Segre maps and entanglement for multipartite systems of indistinguishable particles

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Luminy, January 9, 2012

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Density states

- $(\mathcal{H}, \langle\cdot|\cdot\rangle)$ – Hilbert space,
- $U(\mathcal{H})$ – the group of unitary operators
- $u(\mathcal{H})$ – anti-Hermitian operators
- $u^*(\mathcal{H})$ – Hermitian operators
- The set $\mathcal{D}_1(\mathcal{H})$ of pure states is the image of the map

$$\mathcal{H} \setminus \{0\} \ni x \mapsto \rho_x = \frac{|x\rangle\langle x|}{\|x\|^2} \in \mathcal{D}^1(\mathcal{H}) \subset u^*(\mathcal{H}) \,.$$

- $\mathcal{D}_1(\mathcal{H}) \simeq \mathbb{P}\mathcal{H} = (\mathcal{H} \setminus \{0\})/\mathbb{C}^\times$ – Hilbert projective space
- $\mathcal{D}(\mathcal{H}) = \mathrm{convex}(\mathcal{D}_1(\mathcal{H}))$ – convex body of (mixed) density states

$$\mathcal{D}(\mathcal{H}) = \{\sum_i \lambda_i \rho_{x_i} : \lambda_i \geq 0\,, \quad \sum_i \lambda_i = 1\,, \quad x_i \in \mathcal{H}\}\,.$$

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Composite systems

Let now $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$.

In $\mathcal{H}$ we can distinguish separable (simple) tensors of the form $x = x_1 \otimes x_2$. The corresponding pure states we call separable,

$$\mathcal{S}^1(\mathcal{H}) = \{\rho_x \in \mathcal{D}^1(\mathcal{H}) : x = x_1 \otimes x_2\}.$$

One can show that

$$\rho_{x_1 \otimes x_2} = \rho_{x_1} \otimes \rho_{x_2},$$

i.e.

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Here we use the decomposition

$$u^*(\mathcal{H}) = u^*(\mathcal{H}_1) \otimes u^*(\mathcal{H}_2),$$

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$$(A_1 \otimes A_2)(x_1 \otimes x_2) = A_1 x_1 \otimes A_2 x_2.$$

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By definition, the set $\mathcal{S}(\mathcal{H})$ of separable density states is the convex hull of the set $\mathcal{S}^1(\mathcal{H})$ of separable pure states,

$$\mathcal{S}(\mathcal{H}) = \mathrm{convex}(\mathcal{S}^1(\mathcal{H})) = \{\sum_i \lambda_i \rho_i : \lambda_i \geq 0\,,\ \sum_i \lambda_i = 1\,,\ \rho_i \in \mathcal{S}^1(\mathcal{H})\}\,.$$

The other states are called entangled,

$$\mathcal{E}(\mathcal{H}) = \mathcal{D}(\mathcal{H}) \setminus \mathcal{S}(\mathcal{H})\,.$$

Example

Put $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}^2$. The tensor $x = \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle\right)$ is not simple and the corresponding pure state

$$\begin{aligned}\rho_x \quad = \quad & \frac{1}{2}\left(|0\rangle \otimes |1\rangle\langle 0| \otimes \langle 1| + |1\rangle \otimes |0\rangle\langle 1| \otimes \langle 0|\right. \\ & \left. - |0\rangle \otimes |1\rangle\langle 1| \otimes \langle 0| - |1\rangle \otimes |0\rangle\langle 0| \otimes \langle 1|\right)\end{aligned}$$

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Measure of Entanglement

Question 1: How to decide whether a given state is entangled?

Question 2: How to measure the entanglement?

For pure states we have nice answers:

- Concurrence:

$$c(\rho) = \sqrt{1 - \mathrm{tr}\left(\mathrm{tr}_2(\rho)^2\right)},$$

  where tr_2 is the trace with respect to the second subsystem,

$$\mathrm{tr}_2(A_1 \otimes A_2) = \mathrm{tr}(A_2) \cdot A_1 .$$

- Schmidt rank: the number k of components in any Schmidt decomposition.

Theorem

Any $x \in \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ admits a Schmidt decomposition

$$x = \sum_{i=1}^{k} \lambda_i e_i \otimes f_i\,, \ \lambda_i > 0\,, \ (e_i) \text{ and } (f_i) - \text{orthonormal sets.}\,.$$

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$$x = \sum_{i=1}^{k} \lambda_i e_i \otimes f_i\,,\ \lambda_i > 0\,,\ (e_i) \text{ and } (f_i) - \text{orthonormal sets}..$$

Segre embedding

The tensor product map

$$\otimes : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2\,, \quad (x_1, x_2) \mapsto x_1 \otimes x_2\,,$$

associates the product of rays with a ray, so it induces a canonical embedding on the level of complex projective spaces,

$$\begin{aligned} \mathrm{Seg} : \mathbb{P}\mathcal{H}_1 \times \mathbb{P}\mathcal{H}_2 &\to \mathbb{P}\mathcal{H} = \mathbb{P}(\mathcal{H}_1 \otimes \mathcal{H}_2), \\ \mathrm{Seg} : \mathcal{D}^1(\mathcal{H}_1) \times \mathcal{D}^1(\mathcal{H}_2) &\to \mathcal{D}^1(\mathcal{H}) = \mathcal{D}^1(\mathcal{H}_1 \otimes \mathcal{H}_2), \\ (\rho_{x_1}, \rho_{x_2}) &\mapsto \rho_{x^1 \otimes x^2}\,. \end{aligned}$$

In particular, Seg embeds $\mathbb{CP}^n \times \mathbb{CP}^m$ into $\mathbb{CP}^{nm+n+m}$.

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A pure state ρ on $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is entangled if and only if ρ lies outside the range of the Segre embedding.

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Bosons and Fermions

- Question 3: How to define entanglement for composite systems of indistinguishable particles?
- We work on $\mathcal{H}^{\otimes 2} = \mathcal{H} \otimes \mathcal{H}$ or, more generally, $\mathcal{H}^{\otimes n} = \mathcal{H} \otimes \cdots \otimes \mathcal{H}$ - n-times.
- Question 4: Which symmetric/antisymmetric tensors are 'simple' (separable)?
- Antisymmetric tensors (fermions) are never simple in the standard sense: $x_1 \wedge x_2 = x_1 \otimes x_2 - x_2 \otimes x_1$.
- Symmetric tensors (bosons): $x_1 \otimes x_2 + x_2 \otimes x_1$ or rather $x \otimes x$?
- What about other potential statistics (parastatistics)?

We need a unifying mathematical concept of a simple tensor.

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Multipartite systems of Bosons or Fermion

In the tensor power $\mathcal{H}^{\otimes k} = \underbrace{\mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{k-\text{times}}$ we distinguish the subspaces:

- $\mathcal{H}^{\vee k} = \underbrace{\mathcal{H} \vee \cdots \vee \mathcal{H}}_{k-\text{times}}$ of totally symmetric tensors (bosonic Fock),
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The Hermitian product in $\mathcal{H}$ has an obvious extension to a Hermitian product in $\mathcal{H}^{\otimes k}$, $\mathcal{H}^{\vee k}$ and $\mathcal{H}^{\wedge k}$:

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  Here,

  $$\mathrm{per}(a_{ij}) = \frac{1}{k!} \sum_{\tau \in S_k} \prod_{i=1}^{k} a_{i\tau(i)}$$

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The Hermitian product in $\mathcal{H}$ has an obvious extension to a Hermitian product in $\mathcal{H}^{\otimes k}$, $\mathcal{H}^{\vee k}$ and $\mathcal{H}^{\wedge k}$:

- $$\langle f_1 \otimes \cdots \otimes f_k | g_1 \otimes \cdots \otimes g_k \rangle = \prod_{i=1}^{k} \langle f_i | g_i \rangle \,.$$
- $$\langle f_1 \vee \cdots \vee f_k | g_1 \vee \cdots \vee g_k \rangle = \frac{1}{(k!)^2} \sum_{\sigma,\tau \in S_k} \prod_{i=1}^{k} \langle f_{\sigma(i)} | g_{\tau(i)} \rangle = \frac{1}{k!} \mathrm{per}(\langle f_i | g_j \rangle).$$

  Here,
  $$\mathrm{per}(a_{ij}) = \frac{1}{k!} \sum_{\tau \in S_k} \prod_{i=1}^{k} a_{i\tau(i)}$$
  is the permanent of the matrix $A = (a_{ij})$.
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These Hermitian products can be extended to contractions (inner products) between $\mathcal{H}^{\otimes k}$ ($\mathcal{H}^{\vee k}$, $\mathcal{H}^{\wedge k}$) on one hand, and $\mathcal{H}^{\otimes l}$ ($\mathcal{H}^{\vee l}$, $\mathcal{H}^{\wedge l}$) on the other, $l \leq k$.

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$$\imath_g f = \langle g_1 \otimes \cdots \otimes g_l | f_1 \otimes \cdots \otimes f_l \rangle f_{l+1} \otimes \cdots \otimes f_k .$$

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The S-rank

Definition

Let $u \in \mathcal{H}^{\otimes k}$. By the S-rank of u, we understand the maximum of dimensions of the linear spaces $\imath_{\mathcal{H}}^{k-1}\sigma(u)$, for $\sigma \in S_k$, which are the images of the contraction maps

$$\mathcal{H}^{\otimes(k-1)} \ni \nu \mapsto \imath_\nu \sigma(u) \in \mathcal{H}.$$

Non-zero tensors of minimal S-rank in $\mathcal{H}^{\otimes k}$ (resp., $\mathcal{H}^{\vee k}$, $\mathcal{H}^{\wedge k}$) we will call simple (resp., simple symmetric, simple antisymmetric).

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The S-rank: examples

Assume that (e_i) is an orthonormal set in $\mathcal{H}$.

Example

The tensor $u = e_1 \otimes e_1$ has the S-rank 1:

$$\imath_{ae_1+be_2}\sigma(u) = \imath_{ae_1+be_2}u = \langle ae_1 + be_2 | e_1 \rangle e_1 = ae_1 .$$

Example

The tensor $u_\pm = e_1 \otimes e_2 \pm e_2 \otimes e_1$ has the S-rank 2:

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Example

If (e_i) and (f_i) are orthonormal sets, $\lambda_i > 0$, then the S-rank of $\sum_{i=1}^{r} \lambda_i e_i \otimes f_i$ is r, i.e., the S-rank equals the Schmidt rank.

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Theorem

- *The minimal possible S-rank of a non-zero tensor* $u \in \mathcal{H}^{\otimes k}$ *equals* 1. *A tensor* $u \in \mathcal{H}^{\otimes k}$ *is of S-rank* 1 *if and only if*

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- *The minimal possible S-rank of a non-zero tensor* $u \in \mathcal{H}^{\otimes k}$ *equals* 1. *A tensor* $u \in \mathcal{H}^{\otimes k}$ *is of S-rank* 1 *if and only if*

$$u = f_1 \otimes \cdots \otimes f_k, \quad f_i \in \mathcal{H}, \quad f_i \neq 0.$$

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Entanglement for identical particles

Definition

- A pure state ρ_x on $\mathcal{H}^{\vee k}$ with $x \in \mathcal{H}^{\vee k}$, $x \neq 0$, is called a bosonic simple pure state if x is a simple symmetric, $x = f \otimes \cdots \otimes f$, tensor. If x is not simple symmetric, we call ρ_x a bosonic entangled state.
- A pure state ρ_x on $\mathcal{H}^{\wedge k}$) with $x \in \mathcal{H}^{\wedge k}$, $x \neq 0$, is called a fermionic simple pure state if x is a simple antisymmetric, $x = f_1 \wedge \cdots \wedge f_k$, tensor. If x is not simple antisymmetric, we call ρ_x a fermionic entangled state.
- A mixed state ρ on $\mathcal{H}^{\vee k}$ (resp., on $\mathcal{H}^{\wedge k}$) we call bosonic (fermionic) simple mixed state if it can be written as a convex combination of bosonic (fermionic) simple pure states. In the other case ρ is called bosonic (fermionic) entangled mixed state.

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It is clear that the Segre map for the Bose statistics should be

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\mathcal{H}_\circ & \ni & x & \longmapsto & x^{\otimes k} & \in & \left(\mathcal{H}^{\vee k}\right)_\circ \\
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and for the Fermi statistics:

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Note that the condition $x_1 \wedge \cdots \wedge x_k \neq 0$ does not depend on the choice of the vectors $x_1, \ldots, x_k$ in their projective classes and means that $\rho_{x_1}, \ldots, \rho_{x_k}$ do not lie in a common projective hyperspace.

The subset $\mathcal{H}_\circ^{\times k}$ (resp., $(\mathbb{P}\mathcal{H})_\circ^{\times k}$) is open and dense in $\mathcal{H}^{\times k}$ (resp., $(\mathbb{P}\mathcal{H})^{\times k}$).

Theorem

A bosonic (fermionic) pure state $\rho \in \mathbb{P}(\mathcal{H}^{\vee k})$ (resp., $\rho \in \mathbb{P}(\mathcal{H}^{\wedge k})$) is entangled if and only if it lies outside the range of the Segre map

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Generalized parastatistics

- The subspaces $\mathcal{H}^{\vee k}$ and $\mathcal{H}^{\wedge k}$ form particular irreducible parts of the 'diagonal' representation of the compact group $U(\mathcal{H})$ in the Hilbert space $\mathcal{H}^{\otimes k}$,

$$U(x_1 \otimes \cdots \otimes x_k) = U(x_1) \otimes \cdots \otimes U(x_k)\,.$$

- We can identify the symmetry group S_k with the group of certain unitary operators on the Hilbert space $\mathcal{H}^k$ in the obvious way,

$$\sigma(x_1 \otimes \cdots \otimes x_k) = x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}\,.$$

- The operators of S_k intertwine the unitary action of $U(\mathcal{H})$.
- For $k > 2$, there are other irreducible parts of the above representation of $U(\mathcal{H})$ associated with invariant subspaces of the S_k-action. We shall call them (generalized) k-parastatistics.

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$$\sigma(x_1 \otimes \cdots \otimes x_k) = x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}\,.$$

- The operators of S_k intertwine the unitary action of $U(\mathcal{H})$.
- For $k > 2$, there are other irreducible parts of the above representation of $U(\mathcal{H})$ associated with invariant subspaces of the S_k-action. We shall call them (generalized) k-parastatistics.

Generalized parastatistics

- The subspaces $\mathcal{H}^{\vee k}$ and $\mathcal{H}^{\wedge k}$ form particular irreducible parts of the 'diagonal' representation of the compact group $U(\mathcal{H})$ in the Hilbert space $\mathcal{H}^{\otimes k}$,

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Simple tensors of a given parastatistics

- Any of these k-parastatistics (i.e., any irreducible subspace of the tensor product $\mathcal{H}^{\otimes k}$) is associated with a Young tableau α with k-boxes (chambers). The corresponding irreducible subrepresentation in $\mathcal{H}^{\otimes k}$ we denote $\mathcal{H}^{\alpha}$.
- Any irreducible representation $\mathcal{H}^{\alpha}$ contains cyclic vectors which are of highest weight relative to some choice of a maximal torus and Borel subgroups in $U(\mathcal{H})$. We will call them α-simple tensors or simple tensors in $\mathcal{H}^{\alpha}$.
- Such vectors can be viewed as generalized coherent states or as the 'most classical' states with respect to their correlation properties. These tensors represent the minimal amount of quantum correlations for tensors in $\mathcal{H}^{\alpha}$, namely the quantum correlations forced directly by the particular parastatistics.

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Highest vectors have minimal S-rank

Definition

Consider partitions of k: $k = \lambda_1 + \cdots + \lambda_r$, where $\lambda_1 \geq \cdots \geq \lambda_r \geq 1$. To a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ is associated a *Young diagram* with λ_i boxes in the ith row, the rows of boxes lined up on the left. Define a *tableau* on a given Young diagram to be a numbering of the boxes by the integers $1, \ldots, k$, and denote with $Y(k)$ the set of all Young tableaux with k boxes.

Theorem

To each $\alpha \in Y(k)$ corresponds an irreducible component $\mathcal{H}^\alpha$ in $\mathcal{H}^{\otimes k}$. A tensor $v \in \mathcal{H}^\alpha$ is simple if and only if it has the minimal S-rank among non-zero tensors from $\mathcal{H}^\alpha$. This minimal S-rank equals the number r of rows in the corresponding Young diagram and the simple tensor reads as

$$v = \pi^\alpha \left(e_{\alpha(1)} \otimes \cdots \otimes e_{\alpha(k)} \right) ,$$

where $e_1, \ldots, e_r$ are some linearly independent vectors in $\mathcal{H}$ and $\alpha(i)$ is the number of the row in which the box with the number i appears in the tableaux α.

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Example

For $k = 3$, besides symmetric and antisymmetric tensors associated with the Young tableaux $\alpha_0 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$ and $\alpha_3 = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$, we have two additional irreducible parts associated with the Young tableaux

$$\alpha_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \text{and} \quad \alpha_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} .$$

Hence,

$$\mathcal{H}^{\otimes 3} = \mathcal{H}^{\wedge 3} \oplus \mathcal{H}^{\alpha_1} \oplus \mathcal{H}^{\alpha_2} \oplus \mathcal{H}^{\vee 3},$$

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$$\pi^{\alpha_i} : \mathcal{H}^{\otimes 3} \to \mathcal{H}^{\alpha_i},$$

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Example

The simple tensors (the highest weight vectors) in $\mathcal{H}^{\alpha_1}$ can be written as

$$\begin{aligned} v_\lambda^{\alpha_1} &= \pi^{\alpha_1}\big(x_{\alpha_1(1)} \otimes x_{\alpha_1(2)} \otimes x_{\alpha_1(3)}\big) \\ &= \pi^{\alpha_1}\big(x_1 \otimes x_1 \otimes x_2\big) \\ &= \frac{2}{3}\big(x_1 \otimes x_1 \otimes x_2 - x_2 \otimes x_1 \otimes x_1\big) \\ &= \lambda\big(e_1 \otimes e_1 \otimes e_2 - e_2 \otimes e_1 \otimes e_1\big), \end{aligned}$$

for certain choice of an orthonormal basis e_i in $\mathcal{H}$ and $\lambda \neq 0$.

Analogously, the simple tensors in $\mathcal{H}^{\alpha_2}$ take the form

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Definition

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- A pure state ρ on $\mathcal{H}^{\otimes k}$ obeying a parastatistics α is called a simple pure α-state if ρ is represented by an α-simple tensor in $\mathcal{H}^{\alpha}$. If ρ is not simple α-state, we call it an entangled pure α-state.
- A mixed state ρ on $\mathcal{H}^{\alpha}$ we call a simple α-state if it can be written as a convex combination of simple pure α-states. In the other case, ρ is called an entangled mixed α-state.

Simple pure α-states can be characterized in terms of generalized Segre maps.

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