Problem sheet # 8
K-convexity

Exercise 8.1 A caracterisation of Euclidean balls

Let $K \subset \mathbf{R}^n$ be a convex body such that, for every subspace $E \subset \mathbf{R}^n$, we have $K \cap E = P_E K$ (P_E being the orthogonal projection onto E). Show that K is a Euclidean ball.

Exercise 8.2 A Borel selection theorem

Let $K \subset \mathbf{R}^n$ be a convex body. Show that there is a Borel map $\Theta : \mathbf{R}^n \to K$ with the property that for every $x \in \mathbf{R}^n$, we have $\langle \Theta(x), x\rangle = \max\{\langle z, x\rangle : z \in K\}$. Could a map with this property be non-Borel?

Exercise 8.3 Basic properties of K-convexity constant

Denote by $\mathbf{KC}(K)$ the K-convexity constant associated to a symmetric convex body $K \subset \mathbf{R}^n$.

1. Show that $\mathbf{KC}(T(K)) = \mathbf{KC}(K)$ for every $T \in \mathsf{GL}(n, \mathbf{R})$.
2. Show that $\mathbf{KC}(K) = \mathbf{KC}(K^\circ)$.

Exercise 8.4 Reverse Urysohn inequality

Show that if K is a symmetric convex body in ℓ-position, then

$$\mathrm{vrad}(K) \leqslant w(K) \leqslant C\mathrm{vrad}(K)\log(n).$$

Exercise 8.5 K-convexity for ℓ^p balls

Let $p \geqslant 2$, $B_p^n \subset \mathbf{R}^n$ be the unit ball for $\|\cdot\|_p$, and G be a standard Gaussian vector in $\mathbf{R}^k$. Check the following inequalities, where α_p denotes the L^p norm of a $N(0,1)$ random variable: for every $f_1, \dots, f_n \in L^2(\gamma_k)$,

$$\begin{aligned}
|||\tilde{R}_1\Theta|||^2_{B_p^n} &= \mathbf{E}\left(\sum_{i=1}^n |(R_1 f_i)(G)|^p\right)^{2/p} \\
&\leqslant \left(\sum_{i=1}^n \mathbf{E}\left[|(R_1 f_i)(G)|^p\right]\right)^{2/p} \\
&\leqslant \alpha_p^2 \left(\sum_{i=1}^n \|f_i\|^p_{L^2(\gamma_k)}\right)^{2/p} \\
&\leqslant \alpha_p^2 \mathbf{E}\left[\left(\sum_{i=1}^n |f_i(G)|^p\right)^{2/p}\right] = \alpha_p^2 |||\Theta|||^2_{B_p^n}.
\end{aligned}$$

Conclude that $\mathbf{KC}(B_n^p) \leqslant C\sqrt{p}$, and that $\mathbf{KC}(B_1^n) = \mathbf{KC}(B_\infty^n) \leqslant C\sqrt{\log n}$.
(Hint: *estimate* $d_{BM}(B_\infty^n, B_{\log n}^n)$)