Université Claude Bernard Mathématiques
M2 Phenomena of High Dimension

Final exam – solutions

Warm-up

We first check that $|M - \mathbf{E}X| \leqslant C_0$ for some constant C_0, since

$$|\mathbf{E}X - M| \leqslant \mathbf{E}|X - M| = \int_0^\infty \mathbf{P}(|X - M| \geqslant t)\,\mathrm{d}t \leqslant \int_0^\infty \exp(-t^2/2)\,\mathrm{d}t = C_0.$$

We then write

$$\mathbf{P}(X \geqslant \mathbf{E}[X] + t) \leqslant \mathbf{P}(X \geqslant M + t - C_0) \leqslant \frac{1}{2}\exp(-(t - C_0)^2/2)$$

If $t \geqslant 2C_0$, we have $t - C_0 \geqslant t/2$ and $\mathbf{P}(X \geqslant \mathbf{E}\,[X] + t) \leqslant \frac{1}{2}\exp(-t^2/8)$. Since probabilities are bounded by 1 anyway, the bound $\mathbf{P}(X \geqslant \mathbf{E}\,[X] + t) \leqslant \exp(C_0^2/2) \cdot \exp(-t^2/8)$ holds for every $t \geqslant 0$.

Convex hull of a Gaussian cloud

Let $K = \mathrm{conv}(G_i)$. The function $x \mapsto |x|$ is obviously 1-Lipschitz, and $\mathbf{E}|G_i| = \kappa_n \leqslant \sqrt{n}$. By the warm-up, $\mathbf{P}(|G_i| \geqslant \sqrt{n} + t) \leqslant \exp(-t^2/2)$ for any $t \geqslant 0$. Consequently, by the union bound,

$$\mathbf{P}(K \not\subset (\sqrt{n} + t)B_2^n) = \mathbf{P}(\exists i,\ |g_i| > \sqrt{n} + t) \leqslant N\exp(-t^2/2).$$

Assume $N \geqslant e^n$ and choose $t = 2\sqrt{\log N}$ to obtain that with probability $\geqslant 1 - \frac{1}{N}$, one has

$$K \subset (\sqrt{n} + 2\sqrt{\log N})B_2^n \subset 3\sqrt{\log N}B_2^n.$$

Now for the converse inclusion: for every $x \in S^{n-1}$, the random variables $\langle G_i, x\rangle$ are i.i.d. $N(0,1)$. By a result from the course, we have therefore $\mathbf{E}\max_{1\leqslant i\leqslant N}\langle G_i, x\rangle \geqslant c\sqrt{\log N}$, and again by the warm-up

$$\mathbf{P}\left(w(K,x) \leqslant \sqrt{\log N} - t\right) = \mathbf{P}\left(\max_{1\leqslant i\leqslant N}\langle G_i, x\rangle \leqslant c\sqrt{\log N} - t\right) \leqslant \exp(-t^2/2) = N^{-c^2/8}$$

for the choice $t = \frac{c}{2}\sqrt{\log N}$.

Finally, choose $\varepsilon = c/12$ and let $\mathcal{N}$ a ε-net in S^{n-1} with $\mathrm{card}\,\mathcal{N} \leqslant (36/c)^n$. If we choose $N \geqslant C^n$ for a large enough C, we get $N^{-c^2/8}(36/c)^n \lll 1$. Thereforeby the union bound, with large probability

$$\inf_{x\in\mathcal{N}} w(K,x) \geqslant \frac{c}{2}\sqrt{\log N}.$$

If $K \subset 3\sqrt{\log N}B_2^n$ (another event of large probability), then the function $w(K,\cdot)$ is $3\sqrt{\log N}$-Lipschitz on S^{n-1} and therefore $\inf_{x\in S^{n-1}} w(K,x) \geqslant \frac{c}{2}\sqrt{\log N} - \varepsilon 3\sqrt{\log N} = \frac{c}{4}\sqrt{\log N}$. This inequality is equivalent to the inclusion $\frac{c}{4}\sqrt{\log N} \subset B_2^n$.

Covering in the discrete cube

1. For $\varepsilon \in [0,1]$, let $V(\varepsilon)$ be the number of elements in a ball of radius ε in (Q_n, d). This value does not depend on the center and equals $\sum_{0\leqslant k\leqslant \varepsilon n}\binom{n}{k}$. By the inequality given in the exercise, we have therefore

$$\frac{1}{n+1}2^{nH(\lfloor \varepsilon n\rfloor/n)} \leqslant V(\varepsilon) \leqslant 2^{nH(\lfloor \varepsilon n\rfloor/n)}.$$

In particular, $\lim \frac{1}{n}\log_2 V(\varepsilon) = H(\varepsilon)$. Take a ε-separated set P with maximal cardinality. The balls centered at P with radius $\varepsilon/2$ are disjoint, and therefore $V(\varepsilon/2)\,\mathrm{card}\,P \leqslant 2^n$. This gives the lower bound. By maximality, the balls cented at P with radius ε cover Q_n, and therefore $V(\varepsilon)\,\mathrm{card}\,P \geqslant 2^n$. This gives the upper bound.

2. Let $(x_i)_{1\leqslant i\leqslant N}$ be i.i.d. points uniformly chosen in Q_n, and $A=\bigcup B(x_i,\varepsilon)$. For every $x\in Q_n$, we have

$$\mathbf{P}(x\notin A)=\left(1-\frac{V(\varepsilon)}{2^n}\right)^N\leqslant \exp(-NV(\varepsilon)/2^n)$$

and therefore

$$\mathbf{E}\,\mathrm{card}(Q_n\setminus A)\leqslant 2^n\exp(-NV(\varepsilon)/2^n).$$

If $N>n\log(2)2^n/V(\varepsilon)$, this expectation is <1 and therefore $\mathbf{P}(Q_n=A)>0$. It follows that $N(Q_n,\varepsilon)\leqslant\frac{2^n n\log 2}{V(\varepsilon)}+1$. Together with the lower bound $N(Q_n,\varepsilon)\geqslant 2^n/V(\varepsilon)$, we get the desired limit as $n\to\infty$.

Diameter of random sections

1. For every $O\in\mathsf{O}(n)$, MO^{-1} has the same distribution as M (by rotational invariance of the Gaussian measure). Since $\ker(MO^{-1})=O\ker M$, the distribution of $\ker M$ is invariant under the action of $\mathsf{O}(n)$, therefore it must be $\mu_{n,n-m}$.

2. We use Gordon's lemma in the form "min max" for the processes $X_{(x,y)}=\langle Mx,y\rangle$ and $Y_{(x,y)}=\langle G,x\rangle+\langle G',y\rangle$, indexed by $(x,y)\in L\times S^{m-1}$, where $G'\sim N(0,\mathrm{Id}_m)$ is independent from G. The hypotheses are satisfies as was explained in the main course, and we get

$$\mathbf{E}\min_{y\in L}|My|=\mathbf{E}\min_{y\in L}\max_{x\in S^{m-1}}X_{(x,y)}\geqslant\mathbf{E}\min_{y\in L}\max_{x\in S^{m-1}}Y_{(x,y)}=\kappa_m-w_G(L).$$

For the second part, realize E as $\ker M$ and note that, since L is closed, $E\cap L\neq\emptyset$ if and onlf if $\min_{y\in L}|My|>0$. Since $\phi:M\mapsto\min_{y\in L}|My|$ is a 1-Lipschitz function with respect to the Hilbert–Schmidt distance, we get using the warm up

$$\mathbf{P}(E\cap L\neq\emptyset)=\mathbf{P}(\phi(M)\leqslant 0)\leqslant\exp(-[\mathbf{E}\phi(M)]^2/2)\leqslant\exp(-(\kappa_m-w_G(L))^2/2).$$

3. We apply the previous question with $L=S^{n-1}\cap tK$ for some $t=1/(2w(K))$. We have $w_G(L)\leqslant w_G(tK)=t\kappa_n w(K)\leqslant\kappa_n/2$. Since $\kappa_n\sim\sqrt{n}$ for n large, we have $\kappa_{n/2}>\kappa_n/2$, and by the previous question, we have $E\cap L=\emptyset$ with high probability, which implies $K\cap E\subset 2w(K)B_2^n$.

Problem

1. For $p=1$ or $p=2$, write using Tonelli theorem

$$\begin{aligned}\int_K(1-\|x\|_K^p)^{m/p}\,\mathrm{dvol}(x)&=\int_K\int_{\|x\|_K^p}^1\frac{m}{p}(1-t)^{\frac{m}{p}-1}\,\mathrm{d}t\,\mathrm{dvol}(x)\\&=\mathrm{vol}(K)\frac{m}{p}\int_0^1 t^{n/p}(1-t)^{\frac{m}{p}-1}\,\mathrm{d}t\\&=\mathrm{vol}(K)\frac{m}{p}\frac{(n/p)!(m/p-1)!}{(n/p+m/p)!}.\end{aligned}$$

2. (a) Let $\lambda=\|x\|_{P_EC}$, so $x=\lambda y$ for $y\in P_EC$, and there exists $z\in E^\perp$ such that $w=y+z\in C$. Since C is convex, it contains the convex hull of $\{w\}$ and $C\cap E^\perp$, and in particular $\lambda w+(1-\lambda)C\cap E^\perp=x+\lambda z+(1-\lambda)C\cap E^\perp$, a translate as needed.

   (b) Computing $\mathrm{vol}_n(C)$ using Tonelli theorem gives

$$\mathrm{vol}_n(C)=\int_{P_EC}\mathrm{vol}_{n-k}(C\cap(x+E^\perp))\,\mathrm{d}\,\mathrm{vol_k}\geqslant\mathrm{vol}_{n-k}(C\cap E^\perp)\int_{P_EC}(1-\|x\|_{P_EC})^{n-k}.$$

   By 1), $\mathrm{vol}_n(C)\geqslant\mathrm{vol}_{n-k}(C\cap E^\perp)\,\mathrm{vol}_k(P_EC)\frac{(n-k)!k!}{n!}$.

3. (a) Let $y \in \mathbf{R}^n$. We compute

$$\sup_{x \in K_1 +_2 K_2} \langle x, y \rangle = \sup_{t_1^2+t_2^2 \leqslant 1} \sup_{x_1 \in K_1} \sup_{x_2 \in K_2} t_1 \langle x_1, y \rangle + t_2 \langle x_2, y \rangle = \sup_{t_1^2+t_2^2 \leqslant 1} t_1 \|y\|_{K_1^\circ} + t_2 \|y\|_{K_2^\circ}$$

which equals $\sqrt{\|y\|_{K_1^\circ}^2 + \|y\|_{K_2^\circ}^2}$, giving the announced formula.

(b) Computing $\mathrm{vol}_{2n}(C)$ using Tonelli theorem gives

$$\mathrm{vol}_{2n}(C) = \int_K (1 - \|x\|_K^2)^{n/2} \, \mathrm{vol}_n(K^\circ) = \ (K) \frac{(n/2)!(n/2)!}{n!}$$

using question 1. The map P_E has the form $P_E(x,y) = \left(\frac{x+y}{2}, \frac{x+y}{2}\right)$ and therefore $P_E C = \{\frac{1}{2}(z,z) \, : \, z \in K +_2 K^\circ\}$. Since the map $z \mapsto (z,z)$ has jacobian $(\sqrt{2})^n$ we get $\mathrm{vol}_n(P_E C) = 2^{-n/2} \mathrm{vol}_n(K +_2 K^\circ)$. Finally, note that $E^\perp = \{(z, -z) \, : \, z \in \mathbf{R}^n\}$, so that $C \cap E^\perp$ is the set $\{(z,-z)\}$ with $\|z\|_K^2 + \|z\|_{K^\circ}^2 \leqslant 1$, hence the result.

(c) We have

$$s(K) = \frac{n!}{(n/2)!(n/2)!} \mathrm{vol}_{2n}(C) \geqslant \frac{n!}{(n/2)!(n/2)!} \frac{n!n!}{(2n!)} \mathrm{vol}_n(P_E C) \, \mathrm{vol}_n(C \cap E^\perp).$$

The hinted inequality $\binom{2n}{n} \leqslant 2^n \binom{n}{n/2}$ gives the result. Note that for even n the inequality follows from the fact that $\binom{2n}{n} \leqslant \binom{n}{n/2}^2 \leqslant 2^n \binom{n}{n/2}$

4. (a) Since $s(TK) = s(K)$ for every $T \in \mathsf{GL}(n, \mathbf{R})$, it is enough to show the existence of $T \in \mathsf{GL}(n, \mathbf{R})$ such that $(T(\mathcal{E}))^\circ = T(\mathcal{F}^\circ)$. There is a positive matrix A such that $\mathcal{F} = A(\mathcal{E})$. We check that $T = A^{-1/2}$ works, since $(T(\mathcal{E}))^\circ = A^{1/2} \mathcal{E}^\circ = A^{1/2} A^{-1} \mathcal{F}^\circ = T(\mathcal{F}^\circ)$

(b) It is easily checked that $\frac{1}{\sqrt{2}} \mathcal{E} = \mathcal{E} \cap_2 \mathcal{E} \subset K \cap_2 K^\circ$. On the other hand, if $x \in K \cap_2 K^\circ$, then $|x|^2 \leqslant \|x\|_K \|x\|_{K^\circ} \leqslant \frac{\|x\|_K^2 + \|x\|_{K^\circ}^2}{2} \leqslant \frac{1}{2}$ so $K \cap_2 K^\circ \subset \frac{1}{\sqrt{2}} B_2^n$.

(c) We prove by induction on N the theorem under the assumption that $2^{2^N} \leqslant r \leqslant 2^{2^{N+1}}$. For the base case $N = 0$ (so $2 \leqslant r \leqslant 4$), we can use the bound $s(K) \geqslant r^{-n} s(B_2^n)$ (which follows from $\mathrm{vol}_n(K) \geqslant \mathrm{vol}_n(\mathcal{E}) = r^{-n} \mathrm{vol}_n(\mathcal{F})$ and $\mathrm{vol}_n(K^\circ) \geqslant \mathrm{vol}_n(\mathcal{F}^\circ)$) since $2 \log_2 r \geqslant r$ for $r \in [2,4]$. For the inductive step, apply the induction hypothesis together with (3c) since $\mathcal{E}_0 \subset K \cap_2 K^\circ \subset \mathcal{F}_0$, where the ellipsoids $\mathcal{F}_0 = \frac{1}{\sqrt{2}} B_2^n$ and $\mathcal{E}_0 = \frac{1}{\sqrt{2}} \mathcal{E}$ satisfy $\mathrm{vol}_n(\mathcal{F}_0) / \mathrm{vol}_n(\mathcal{E}_0) = (\sqrt{r})^n$. Note that $2^{2^{N-1}} \leqslant \sqrt{r} \leqslant 2^{2^N}$. By induction, we get

$$s(K) \geqslant 2^{-n} s(K \cap_2 K^\circ) \geqslant 2^{-n} (2 \log_2 \sqrt{r})^{-n} = (2 \log_2 r)^{-n}.$$