \quad \sum_i L_{ij} = 0$$

Classical problem — general dynamics

$$L \longrightarrow L(t)$$

$$\frac{d}{dt}\, p_i(t) = \sum_j L_{ij}(t)\, p_j(t)$$

The classical problem is as hard as the quantum one!

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Memory kernel vs. local generator

Laplace transform

$$\widetilde{f}(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$\Lambda(t) \quad \longleftrightarrow \quad \widetilde{\Lambda}(s)$$

$$\mathcal{L}(t) \; := \; \frac{d}{dt}\Lambda(t) \cdot \Lambda(t)^{-1}$$

$$\widetilde{\mathcal{K}}(s) = \left[s\widetilde{\Lambda}(s) - \mathbb{1} \right] \widetilde{\Lambda}(s)^{-1}$$

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Bernstein theorem

A function $f : [0, \infty) \to \mathbb{R}$ is **completely monotone** if

$$(-1)^k \frac{d^k}{dt^k} f(t) \geq 0 \ ,$$

for all $t \in [0, \infty)$ and $k = 0, 1, 2, \ldots$.

f is completely monotone iff f is a Laplace transform of g

$$f(s) = \int_0^\infty e^{-st} g(t) dt \ ,$$

with $g(t) \geq 0$ and $s \geq 0$.

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Quantum Bernstein theorem

A map $\Lambda(t)$ is **completely monotone** if

$$(-1)^k \frac{d^k}{dt^k} \Lambda(t) \quad \text{is completely positive}$$

for all $t \in [0, \infty)$ and $k = 0, 1, 2, \ldots$.

Λ is completely monotone iff Λ is a Laplace transform of Φ

$$\Lambda(s) = \int_0^\infty e^{-st} \Phi(t) dt \ ,$$

with $\Phi(t)$ completely positive and $s \geq 0$.

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Quantum Bernstein theorem

$$\frac{d}{dt}\Lambda(t) \;=\; \int_0^t \mathcal{K}(t-u)\Lambda(u)\,du$$

If $\mathcal{K}(t)$ is a legitimate memory kernel, then

$$\frac{1}{s-\widetilde{\mathcal{K}}(s)} \qquad (s \geq 0)$$

is **completely monotone**.

Markovianity

Given $\Lambda(t)$

is it Markovian or non-Markovian?

What is the definition of Markovianity?

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Divisible maps

$\Lambda(t)$ – dynamical map

$\Lambda(t)$ is divisible iff

$$\Lambda(t) = V(t,s)\Lambda(s)$$

$$t \geq s \geq 0$$

$V(t,s)$ completely positive maps for all $t \geq s$

Divisible maps

$$V(t,s)V(s,u) = V(t,u)$$

$V(t,s)$ – completely positive

$$\frac{d}{dt}V(t,s) = \mathcal{L}(t)V(t,s) \ , \quad V(s,s) = \mathbb{1}$$

$\mathcal{L}(t)$ – legitimate GKSL generator for all t

Markovianity $=$ Divisibility

How to check for divisibility?

$\Lambda(t)$ – dynamical map

Is $\Lambda(t)$ divisible?

$$\mathcal{L}(t) := \frac{d\Lambda(t)}{dt} \cdot \Lambda(t)^{-1}$$

$\Lambda(t)$ is divisible iff $\mathcal{L}(t)$ has a GKSL form for all $t \geq 0$

$$\mathcal{L}(t)\rho = -i[H(t),\rho] + \sum_\alpha \left(V_\alpha(t)\rho V_\alpha(t)^\dagger - \frac{1}{2}\{V_\alpha(t)^\dagger V_\alpha(t),\rho\} \right)$$

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Example: decoherence of qubit

$$\mathcal{L}(t)\rho = \frac{1}{2}\gamma(t)\Big[\sigma_z\rho\sigma_z - \rho\Big]$$

$$\Lambda(t) = \exp\left(\int_0^t \mathcal{L}(u)\,du\right)$$

$$\rho(t) \;=\; \begin{pmatrix} \rho_{00} & \rho_{01}e^{-\Gamma(t)} \\ \rho_{10}e^{-\Gamma(t)} & \rho_{11} \end{pmatrix}$$

$$\Gamma(t) = \int_0^t \gamma(u)\,du$$

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$\Gamma(t) \geq 0 \quad \longrightarrow \quad \Lambda(t)$ dynamical map

$\gamma(t) \geq 0 \quad \longrightarrow \quad \Lambda(t)$ divisible dynamical map

$\gamma = \text{const.} \geq 0 \quad \longrightarrow \quad \Lambda(t)$ dynamical semigroup

Markovianity vs. distinguishability

$$D[\rho_1, \rho_2] := \frac{1}{2}||\rho_1 - \rho_2||_1$$

$$||A||_1 := \mathrm{Tr}|A| = \mathrm{Tr}\sqrt{AA^\dagger}$$

$$D[\rho_1, \rho_2] = \text{distinguishibility of } \rho_1 \text{ and } \rho_2$$

Markovianity vs. distinguishability

$\Lambda(t)$ – dynamical map

$$\rho_1(t) = \Lambda(t)\rho_1 \ , \qquad \rho_2(t) = \Lambda(t)\rho_2$$

$$D[\rho_1(t), \rho_2(t)] \ \leq \ D[\rho_1, \rho_2]$$

$\Lambda(t)$ – Markovian dynamical map

$$\frac{d}{dt} D[\rho_1(t), \rho_2(t)] \ \leq \ 0$$

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Markovianity vs. distinguishability

Breuer, Lane and Piilo, PRL 2008

$$\sigma[\rho_1, \rho_2; t] := \frac{d}{dt} D[\rho_1(t), \rho_2(t)] = \text{flux of information}$$

$\Lambda(t)$ is Markovian iff $\sigma[\rho_1, \rho_2; t] \leq 0$

Divisibility $\Longrightarrow$ $\sigma[\rho_1, \rho_2; t] \leq 0$

$\sigma[\rho_1, \rho_2; t] \leq 0$ does not imply Divisibility

D.C., A. Kossakowski, and A. Rivas, Phys. Rev. A (2011)

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Markovianity and contractivity

Λ – dynamical map

$$||\Lambda(t)||_1 = \sup_{||\rho||_1=1} ||\Lambda(t)\rho||_1 = 1$$

If $\Lambda(t)$ is a dynamical map, then

$$||\Lambda(t)a||_1 \leq ||a||_1 \ .$$

If $\Lambda(t)$ is a divisible dynamical map, then

$$\frac{d}{dt}||\Lambda(t)a||_1 \leq 0$$

for all $a \in \mathcal{B}(\mathcal{H})$.

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Markovianity and contractivity

$\Lambda(t)$ is divisible if and only if

$$\frac{d}{dt} ||(\mathbb{1} \otimes \Lambda(t))A||_1 \leq 0$$

for all $A^\dagger = A \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$

D.C., A. Kossakowski, and A. Rivas, Phys. Rev. A (2011)

Example: decoherence of qubit

$$\mathcal{L}(t)\rho = \frac{1}{2}\gamma(t)\Big[\sigma_z\rho\sigma_z - \rho\Big]$$

$$\mathcal{L}(t)\sigma_x = -\gamma(t)\sigma_x$$

$$\Lambda(t)\sigma_x = e^{-\Gamma(t)}\sigma_x \;; \quad \Gamma(t) = \int_0^t \gamma(u)\,du$$

$$\frac{d}{dt}\,||\Lambda(t)\sigma_x||_1 = \frac{d}{dt}\,e^{-\Gamma(t)}||\sigma_x||_1 = -2\gamma(t)\,e^{-\Gamma(t)}$$

$$\gamma(t) \geq 0 \quad \Longleftrightarrow \quad \frac{d}{dt}\,||\Lambda(t)\sigma_x|| \leq 0$$

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Markovianity vs. fidelity

$$F(\rho,\sigma) = \left(\mathrm{Tr}\left[\sqrt{\sqrt{\rho}\,\sigma\,\sqrt{\rho}}\right]\right)^2$$

$$F(\rho,\sigma) \leq F(\rho(t),\sigma(t))$$

If $\Lambda(t)$ is a divisible map, then

$$\frac{d}{dt} F(\rho(t),\sigma(t)) \geq 0 \ .$$

Markovianity vs. entropy

$$S(\rho\,||\,\sigma) = \mathrm{Tr}\Big[\rho(\log\rho - \log\sigma)\Big]$$

$$S(\rho(t)\,||\,\sigma(t)) \leq S(\rho\,||\,\sigma)$$

If $\Lambda(t)$ is a divisible map, then

$$\frac{d}{dt}\,S(\rho(t)\,||\,\sigma(t)) \leq 0\ .$$

Markovianity vs. entropy

The same works for relative Rényi and Tsallis entropies

$$S_\alpha(\rho\,||\,\sigma) = \frac{1}{\alpha - 1}\log\left[\mathrm{Tr}\,\rho^\alpha\sigma^{1-\alpha}\right]\ ; \qquad \alpha\in[0,1)\cup(1,\infty)$$

$$T_q(\rho\,||\,\sigma) = \frac{1}{1-q}\left[1 - \mathrm{Tr}\,\rho^q\sigma^{1-q}\right]\ ; \qquad q\in[0,1)$$

$$\lim_{\alpha\to1} S_\alpha(\rho\,||\,\sigma) = \lim_{q\to1} T_q(\rho\,||\,\sigma) = S(\rho\,||\,\sigma)$$

Markovianity vs. entanglement

W – an arbitrary density matrix in $\mathcal{H} \otimes \mathcal{H}'$

$$W(t) = (\Lambda(t) \otimes \mathbb{1})W$$

If $\mathcal{E}$ is an entanglement measure then

$$\mathcal{E}[(\Phi \otimes \Phi')W] \leq \mathcal{E}[W] \ ,$$

If $\mathcal{E}$ is an entanglement measure then

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If $\Lambda(t)$ is divisible then

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Heisenberg picture

$$\mathrm{Tr}(\Lambda^* a \cdot \rho) = \mathrm{Tr}(a \cdot \Lambda\rho)$$

$$\Lambda(t) \text{ trace preserving} \quad \Longleftrightarrow \quad \Lambda^*(t) \text{ unital}$$

$$||\Lambda^*(t)a|| \leq ||a|| \ , \quad a \in \mathcal{B}(\mathcal{H})$$

$$||a|| \ \text{ operator norm in } \mathcal{B}(\mathcal{H})$$

If $\Lambda(t)$ is divisible, then

$$\frac{d}{dt} ||\Lambda^*(t)a|| \leq 0 \ ,$$

Conclusions

- characterization of admissible $\mathcal{K}(t)$ and $\mathcal{L}(t)$ is an open problem
- Markovianity (= divisibility) is fully characterized in terms of $\mathcal{L}(t)$ (but not of $\mathcal{K}(t)$!)
- Markovianity implies extra condition for monotonicity of several quantities like:
  - distinguishability,
  - fidelity,
  - relative entropy,
  - entanglement measures, ...

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