# Segre maps and entanglement for multipartite systems of indistinguishable particles

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J.Grabowski (IMPAN)

Segre maps and entanglement

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- Density states and composite systems
- Entanglement and measure of entanglement
- Segre embedding
- Multipartite systems of bosons and fermions
- Multipartite Hermitian product and contractions
- The S-rank and simple tensors
- Entanglement for indistinguishable particles
- Bosonic and fermionic Segre maps
- Generalized parastatistics and simple tensors
- Entanglement for arbitrary parastatistics
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- $U(\mathcal{H})$  the group of unitary operators
- $u(\mathcal{H})$  anti-Hermitian operators
- $u^*(\mathcal{H})$  Hermitian operators
- The set  $\mathcal{D}_1(\mathcal{H})$  of pure states is the image of the map

$$\mathcal{H}\setminus\{0\}
i x\mapsto 
ho_x=rac{|x
angle\!\langle x|}{||x||^2}\in\mathcal{D}^1(\mathcal{H})\subset u^*(\mathcal{H})\,.$$

•  $\mathcal{D}_1(\mathcal{H})\simeq \mathbb{P}\mathcal{H}=(\mathcal{H}\setminus\{0\})/\mathbb{C}^{\times}$  – Hilbert projective space

•  $\mathcal{D}(\mathcal{H}) = \operatorname{convex}(\mathcal{D}_1(\mathcal{H})) - \text{convex body of (mixed) density states}$ 

$$\mathcal{D}(\mathcal{H}) = \{\sum_i \lambda_i 
ho_{\mathsf{x}_i} : \lambda_i \geq 0\,, \quad \sum_i \lambda_i = 1\,, \quad \mathsf{x}_i \in \mathcal{H}\}\,.$$

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## Let now $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ .

In  $\mathcal{H}$  we can distinguish separable (simple) tensors of the form  $x = x_1 \otimes x_2$ . The corresponding pure states we call separable,

$$\mathcal{S}^1(\mathcal{H}) = \{ \rho_x \in \mathcal{D}^1(\mathcal{H}) : x = x_1 \otimes x_2 \}.$$

One can show that

 $\rho_{x_1\otimes x_2}=\rho_{x_1}\otimes\rho_{x_2}\,,$ 

i.e.

$$\mathcal{S}^1(\mathcal{H}) = \{ \rho \in \mathcal{D}^1(\mathcal{H}) : \rho = \rho_{x_1} \otimes \rho_{x_2}, \quad x_i \in \mathcal{H}_i \}.$$

Here we use the decomposition

$$u^*(\mathcal{H}) = u^*(\mathcal{H}_1) \otimes u^*(\mathcal{H}_2)$$
,

where

$$(A_1\otimes A_2)(x_1\otimes x_2)=A_1x_1\otimes A_2x_2$$

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ho_i : \lambda_i \ge 0 \ , \ \sum_i \lambda_i = 1 \ , \ 
ho_i \in \mathcal{S}^1(\mathcal{H}) \} \ .$$

The other states are called entangled,

 $\mathcal{E}(\mathcal{H}) = \mathcal{D}(\mathcal{H}) \setminus \mathcal{S}(\mathcal{H})$  .

### Example

Put  $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}^2$ . The tensor  $x = \frac{1}{\sqrt{2}} (|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)$  is not simple and the corresponding pure state

# $egin{aligned} & p_{\mathbf{x}} & = & rac{1}{2} \left( \left| 0 ight angle \otimes \left| 1 ight angle \left\langle 0 ight| \otimes \left\langle 1 ight| + \left| 1 ight angle \otimes \left| 0 ight angle \left\langle 1 ight| \otimes \left\langle 0 ight| & \ - & \left| 0 ight angle \otimes \left| 1 ight angle \left\langle 1 ight| \otimes \left\langle 0 ight| - & \left| 1 ight angle \otimes \left| 0 ight angle \left\langle 0 ight angle & \left\langle 1 ight angle \end{aligned}$

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 $\begin{array}{ll} |0\rangle \otimes |1\rangle \langle 0| \otimes \langle 1| + |1\rangle \otimes |0\rangle \langle 1| \otimes \langle 0| \right) \frac{1}{2} &= -\frac{1}{2} \langle 0| \otimes \langle 1| \otimes \langle 0| - \frac{1}{2} \rangle \\ &- \langle 0| \otimes \langle 0| \otimes \langle 1| - |0\rangle \otimes |1\rangle \langle 1| \otimes \langle 0| - \frac{1}{2} \rangle \end{array}$ 

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ho_{\mathrm{X}} &=& rac{1}{2} \left( |0
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angle \langle 0| \otimes \langle 1| + |1
angle \otimes |0
angle \langle 1| \otimes \langle 0| \ &- |0
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Question 1: How to decide whether a given state is entangled? Question 2: How to measure the entanglement?

For pure states we have nice answers

• Concurrence:

$$c(
ho)=\sqrt{1-\mathrm{tr}\left(\mathrm{tr}_{\,2}(
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ight)}\,,$$

where  $\mathrm{tr}_2$  is the trace with respect to the second subsystem,

$$\operatorname{tr}_2(A_1\otimes A_2) = \operatorname{tr}(A_2)\cdot A_1.$$

• Schmidt rank: the number k of components in any Schmidt decomposition.

### l heorem

 $\mathcal{H}_1\otimes\mathcal{H}_2$  admits a Schmidt decomposition .

 $\chi = \sum \lambda_i e_i \otimes f_i \ , \ \lambda_i > 0 \ , \ (e_i)$  and  $(f_i)$  – orthonormal sets.

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## $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ admits a Schmidt decomposition .

 $\mathsf{x}=\sum \lambda_l \mathsf{e}_l\otimes \mathsf{f}_l\,,\;\lambda_l>0\,,\;(\mathsf{e}_l)$  and  $(f_l)$  – orthonormal sets.

Question 1: How to decide whether a given state is entangled? Question 2: How to measure the entanglement?

For pure states we have nice answers:

• Concurrence:

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In particular, Seg embeds  $\mathbb{CP}^n \times \mathbb{CP}^m$  into  $\mathbb{CP}^{nm+n+m}$ .

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A pure state  $\rho$  on  $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$  is entangled if and only if  $\rho$  lies outside the range of the Segre embedding.

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- We work on  $\mathcal{H}^{\otimes 2} = \mathcal{H} \otimes \mathcal{H}$  or, more generally,  $\mathcal{H}^{\otimes n} = \mathcal{H} \otimes \cdots \otimes \mathcal{H}$  *n*-times.
- Question 4: Which symmetric/antisymmetric tensors are 'simple' (separable)?
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- Symmetric tensors (bosons):  $x_1 \otimes x_2 + x_2 \otimes x_1$  or rather  $x \otimes x$ ?
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$$\pi_k^{\vee}(f_1 \otimes \cdots \otimes f_k) = \frac{1}{k!} \sum_{\sigma \in S_k} f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(k)},$$

$$\pi_k^{\wedge}(f_1\otimes\cdots\otimes f_k)=rac{1}{k!}\sum_{\sigma\in S_k}(-1)^{\sigma}f_{\sigma(1)}\otimes\cdots\otimes f_{\sigma(k)}$$

The Hermitian product in  $\mathcal{H}$  has an obvious extension to a Hermitian product in  $\mathcal{H}^{\otimes k}$ ,  $\mathcal{H}^{\vee k}$  and  $\mathcal{H}^{\wedge k}$ :

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$$\langle f_1 \otimes \cdots \otimes f_k | g_1 \otimes \cdots \otimes g_k \rangle = \prod_{i=1}^k \langle f_i | g_i \rangle \cdot \langle f_1 \vee \cdots \vee f_k | g_1 \vee \cdots \vee g_k \rangle = \frac{1}{(k!)^2} \sum_{\sigma_i \tau \in S_k} \prod_{i=1}^k \langle f_{\sigma(i)} | g_{\tau(i)} \rangle = \frac{1}{k!} \operatorname{per}(\langle f_i | g_j \rangle).$$

Here,

$$\operatorname{per}(a_k) = \frac{1}{k!} \sum_{\tau \in S_k} \prod_{i=1}^k a_{i\tau(i)}$$

is the permanent of the matrix  $A = (a_{ij})$ .

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# $\langle f_1 \wedge \cdots \wedge f_k | g_1 \wedge \cdots \wedge g_k \rangle = rac{1}{k!} \det(\langle f_i | g_j \rangle).$

J.Grabowski (IMPAN)

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The Hermitian product in  $\mathcal{H}$  has an obvious extension to a Hermitian product in  $\mathcal{H}^{\otimes k}$ ,  $\mathcal{H}^{\vee k}$  and  $\mathcal{H}^{\wedge k}$ :

$$\langle f_1 \otimes \cdots \otimes f_k | g_1 \otimes \cdots \otimes g_k \rangle = \prod_{i=1}^k \langle f_i | g_i \rangle .$$

$$\langle f_1 \vee \cdots \vee f_k | g_1 \vee \cdots \vee g_k \rangle = \frac{1}{(k!)^2} \sum_{\sigma, \tau \in S_k} \prod_{i=1}^k \langle f_{\sigma(i)} | g_{\tau(i)} \rangle = \frac{1}{k!} \operatorname{per}(\langle f_i | g_j \rangle).$$

$$\operatorname{por}(a_{ij}) = \frac{1}{k!} \sum_{\tau \in S_k} \prod_{i=1}^{l} a_{i\tau(i)}$$

is the permanent of the matrix  $A=(a_{ij}).$ 

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J.Grabowski (IMPAN)

Segre maps and entanglement

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The Hermitian product in  $\mathcal{H}$  has an obvious extension to a Hermitian product in  $\mathcal{H}^{\otimes k}$ ,  $\mathcal{H}^{\vee k}$  and  $\mathcal{H}^{\wedge k}$ :

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$$\langle f_1 \otimes \cdots \otimes f_k | g_1 \otimes \cdots \otimes g_k \rangle = \prod_{i=1}^k \langle f_i | g_i \rangle.$$
  
•  $\langle f_1 \vee \cdots \vee f_k | g_1 \vee \cdots \vee g_k \rangle = \frac{1}{(k!)^2} \sum_{\sigma, \tau \in S_k} \prod_{i=1}^k \langle f_{\sigma(i)} | g_{\tau(i)} \rangle = \frac{1}{k!} \operatorname{per}(\langle f_i | g_j \rangle).$ 

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J.Grabowski (IMPAN)

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$$\langle f_1 \wedge \cdots \wedge f_k | g_1 \wedge \cdots \wedge g_k \rangle = \frac{1}{k!} \det(\langle f_i | g_j \rangle).$$

These Hermitian products can be extended to contractions (inner products) between  $\mathcal{H}^{\otimes k}$   $(\mathcal{H}^{\vee k}, \mathcal{H}^{\wedge k})$  on one hand, and  $\mathcal{H}^{\otimes l}$   $(\mathcal{H}^{\vee l}, \mathcal{H}^{\wedge l})$  on the other,  $l \leq k$ .

• For  $f = f_1 \otimes \cdots \otimes f_k \in \mathcal{H}^{\otimes k}$  and  $g = g_1 \otimes \cdots \otimes g_l \in \mathcal{H}^{\otimes l}$ ,

 $i_g f = \langle g_1 \otimes \cdots \otimes g_l | f_1 \otimes \cdots \otimes f_l \rangle f_{l+1} \otimes \cdots \otimes f_k$ 

• If  $f \in \mathcal{H}^{\vee k}$   $(f \in \mathcal{H}^{\wedge k})$  and  $g \in \mathcal{H}^{\vee l}$   $(g \in \mathcal{H}^{\wedge l})$ , then  $\imath_g f \in \mathcal{H}^{\vee (k-l)}$  $(\imath_g f \in \mathcal{H}^{\wedge (k-l)})$ .

In particular,

 $-t_{g_1\vee\cdots\vee g_{k-1}}f_1\vee\cdots\vee f_k = \frac{1}{k!}\sum_{j=1}(g_1\vee\cdots\vee g_{k-1})f_1\vee\cdots\vee f_k)f_j,$ 

 $\iota_{g_1\wedge\cdots\wedge g_{k-1}}f_1\wedge\cdots\wedge f_k=rac{1}{k!}\sum_{j=1}^k (-1)^{k-j}\langle g_1\wedge\cdots\wedge g_{k-1}|f_1\wedge \overset{j}{\cdots}\wedge f_k
angle f_j.$ 

These Hermitian products can be extended to contractions (inner products) between  $\mathcal{H}^{\otimes k}$  ( $\mathcal{H}^{\vee k}$ ,  $\mathcal{H}^{\wedge k}$ ) on one hand, and  $\mathcal{H}^{\otimes l}$  ( $\mathcal{H}^{\vee l}$ ,  $\mathcal{H}^{\wedge l}$ ) on the other,  $l \leq k$ .

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• If  $f \in \mathcal{H}^{\vee k}$   $(f \in \mathcal{H}^{\wedge k})$  and  $g \in \mathcal{H}^{\vee l}$   $(g \in \mathcal{H}^{\wedge l})$ , then  $\imath_g f \in \mathcal{H}^{\vee (k-l)}$  $(\imath_g f \in \mathcal{H}^{\wedge (k-l)})$ .

In particular,

 $-\epsilon_{g_1} \vee \cdots \vee \epsilon_{g_{k-1}} f_1 \vee \cdots \vee f_k = \frac{1}{k!} \sum_{i=1} \left( g_1 \vee \cdots \vee g_{k-1} \right) f_1 \vee \cdots \vee f_k \right) f_j.$ 

 $\iota_{\mathbf{g}_1\wedge\cdots\wedge\mathbf{g}_{k-1}}f_1\wedge\cdots\wedge f_k = rac{1}{k!}\sum_{j=1}^k (-1)^{k-j} \langle g_1\wedge\cdots\wedge g_{k-1}|f_1\wedge \stackrel{j}{\cdots}\wedge f_k 
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• For  $f = f_1 \otimes \cdots \otimes f_k \in \mathcal{H}^{\otimes k}$  and  $g = g_1 \otimes \cdots \otimes g_l \in \mathcal{H}^{\otimes l}$ ,

 $i_g f = \langle g_1 \otimes \cdots \otimes g_l | f_1 \otimes \cdots \otimes f_l \rangle f_{l+1} \otimes \cdots \otimes f_k$ 

• If  $f \in \mathcal{H}^{\vee k}$   $(f \in \mathcal{H}^{\wedge k})$  and  $g \in \mathcal{H}^{\vee l}$   $(g \in \mathcal{H}^{\wedge l})$ , then  $\imath_g f \in \mathcal{H}^{\vee (k-l)}$  $(\imath_g f \in \mathcal{H}^{\wedge (k-l)})$ .

In particular,

 $\langle \langle g_1 \vee \cdots \vee g_{k-1} f_1 \rangle \vee \cdots \vee \langle f_k = rac{1}{k!} \sum_{i=1} \langle g_1 \vee \cdots \vee g_{k-1} | f_1 \rangle \vee \cdots \vee \langle f_k \rangle f_j,$ 

 $\iota_{\mathbf{g}_1\wedge\cdots\wedge\mathbf{g}_{k+1}}f_1\wedge\cdots\wedge f_k = rac{1}{k!}\sum_{i=1}^k (-1)^{k-j} \langle g_1\wedge\cdots\wedge g_{k+1}|f_1\wedge\cdots\wedge f_k 
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These Hermitian products can be extended to contractions (inner products) between  $\mathcal{H}^{\otimes k}$  ( $\mathcal{H}^{\vee k}$ ,  $\mathcal{H}^{\wedge k}$ ) on one hand, and  $\mathcal{H}^{\otimes l}$  ( $\mathcal{H}^{\vee l}$ ,  $\mathcal{H}^{\wedge l}$ ) on the other,  $l \leq k$ .

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• If  $f \in \mathcal{H}^{\vee k}$   $(f \in \mathcal{H}^{\wedge k})$  and  $g \in \mathcal{H}^{\vee l}$   $(g \in \mathcal{H}^{\wedge l})$ , then  $\imath_g f \in \mathcal{H}^{\vee (k-l)}$  $(\imath_g f \in \mathcal{H}^{\wedge (k-l)})$ .

• In particular,

$$i_{g_1 \vee \cdots \vee g_{k-1}} f_1 \vee \cdots \vee f_k = \frac{1}{k!} \sum_{j=1}^k \langle g_1 \vee \cdots \vee g_{k-1} | f_1 \vee \overset{\vee}{\cdots} \vee f_k \rangle f_j,$$

$$i_{g_1 \wedge \cdots \wedge g_{k-1}} f_1 \wedge \cdots \wedge f_k = \frac{1}{k!} \sum_{j=1}^k (-1)^{k-j} \langle g_1 \wedge \cdots \wedge g_{k-1} | f_1 \wedge \overset{\vee}{\cdots} \wedge f_k \rangle f_j.$$

These Hermitian products can be extended to contractions (inner products) between  $\mathcal{H}^{\otimes k}$  ( $\mathcal{H}^{\vee k}$ ,  $\mathcal{H}^{\wedge k}$ ) on one hand, and  $\mathcal{H}^{\otimes l}$  ( $\mathcal{H}^{\vee l}$ ,  $\mathcal{H}^{\wedge l}$ ) on the other,  $l \leq k$ .

• For  $f = f_1 \otimes \cdots \otimes f_k \in \mathcal{H}^{\otimes k}$  and  $g = g_1 \otimes \cdots \otimes g_l \in \mathcal{H}^{\otimes l}$ ,

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• If  $f \in \mathcal{H}^{\vee k}$   $(f \in \mathcal{H}^{\wedge k})$  and  $g \in \mathcal{H}^{\vee l}$   $(g \in \mathcal{H}^{\wedge l})$ , then  $\imath_g f \in \mathcal{H}^{\vee (k-l)}$  $(\imath_g f \in \mathcal{H}^{\wedge (k-l)})$ .

• In particular,

$$i_{g_1 \lor \dots \lor g_{k-1}} f_1 \lor \dots \lor f_k = rac{1}{k!} \sum_{j=1}^k \langle g_1 \lor \dots \lor g_{k-1} | f_1 \lor \stackrel{\checkmark}{\dots} \lor f_k 
angle f_j,$$
  
 $g_1 \land \dots \land g_{k-1} f_1 \land \dots \land f_k = rac{1}{k!} \sum_{j=1}^k (-1)^{k-j} \langle g_1 \land \dots \land g_{k-1} | f_1 \land \stackrel{\checkmark}{\dots} \land f_k 
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# Contractions

These Hermitian products can be extended to contractions (inner products) between  $\mathcal{H}^{\otimes k}$  ( $\mathcal{H}^{\vee k}$ ,  $\mathcal{H}^{\wedge k}$ ) on one hand, and  $\mathcal{H}^{\otimes l}$  ( $\mathcal{H}^{\vee l}$ ,  $\mathcal{H}^{\wedge l}$ ) on the other,  $l \leq k$ .

• For  $f = f_1 \otimes \cdots \otimes f_k \in \mathcal{H}^{\otimes k}$  and  $g = g_1 \otimes \cdots \otimes g_l \in \mathcal{H}^{\otimes l}$ ,

 $i_g f = \langle g_1 \otimes \cdots \otimes g_l | f_1 \otimes \cdots \otimes f_l \rangle f_{l+1} \otimes \cdots \otimes f_k$ .

- If  $f \in \mathcal{H}^{\vee k}$   $(f \in \mathcal{H}^{\wedge k})$  and  $g \in \mathcal{H}^{\vee l}$   $(g \in \mathcal{H}^{\wedge l})$ , then  $\imath_g f \in \mathcal{H}^{\vee (k-l)}$  $(\imath_g f \in \mathcal{H}^{\wedge (k-l)})$ .
- In particular,

$$i_{g_1 \vee \cdots \vee g_{k-1}} f_1 \vee \cdots \vee f_k = \frac{1}{k!} \sum_{j=1}^k \langle g_1 \vee \cdots \vee g_{k-1} | f_1 \vee \cdots \vee f_k \rangle f_j,$$
  
 $g_1 \wedge \cdots \wedge g_{k-1} f_1 \wedge \cdots \wedge f_k = \frac{1}{k!} \sum_{j=1}^k (-1)^{k-j} \langle g_1 \wedge \cdots \wedge g_{k-1} | f_1 \wedge \cdots \wedge f_k \rangle f_j.$ 

# Contractions

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- If  $f \in \mathcal{H}^{\vee k}$   $(f \in \mathcal{H}^{\wedge k})$  and  $g \in \mathcal{H}^{\vee l}$   $(g \in \mathcal{H}^{\wedge l})$ , then  $\imath_g f \in \mathcal{H}^{\vee (k-l)}$  $(\imath_g f \in \mathcal{H}^{\wedge (k-l)})$ .
- In particular,

$$egin{aligned} & f_{g_1 ee \cdots ee g_{k-1}} f_1 ee \cdots ee f_k = rac{1}{k!} \sum_{j=1}^k \langle g_1 ee \cdots ee g_{k-1} | f_1 ee \stackrel{j}{\cdots} ee f_k 
angle f_j, \ & g_1 \wedge \cdots \wedge g_{k-1} f_1 \wedge \cdots \wedge f_k = rac{1}{k!} \sum_{j=1}^k (-1)^{k-j} \langle g_1 \wedge \cdots \wedge g_{k-1} | f_1 \wedge \stackrel{j}{\cdots} \wedge f_k 
angle f_j. \end{aligned}$$

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### Definition

Let  $u \in \mathcal{H}^{\otimes k}$ . By the S-rank of u, we understand the maximum of dimensions of the linear spaces  $\imath_{\mathcal{H}}^{k-1}\sigma(u)$ , for  $\sigma \in S_k$ , which are the images of the contraction maps

$$\mathcal{H}^{\otimes (k-1)} \ni \nu \mapsto \imath_{\nu} \sigma(u) \in \mathcal{H}.$$

Non-zero tensors of minimal S-rank in  $\mathcal{H}^{\otimes k}$  (resp.,  $\mathcal{H}^{\vee k}$ ,  $\mathcal{H}^{\wedge k}$ ) we will call simple (resp., simple symmetric, simple antisymmetric).

Note that for  $u \in \mathcal{H}^{\lor k}$  (resp.,  $u \in \mathcal{H}^{\land k}$ ), the S-rank of u equals the dimension of the linear space which is the image of the contraction map,

$$\mathcal{H}^{\vee(k-1)} \ni \nu \mapsto \imath_{\nu} u \in \mathcal{H},$$

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Let  $u \in \mathcal{H}^{\otimes k}$ . By the S-rank of u, we understand the maximum of dimensions of the linear spaces  $\iota_{\mathcal{H}}^{k-1}\sigma(u)$ , for  $\sigma \in S_k$ , which are the images of the contraction maps

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### Definition

Let  $u \in \mathcal{H}^{\otimes k}$ . By the S-rank of u, we understand the maximum of dimensions of the linear spaces  $\iota_{\mathcal{H}}^{k-1}\sigma(u)$ , for  $\sigma \in S_k$ , which are the images of the contraction maps

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 $\mathcal{H}^{\vee(k-1)} \ni \nu \mapsto \imath_{\nu} u \in \mathcal{H},$ 

$$\mathcal{H}^{\wedge (k-1)} \ni \nu \mapsto \imath_{\nu} u \in \mathcal{H}.$$

Assume that  $(e_i)$  is an orthonormal set in  $\mathcal{H}$ .

Example

The tensor  $u=e_1\otimes e_1$  has the S-rank 1:

 $\iota_{ae_1+be_2}\sigma(u) = \iota_{ae_1+be_2}u = \langle ae_1+be_2|e_1
angle e_1 = ae_1$  .

Example

The tensor  $u_{\pm}=e_1\otimes e_2\pm e_2\otimes e_1$  has the S-rank 2:

 $egin{aligned} & u_{ae_1+be_2}\sigma(u) &= \pm u_{ae_1+be_2}u_{\pm} = \ \pm (ae_1+be_2|e_1)e_2 \pm (ae_1+be_2|e_2)e_1 = \pm ae_2 \pm be_1 \,. \end{aligned}$ 

Example

 $f_i(e_i)$  and  $(f_i)$  are orthonormal sets,  $\lambda_i > 0,$  then the S-rank of .

Assume that  $(e_i)$  is an orthonormal set in  $\mathcal{H}$ .

#### Example

The tensor  $u = e_1 \otimes e_1$  has the S-rank 1:

$$\imath_{ae_1+be_2}\sigma(u)=\imath_{ae_1+be_2}u=\langle ae_1+be_2|e_1
angle e_1=ae_1$$
 .

### Example

The tensor  $u_{\pm}=e_1\otimes e_2\pm e_2\otimes e_1$  has the S-rank 2:

$$\begin{split} \imath_{ae_1+be_2}\sigma(u) &= \pm \imath_{ae_1+be_2}u_{\pm} = \\ &\pm \langle ae_1+be_2|e_1\rangle e_2 \pm \langle ae_1+be_2|e_2\rangle e_1 = \pm ae_2 \pm be_1 \,. \end{split}$$

#### Example

If  $(e_i)$  and  $(f_i)$  are orthonormal sets,  $\lambda_i > 0$ , then the S-rank of  $\sum_{i=1}^r \lambda_i e_i \otimes f_i$  is r, i.e., the S-rank equals the Schmidt rank.

J.Grabowski (IMPAN)

Segre maps and entanglement

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### Example

The tensor  $u = e_1 \otimes e_1$  has the S-rank 1:

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#### Theorem

 The minimal possible S-rank of a non-zero tensor u ∈ H<sup>⊗k</sup> equals 1. A tensor u ∈ H<sup>⊗k</sup> is of S-rank 1 if and only if

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## Definition

- A pure state p<sub>x</sub> on H<sup>∨k</sup> with x ∈ H<sup>∨k</sup>, x ≠ 0, is called a bosonic simple pure state if x is a simple symmetric, x = f ⊗ ···· ⊗ f, tensor.
   If y is not simple symmetric, we call by a basened entropy of the symmetric.
  - A pure state (c) on (2011) with (c) is 2012, (c) sole (c) is called a free from one should pure state (c) on (2011) with (c) is 2010 problem in the problem in the pure called a free from (c) there is not some from the problem in the problem in the problem is called by the formation (c) and (c) is not some place in the problem in the problem is called by the formation (c) and (c) is a problem in the problem in the problem is called by the formation (c) and (c) is a problem in the problem in the problem in the problem is called by the formation (c).

- A pure state ρ<sub>x</sub> on H<sup>∨k</sup> with x ∈ H<sup>∨k</sup>, x ≠ 0, is called a bosonic simple pure state if x is a simple symmetric, x = f ⊗ ··· ⊗ f, tensor. If x is not simple symmetric, we call ρ<sub>x</sub> a bosonic entangled state.
- A pure state ρ<sub>x</sub> on H<sup>∧k</sup>) with x ∈ H<sup>∧k</sup>, x ≠ 0, is called a fermionic simple pure state if x is a simple antisymmetric, x = f<sub>1</sub> ∧ · · · ∧ f<sub>k</sub>, tensor. If x is not simple antisymmetric, we call ρ<sub>x</sub> a fermionic entangled state.
- A mixed state ρ on H<sup>VK</sup> (resp., on H<sup>AK</sup>) we call bosonic (fermionic) simple mixed state if it can be written as a convex combination of bosonic (fermionic) simple pure states. In the other case, ρ is called bosonic (fermionic) entangled mixed state.

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where  $\mathcal{H}_{\circ}^{\times k} = \mathcal{H}^{\times k} \setminus \{(x_1, \ldots, x_k) : x_1 \wedge \cdots \wedge x_k = 0\}$ 

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Note that the condition  $x_1 \wedge \cdots \wedge x_k \neq 0$  does not depend on the choice of the vectors  $x_1, \ldots, x_k$  in their projective classes and means that  $\rho_{x_1}, \ldots, \rho_{x_k}$  do not lie in a common projective hyperspace.

The subset  $\mathcal{H}_{\circ}^{\times k}$  (resp.,  $(\mathbb{P}\mathcal{H})_{\circ}^{\times k}$ ) is open and dense in  $\mathcal{H}^{\times k}$  (resp.,  $(\mathbb{P}\mathcal{H})^{\times k}$ ).

#### Theorem

A bosonic (fermionic) pure state  $\rho \in \mathbb{P}(\mathcal{H}^{\vee k})$  (resp.,  $\rho \in \mathbb{P}(\mathcal{H}^{\wedge k})$ ) is entangled if and only if it lies outside the range of the Segre map

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$$U(x_1 \otimes \cdots \otimes x_k) = U(x_1) \otimes \cdots \otimes U(x_k)$$
.

• We can identify the symmetry group  $S_k$  with the group of certain unitary operators on the Hilbert space  $\mathcal{H}^k$  in the obvious way,

$$\sigma(x_1\otimes\cdots\otimes x_k)=x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(k)}.$$

- The operators of  $S_k$  intertwine the unitary action of  $U(\mathcal{H})$ .
- For k > 2, there are other irreducible parts of the above representation of U(H) associated with invariant subspaces of the S<sub>k</sub>-action. We shall call them (generalized) k-parastatistics.

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- Any of these k-parastatistics (i.e., any irreducible subspace of the tensor product H<sup>⊗k</sup>) is associated with a Young tableau α with k-boxes (chambers). The corresponding irreducible subrepresentation in H<sup>⊗k</sup> we denote H<sup>α</sup>.
- Any irreducible representation H<sup>α</sup> contains cyclic vectors which are of highest weight relative to some choice of a maximal torus and Borel subgroups in U(H). We will call them α-simple tensors or simple tensors in H<sup>α</sup>.
- Such vectors can be viewed as generalized coherent states or as the 'most classical' states with respect to their correlation properties. These tensors represent the minimal amount of quantum correlations for tensors in H<sup>a</sup>, namely the quantum correlations forced directly by the particular parastatistics.

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#### Definition

Consider partitions of k:  $k = \lambda_1 + \cdots + \lambda_r$ , where  $\lambda_1 \ge \cdots \ge \lambda_r \ge 1$ . To a partition  $\lambda = (\lambda_1, \ldots, \lambda_r)$  is associated a *Young diagram* with  $\lambda_i$  boxes in the *i*th row, the rows of boxes lined up on the left. Define a *tableau* on a given Young diagram to be a numbering of the boxes by the integers  $1, \ldots, k$ , and denote with Y(k) the set of all Young tableaux with k boxes.

#### Theorem

To each  $\alpha \in Y(k)$  corresponds an irreducible component  $\mathcal{H}^{\alpha}$  in  $\mathcal{H}^{\otimes k}$ . A tensor  $v \in \mathcal{H}^{\alpha}$  is simple if and only if it has the minimal S-rank among non-zero tensors from  $\mathcal{H}^{\alpha}$ . This minimal S-rank equals the number r of rows in the corresponding Young diagram and the simple tensor reads as  $v = \pi^{\alpha} \left( e_{\alpha(1)} \otimes \cdots \otimes e_{\alpha(k)} \right),$ 

where  $e_1, \ldots, e_r$  are some linearly independent vectors in  $\mathcal{H}$  and  $\alpha(i)$  is the number of the row in which the box with the number *i* appears in the tableaux  $\alpha$ .

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J.Grabowski (IMPAN)

For k = 3, besides symmetric and antisymmetric tensors associated with

the Young tableaux  $\alpha_0 = 1 2 3$  and  $\alpha_3 = \frac{1}{2}$ , we have two

additional irreducible parts associated with the Young tableaux

$$\alpha_1 = \begin{array}{c}
1 & 2 \\
3 & 
\end{array}$$
 and  $\alpha_2 = \begin{array}{c}
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Hence,

$$\mathcal{H}^{\otimes 3} = \mathcal{H}^{\wedge 3} \oplus \mathcal{H}^{\alpha_1} \oplus \mathcal{H}^{\alpha_2} \oplus \mathcal{H}^{\vee 3}\,,$$

with

$$\pi^{\alpha_i}: \mathcal{H}^{\otimes 3} \to \mathcal{H}^{\alpha_i},$$

 $\pi^{\alpha_1}(x_1 \otimes x_2 \otimes x_3) = \frac{1}{3}(x_1 \otimes x_2 \otimes x_3 + x_2 \otimes x_1 \otimes x_3 - x_3 \otimes x_2 \otimes x_1 - x_3 \otimes x_1 \otimes x_2),$ and  $\pi^{\alpha_2}(x_1 \otimes x_2 \otimes x_3) = \frac{1}{3}(x_1 \otimes x_2 \otimes x_3 + x_3 \otimes x_2 \otimes x_1 - x_2 \otimes x_1 \otimes x_3 - x_2 \otimes x_3 \otimes x_1).$ 

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The simple tensors (the highest weight vectors) in  $\mathcal{H}^{\alpha_1}$  can be written as

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for certain choice of an orthonormal basis  $e_i$  in  $\mathcal{H}$  and  $\lambda \neq 0$ . Analogously, the simple tensors in  $\mathcal{H}^{\alpha_2}$  take the form

$$\begin{array}{lll} v_{\lambda}^{\alpha_2} & = & \pi^{\alpha_2}(x_{\alpha_2(1)}\otimes x_{\alpha_2(2)}\otimes x_{\alpha_2(3)}) \\ & = & \pi^{\alpha_2}(x_1\otimes x_2\otimes x_1) \\ & = & \lambda(e_1\otimes e_2\otimes e_1 - e_2\otimes e_1\otimes e_1) \end{array}$$

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\mathbf{v}_{\boldsymbol{\lambda}}^{\boldsymbol{\alpha_{1}}} &= \pi^{\alpha_{1}}(x_{\alpha_{1}(1)} \otimes x_{\alpha_{1}(2)} \otimes x_{\alpha_{1}(3)}) \\
&= \pi^{\alpha_{1}}(x_{1} \otimes x_{1} \otimes x_{2}) \\
&= \frac{2}{3}(x_{1} \otimes x_{1} \otimes x_{2} - x_{2} \otimes x_{1} \otimes x_{1}) \\
&= \lambda(e_{1} \otimes e_{1} \otimes e_{2} - e_{2} \otimes e_{1} \otimes e_{1}),
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$$\begin{aligned} \nu_{\lambda}^{\alpha_{1}} &= \pi^{\alpha_{1}} (x_{\alpha_{1}(1)} \otimes x_{\alpha_{1}(2)} \otimes x_{\alpha_{1}(3)}) \\ &= \pi^{\alpha_{1}} (x_{1} \otimes x_{1} \otimes x_{2}) \\ &= \frac{2}{3} (x_{1} \otimes x_{1} \otimes x_{2} - x_{2} \otimes x_{1} \otimes x_{1}) \\ &= \lambda (e_{1} \otimes e_{1} \otimes e_{2} - e_{2} \otimes e_{1} \otimes e_{1}), \end{aligned}$$

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$$\begin{array}{lll} v_{\lambda}^{\alpha_2} & = & \pi^{\alpha_2}(x_{\alpha_2(1)}\otimes x_{\alpha_2(2)}\otimes x_{\alpha_2(3)}) \\ & = & \pi^{\alpha_2}(x_1\otimes x_2\otimes x_1) \\ & = & \lambda(e_1\otimes e_2\otimes e_1 - e_2\otimes e_1\otimes e_1) \end{array}$$

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The simple tensors (the highest weight vectors) in  $\mathcal{H}^{\alpha_1}$  can be written as

$$egin{array}{rcl} & x_{\lambda} & = & \pi^{lpha_1}(x_{lpha_1(1)}\otimes x_{lpha_1(2)}\otimes x_{lpha_1(3)}) \ & = & \pi^{lpha_1}(x_1\otimes x_1\otimes x_2) \ & = & rac{2}{3}(x_1\otimes x_1\otimes x_2-x_2\otimes x_1\otimes x_1) \ & = & \lambda(e_1\otimes e_1\otimes e_2-e_2\otimes e_1\otimes e_1)\,, \end{array}$$

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#### Definition

- We say that a pure state ρ<sub>ν</sub> on H<sup>⊗k</sup> obeys a parastatistics α ∈ Y(k) (is a pure α-state for short) if v ∈ H<sup>α</sup>, i.e. ρ is a pure state on the Hilbert space H<sup>α</sup>.
- A pure state ρ on H<sup>Sk</sup> obeying a parastatistics α is called a simple pure α-state if ρ is represented by an α-simple tensor in H<sup>α</sup>. If ρ is not simple α-state, we call it an entangled pure α-state.
- A mixed state ρ on H<sup>α</sup> we call a simple α-state if it can be written as a convex combination of simple pure α-states. In the other case, ρ is called an entangled mixed α-state.

Simple pure  $\alpha$ -states can be characterized in terms of generalized Segre maps.

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Segre maps and entanglement

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