Université Claude Bernard | Mathématiques
M2 Phenomena of High Dimension

Problem sheet # 7
Gaussian processes

Exercise 7.1 Warm-up
What can be said of centered jointly Gaussian variables $(X_1, \cdots, X_n)$ for which $\mathbf{E}\max X_i = 0$?

Exercise 7.2 Lower bounds on norms of Gaussian matrices
Let G be a $n \times n$ random matrix with i.i.d. $N(0,1)$ entries.

1. Show that G has the same distribution as OAP, where O, A, P are independent random matrices, with O and P being Haar-distributed on $\mathsf{O}(n)$ and A a bidiagonal matrix with independent entries with distributions $a_{i,i} \sim \chi^{(n+1-i)}$ and $a_{i,i+1} \sim \chi^{(n-i)}$ (the $\chi^{(k)}$ distribution is the distribution of $|G|$ where G is a standard Gaussian vector in $\mathbf{R}^k$).

2. Derive the inequality (κ_n being the expectation of a $\chi^{(n)}$ variable)
$$\mathbf{E}\|G\|_{\mathrm{op}} \geqslant \left\| \begin{pmatrix} \kappa_n & \kappa_{n-1} & 0 & \cdots & 0 \\ 0 & \kappa_{n-1} & \kappa_{n-2} & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \kappa_2 & \kappa_1 \\ 0 & \cdots & \cdots & 0 & \kappa_1 \end{pmatrix} \right\|_{\mathrm{op}}.$$

3. Conclude that $\mathbf{E}\|G\|_{\mathrm{op}}$ is equivalent to $2\sqrt{n}$ as $n \to \infty$ (the upper bound appeared in the main course).

Exercise 7.3 Some inequalities for Gaussian measure

1. Let X_N be a random vector uniformly distributed in the Euclidean ball $\sqrt{N}B_2^N$, and $\pi_{N,n}$ be the orthogonal projection from $\mathbf{R}^N$ onto its subspace $\mathbf{R}^n$. Show that as $N \to \infty$, $\pi_{N,n}(X_N)$ converges in distribution towards a standard Gaussian vector in $\mathbf{R}^n$.

2. Deduce the following inequality for the standard Gaussian mesaure γ_n: for any compact sets $K, L \subset \mathbf{R}^n$ and $\lambda \in [0,1]$,
$$\gamma_n(\lambda K + (1-\lambda)L) \geqslant \gamma_n(K)^\lambda \gamma_n(L)^{1-\lambda}.$$

3. Let $K \subset \mathbf{R}^n$ be a symmetric convex body. Define $f : \mathbf{R} \to \mathbf{R}$ by $f(t) = \gamma_{n-1}(\{y \in \mathbf{R}^{n-1} \,:\, (t,y) \in K\})$. Show that $\log f$ is a concave function. Deduce that
$$\int_0^t f \,\mathrm{d}\gamma_1 \geqslant 2\gamma_1([0,t]) \int_0^\infty f \,\mathrm{d}\gamma_1.$$

4. Show the following: if K a symmetric convex body and $L = \{x \in \mathbf{R}^n \,:\, |\langle x, u\rangle| \leqslant a\}$ for some $u \in \mathbf{R}^n$ and $a > 0$, then $\gamma_n(K \cap L) \geqslant \gamma_n(K)\gamma_n(L)$.

Exercise 7.4 Covering with centers outside
If K, L are symmetric convex bodies in $\mathbf{R}^n$, denote by $N(K,L,\varepsilon) = \inf\{\mathrm{card}(A) \,:\, A \subset K \subset A + \varepsilon L\}$ and $N'(K,L,\varepsilon) = \inf\{\mathrm{card}(A) \,:\, A \subset \mathbf{R}^n,\ K \subset A + \varepsilon L\}$. Show that $N(K,L,\varepsilon) \leqslant N'(K,L,\varepsilon/2)$. Can you give an example with $N(K,L,\varepsilon) \neq N'(K,L,\varepsilon)$?

Exercise 7.5 Exact duality for ellipsoids
Show that if $\mathcal{E}$ and $\mathcal{F}$ are ellipsoids in $\mathbf{R}^n$, then $N(\mathcal{E},\mathcal{F},\varepsilon) = N(\mathcal{F}^\circ,\mathcal{E}^\circ,\varepsilon)$ for every ε.

Exercise 7.6 Covering balls by cubes
Deduce from the dual Sudakov inequality that for every $\varepsilon > 0$, there is a polynomial P such that $N(B_2^n, B_\infty^n, \varepsilon) \leqslant P(n)$ for every n. Can you give a direct proof of this fact, say for $\varepsilon = 1/2$?