Two-point symmetrization and convexity

Guillaume AUBRUN and Matthieu FRADELIZI

Abstract

We prove a conjecture of R. Schneider: the spherical caps are the only spherically convex bodies of the sphere which remain spherically convex after any two-point symmetrization. More generally, we study the relationships between convexity and two-point symmetrization in the Euclidean space and on the sphere.

Introduction

In the following, n is an integer, $n \geq 2$, the Euclidean norm on $\mathbb{R}^n$ is denoted by $|.|$ and the Euclidean unit sphere by S^{n-1}. Let H be an affine hyperplane, denote by σ_H the reflection (the orthogonal symmetry) with respect to H and by H^+ and H^- the two closed half-spaces delimited by H. The two-point symmetrization $\tau_H K$ of a subset K of $\mathbb{R}^n$ with respect to H is defined as follows (see Figure 1)

$$\tau_H K = ((K \cap \sigma_H K) \cap H^-) \cup ((K \cup \sigma_H K) \cap H^+)$$

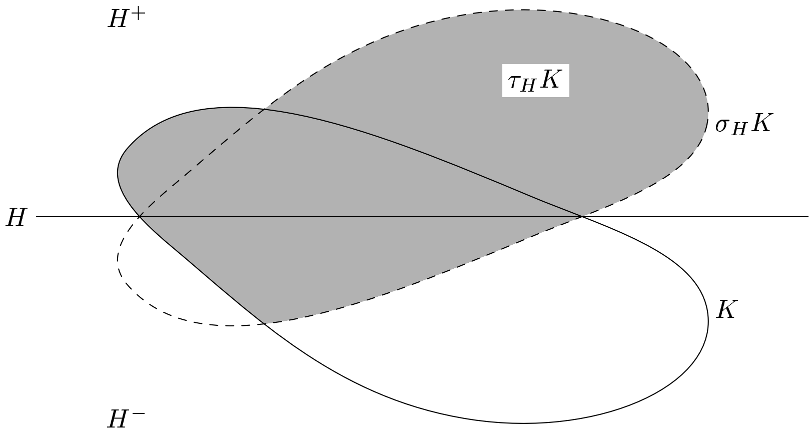

Figure 1: Two-point symmetrization

The set obtained in this way does not look more symmetric than the original one, hence the terminology "two-point symmetrization" may seem imperfect. Many other names have been proposed, such as compression, two-point rearrangement or

Mathematics Subject Classification (2000): 52A55, 52A20.
Key words: Two-point symmetrization, spherical cap, revolution symmetry.

polarization (see [BL], [BuS] and [BrS]). It can also be defined in other settings like in discrete or hyperbolic spaces.

Note that if we switch H^+ and H^-, the result of the symmetrization is the same, up to a reflection with respect to H. In particular, since we are only interested in geometric properties of $\tau_H K$, such as being convex, the choice of H^+ and H^- is never important.

If $A \subset S^{n-1}$ and $0 \in H$ then $\tau_H A \subset S^{n-1}$ and we get the usual two-point symmetrization on the sphere (see [B]). A subset A of S^{n-1} is said to be spherically convex if the cone $\mathbb{R}^+ A$ generated by A is convex in $\mathbb{R}^n$; if moreover $\mathbb{R}^+ A$ is closed with nonempty interior we say that A is a spherical convex body (recall that a convex body in $\mathbb{R}^n$ is a convex, compact set with nonempty interior). Remark that if a proper subset of S^{n-1} is a spherical convex body then it is contained in a hemisphere. Note that the two-point symmetrization with respect to a hyperplane passing through 0 maps spherical caps to spherical caps.

Among many results, the two-point symmetrization was used to prove the isoperimetric inequality on the sphere (see [B]). More recently ([S2]), R. Schneider used it to prove a spherical analogue of the Blaschke-Santaló inequality. The main purpose of this article is to prove the following theorem, conjectured by R. Schneider to shorten his proof.

Theorem 1 *Let A be a spherical convex body of S^{n-1} such that for all hyperplanes H through the origin, $\tau_H A$ is spherically convex. Then A is a spherical cap.*

We also prove the analogous result in $\mathbb{R}^n$.

Theorem 2 *Let K be a closed n-dimensional convex proper subset of $\mathbb{R}^n$, such that for any affine hyperplane H, $\tau_H K$ is convex. Then K is either a Euclidean ball or a half-space.*

Note that in both theorems, the reciprocal statement obviously holds. In the second part of the article, we prove other results when the body is assumed to remain convex after two-point symmetrizations with respect to smaller sets of hyperplanes.

After this note has been written, we learnt from R. Schneider that G. Bianchi had already outlined to him a proof of his conjecture; actually his proof is closely related to part 2 of our paper.

1 Proof of the main theorems

Let K be a closed convex set in $\mathbb{R}^n$ and ∂K its boundary. For a point x of ∂K, we denote by $N_K(x)$ the set of outer unit normal vectors of K at x. A point x of ∂K is said to be regular if $N_K(x)$ is a singleton. In this case we abusively denote this unique vector by the same notation $N_K(x)$. Recall that the set of singular points (i.e. non-regular) has $(n-1)$-dimensional Hausdorff measure zero (cf [S1], page 73). In particular, regular points are dense in ∂K.

We first prove the following proposition.

Proposition 1 *Let K be a closed convex n-dimensional subset of $\mathbb{R}^n$ and H be an affine hyperplane. Then $\tau_H K$ is convex if and only if given two distinct points of ∂K symmetric with respect to H, there exist two supporting hyperplanes (one at each point) symmetric with respect to H.*

Remark: If $\tau_H K$ is convex and the two points x and $y = \sigma_H x$ from Proposition 1 are regular, it implies that $N_K(x) = \sigma_{H_0} N_K(y)$ where $H_0 = H - \frac{x+y}{2}$ passes through the origin.

Proof of Proposition 1: First note that $\partial K \cap \partial \sigma_H K \subset \partial \tau_H K$. Indeed, if $K \subset H^+$ (or H^-) then $\tau_H K = K$ (or $\sigma_H K$) and the inclusion is obvious. If not then $K \cap \sigma_H K$ has a non empty interior. For x in $\partial K \cap \partial \sigma_H K$ there exist two supporting hyperplanes T_1 and T_2 at x respectively of K and $\sigma_H K$, such that $K \subset T_1^+$ and $\sigma_H K \subset T_2^+$. Hence $K \cap \sigma_H K \subset T_1^+ \cap T_2^+$. This implies that $T_1^+ \cup T_2^+$ cannot be the whole space. Therefore $x \in \partial \tau_H K$ because $\tau_H K \subset T_1^+ \cup T_2^+$.

We first show the easiest part. Assuming that $\tau_H K$ is convex, let x and y be distinct points in ∂K such that $y = \sigma_H x$, with $x \in H^+$. We have $x \in \partial K \cap \partial \sigma_H K \subset \partial \tau_H K$. As $\tau_H K$ is convex, it admits a supporting hyperplane T at x. Thus T supports $K \cap H^+$ and $\sigma_H K \cap H^+$ at x and this implies that T supports K and $\sigma_H K$ at x. Hence $\sigma_H T$ supports K at $y = \sigma_H x$ and we are done.

The converse is a direct consequence of the following characterization of convexity due to Tietze (see Valentine [V], pp 51–53): an open connected set C in $\mathbb{R}^n$ is convex if and only if C admits a *local supporting hyperplane* at every point x of its boundary, *i.e.* there is a neighbourhood V of x such that $C \cap V$ admits a supporting hyperplane at x.

Let x be a boundary point of $\tau_H K$. Then either x belongs to ∂K and $\partial \sigma_H K$ or it belongs to only one of these sets. In the second case the existence of a local supporting hyperplane is clear. Let us consider the first case, *i.e.* $x \in \partial K \cap \partial \sigma_H K$.

If $x \in H$ let $u = N_{H^-}(x)$ be the unit outer vector of H^- and take $w \in N_K(x)$. There are again two subcases. If $\langle u, w \rangle \leq 0$ then let us prove that $x + w^\perp$ is a supporting hyperplane of $\tau_H K$. It is enough to prove that for all $z \in K \cup (\sigma_H K \cap H^+)$ we have $\langle z - x, w \rangle \leq 0$. Since $w \in N_K(x)$ the inequality is satisfied for $z \in K$. For $z \in \sigma_H K \cap H^+$, there exists $\alpha \geq 0$ such that $z - \sigma_H z = \alpha u$. Since $\langle u, w \rangle \leq 0$ and $\sigma_H z \in K$, we get $\langle z - x, w \rangle = \langle z - \sigma_H z, w \rangle + \langle \sigma_H z - x, w \rangle \leq 0$. Now if $\langle u, w \rangle \geq 0$ then the same proof gives that $\sigma_H(x + w^\perp)$ is a supporting hyperplane of $\tau_H K$.

Notice that without using the hypothesis we already proved that the defaults of convexity of the boundary of $\tau_H K$ can only occur at a point x of $\partial K \cap \partial(\sigma_H K) \setminus H$. But this case is settled by the hypothesis since $y := \sigma_H x \in \partial K$, hence there exists a supporting hyperplane of K at x such that $\sigma_H T$ supports K at y. This means that T supports $K \cup \sigma_H K$ and consequently $\tau_H K$ at x. Therefore K is convex.

We now prove Theorem 2.

Proof of Theorem 2: Let K be a convex body satisfying the hypotheses of Theorem 2. We know from the remark after Proposition 1 that for any couple (x, y) of distinct regular points of ∂K, $N_K(x) = \sigma_{H_0} N_K(y)$, where $H_0 = (x - y)^\perp$. We can rewrite this in the following form : $N_K(x) - N_K(y)$ and $x - y$ are collinear, hence there exists a scalar $\lambda(x, y)$ such that $N_K(x) - N_K(y) = \lambda(x, y)(x - y)$. For three regular points x, y and z of ∂K, we have

$$\begin{cases} N_K(x) - N_K(y) = \lambda(x,y)(x-y) \\ N_K(y) - N_K(z) = \lambda(y,z)(y-z) \\ N_K(z) - N_K(x) = \lambda(z,x)(z-x) \end{cases}$$

Adding the three equalities we get $(\lambda(x,y) - \lambda(z,x))x + (\lambda(y,z) - \lambda(x,y))y + (\lambda(z,x) - \lambda(y,z))z = 0$.

If all regular points of ∂K are collinear then K is a half-plane in $\mathbb{R}^2$ and there is nothing to prove. So we can assume that there exist three non-collinear (hence affinely independent) regular points x, y and z; then the previous discussion implies that $\lambda(x,y) = \lambda(y,z) = \lambda(z,x)$, and necessarily the function λ is constant on couples of regular points ; let λ be the value of this constant. Going back to the definition of $\lambda(x,y)$, this shows that the point $\lambda x - N_K(x)$ does not depend on the choice of the regular point x ; let $e = \lambda x - N_K(x)$ be this point. Then $N_K(x) = \lambda x - e$, hence $|\lambda x - e| = 1$. By density of the regular points, this is valid for any boundary point.

If $\lambda = 0$ then K has to be a half-space. Now assume $\lambda \neq 0$. As K is an n-dimensional convex set, it is necessarily the Euclidean ball of center e/λ and radius $1/|\lambda|$. This concludes the proof.

The proof of Theorem 1 follows exactly the same scheme. Here are the main points.

Proof of Theorem 1: Let $A \subset S^{n-1}$ be a subset satisfying the hypotheses of Theorem 1. In the case $A = S^{n-1}$ or if $n = 2$ there is nothing to prove; thus in the following we assume that $n \geq 3$ and that A is a spherical convex body contained in a hemisphere. By ∂A, we mean the relative boundary of A in S^{n-1}. Let $C = \mathbb{R}^+ A$ be the cone generated by A ; C is convex and for any hyperplane H containing the origin, $\tau_H C$ is convex. Let x and y be two distinct points from ∂A, regular as points of ∂C. The origin belongs to the perpendicular bisector hyperplane H of the segment $[x, y]$, hence from the remark after Proposition 1 we have $N_C(x) = \sigma_H N_C(y)$.

The remainder of the proof is identical : we get $|\lambda x - e| = 1$ for all x in ∂A. Note that the point e cannot be the origin, since $N_C(x)$ is orthogonal to x. Hence ∂A is included in the intersection of two non-concentric spheres, therefore the spherical convex body A is necessarily a spherical cap.

2 Smaller families of hyperplanes

The aim of this section is to characterize the set of closed n-dimensional convex subsets of $\mathbb{R}^n$ which remain convex after the two-point symmetrizations with respect to any hyperplane belonging to some fixed natural families of hyperplanes.

To state our result, we need the following notion. Let K be a closed subset of $\mathbb{R}^n$ and E a subspace. We say that K has a *revolution symmetry* around E if every section of K with a translate of $E^\perp$ is a (possibly empty) Euclidean ball with center in E. This is equivalent to saying that K is equal to its Schwarz-symmetral with respect to E and also to the existence of a function $f : E \to \mathbb{R}$ such that

$$K = \{x \in \mathbb{R}^n \text{ s.t. } |x - P_E x| \leq f(P_E x)\} .$$

Note that if E is a line of $\mathbb{R}^3$, this is the usual notion of revolution body. If E is a hyperplane, the revolution symmetry is just the usual orthogonal symmetry. If E is reduced to a single point, only Euclidean balls centered at that point have a revolution symmetry around E.

We will prove the following theorem.

Theorem 3 *Let E be a subspace of $\mathbb{R}^n$ and K be a convex body in $\mathbb{R}^n$. Then $\tau_H K$ is convex for every affine hyperplane H containing a translate of E if and only if K has a revolution symmetry around a translate of E.*

Proof: If K has a revolution symmetry around a translate of E, then two-point symmetrizations with respect to a hyperplane H containing E either fix K or map it to $\sigma_H K$, so we obviously obtain a convex set.

Now we deal with the other direction : let K be a convex body in $\mathbb{R}^n$ satisfying the hypotheses of Theorem 3.

Case 1: $n = 2$ and E is a line in $\mathbb{R}^2$
We have to prove that K is symmetric with respect to a line parallel to E. Using the decomposition $\mathbb{R}^2 = E \oplus E^\perp$, there exists $a < b \in \mathbb{R}$ and $f, g : [a, b] \to \mathbb{R}$, with $f \geq g$, f concave and g convex such that we may write

$$K = \{(t, s) \in [a, b] \times \mathbb{R};\ g(t) \leq s \leq f(t),\ \forall t \in [a, b]\} .$$

Let u be a point of (a, b) such that f and g are both differentiable at u. Then $x = (u, f(u))$ and $y = (u, g(u))$ are regular points of ∂K and their bisector line, $D := \{(t, \frac{f(u)+g(u)}{2});\ t \in \mathbb{R}\}$, is parallel to E. Hence by Proposition 1 the tangent lines of K at x and y are symmetric with respect to D, which means that $g'(u) = -f'(u)$. Since g and $-f$ are convex on $[a, b]$, we can write

$$g(t) - g(a) = \int_a^t g'(u)du = -\int_a^t f'(u)du = f(a) - f(t) \quad \forall\, t \in [a, b] .$$

Therefore K is symmetric with respect to the line $\{(t, \frac{f(a)+g(a)}{2});\ t \in \mathbb{R}\}$, which is parallel to E.

Case 2: $n \geq 2$ and E is a hyperplane in $\mathbb{R}^n$
Assume for simplicity that E contains the origin. Let x, y be two points of ∂K such that $K \cap (E^\perp + x) = [x, y]$. Let us prove that K is symmetric with respect to the affine hyperplane parallel to E passing through $\frac{x+y}{2}$. Let P be any affine 2-plane containing the segment $[x, y]$. For any affine hyperplane H parallel to E we have that $(\tau_H K) \cap P = \tau_{H \cap P}(K \cap P)$ is convex. By case 1, this immediately implies that $K \cap P$ is symmetric with respect to a line parallel to $E \cap P$. Because of the choice of x and y this line has to be $E \cap P + \frac{x+y}{2}$. This implies that $K \cap P$ is symmetric with respect to the affine hyperplane $E + \frac{x+y}{2}$. It follows that K is symmetric with respect to $E + \frac{x+y}{2}$.

Case 3: general case

We first need the following lemma :

Lemma 1 *Let K be a convex body in $\mathbb{R}^n$ which admits a hyperplane of symmetry in every direction. Then K is a Euclidean ball.*

Proof of Lemma 1: First choose a basis $(e_1, \ldots, e_n)$ and for each i, let H_i be an (affine) hyperplane of symmetry of K orthogonal to e_i. Then the intersection of the H_i's is a singleton $\{x\}$, and x is a center of symmetry of K (we can assume x to be the origin) . Thus all the symmetry hyperplanes must pass through x, and so K is invariant under all orthogonal symmetries, therefore under the whole orthogonal group $O(n)$. So K is a Euclidean ball.

We now return to case 3. From case 2, for any hyperplane H containing E, K is symmetric with respect to a translate of H. This implies that for any affine subspace F parallel to $E^\perp$, the convex set $K \cap F$ admits a hyperplane (in F) of symmetry in every direction, so by Lemma 1 it is a Euclidean ball in F. It remains to prove that the centers of these balls lie on a translate of E. But if K is symmetric with respect to an hyperplane H containing E, H must pass through all theses centers. As (again by case 2) such hyperplanes exist in any direction, the centers necessarily lie on a translate of E. Hence K admits a revolution symmetry around a translate of E and the proof is complete.

Remark 1 : Note that Theorem 3 is a generalization of Theorem 2 in the bounded case (take $E = \{0\}$). The boundedness requirement in Theorem 3 is important to avoid the situation of the half-spaces.

Remark 2 : A natural question to generalize Theorem 1 would be : "what are the convex bodies in $\mathbb{R}^n$ which remain convex after any two-point symmetrization with respect to hyperplanes containing a fixed point (say, the origin) ? ". Unfortunately, the answer does not seem to be very nice. Such a body may even fail to have any symmetry, as it can be checked thanks to Proposition 1 on the following example (see Figure 2).

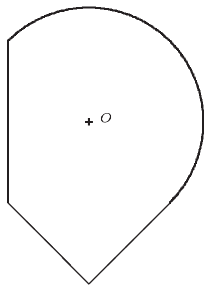

Figure 2: A convex body in $\mathbb{R}^2$ remaining convex after two-point symmetrizations with respect to hyperplanes through the origin

References

[B] Benyamini, Y. (1984). Two-point symmetrization, the isoperimetric inequality on the sphere and some applications. *Texas functional analysis seminar 1983–1984*, 39–51, *Longhorn Notes.*

[BL] Bollobás, B. and Leader, I.B. (1991). Compressions and isoperimetric inequalities. *J. Combin. Theory Ser.*, **A 56**, 47–62.

[BrS] Brock, F. and Solynin, A.Yu. (2000). An approach to symmetrization via polarization. *Trans. Amer. Math. Soc.*, **352**, 1759–1796.

[BuS] Burchard, A. and Schmuckenschläger, M. (2002). Comparison theorems for exit times. *Geom. Funct. Anal.*, **11**, 651–692.

[S1] Schneider, R. (1993). *Convex bodies : the Brunn-Minkowski theory.* Encyclopedia of Mathematics and its Applications, 44, Cambridge University Press, Cambridge.

[S2] Schneider, R. (2002). *A spherical analogue of the Blaschke-Santaló inequality.* Conference given during PIMS Thematic Programme on Asymptotic Geometric Analysis in Vancouver, summer 2002.

[V] VALENTINE, F.A. (1964). *Convex sets.* McGraw-Hill, New-York.

GUILLAUME AUBRUN
Université de Paris 6, Institut de Mathématiques, Équipe d'Analyse Fonctionnelle, Boite 186, 4 place Jussieu, 75005 PARIS
e-mail: `aubrun@ccr.jussieu.fr`

MATTHIEU FRADELIZI
Université de Marne-la-Vallée, Equipe d'Analyse et de Mathématiques Appliquées, Cité Descartes, 5 Bd Descartes, Champs sur Marne, 77454 Marne-la-Vallée Cedex 2, FRANCE
e-mail: `fradeliz@math.univ-mlv.fr`