How many singlets are needed to create a bipartite state via LOCC ?

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[PRL 106, 130503 (2011)]

Entanglement

- cannot be created or increased by local operations (LO) & classical communication (CC)

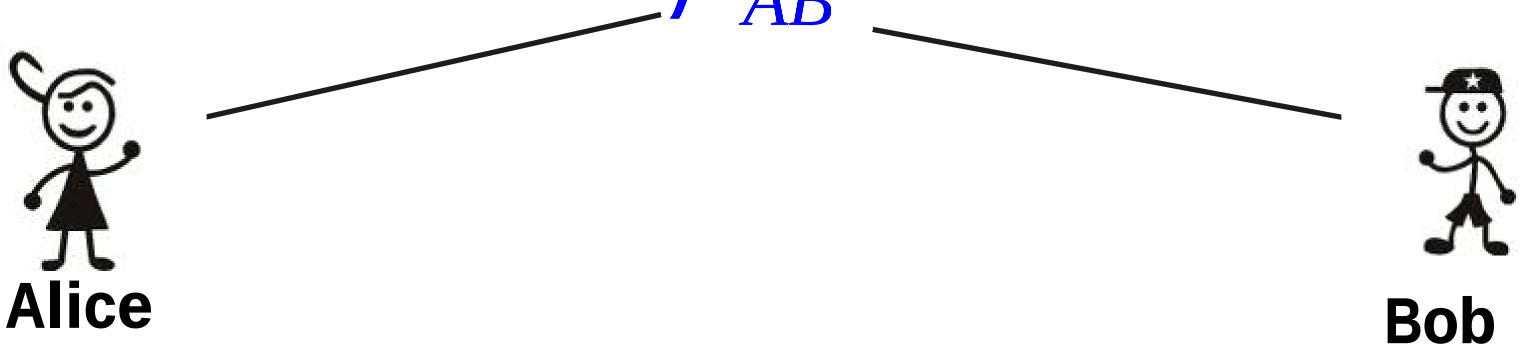
- **Entanglement manipulation :**

  the conversion of entanglement from one form to another via LOCC by 2 distant parties

- Asymptotic entanglement manipulation protocols:
  - Entanglement distillation
  - Entanglement dilution

Asymptotic Entanglement Distillation

$$\rho_{AB}^{\otimes n}$$



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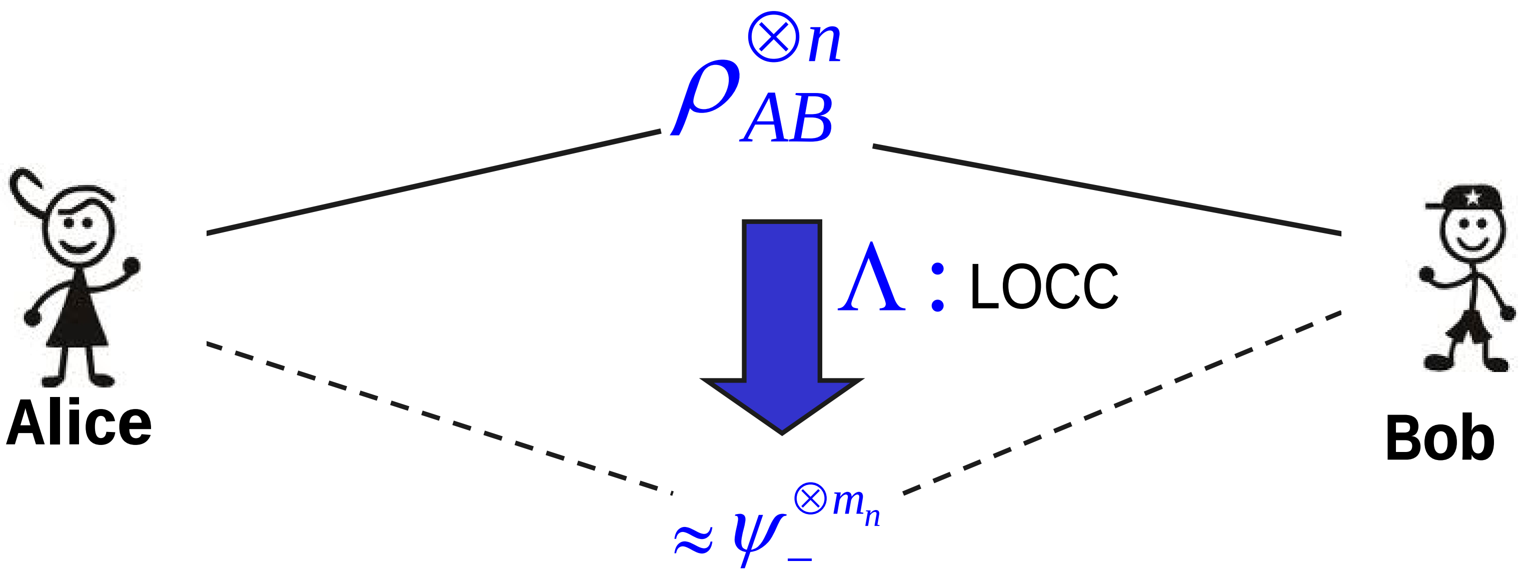
Asymptotic Entanglement Distillation

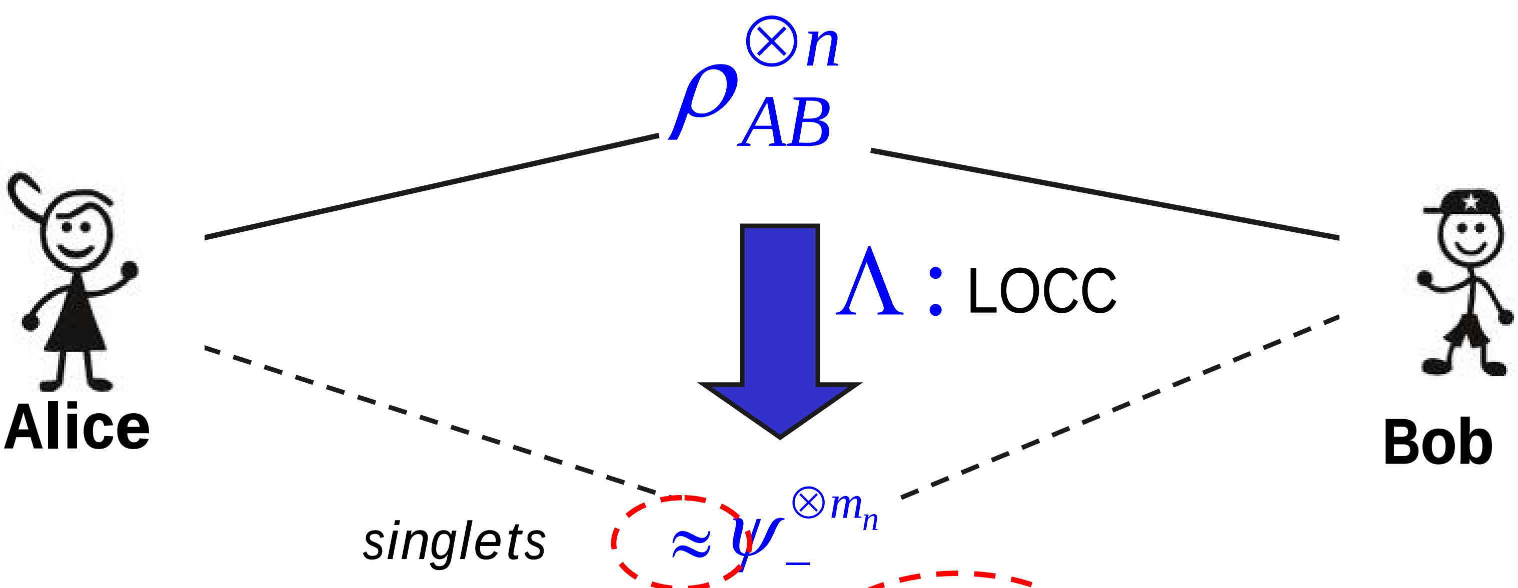


singlet

$$\psi_- = |\psi_-\rangle\langle\psi_-|$$

$$|\psi_-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right) \in \mathbf{C}^2 \otimes \mathbf{C}^2$$

Entanglement Distillation



- **Heuristically:**

$$F\left(\Lambda\left(\rho_{AB}^{\otimes n}\right), \psi_{-}^{\otimes m_n}\right) \xrightarrow[n\to\infty]{} 1$$

$$\limsup_{n\to\infty} \frac{m_n}{n} =$$ *the maximum number of singlets that can be extracted from each copy of the state* ρ_{AB}

= "*distillable entanglement*" $E_D(\rho_{AB})$

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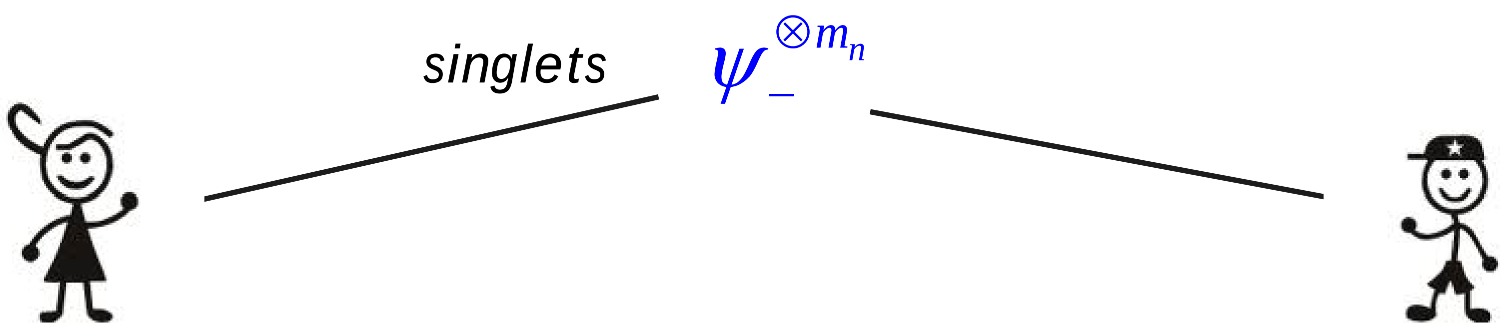
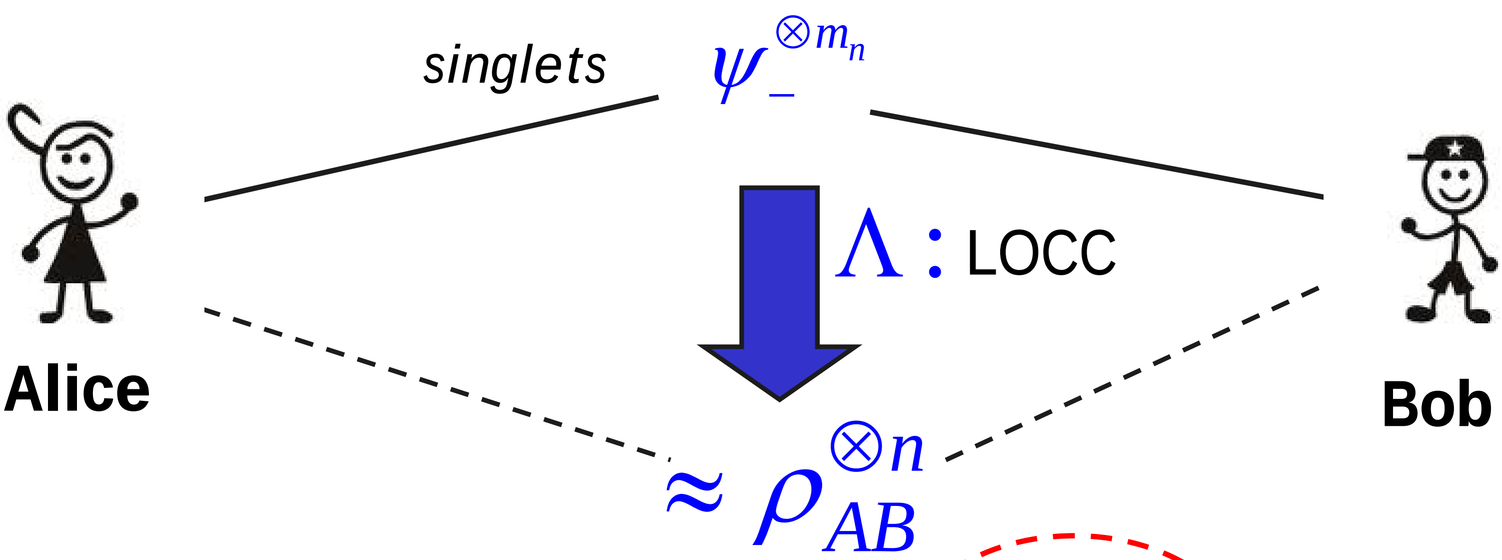
Asymptotic Entanglement Dilution



Entanglement Dilution



$$F\left(\Lambda\left(\psi_-^{\otimes m_n}\right), \rho_{AB}^{\otimes n}\right) \underset{n\to\infty}{\longrightarrow} 1$$

- **Heuristically:**

$$\liminf_{n\to\infty} \frac{m_n}{n} =$$ *the minimum number of Bell states needed to create a copy of the state* ρ_{AB}

= "*entanglement cost*" $E_C(\rho_{AB})$

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Asymptotic Entanglement Manipulation of Pure States

$$\rho_{AB} = |\varphi_{AB}\rangle\langle\varphi_{AB}|$$

$$E_C(\rho_{AB}) = E_D(\rho_{AB}) = E\left(|\varphi_{AB}\rangle\right) := S(\rho_A) = S(\rho_B)$$

Entropy of entanglement

$$S(\rho_A) = -\mathrm{Tr}(\rho_A \log_2 \rho_A)$$ = von Neumann entropy of its reduced state ρ_A

- entanglement of pure states is asymptotically reversible
- This is not true for mixed states.

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Asymptotic Entanglement Dilution

Entanglement Cost of a mixed state

[Hayden, Horodecki & Terhal]

$$E_C(\rho_{AB}) = E_F^{\infty}(\rho_{AB}) := \lim_{n\to\infty}\frac{1}{n}E_F(\rho_{AB}^{\otimes n})$$

Entanglement of Formation of a bipartite state ω_{AB}

$$E_F(\omega_{AB}) = \min_{\substack{\{p_i, |\varphi_A^i\rangle\} \\ \omega_{AB}=\sum_i p_i |\varphi_{AB}^i\rangle\langle\varphi_{AB}^i|}} \sum_i p_i S(\varphi_A^i)$$

convex roof extension

$$S(\varphi_A^i) \rightarrow E(|\varphi_{AB}^i\rangle)$$

$$\varphi_A^i = \mathrm{Tr}_B |\varphi_{AB}^i\rangle\langle\varphi_{AB}^i|$$

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$E_C(\rho_{AB}),\ E_D(\rho_{AB})$: evaluated in the asymptotic i.i.d. framework

multiple, identical copies of the initial state —**LOCC**→ multiple, identical copies of the desired target state

e.g. in ent. distillation

$\rho_{AB}^{\otimes n}$ *: no correlations*

with asymptotically vanishing error

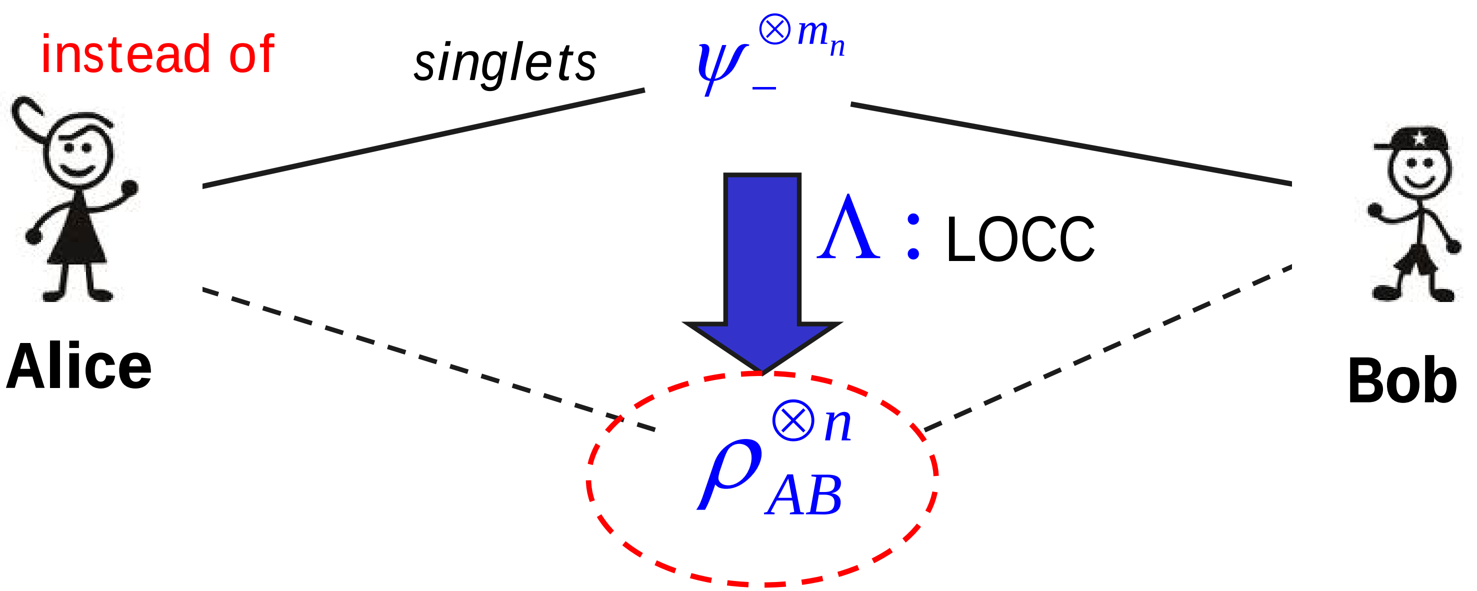
$n \to \infty$

- in practice: difficult to achieve due to noise
- typically one doesn't know how such a limit is approached

A more relevant scenario

~~Asymptotic i.i.d. framework~~ → One-shot scenario

- E.g. Entanglement Dilution



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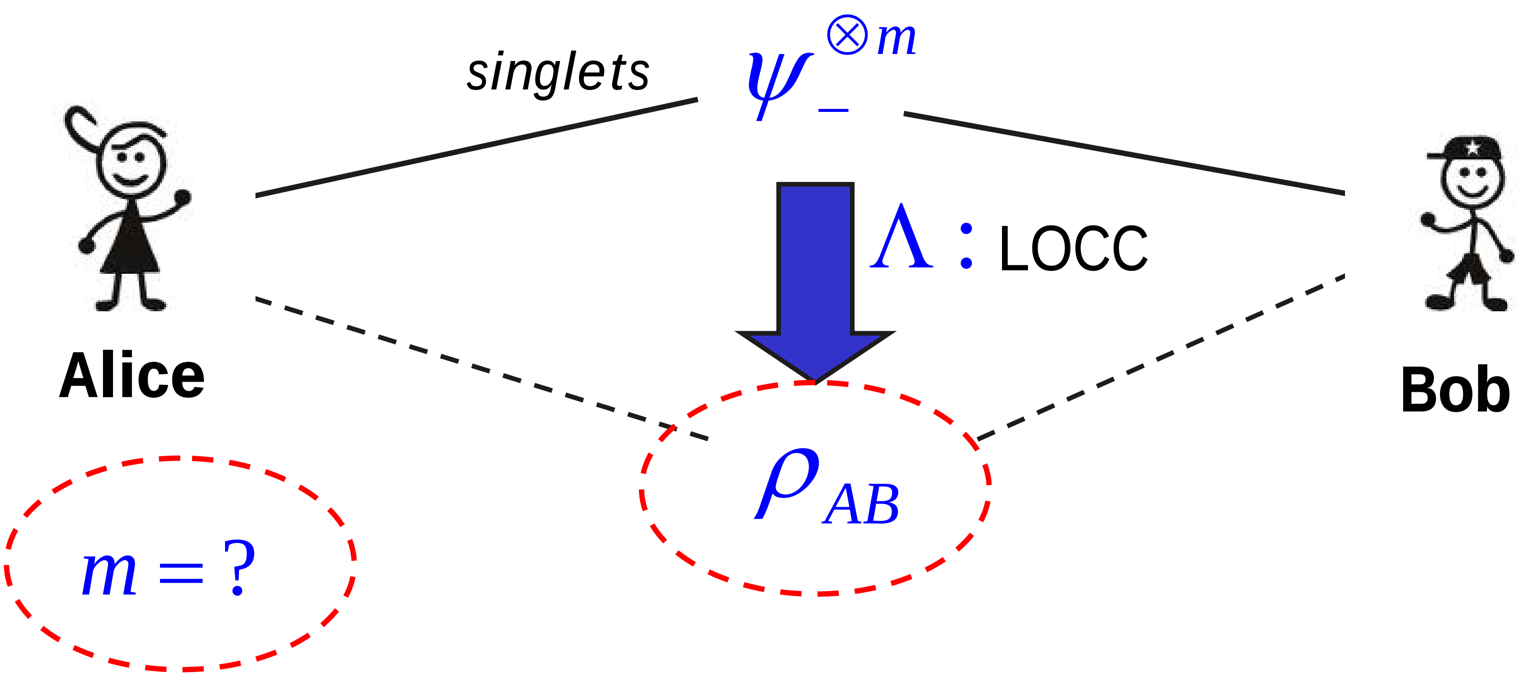
A more relevant scenario

~~Asymptotic i.i.d. framework~~ → One-shot scenario

- E.g. One-shot Entanglement Dilution



How many singlets are needed to create a single copy of a bipartite state ρ_{AB} *via LOCC ?*

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Outline of the talk

Question

How many singlets are needed to create a single copy of ρ_{AB} via LOCC ?

- One-shot entanglement dilution
- Some relevant entropic quantities

Question

Is there an operational interpretation of the Schmidt number of ρ_{AB} ?

- Definition & properties of the Schmidt number of ρ_{AB}

- Answering the questions

- One-shot entanglement distillation & entanglement spread
- Open questions

Schmidt number

- A bipartite pure state $|\psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$

- **Schmidt decomposition**

$$|\psi_{AB}\rangle = \sum_{i=1}^{d} \sqrt{\lambda_i}\, |i_A\rangle |i_B\rangle$$

Schmidt coeffs. $\lambda_i \geq 0 \; ; \; \sum_{i=1}^{d} \lambda_i = 1$ $\quad d = \min\{d_A, d_B\}$

- Schmidt rank : $r(\psi_{AB})$
  - = no.of non-zero Schmidt coeffs.
  - = rank of ρ_A = rank of ρ_B
  - $\rho_A = \mathrm{Tr}_B |\psi_{AB}\rangle\langle\psi_{AB}|$

- A pure state $|\psi_{AB}\rangle$ is entangled if and only if $r(\psi_{AB}) > 1.$

- A necessary condition for $|\psi_{AB}\rangle \overset{\text{LOCC}}{\longrightarrow} |\varphi_{AB}\rangle$: $r(\psi_{AB}) \geq r(\varphi_{AB}).$

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Schmidt number : mixed states

[Terhal & Horodecki]

$$\rho_{AB} \leftrightarrow \mathcal{E} = \left\{ p_i, \left| \psi_i^{AB} \right\rangle \right\}$$ an ensemble of pure states

$$\rho_{AB} = \sum_i p_i \left| \psi_i^{AB} \right\rangle \left\langle \psi_i^{AB} \right|$$

$$r_{\max}\left(\mathcal{E}\right) = \max_i r(\psi_i^{AB})$$ = max. Schmidt rank of the pure states in $\mathcal{E}$

Schmidt number of ρ_{AB} : $N(\rho_{AB}) := \min_{\mathcal{E}} \left\{ r_{\max}\left(\mathcal{E}\right) \right\}$

- In any $\mathcal{E} = \left\{ p_i, \left| \psi_i^{AB} \right\rangle \right\}$ of ρ_{AB}, $\exists$ at least one $\left| \psi_i^{AB} \right\rangle$ such that

$$r(\psi_i^{AB}) \geq N(\rho_{AB})$$

- $\exists$ at least one $\mathcal{E} = \left\{ p_i, \left| \psi_i^{AB} \right\rangle \right\}$ of ρ_{AB}, such that $\forall \left| \psi_i^{AB} \right\rangle \in \mathcal{E}$,

$$r(\psi_i^{AB}) \leq N(\rho_{AB})$$

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Schmidt number contd.

- For a pure state $\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|: \quad N(\rho_{AB}) = r(\psi_{AB})$
- For a separable state $N(\rho_{AB}) = 1.$

- Schmidt number is an entanglement monotone

$$\forall \Lambda \in \mathrm{LOCC} \quad N(\Lambda(\rho_{AB})) \leq N(\rho_{AB})$$

- **Classification of density matrices:**

$\mathcal{H}_n$: finite-dl Hilbert space; $\rho_{AB} \in \mathcal{D}\left(\mathcal{H}_n \otimes \mathcal{H}_n\right)$

set of density matrices

$$S_k := \left\{\rho_{AB} \in \mathcal{D}\left(\mathcal{H}_n \otimes \mathcal{H}_n\right);\ N(\rho_{AB}) \leq k\right\} \qquad S_{k-1} \subset S_k$$

$S_1 =$ set of separable density matrices

→ **Classification of linear maps**

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Schmidt number contd.

- **Classification of linear maps:**

- A linear Hermiticity-preserving map Λ_k is k -positive if & only if

$$\left(I \otimes \Lambda_k\right)\rho_{AB} \geq 0 \qquad \forall \rho_{AB} \in S_k$$

Λ_1 : positive map

separable states ⟷ positive maps *[Horodecki]*

- A state ρ_{AB} is entangled if & only if $\exists$ a Λ_1 such that

$$\left(I \otimes \Lambda_1\right)\rho_{AB} < 0$$

Schmidt number ⟷ k -positive maps

- A state ρ_{AB} has Schmidt number $\geq k+1,$ if $\exists$ a Λ_k such that

$$\left(I \otimes \Lambda_k\right)\rho_{AB} < 0$$

Schmidt number -- Summary

- Schmidt number of ρ_{AB} is an entanglement monotone
- S_k can be characterized by k -positive maps
- can be detected by k-Schmidt number witnesses

- There is a ‘zoo’ of entanglement monotones!

(Q) Does the Schmidt number have an operational significance ?

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Outline of the talk

Question

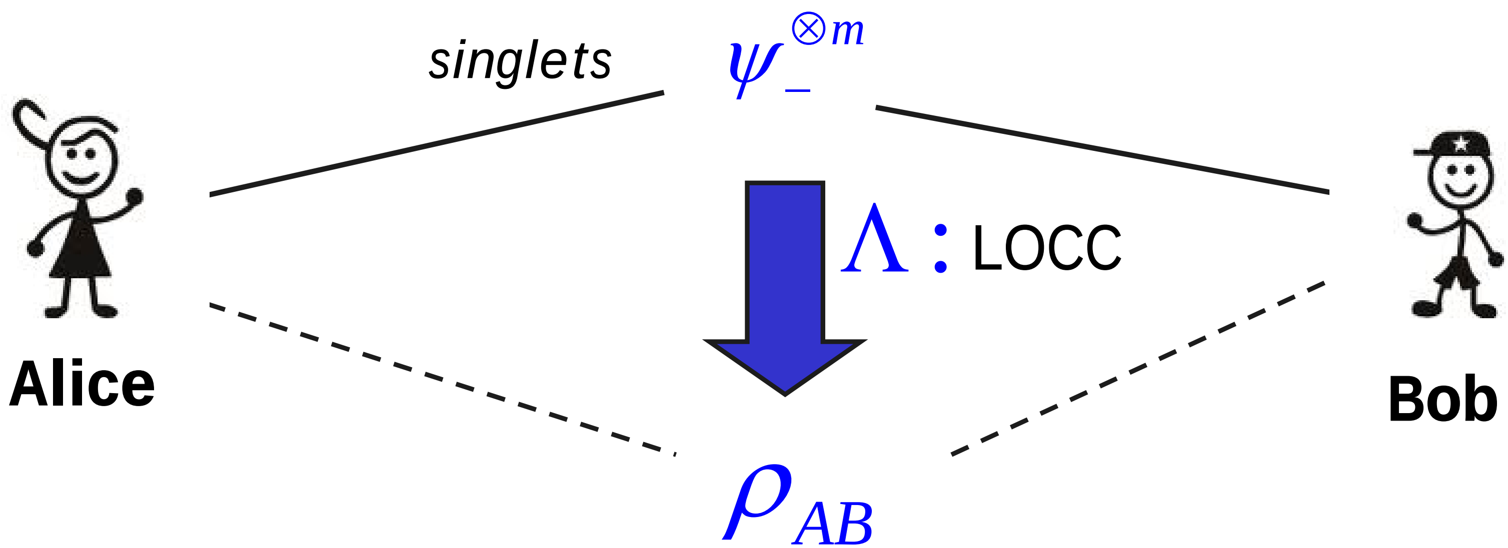
How many singlets are needed to create a single copy of ρ_{AB} via LOCC ?

- One-shot entanglement dilution

Question

Is there an operational interpretation of the Schmidt number of ρ_{AB} ?

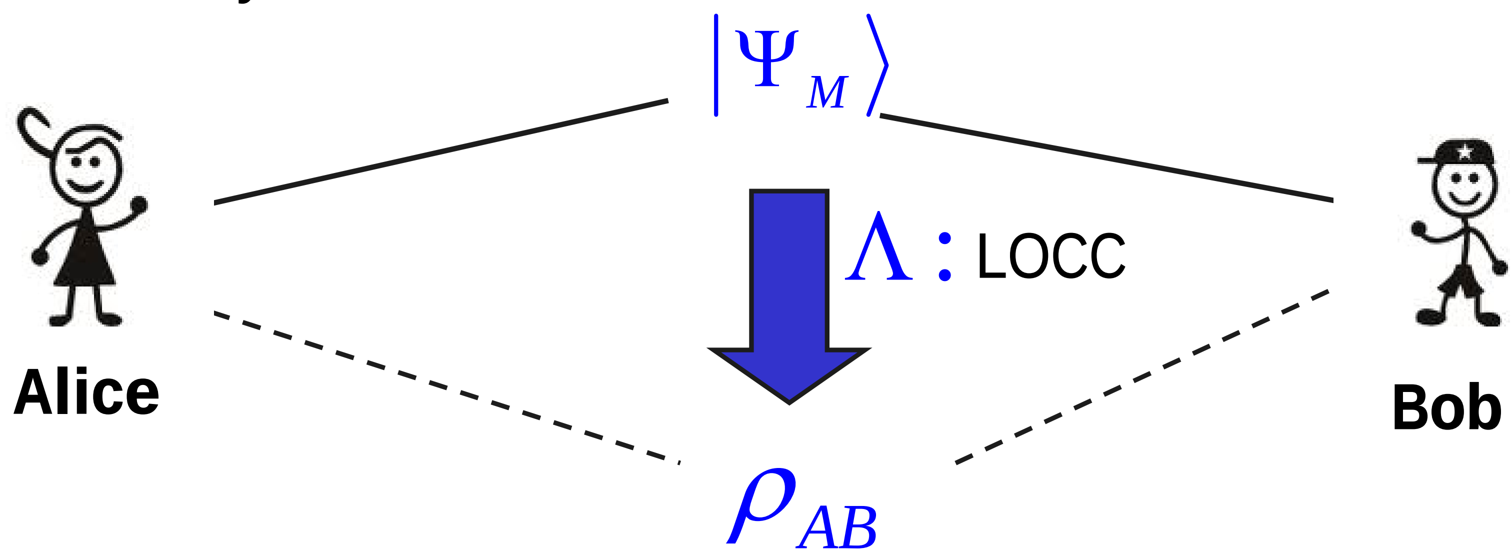
One-shot Entanglement Dilution



$$m = ?$$

One-shot Entanglement Dilution

- Equivalently,

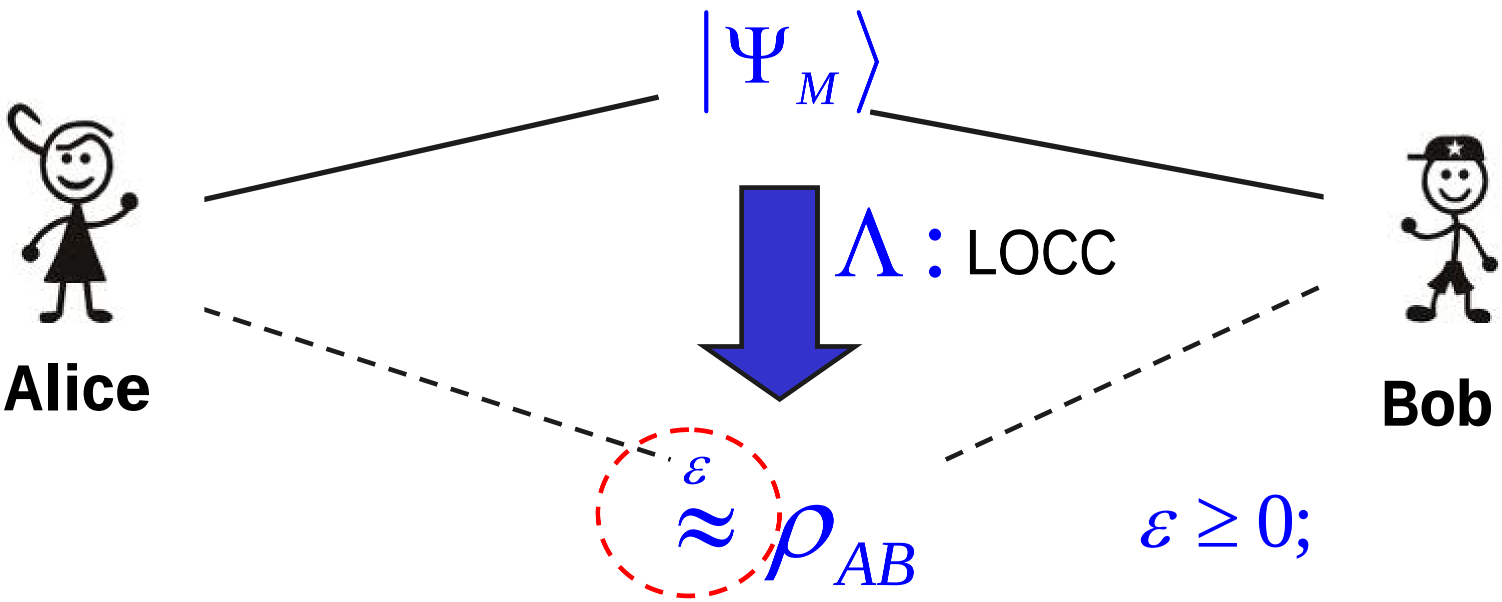


$$|\Psi_M\rangle = \frac{1}{\sqrt{M}} \sum_{i=1}^{M} |i\rangle|i\rangle$$

a *maximally entangled state* of Schmidt rank M,

$$\log M = ?$$

One-shot Entanglement Dilution



$\varepsilon \geq 0;$

$F(\Lambda(\Psi_M), \rho_{AB}) \geq 1 - \varepsilon$

- **Dilution fidelity**

$$\mathrm{F}_{\mathrm{dil}}(M, \rho_{AB}) := \max_{\Lambda \in \mathrm{LOCC}} F(\Lambda(\Psi_M), \rho_{AB})$$

One-shot ε -error entanglement cost

$$E_{C,\varepsilon}^{(1)}(\rho_{AB}) := \min\{\log M : \mathrm{F}_{\mathrm{dil}}(M, \rho_{AB}) \geq 1 - \varepsilon\}$$

One-shot ε -error entanglement cost

$$E_{C,\varepsilon}^{(1)}(\rho_{AB}) = ?$$

- **Theorem:** For any $\varepsilon \geq 0$,

$$E_{C,\varepsilon}^{(1)}(\rho_{AB}) \approx \min_{\mathcal{E}} H_0^{\varepsilon}(\rho_{RA} \mid R)$$

$$\rho_{AB} \leftrightarrow \mathcal{E} = \left\{ p_i, \left| \psi_i^{AB} \right\rangle \right\}$$

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- **Definition of** $H_0^{\varepsilon}(\rho_{RA}|R):$

$$\rho_{AB} \leftrightarrow \mathcal{E} = \left\{ p_i, \left|\psi_i^{AB}\right\rangle \right\} \qquad \rho_{AB} = \sum_i p_i \left|\psi_i^{AB}\right\rangle\left\langle\psi_i^{AB}\right|$$

- **Tripartite c-q state:**

$$\rho_{RAB} = \sum_i p_i \left|i_R\right\rangle\left\langle i_R\right| \otimes \left|\psi_i^{AB}\right\rangle\left\langle\psi_i^{AB}\right|$$

$R:$ an auxiliary classical system $\qquad \left\{\left|i_R\right\rangle\right\} \in \mathcal{H}_R$

$$\rho_{RA} = \sum_i p_i \left|i_R\right\rangle\left\langle i_R\right| \otimes \psi_i^A \qquad \psi_i^A = \mathrm{Tr}_B \left|\psi_i^{AB}\right\rangle\left\langle\psi_i^{AB}\right|$$

▪ Definition of $H_0^{\varepsilon}(\rho_{RA} | R)$:

▪ **Quantum relative Renyi entropy of order** $\alpha \neq 1$

$$S_\alpha(\rho \| \sigma) := \frac{1}{\alpha - 1} \log \left(\mathrm{Tr}\,(\rho^\alpha \sigma^{1-\alpha}) \right)$$

$$S_0(\rho \| \sigma) := \lim_{\alpha \to 0^+} S_\alpha(\rho \| \sigma) = -\log \left(\mathrm{Tr}\,(\Pi_\rho \sigma) \right)$$

$\rho^0 \equiv \Pi_\rho$ projection onto support of ρ

▪ Definition of $H_0(\rho_{RA} | R)$:

$$H_0(\rho_{RA} | R) = \max_{\sigma_R} H_0(\rho_{RA} | \sigma_R)$$

$$H_0(\rho_{RA} | \sigma_R) = -S_0(\rho_{RA} \| \sigma_R \otimes I_A)$$

$$H_0(\rho_{RA} | R) = \max_{\sigma_R} \log \left(\mathrm{Tr} \left(\Pi_{\rho_{RA}} (\sigma_R \otimes I_A) \right) \right)$$

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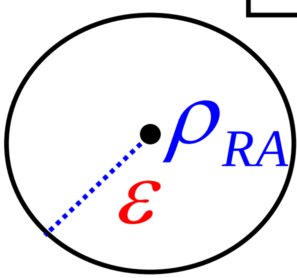
Definition of $H_0^{\varepsilon}(\rho_{RA} \mid R)$: *'smoothed' entropy*

$$H_0(\rho_{RA} \mid R) = \max_{\sigma_R} \log\left(\mathrm{Tr}\left(\pi_{\rho_{RA}}(\sigma_R \otimes I_A)\right)\right)$$

$$H_0^{\varepsilon}(\rho_{RA} \mid R) := \min_{\bar{\rho}_{RA} \in B^{\varepsilon}(\rho_{RA})} H_0(\bar{\rho}_{RA} \mid R)$$



$$B^{\varepsilon}(\rho_{RA}) := \{\bar{\rho}_{RA} : \ \|\rho_{RA} - \bar{\rho}_{RA}\|_1 \leq \varepsilon\}$$

- **Theorem:** For any $\varepsilon \geq 0$,

$$E_{C,\varepsilon}^{(1)}(\rho_{AB}) \approx \min_{\mathcal{E}} H_0^{\varepsilon}(\rho_{RA} \mid R)$$

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One-shot ε -error entanglement cost

- **Theorem:** For any $\varepsilon \geq 0$,

$$E_{C,\varepsilon}^{(1)}(\rho_{AB}) \approx \min_{\mathcal{E}} H_0^{\varepsilon}(\rho_{RA} | R)$$

$$\min_{\mathcal{E}} H_0^{2\sqrt{\varepsilon}}(\rho_{RA} | R) \leq E_{C,\varepsilon}^{(1)}(\rho_{AB}) \leq \min_{\mathcal{E}} H_0^{\varepsilon}(\rho_{RA} | R)$$

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$\varepsilon = 0:$ One-shot 0-error entanglement cost

- **Corollary:**

$$E_C^{(1)}(\rho_{AB}) = \min_{\mathcal{E}} H_0(\rho_{RA} \mid R)$$
$$= \min_{\mathcal{E}} \max_{\sigma_R} \log\left(\mathrm{Tr}\left(\pi_{\rho_{RA}}(\sigma_R \otimes I_A)\right)\right)$$
$$= \log N(\rho_{AB}) \qquad (*)$$

Schmidt number

- **Answer:** The minimum number of singlets needed to create a single copy of ρ_{AB} is $\log N(\rho_{AB})$

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Key ingredient of the proof

$$E_{C,\varepsilon}^{(1)}(\rho_{AB}) := \min\{\log M : \mathrm{F}_{\mathrm{dil}}(M, \rho_{AB}) \geq 1-\varepsilon\}$$

- Lemma: *[G.Bowen & ND ; M. Hayashi]*

$$\mathrm{F}_{\mathrm{dil}}(M, \rho_{AB}) = \max_{\substack{\mathcal{E}=\{p_i, |\psi_i^{AB}\rangle\} \\ \rho_{AB}=\sum_i p_i |\psi_i^{AB}\rangle\langle\psi_i^{AB}|}} \sum_i p_i \sum_{k=1}^{M} \lambda_k^{(i)} \quad \text{..............(1)}$$

k-th Schmidt coeff. of $|\psi_i^{AB}\rangle$

- $\mathrm{F}_{\mathrm{dil}}(M, \rho_{AB}) \geq$ RHS of (1) — *'quantum scissors effect'*
- $\mathrm{F}_{\mathrm{dil}}(M, \rho_{AB}) \leq$ LHS of (1) — *Lo-Popescu theorem*

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Open Questions

(Q) Is $\log N(\rho_{AB})$ always additive ?

$$\log N(\rho_{AB}^{\otimes n}) \stackrel{?}{=} n \log N(\rho_{AB})$$ *

(A) No! *[Terhal & Horodecki]*

- **Isotropic state:** *Invariant under* $U \otimes U^*$

$$\rho_{AB} = \frac{1-f}{M^2-1}\left(I - |\Psi_M\rangle\langle\Psi_M|\right) + f\,|\Psi_M\rangle\langle\Psi_M|$$

$$0 \le f \le 1$$

- **Proved:** $f = 1/\sqrt{2}$; $N(\rho_{AB}^{\otimes 2}) = N(\rho_{AB})$

[Terhal & Horodecki]

$$\log N(\rho_{AB}^{\otimes 2}) = \log N(\rho_{AB})$$

- (Q) Can one understand this operationally ?

- **Isotropic state**

$$\rho_{AB} = \frac{1-f}{M^2-1}\left(I - |\Psi_M\rangle\langle\Psi_M|\right) + f\,|\Psi_M\rangle\langle\Psi_M|$$

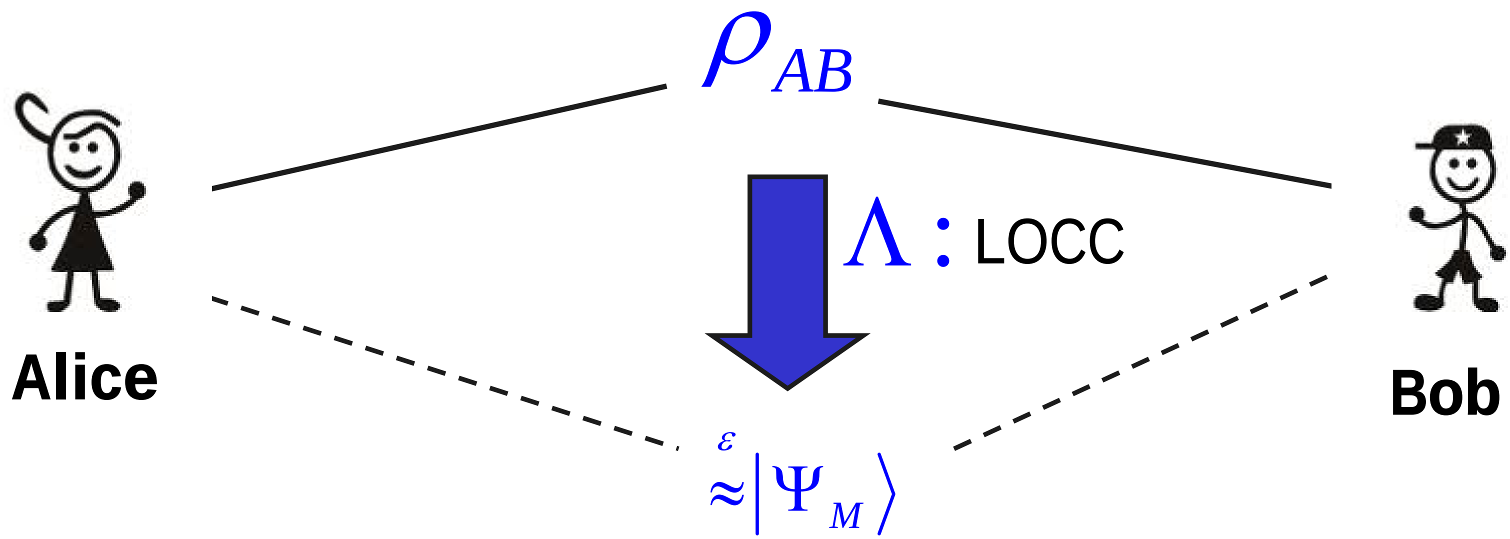
$$0 \le f \le 1$$

- **Numerical evidence:** $f = \sqrt{3}/2\,;\quad N(\rho_{AB}^{\otimes 2}) = N(\rho_{AB})$

$$\log N(\rho_{AB}^{\otimes 2}) = \log N(\rho_{AB})$$

- (Q) Can one prove this ?

One-shot Entanglement Distillation



$$E_{D,\varepsilon}^{(1)}(\rho_{AB}) := \max\left\{\log M : \exists\Lambda : \text{LOCC},\ \mathrm{F}(\Lambda(\rho_{AB}), \Psi_M) \geq 1-\varepsilon\right\}$$

One-shot ε -error distillable entanglement

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For a pure state $\rho_{AB} = |\varphi_{AB}\rangle\langle\varphi_{AB}|;$

- **Theorem: One-shot** ε **-error distillable entanglement**

$$E_{D,\varepsilon}^{(1)}(\rho_{AB}) \approx S_{\min}^{\varepsilon}(\varphi_A)$$

$$S_{\min}^{\varepsilon}(\varphi_A) \le E_{D,\varepsilon}^{(1)}(\rho_{AB}) \le S_{\min}^{\varepsilon'}(\varphi_A) - \log(1-2\sqrt{\varepsilon})$$

$$S_{\min}^{\varepsilon}(\varphi_A) = \max_{\omega_A \in B_{\varepsilon}(\varphi_A)} S_{\min}(\omega_A)$$

smoothed min-entropy

$$S_{\min}(\omega_A) := \log\left(\lambda_{\max}(\omega_A)\right)$$

min-entropy

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- **For a pure state** $\rho_{AB} = |\varphi_{AB}\rangle\langle\varphi_{AB}|;$

$$E_{D,\varepsilon}^{(1)}(\rho_{AB}) \approx S_{\min}^{\varepsilon}(\varphi_A)$$

$$\varepsilon = 0$$

min-entropy

$$E_D^{(1)}(\varphi_{AB}) = S_{\min}(\varphi_A) := -\log \lambda_{\max}(\varphi_A)$$

$$E_{C,\varepsilon}^{(1)}(\rho_{AB}) \approx S_{\max}^{\varepsilon}(\varphi_A)$$

max-entropy

$$E_C^{(1)}(\varphi_{AB}) = S_{\max}(\varphi_A) := \log\left(\mathrm{rank}(\varphi_A)\right)$$

- The differences :

$$\Delta(\varphi_{AB}) = E_C^{(1)}(\varphi_{AB}) - E_D^{(1)}(\varphi_{AB})$$

$$\varepsilon = 0$$

Operational significance ?

$$\Delta_\varepsilon(\varphi_{AB}) = E_{C,\varepsilon}^{(1)}(\varphi_{AB}) - E_{D,\varepsilon}^{(1)}(\varphi_{AB})$$

$$\varepsilon \geq 0$$

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For a pure state $\rho_{AB} = |\varphi_{AB}\rangle\langle\varphi_{AB}|;$

[Hayden, Winter; Harrow]

- The difference :

$$\Delta(\varphi_{AB}) = E_C^{(1)}(\varphi_{AB}) - E_D^{(1)}(\varphi_{AB})$$ **= entanglement spread**

$$\Delta_\varepsilon(\varphi_{AB}) = E_{C,\varepsilon}^{(1)}(\varphi_{AB}) - E_{D,\varepsilon}^{(1)}(\varphi_{AB})$$ **=** ε - perturbed entanglement spread

$\Delta(\varphi_{AB})$: lower bound to the classical communication cost of creating φ_{AB} from singlets via LOCC

$\Delta_\varepsilon(\varphi_{AB})$: .... to a fidelity $\geq (1-\varepsilon)$.

(Q) Do these differences for a mixed state ρ_{AB} have have similar operational interpretations ?

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SUMMARY

- One-shot entanglement dilution
  - Minimum number of singlets needed to create a single copy of ρ_{AB}

    $E_C^{(1)}(\varphi_{AB}) = \log\big(N(\rho_{AB})\big)$ *[logarithm of the Schmidt number]*

  - For an accuracy $\varepsilon \geq 0$: $E_{C,\varepsilon}^{(1)}(\rho_{AB}) \approx \min_{\mathcal{E}} H_0^{\varepsilon}(\rho_{RA} \mid R)$

  - For a pure state $\rho_{AB} = |\varphi_{AB}\rangle\langle\varphi_{AB}|$;

$$E_C^{(1)}(\varphi_{AB}) = S_{\max}(\varphi_A) := \log\big(\mathrm{rank}(\varphi_A)\big)$$

- One-shot entanglement distillation
  - For a pure state $\rho_{AB} = |\varphi_{AB}\rangle\langle\varphi_{AB}|$;

$$E_D^{(1)}(\varphi_{AB}) = S_{\min}(\varphi_A) := -\log\big(\lambda_{\max}(\varphi_A)\big)$$

  - For an accuracy $\varepsilon \geq 0$: $E_{D,\varepsilon}^{(1)}(\rho_{AB}) \approx S_{\min}^{\varepsilon}(\varphi_A)$

- Open questions: Schmidt number, entanglement spread

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Retrieving the asymptotic entanglement cost

$$E_C(\rho_{AB}) = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}E^{(1)}_{C,\varepsilon}(\rho_{AB}^{\otimes n})$$

Our Theorem $\Rightarrow$ $$.... \leq E^{(1)}_{C,\varepsilon}(\rho_{AB}^{\otimes n}) \leq ....$$

$S_0(\rho\|\sigma) \leq S(\rho\|\sigma)$

Fannes' inequality

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}$$

Generalized Stein's lemma

$$E_C(\rho_{AB}) = E_F^{\infty}(\rho_{AB})$$

[Hayden, Horodecki, Terhal]