Problem sheet # 5

More on volume in high dimension; polytopes

Exercise 5.1 Polytopes

1. Let $K \subset \mathbf{R}^n$ be a convex body. Show that K is the convex hull of a finite set iff it is the intersection of finitely many half-spaces.
2. Let $P \subset \mathbf{R}^n$ be polytope with v vertices and f facets, with 0 in the interior of P. Show that P° is a polytope with v facets and f vertices.

Exercise 5.2 Octahedron as a section of the cube

Fix a integer n. Show that there exist N and a n-dimensional subspace $E \subset \mathbf{R}^N$ such that $d_{BM}(B_1^n, E \cap B_\infty^N) = 1$. What is the minimal such N?

Exercise 5.3 Zonotopes

A zonotope in $\mathbf{R}^n$ is a polytope which can be written as the Minkoswki addition of finitely many segments. For example the cube $[-1,1]^n$ is a zonotope since

$$[-1,1]^n = [-e_1, e_1] + \cdots + [-e_n, e_n].$$

1. Show that a convex body is a zonotope iff it is of the form $A(B_\infty^N)$ for some integer N and some linear map $A : \mathbf{R}^N \to \mathbf{R}^n$.
2. Show that every centrally symmetric polygon (i.e. planar polytope) is a zonotope.
3. Show that the octahedron B_1^3 is not a zonotope.

Exercise 5.4 Mean width = perimeter for planar sets

Show that the perimeter of any planar convex body is equal to 2π times its mean width.

Therefore, the Urysohn inequality in dimension 2 is equivalent to the isoperimetric inequality.

Exercise 5.5 Another proof of Urysohn inequality

Show that the mean width of a convex body does not increase under Steiner symmetrisations. Deduce an alternative proof of Urysohn inequality.

Exercise 5.6 Simplex

Show that there exists a simplex $\Delta_n \in \mathbf{R}^n$ with the property that $\Delta^\circ = -\Delta$

Exercise 5.7 Mean width and volume radius for standard sets

Compute an equivalent as $n \to \infty$ of both the mean width and the volume radius of the following convex bodies: B_2^n, B_1^n, B_∞^n and Δ_n (from the previous exercise).

Exercise 5.8 Intersection of two cubes

It is possible to have for every n a transformation $O \in \mathsf{O}(n)$ such that $d_{BM}(B_2^n, B_\infty^n \cap O(B_\infty^n)) \leqslant C$ for some absolute constant C ?

Exercise 5.9 Vertices and facets of symmetric polytopes

1. For integers m and k, how many vertices and facets has the polytope $B_1^m \times \cdots \times B_1^m \subset (\mathbf{R}^m)^k$?
2. Fix $0 < \theta < 1$. Show the existence of a symmetric polytope $P_n \subset \mathbf{R}^n$ with v_n vertices and f_n facets, with $\log v_n \sim \theta n$ and $\log f_n \sim (1-\theta)n$ as n tends to infinity.

Exercise 5.10 A variant on the volume ratio theorem

Let $K \subset \mathbf{R}^n$ be a symmetric convex body with $B_2^n \subset K$, and $A = \mathrm{vrad}(K)$.

1. Show that (μ being the Haar on $\mathsf{O}(n)$),

$$\int_{\mathsf{O}(n)} \int_{S^{n-1}} \left(\frac{1}{\|U(\theta)\|_K \|\theta\|_K} \right)^n \mathrm{d}\sigma(\theta)\mathrm{d}\mu(U) = A^{2n}.$$

2. Deduce the existence of $U_0 \in \mathsf{O}(n)$ such that

$$\int_{S^{n-1}} \frac{1}{N(\theta)^{2n}} d\sigma(\theta) \leqslant A^{2n},$$

   where $N(x) = \frac{1}{2}(\|x\|_K + \|U_0(x)\|_K)$.

3. Let $x \in S^{n-1}$. Show that if $N(x) \leqslant t$, then $N(y) \leqslant 2t \quad \forall y \in C(x,t) := \{y \in S^{n-1} : |y - x| \leqslant t\}$.

4. Show that $N \geqslant 1/(12A^2)$ and deduce that

$$B_2^n \subset K \cap U_0(K) \subset 12A^2 \cdot B_2^n.$$

Exercise 5.11 2-concavity of p-norm

1. (warm-up) Show the following inequality: $\sqrt{(\mathbf{E}X)^2 + (\mathbf{E}Y)^2} \leqslant \mathbf{E}\sqrt{X^2 + Y^2}$.

2. Show that for any random variables $X_1, \ldots, X_n$ and every $1 \leqslant p \leqslant 2$, we have

$$\left(\sum \|X_k\|_{L^p}^2 \right)^{1/2} \leqslant \left\| \left(\sum X_k^2 \right)^{1/2} \right\|_{L^p}.$$