Properties of typical output states from product channels

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Contents

1) Background
2) Random states and channels
3) Average output entropy per channel use
4) Typical behavior of output states from multi-product channels
5) Special cases
6) Typical behavior of partial transpose output states
7) Future directions

Background on states

Quantum system is described by a Hilbert space $\mathcal{H}$.

Pure state $|\psi\rangle$ is a unit vector in $\mathcal{H}$:

$$\langle\psi|\psi\rangle = \|\psi\|^2 = 1$$

Mixed state ρ is a trace one positive semidefinite operator on $\mathcal{H}$. For example

$$\rho = \sum_{j=1}^{n} p_j\,|\psi_j\rangle\langle\psi_j|, \quad \sum_j p_j = 1$$

$$\mathcal{L}(\mathcal{H}) = \{\text{bounded operators on } \mathcal{H}\}$$
$$\mathcal{S}(\mathcal{H}) = \{\text{density matrices on } \mathcal{H}\}$$

Background on entanglement

A compound or bipartite system has state space $\mathcal{H}_1 \otimes \mathcal{H}_2$.
There are product states in $\mathcal{H}_1 \otimes \mathcal{H}_2$ of the form

$$|\psi_1\rangle \otimes |\psi_2\rangle$$

as well as non-product states, for example

$$\frac{1}{\sqrt{2}}\Big[|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle\Big]$$

For mixed states the relevant property is *separability*: the state $\rho_{12} \in \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is separable if

$$\rho_{12} = \sum_j p_j \, \sigma_j \otimes \tau_j$$

with states $\sigma_j \in \mathcal{S}(\mathcal{H}_1)$, $\tau_j \in \mathcal{S}(\mathcal{H}_2)$, and $\sum p_j = 1$.

Background on PPT

If a state ρ_{12} is separable then the partial transpose is positive:

$$\begin{aligned} \rho_{12} &= \sum_j p_j \, \sigma_j \otimes \tau_j \\ (I \otimes \mathcal{T})\rho_{12} &= \sum_j p_j \, \sigma_j \otimes \tau_j^T \end{aligned}$$

The converse is true for low dimensional states, but false in general.

Since PPT is much easier to check than separability, it provides a convenient proxy for measuring entanglement ...

Background on entropy

The von Neumann entropy of the state ρ is

$$S(\rho) = -\text{Tr}\,\rho \log \rho, \quad \rho \in \mathcal{S}(\mathcal{H})$$

and the related Renyi entropies are defined for $r > 0, r \neq 1$ by

$$S_r(\rho) = \frac{1}{1-r} \log \text{Tr}\,\rho^r$$

It follows that

$$\lim_{r \to 1} S_r(\rho) = S(\rho)$$

Entropy measures the information content or disorder of a state. Thus $S_r(\rho) \geq 0$, with equality if and only if $\rho = |\psi\rangle\langle\psi|$ is in a pure state.

Background on channels

A quantum channel on $\mathcal{H}$ is a completely positive trace preserving (CPTP) map

$$\mathcal{A} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$$

The operational meaning of the channel can be seen using the Lindbald-Stinespring representation: there is some ancillary space $\mathcal{E}$ (the environment), some unitary map $U \in \mathcal{L}(\mathcal{H} \otimes \mathcal{E})$, and some state $\omega \in \mathcal{S}(\mathcal{E})$, such that for all $M \in \mathcal{L}(\mathcal{H})$

$$\mathcal{A}(M) = \mathrm{Tr}_{\mathcal{E}} \Big[U \, (M \otimes \omega) \, U^* \Big]$$

where $\mathrm{Tr}_{\mathcal{E}}$ denotes the partial trace over the environment.

[...where $U = e^{-iHt}$ is the unitary dynamics generated by a Hamiltonian operator H which entangles $\mathcal{H}$ and $\mathcal{E}$]

Background on channels

More generally a quantum channel is a CPTP map between different spaces

$$\mathcal{A} : \mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{H}_2)$$

Questions of interest include:

- what is the maximum rate of information transfer using $\mathcal{A}$?
- how noisy is a typical output state $\mathcal{A}(|\psi\rangle\langle\psi|)$ from the channel?
- what is the entanglement of a typical output state $(I \otimes \mathcal{A})(|\psi\rangle\langle\psi|)$?

Random states and random channels in high-dimensional spaces have played an important role in recent advances in QIT.

Randomness for pure states is straightforward: the unit vectors $\|\psi\| = 1$ in $\mathbb{C}^d$ can be identified with the real sphere S^{2d-1}. Normalized uniform measure on S^{2d-1} provides the probability measure.

Alternatively, normalized Haar measure on the unitary group $\mathcal{U}(d)$ descends to the uniform measure on $|\psi\rangle$ via the representation

$$|\psi\rangle = U\,|\psi_0\rangle$$

where $|\psi_0\rangle$ is some fixed unit state.

Background on random channels

Random channels are also easy to define using the Lindblad-Stinespring representation. A channel $\mathcal{A} : \mathcal{L}(\mathbb{C}^d) \to \mathcal{L}(\mathbb{C}^n)$ is derived from an isometric embedding $W : \mathbb{C}^d \to \mathbb{C}^k \otimes \mathbb{C}^n$, with $W^* W = I$, where $\mathcal{E} = \mathbb{C}^n$ is the environment. The relation is

$$\mathcal{A}(\rho) = \mathrm{Tr}_{\mathbb{C}^n} W \rho W^*$$

Every such embedding W can be written as $W = UW_0$ for some unitary matrix $U \in \mathcal{U}(kn)$. Then the normalized Haar measure on $\mathcal{U}(kn)$ provides a probability measure on embeddings, and hence on random channels.

Random mixed states

Not so clear how to define random mixed states. One way is to consider reduced density matrices $\rho = \text{Tr}_2 |\psi\rangle\langle\psi|$ where the $|\psi\rangle$ are random pure states on a bipartite space.

More generally: apply a channel $\mathcal{A}$ to random pure states $|\psi\rangle$, then the outputs $\mathcal{A}(|\psi\rangle\langle\psi|)$ are random mixed states.

Our focus: consider a fixed channel $\mathcal{A} : \mathcal{L}(\mathbb{C}^d) \to \mathcal{L}(\mathbb{C}^d)$. Take many independent copies $\mathcal{A}^{\otimes n}$ applied to random input states $|\psi\rangle \in \mathcal{S}(\mathbb{C}^{dn})$. Then analyze properties of typical output states $\mathcal{A}^{\otimes n}(|\psi\rangle\langle\psi|)$ using a regularized version of output entropy.

Idea: average behavior becomes typical in high dimensions. Analagous to the convergence of the empirical entropy for a classical random sequence.

Why random?

There is often a concentration of measure phenomenon which allows properties of ‘typical’ states and channels to be computed using average values.

Example: fixed unit vector $|\psi_0\rangle \in \mathbb{C}^n$, random unit vector $|\psi\rangle$, then

$$\mathbb{E}\left[|\langle\psi|\psi_0\rangle|^2\right] = \frac{1}{n}$$

Concentration of measure:

$$P\left(\left||\langle\psi|\psi_0\rangle|^2 - \frac{1}{n}\right| > a\right) \leq 2e^{-C\,n\,a^2}$$

The basic result used to derive such bounds is

Theorem (Levy's Lemma)

Let $f : S^k \to \mathbb{R}$ be a function with Lipschitz constant η (with respect to the Euclidean norm), and $X \in S^k$ be chosen uniformly at random. Then

$$P\left(|f(X) - \mathbb{E}[f(X)]| > \alpha\right) \le 2 \exp\left(-C(k+1)\alpha^2/\eta^2\right)$$

for an absolute constant $C > 0$ (may be chosen as $C = (9\pi^3 \ln 2)^{-1}$).

Then the above result about state overlap follows by applying Levy's Lemma with

$k = 2n - 1$ (real dimension of the unit sphere in $\mathbb{C}^n$),
$X = |\psi\rangle$ a random state, and
$f(X) = |\langle\psi|\psi_0\rangle|^2$.

The Lipschitz constant is $\eta = 2$, since

$$|\langle\psi|\psi_0\rangle|^2 - |\langle\phi|\psi_0\rangle|^2 \leq 2\|\psi - \phi\|_2$$

By a *random state* here we mean $|\psi\rangle = U|\psi_0\rangle$, where U is a random unitary (Haar measure on unitary group).

Channels

A channel $\mathcal{A}$ can be described in Kraus form

$$\mathcal{A}(\rho) = \sum_{i=1}^{k} A_k \, \rho \, A_k^*, \quad \sum_i A_k^* \, A_k = I$$

Alternatively it is given by the Choi matrix

$$Choi(\mathcal{A}) = (I \otimes \mathcal{A})(|\mathrm{ME}\rangle\langle\mathrm{ME}|)$$

where $|\mathrm{ME}\rangle$ is the maximally entangled state

$$|\mathrm{ME}\rangle = \sum_i |i\rangle \otimes |i\rangle$$

Schatten norms

For $r \geq 1$ define the Schatten norm of the state ρ

$$\|\rho\|_r = (\mathrm{Tr}\,\rho^r)^{1/r}$$

and the corresponding Renyi entropy

$$S_r(\rho) = \frac{r}{1-r} \log \|\rho\|_r = \frac{1}{1-r} \log \mathrm{Tr}\,\rho^r$$

The entropy of an output state from the channel is

$$S_r(\mathcal{A}(|\psi\rangle\langle\psi|)) = \frac{1}{1-r} \log \mathrm{Tr}\,(\mathcal{A}(|\psi\rangle\langle\psi|))^r$$

We will be concerned with the output entropy of a *typical state* from a high-dimensional channel $\mathcal{A}$.

Concentration of measure will apply, and so typical output states will have entropy close to the average value:

$$\mathbb{E}[S_r(\mathcal{A}(|\psi\rangle\langle\psi|))]$$

Average output entropy

Fix a channel $\mathcal{A}$, define the *average output entropy* for one channel use

$$\overline{S}_r(\mathcal{A}) = \mathbb{E}[S_r(\mathcal{A}(|\psi\rangle\langle\psi|))]$$

where average runs over random input states.

We are mostly interested in the regularized version: take many copies of the channel $\mathcal{A}^{\otimes n}$ with $n \to \infty$.

What is typical behavior of $\mathcal{A}^{\otimes n}(|\psi\rangle\langle\psi|)$ for a random input state $|\psi\rangle$?

Regularized

Define the regularized average entropy:

$$\overline{S}_r^{reg}(\mathcal{A}) = \liminf_{n\to\infty} \frac{1}{n} \overline{S}_r(\mathcal{A}^{\otimes n}) = \liminf_{n\to\infty} \frac{1}{n} \mathbb{E}[S_r(\mathcal{A}^{\otimes n}(|\psi\rangle\langle\psi|))]$$

Why lim inf? no easy subadditivity bound to get the existence of the limit.

Difficult to directly compute the expected value of the entropy. So try to get it another way:

$$\begin{aligned} \overline{S}_r(\mathcal{A}) &= \frac{1}{1-r} \mathbb{E}[\log(\mathcal{A}(|\psi\rangle\langle\psi|))^r] \\ \beta_r(\mathcal{A}) &= \frac{1}{1-r} \log \mathbb{E}[\mathrm{Tr}\,(\mathcal{A}(|\psi\rangle\langle\psi|))^r] \end{aligned}$$

Goal

Again the regularized version:

$$\beta_r^{reg}(\mathcal{A}) = \liminf_{n \to \infty} \frac{1}{n} \beta_r(\mathcal{A}^{\otimes n})$$

Goal: use concentration of measure to show that for a typical sequence of input states $|\psi_n\rangle \in (\mathbb{C}^d)^{\otimes n}$,

$$\frac{1}{n} S_r(\mathcal{A}^{\otimes n}(|\psi_n\rangle\langle\psi_n|)) \to \overline{S}_r^{reg}(\mathcal{A}) = \beta_r^{reg}(\mathcal{A})$$

The method will be to prove convergence first for the easier quantity $\beta_r(\mathcal{A}^{\otimes n})$, then relate this to the average entropy.

Lipschitz constants

The quantity $\beta_r(\mathcal{A})$ is more tractable, so we want to relate it to the average Renyi entropy. Jensen's inequality goes one way:

$$\overline{S}_r(\mathcal{A}) \geq \beta_r(\mathcal{A})$$

The regularized quantities can be related by applying a concentration of measure argument. This requires some estimates of Lipschitz constants and averages. Assume henceforth that d is both input and output dimension of $\mathcal{A}$. Define the function f_n by

$$f_n(\psi) = \text{Tr } \left(\mathcal{A}^{\otimes n}(|\psi\rangle\langle\psi|)\right)^r$$

The Lipschitz constant depends on n. Assume there is $\kappa(r) < 1$ such that

$$|f_n(\psi) - f_n(\phi)| \leq C\kappa(r)^n \, \|\psi - \phi\|_2$$

Also assume there is $\alpha(r) > 0$ such that

$$\mathbb{E}[f_n(\psi)] \geq C\alpha(r)^n$$

Concentration

Theorem

Suppose that $\sqrt{2d}\alpha(r) > \kappa(r)$. *Then*

$$\left| \frac{1}{n} \overline{S}_r(\mathcal{A}^{\otimes n}) - \frac{1}{n} \beta_r(\mathcal{A}^{\otimes n}) \right| \to 0$$

as $n \to \infty$. *Furthermore, if the limit*

$$\beta_r^{reg}(\mathcal{A}) = \lim_{n \to \infty} \frac{1}{n} \beta_r(\mathcal{A}^{\otimes n})$$

exists, then for a.e. sequence of input states $\{|\psi_n\rangle\}$ *the entropies* $\{\frac{1}{n} S_r(\mathcal{A}^{\otimes n}(|\psi_n\rangle\langle\psi_n|))$ *converge to* $\overline{S}_r^{reg}(\mathcal{A})$.

Concentration

So if we can

- upper bound the Lipschitz constant of $\text{Tr}\left(\mathcal{A}^{\otimes n}(|\psi\rangle\langle\psi|)\right)^r$ by $C\kappa(r)^n$ and
- lower bound the expected value of $\text{Tr}\left(\mathcal{A}^{\otimes n}(|\psi\rangle\langle\psi|)\right)^r$ by $C\alpha(r)^n$ and show that
- $\sqrt{2d}\alpha(r) > \kappa(r)$

then we get concentration of measure and hence convergence.

Concentration

So progress depends on two things:

- compute expected value $\text{Tr}\,(\mathcal{A}^{\otimes n}(|\psi\rangle\langle\psi|))^r$: for integer r this is feasible
- estimate Lipschitz constant of $\text{Tr}\,(\mathcal{A}^{\otimes n}(|\psi\rangle\langle\psi|))^r$: for some simple channels this is feasible

β_r as a sum

The quantity $\beta_r(\mathcal{A})$ can be analyzed when r is a positive integer, $r \geq 2$. Note that

$$\begin{aligned} \mathrm{Tr}\,(\mathcal{A}(|\psi\rangle\langle\psi|))^r &= \mathrm{Tr}\left(\sum_{a,b,x,y} |a\rangle\langle a|\mathcal{A}\Big(|x\rangle\langle x|\psi\rangle\langle\psi|y\rangle\langle y|\Big)|b\rangle\langle b|\right)^r \\ &= \sum_{a_i,x_i,y_i} \prod_{i=1}^{r} \mathcal{A}_{a_i x_i y_i a_{i+1}} \prod_{j=1}^{r} \langle x_j|\psi\rangle\langle\psi|y_j\rangle \end{aligned}$$

where the matrix elements are

$$\mathcal{A}_{axyb} = \langle a|\mathcal{A}(|x\rangle\langle y|)|b\rangle$$

The channel piece is separated from the input state. Can evaluate average over $|\psi\rangle$ using the Weingarten calculus.

Weingarten calculus

$$\mathbb{E}\left[\prod_{j=1}^{r}\langle x_j|U|0\rangle\langle 0|U^*|y_j\rangle\right] = C_{k,r}\sum_{\alpha\in Sym(r)}\delta_{x_{\alpha(1)}y_1}\dots\delta_{x_{\alpha(r)}y_r}$$

where $Sym(r)$ is the symmetric group on r letters, and the constant is

$$C_{k,r} = \sum_{\gamma\in Sym(r)} Wg(k,\gamma) = \prod_{j=0}^{r-1}\frac{1}{k+j} \simeq k^{-r}$$

β_r as a sum

At the end of the day we get

$$\mathbb{E}[\mathrm{Tr}\,(\mathcal{A}(|\psi\rangle\langle\psi|))^r] = C_{k,r} \sum_{\alpha \in Sym(r)} Q_{\mathcal{A}}(\alpha)$$

where

$$Q_{\mathcal{A}}(\alpha) = \sum_{x_i} \mathrm{Tr} \prod_{i=1}^{r} \mathcal{A}_{x_i, x_{\alpha(i)}}, \quad \mathcal{A}_{x,y} = \mathcal{A}(|x\rangle\langle y|)$$

Note that $\mathcal{A}(|x\rangle\langle y|)$ are the blocks of the Choi matrix of $\mathcal{A}$.

Product channel property

Key property is: for any permutation α

$$Q_{\mathcal{A}\otimes\mathcal{B}}(\alpha) = Q_{\mathcal{A}}(\alpha)\, Q_{\mathcal{B}}(\alpha)$$

Hence

$$\mathbb{E}[\mathrm{Tr}\, \left(\mathcal{A}^{\otimes n}(|\psi\rangle\langle\psi|)\right)^{r}] = C_{k,r} \sum_{\alpha \in Sym(r)} Q_{\mathcal{A}}(\alpha)^{n}$$

As $n \to \infty$, the largest term dominates.

Theorem

For integer $r \geq 2$ let

$$Q_{max} = \max_{\alpha \in Sym(r)} |Q_{\mathcal{A}}(\alpha)|.$$

Then

$$\beta_r^{reg}(\mathcal{A}) = \frac{r \log d - \log Q_{max}}{r - 1}$$

In some situations the $\liminf$ in the definition of $\beta_r^{reg}(\mathcal{A})$ can be replaced by lim.

Theorem

If the maximum Q_{max} is attained for a unique α, or if the channel $\mathcal{A}$ is entrywise positive, then

$$\beta_r^{reg}(\mathcal{A}) = \lim_{n \to \infty} \frac{1}{n} \beta_r(\mathcal{A}^{\otimes n})$$

Two special permutations

When $\alpha = id$ is the identity permutation,

$$Q_{\mathcal{A}}(id) = \mathrm{Tr}\,(\mathcal{A}(I))^r$$

When $\alpha = (12 \cdots r)$,

$$Q_{\mathcal{A}}(12 \cdots r) = \mathrm{Tr}\,(Choi(\mathcal{A})^r)$$

For $r = 2$ these are the only permutations.

Proposition

$$\beta_2^{reg}(\mathcal{A}) = 2 \log d - \log \max\{Tr(\mathcal{A}(I))^2, Tr(Choi(\mathcal{A})^2)\}$$

Special case: Entanglement-breaking

An E-B channel has the form

$$\mathcal{A}(\rho) = \sum_k \sigma_k \,\mathrm{Tr}\,(X_k \rho)$$

where $\{\sigma_k\}$ are density matrices, and $\{X_k\}$ are POVM. Suppose in addition that for all m and all $k_1, \ldots, k_m$,

$$\mathrm{Tr}\left(\prod_{i=1}^{m} \sigma_{k_i}\right) \geq 0$$

Proposition

For all E-B channels satisfying this condition, all integer $r \geq 2$,

$$\beta_2^{reg}(\mathcal{A}) = \frac{r \log d - \log Tr(\mathcal{A}(I))^r}{r-1}$$

Special case: Werner-Holevo

Recall

$$\mathcal{WH}(\rho) = \frac{1}{d-1}\left[\mathrm{Tr}\,\rho\, I - \rho^T\right]$$

Can show that

$$\mathrm{Tr}\left(Choi(\mathcal{WH})^r\right) \leq d$$

hence

$$Q_{\mathcal{WH}}(12\cdots r) \leq Q_{\mathcal{WH}}(id)$$

So for $r = 2$ get maximal output entropy:

$$\overline{S}_2^{reg}(\mathcal{WH}) = \log d$$

Other values of r??

Special case: qubit depolarizing

$$\Delta_\lambda(\rho) = \lambda\rho + \frac{1-\lambda}{2}\,\mathrm{Tr}\,\rho\, I$$

Main result: maximum is achieved at either identity or cycle $(12\cdots r)$. So

$$Q_{\Delta_\lambda}(\alpha) \leq \max\{Q_{\Delta_\lambda}(id), Q_{\Delta_\lambda}(12\cdots r)\} \quad (1)$$

$$= \max\{2, \mathrm{Tr}\,(Choi(\Delta_\lambda)^r)\} \quad (2)$$

Proposition

For all integer $r \geq 2$, all $0 \leq \lambda \leq 1$,

$$\beta_r^{reg}(\Delta_\lambda) = \frac{1}{r-1}\,\min\{1, 2r - \log[(1+3\lambda)^r + 3(1-\lambda)^r]\}$$

Special case: depolarizing channel

Also have concentration of measure for interval of values of λ.
Recall condition for concentration: need

$$\sqrt{2d}\,\alpha(r) > \kappa(r)$$

where

$$\alpha(r) = \frac{Q_{max}}{d^r}$$

and $\kappa(r)$ is growth rate of Lipschitz constant for $\mathrm{Tr}\,(\Delta_\lambda)^{\otimes n}(|\psi\rangle\langle\psi|))^r$.

For qubit depolarizing get explicit bound

$$\kappa(r) \leq \lambda + \frac{1-\lambda}{\sqrt{2}}$$

Special case: qubit depolarizing

Proposition

For all integer $r \geq 2$, there are $0 < c_r < d_r < 1$ such that for all $\lambda \in [0, c_r] \cup [d_r, 1]$,

$$\overline{S}_r^{reg}(\Delta_\lambda) = \beta_r^{reg}(\Delta_\lambda)$$

Typical behavior also holds in this range.

Note: as $r \to \infty$, $c_r \to 1/3$, $d_r \to 1$.

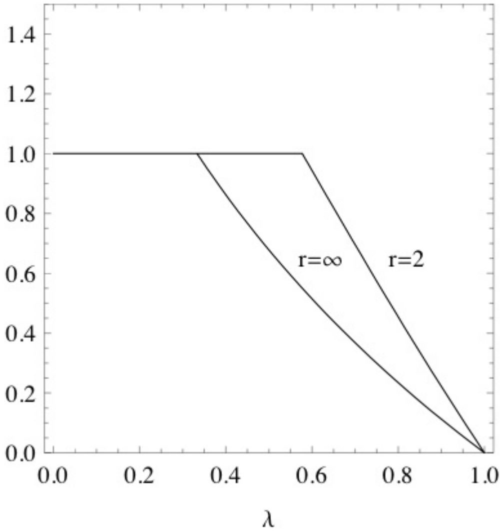

Figure: $\beta_2^{reg}(\Delta_\lambda)$ and $\beta_\infty^{reg}(\Delta_\lambda)$.

r	c_r	d_r
2	.577	.732
3	.5	.835
4	.458	.878
10	.381	.953
100	.338	.995

Table: Range parameters for qubit depolarizing

Qubit depolarizing

Conjecture that for all λ, for all r

$$\overline{S}_r^{reg}(\Delta_\lambda) = \beta_r^{reg}(\Delta_\lambda)$$

and this is empirical entropy for typical output state as $n \to \infty$.

Conjecture that same holds for all qubit channels, also depolarizing in higher dimensions.

Technical obstacles: compute Q_{max}.

PPT for typical mixed states

$$\begin{aligned}\mathcal{I} &= \text{identity map} : \mathcal{I}(\rho) = \rho \\ \mathcal{T} &= \text{transpose map} : \mathcal{T}(\rho) = \rho^T \\ \mathcal{P} &= \text{trace map} : \mathcal{P}(\rho) = \frac{\text{Tr}\,(\rho)}{d} I\end{aligned}$$

Test for separability of bipartite state: ρ_{12} is entangled if $(\mathcal{I} \otimes \mathcal{T})\rho_{12}$ is not positive semidefinite.

Define $\mathcal{A}_{d_1,d_2,d_3} = \mathcal{I}_{d_1} \otimes \mathcal{T}_{d_1} \otimes \mathcal{P}_{d_3}$ on tripartite space. So

$$\mathcal{A}_{d_1,d_2,d_3}(|\psi\rangle\langle\psi|) = (\mathcal{I}_{d_1} \otimes \mathcal{T}_{d_1})\rho_{12} \otimes \frac{1}{d_3} I$$

where ρ_{12} is the mixed state obtained by tracing over third factor. So with a random state $|\psi\rangle$, this is a random mixed state on the first two factors. Thus $\mathcal{A}_{d_1,d_2,d_3}(|\psi\rangle\langle\psi|)$ is the partial transpose of a random mixed state.

Also define $\pi = (12 \cdots r) \in Sym(r)$.

$$\mathbb{E}\left[\mathrm{Tr}\,\mathcal{A}_{d_1,d_2,d_3}(|\psi\rangle\langle\psi|)^r\right] = \sum_{\alpha} d_1^{|\alpha\pi^{-1}|-r}\, d_2^{|\alpha\pi|-r}\, d_3^{|\alpha|-2r+1}$$

where $|\alpha|$ is number of conjugacy classes in α.

Now consider $d_1 = d_2 = d$ and $d_3 = d^s$, and $d \to \infty$. Fix $r = 2$, then

$$\begin{aligned} s < 2 \quad &\Rightarrow \quad \beta_2^{reg} = \log 4 \\ s > 2 \quad &\Rightarrow \quad \beta_2^{reg} = \log(2+s) \end{aligned}$$

So transition point at $s = 2$. Conjecture this is the transition from a.s. PPT to a.s. non-PPT.

[Ref: recent work of Aubrun, Szarek and Ye]

PPT for depolarizing channel

Now consider a product of qubit depolarizing channels applied to a random input state:

$$\rho_{12} = (\Delta_\lambda^{\otimes n} \otimes \Delta_\lambda^{\otimes m})(|\psi\rangle\langle\psi|)$$

For $\lambda \leq 1/3$ we know that Δ_λ is entanglement-breaking, and hence ρ_{12} is always separable.

What happens for $\lambda > 1/3$? Is the state entangled a.s. for $\lambda > 1/3$?

Address this by looking at PPT behavior of ρ_{12} as $n, m \to \infty$. Take $n = nx$ and $m = n(1-x)$ with $0 < x < 1$. Examine entropy as a function of x.

$$\beta_r(\lambda) = \frac{1}{1-r} \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\mathrm{Tr}\,(\Delta_\lambda^{\otimes nx} \otimes \mathcal{T}\Delta_\lambda^{\otimes n(1-x)})(|\psi\rangle\langle\psi|)^r$$

Future directions

Entanglement of typical output state? PPT transition for $\mathcal{A}^{\otimes nx} \otimes (\mathcal{T} \circ \mathcal{A})^{\otimes n(1-x)}$

$$\max_{\alpha \in Sym(r)} Q_{\mathcal{A}}(\alpha)^x \, Q_{\mathcal{A}}(\gamma \alpha \gamma^{-1})^{1-x}$$

where γ is the permutation

$$\gamma : 1 \to r, \, 2 \to r-1, \, 3 \to r-2, \dots$$

e.g. Δ_λ is E-B for $\lambda \leq 1/3$. Is entanglement non-zero for typical output state for $\lambda > 1/3$?