Problem sheet # 3

The concentration of measure phenomenon

Exercise 3.1 Kissing numbers

For $n \geqslant 1$, let K_n be the maximal number of unit balls in $\mathbf{R}^n$ with pairwise disjoint interiors which are tangent to B_2^n. Compute K_1 and K_2, check that $K_3 \geqslant 12$ (there is equality) and prove the bounds

$$\left(\frac{2}{\sqrt{3}} + o(1)\right)^n \leqslant K_n \leqslant (2 + o(1))^n$$

by considering an equivalent packing problem on S^{n-1}.

Remark. The exact value of K_n is known only for $n \in \{1, 2, 3, 4, 8, 24\}$.

Exercise 3.2 Nets and convex hull

Let $\mathcal{N} \subset S^{n-1}$ and $\theta \in (0, \pi/2)$. Show that $\mathcal{N}$ is a θ-net in (S^{n-1}, g) if and only if $(\cos\theta) B_2^n \subset \operatorname{conv} \mathcal{N}$.

Exercise 3.3 Isoperimetry: sphere vs Euclidean space

Show that the isoperimetric inequality on $\mathbf{R}^{n-1}$ can be deduced from the isoperimetric inequality on S^{n-1}.

Exercise 3.4 Tricks with concentration

Let X be random variable and $a \in \mathbf{R}$ such that for every $t \geqslant 0$,

$$\mathbf{P}(|X - a| \geqslant t) \leqslant C \exp(-\alpha t^2).$$

Show the following inequalities, where C_i and $\alpha_i > 0$ depend only on C and α

1. $\mathbf{P}(|X - \mathbf{E}X| \geqslant t) \leqslant C_1 \exp(-\alpha_1 t^2)$,
2. $\mathbf{P}(|X - M_X| \geqslant t) \leqslant C_2 \exp(-\alpha_2 t^2)$, where M_X is a median of X,
3. (assuming $X \geqslant 0$) $\mathbf{P}(|X - \sqrt{\mathbf{E}X^2}| \geqslant t) \leqslant C_3 \exp(-\alpha_3 t^2)$.

Exercise 3.5 An alternative argument for Gaussian concentration

The goal of this exercise is to show that the following: if $G = (G_1, \ldots, G_n)$ are i.i.d. $N(0,1)$ random variables and $f : \mathbf{R}^n \to \mathbf{R}$ is 1-Lipschitz, then for every $t \geqslant 0$,

$$\mathbf{P}\left(|f(G) - \mathbf{E}f(G)| \geqslant t\right) \leqslant 2e^{-\frac{2t^2}{\pi^2}}.$$

1. Show that we can assume that f is C^1 and $\mathbf{E}f(G) = 0$.
2. Let H be an independant copy of G, and for $0 \leqslant \theta \leqslant \pi/2$, define $G_\theta = G\sin(\theta) + H\cos(\theta)$. Show that for every θ, $(G_\theta, \frac{\mathrm{d}}{\mathrm{d}\theta} G_\theta)$ has the same distribution as (G, H).
3. Show that for every convex function $\psi : \mathbf{R} \to \mathbf{R}$ we have
$$\mathbf{E}\left[\psi(f(G))\right] \leqslant \mathbf{E}\left[\psi(f(G) - f(H))\right] = \mathbf{E}\left[\psi\left(\int_0^{\pi/2} \langle \nabla f(G_\theta), \frac{\mathrm{d}}{\mathrm{d}\theta} G_\theta \rangle \,\mathrm{d}\theta\right)\right] \leqslant \mathbf{E}\left[\psi\left(\frac{\pi}{2}\langle \nabla f(G), H\rangle\right)\right].$$
4. Apply the previous inequality to $\psi : x \mapsto \exp(\lambda x)$ for $\lambda \geqslant 0$, and deduce that
$$\mathbf{E}\left[\exp(\lambda f(G))\right] \leqslant \exp(\pi^2 \lambda^2 / 8).$$
5. Conclude.

Exercise 3.6 Komatsu inequalities

1. Show that for every $x \geqslant 0$

$$\frac{2}{x+\sqrt{x^2+4}} \leqslant e^{x^2/2} \int_x^\infty e^{-t^2/2}\,\mathrm{d}t \leqslant \frac{2}{x+\sqrt{x^2+2}} \tag{1}$$

as follows: if $f_-(x)$, $f(x)$ and $f_+(x)$ denote the left, middle and right member of (1), show that $f_-'(x) \geqslant xf_-(x) - 1$, $f'(x) = xf_-(x) - 1$, $f_+'(x) \leqslant xf_+(x) - 1$

2. Show that if G is a $N(0,1)$ random variable, $\mathbf{P}(G > t) \leqslant \frac{1}{2}\exp(-t^2/2)$ for every $t \geqslant 0$.

Exercise 3.7 Median of a $\chi^2(n)$ distribution

Let $G = (G_1, \ldots, G_n)$ be a standard Gaussian vector in $\mathbf{R}^n$. The random variable $X = |G|^2 = G_1^2 + \cdots + G_n^2$ follows a $\chi^2(n)$ distribution. Denote by M_X a median of X. We are going to prove that

$$n - 2/3 \leqslant M_X \leqslant n. \tag{2}$$

1. Check that the density of X is proportional to $x \mapsto x^{\frac{n}{2}-1}e^{-\frac{x}{2}}$.
2. Consider the random variable $Y = \log(X/n)$ with density g. Compute g, check that $g(y) \leqslant g(-y)$ for every $y \geqslant 0$ and conclude that $M_Y \leqslant 0$ and $M_X \leqslant n$.
3. Consider the random variable $Z = (\frac{X}{n-2/3})^{1/3}$ with density h. Compute h, check that $h(1-t) \leqslant h(1+t)$ for $t \in [0,1]$ and conclude that $M_Z \geqslant 1$ and $M_X \geqslant n - 2/3$.

Exercise 3.8 Ehrhard inequality

Denote $\Phi(t) = \mathbf{P}(X \leqslant t)$ for $X \sim N(0,1)$. The following inequality is called the Ehrhard inequality: for any Borel sets A, $B \subset \mathbf{R}^n$ and $t \in [0,1]$,

$$\Phi^{-1}(\gamma_n(((1-t)A + tB) \geqslant (1-t)\Phi^{-1}(\gamma_n(A)) + t\Phi^{-1}(\gamma_n(B)). \tag{3}$$

1. Check that there is equality when A and B are half-spaces with $A \subset B$ or $B \subset A$.
2. Deduce the Gaussian isoperimetric inequality from (3) by choosing $B = \frac{r}{t}B_2^n$ and taking $t \to 0$.
3. We are going to show that for any convex function $F : \mathbf{R}^n \to \mathbf{R}$, if G is a standard Gaussian vector in $\mathbf{R}^n$, then $M_{F(G)} \leqslant \mathbf{E}F(G)$, where $M_{F(G)}$ denotes the median.
   (a) Using (3), show that the function $g : t \mapsto \Phi^{-1}(\mathbf{P}(F(G) \leqslant t))$ is concave on $\mathbf{R}$.
   (b) Deduce that there exists $\alpha > 0$ such that $g(t) \leqslant \alpha(t - M_{F(G)})$ for every $t \in \mathbf{R}$.
   (c) Conclude that $M_{F(G)} \leqslant \mathbf{E}F(G)$.
   (d) Give an alternative proof of the upper bound in (2).