Exposed positive maps - a sufficient condition

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[1] A common work in collaboration with Dariusz Chruściński

Definition of exposedness

A point x an the boundary of a convex set A is **exposed**, if there exists a hypersurface of codimension 1, which intersects the set A is exactly one point x.

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Exposed points form a dense subset in the set of extremal points.

A sufficient condition of optimality

For a witness $W \in \mathcal{W}_1(\mathbb{C}^n \otimes \mathbb{C}^m)$ one defines a set $\mathcal{P}_W = \{e \otimes f : \langle e \otimes f | W | e \otimes f \rangle = 0\}$.

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If $\mathrm{span}\mathcal{P}_W = \mathbb{C}^n \otimes \mathbb{C}^m$, then the map Φ is optimal (subtracting something completely positive from it one cannot get a positive map).

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Theorem: If a map Φ is irreducible on its image and
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The hermicity condition:
$\forall a \ X\Phi(a) = \Phi(a)X^\dagger \implies \forall a \ X\Phi(a) = \Phi(a)X$.
From irreducibility on the image one has $X \sim I$.

Reduction map and Robertson map

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- Robertson map $\mathcal{R} : \mathcal{B}(\mathbb{C}^4) \to \mathcal{B}(\mathbb{C}^4)$:

$$\mathcal{R}(\rho) = \mathrm{Tr}(\rho)I - \rho - \sigma\rho^T\sigma^\dagger$$

where $\sigma = \sigma_y \otimes I_2$

The generalized Robertson map

The generalized Robertson map $\Phi : \mathcal{B}(\mathbb{C}^{2n}) \to \mathcal{B}(\mathbb{C}^{2n})$ is given by the formula:

$$\Phi(|\varphi_1 \oplus \varphi_2\rangle\langle\varphi_1 \oplus \varphi_2|) =$$

$$\begin{bmatrix} |\varphi_2|^2 I_n & -|\varphi_1\rangle\langle\varphi_2| + |\varphi_2\rangle\langle\varphi_1| - \langle\varphi_1|\varphi_2\rangle I_n \\ -|\varphi_2\rangle\langle\varphi_1| + |\varphi_1\rangle\langle\varphi_2| - \langle\varphi_2|\varphi_1\rangle I_n & |\varphi_1|^2 I_n \end{bmatrix}$$

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$$= \begin{bmatrix} |\varphi_2|^2 & -\langle\varphi_1|\varphi_2\rangle \\ -\langle\varphi_2|\varphi_1\rangle & |\varphi_1|^2 \end{bmatrix} \otimes I_n$$
$$+ \begin{bmatrix} 0 & -|\varphi_1\rangle\langle\varphi_2| + |\varphi_2\rangle\langle\varphi_1| \\ -|\varphi_2\rangle\langle\varphi_1| + |\varphi_1\rangle\langle\varphi_2| & 0 \end{bmatrix}$$

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If $\Phi(|\varphi_1 \oplus \varphi_2\rangle\langle\varphi_1 \oplus \varphi_2|)|\psi_1 \oplus \psi_2\rangle = 0$, then:

1. $\phi_1 || \phi_2$
2. $\psi_1, \psi_2 \in \mathrm{span}\{\varphi_1, \varphi_2\}$

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$\Phi(|[\alpha,\beta]\otimes\varphi\rangle\langle[\alpha,\beta]\otimes\varphi|)\psi = 0 \iff$

- $\psi = [\alpha,\beta]\otimes\varphi$
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We introduce two linear subspaces in $(\mathbb{C}^{2n})^{\otimes 3}$:

- $\mathrm{span}\{([\alpha,\beta]^*\otimes\varphi^*)\otimes([\alpha,\beta]\otimes\varphi)\otimes([\alpha,\beta]\otimes\varphi)\}$
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Rearranging the factors of tensor product:

- $W_1 = S_{23}\otimes S_{23}$
- $W_2 = S_{13}\otimes T_{13}$

The dimensions are:

- $\dim W_1 = 3n^2(n+1)$
- $\dim W_2 = 6n(n^2-1)$
- $\dim W_1 \cup W_2 = 7n^3 + n^2 - 2n$

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Consider an arbitrary orthonormal basis $\{g_k\}$ in the space $\mathbb{C}^n$. On $n(n-1)$ vectors $G_{kl} = (g_k \oplus g_l)^* \otimes (g_k \oplus g_l) \otimes (g_k \oplus g_l)$, where $k \neq l$, the map $\tilde{\Phi}$ vanishes.

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We take n such basis, encoded by n unitary matrices. If $U^{(m_1)}/U^{(m_2)}$ is not a permutation matrix (*), then the vectors $G_{kl}^{(m)}$ coming from Us are linearly independent.

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We define a family $U^{(m)}$ as $V^m U V^{-m}$, where $V = \mathrm{diag}\{e^{i\alpha_1}, \ldots, e^{i\alpha_n}\}$

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If the following assumptions are fulfilled:

1. $u_{ij} \neq 0$
2. the matrix of modules of elements of U is nonsingular
3. $\forall i,j \frac{\alpha_i - \alpha_j}{2\pi} \neq \frac{p}{q}$ for $|q| < n$.

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We are looking for elements of intersection taking an arbitrary combination $G_{kl}^{(m)}$ and requiring that it be orthogonal to $n^2(n-1)$ vectors from the basis of orthogonal complement of $\mathrm{span}(W_1 \cup W_2)$. We have $n^2(n-1)$ equations for $n^2(n-1)$ variables.

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We have to show an unitary matrix U and vector $\vec{\alpha}$ of phases for which the matrix with entries:

$$A_{i,j<h;m,k\neq l} = e^{-im\alpha_i} e^{im(\alpha_j-\alpha_h)} e^{-im\alpha_l} U^*_{ik} \left(U_{jk} U_{hl} - U_{hk} U_{jl}\right)$$
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A generic U and generic α has the above property.

Publications

- Criterion and exposedness of transposition: arXiv:1111.2012
- Exposedness of the generalized Robertson map: soon on arXiv

Thank you for your attention

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