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CATALYSIS IN THE TRACE CLASS AND WEAK TRACE CLASS IDEALS

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Abstract. Given operators A, B in some ideal $\mathcal{I}$ in the algebra $\mathcal{L}(H)$ of all bounded operators on a separable Hilbert space H, can we give conditions guaranteeing the existence of a trace-class operator C such that $B \otimes C$ is submajorized (in the sense of Hardy–Littlewood) by $A \otimes C$? In the case when $\mathcal{I} = \mathcal{L}_1$, a necessary and almost sufficient condition is that the inequalities $\mathrm{Tr}(B^p) \leq \mathrm{Tr}(A^p)$ hold for every $p \in [1, \infty]$. We show that the analogous statement fails for $\mathcal{I} = \mathcal{L}_{1,\infty}$ by connecting it with the study of Dixmier traces.

1. Introduction

Let H be an infinite-dimensional separable Hilbert space, $\mathcal{L}(H)$ be the algebra of all bounded operators on H and $\mathcal{C}_0 = \mathcal{C}_0(\mathcal{H})$ the set of compact operators.

Given $A \in \mathcal{C}_0$, we denote by $\mu(A) := \{\mu(k, A)\}_{k \geq 0}$ the sequence of singular values of the operator A (that is, eigenvalues of the operator $|A|$) arranged in decreasing order and taken with multiplicities (if any). We say that $B \in \mathcal{C}_0$ is submajorized by $A \in \mathcal{C}_0$ in the sense of Hardy–Littlewood (written $B \prec\prec A$) if for every integer n

$$\sum_{k=0}^{n} \mu(k, B) \leq \sum_{k=0}^{n} \mu(k, A).$$

If $A, B \in \mathcal{C}_0$ are such that $B \prec\prec A$, then $B \otimes C \prec\prec A \otimes C$ for every $C \in \mathcal{C}_0$.[1] The converse does not hold, even in the finite-dimensional setting: if A, B, C are such that $\mu(A) = (0.5, 0.25, 0.25, 0, \cdots)$, $\mu(B) = (0.4, 0.4, 0.1, 0.1, 0, \cdots)$ and $\mu(C) = (0.6, 0.4, 0, \cdots)$, one checks easily that $B \otimes C \prec\prec A \otimes C$ while B is not submajorized by A. This example appears in [7] and is related to the phenomenon of *catalysis* in quantum information theory (the operator C being called a catalyst). This corresponds to the situation where the transformation of some quantum state (in that case, B) into another quantum state (in that case, A) is only possible in

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[1] Suppose first that $C \geq 0$ has finite rank. That is, $C = \sum_{k=0}^{n-1} \mu(k, C) p_k$, where p_k, $0 \leq k < n$, are pairwise orthogonal rank one projections. Set $A_k = A \otimes \mu(k, C) p_k$ and $B_k = B \otimes \mu(k, C) p_k$. It is immediate that $B_k \prec\prec A_k$ for $0 \leq k < n$. It follows from Lemma 2.3 in [4] that $\sum_{k=0}^{n-1} B_k \prec\prec \sum_{k=0}^{n-1} A_k$ or, equivalently, $B \otimes C \prec\prec A \otimes C$. For an arbitrary C, the assertion follows by approximation.

the presence of an extra quantum state (in that case, C), although the latter is not consumed in the process. It is argued in [7] that this phenomenon can be used to improve the efficiency of entanglement concentration procedures.

In the following we restrict ourselves to A, B being positive elements in $\bigcap_{p>1} \mathcal{L}_p$ ($\mathcal{L}_p$ denoting the Schatten–von Neumann ideal) and compare the following statements:

(i) There exists a nonzero $C \in \mathcal{L}_1$ such that $B \otimes C \prec\prec A \otimes C$.
(ii) For every $p > 1$, we have $\mathrm{Tr}(B^p) \leq \mathrm{Tr}(A^p)$.

One checks that (i) implies (ii). This follows from the monotonicity of $A \mapsto \mathrm{Tr}(A^p)$ with respect to submajorization and from the formula

$$\mathrm{Tr}(S \otimes T) = \mathrm{Tr}(S) \cdot \mathrm{Tr}(T), \quad S, T \in \mathcal{L}_1.$$

There is some hope to reverse the implication (i) $\Rightarrow$ (ii) if we allow closure of the set

$$\{B : \exists C \in \mathcal{L}_1 \text{ such that } B \otimes C \prec\prec A \otimes C\}$$

with respect to some topology (for the finite-dimensional case, see [1, 9, 15]).

To explain why some closure is needed, we give an example of a pair A, B of positive operators satisfying (ii) but not (i). Consider positive operators[2] with $\mu(A) \neq \mu(B)$ and such that $\mathrm{Tr}(B^p) \leq \mathrm{Tr}(A^p)$ for $p \in (1, \infty)$, while $\mathrm{Tr}(B^{p_0}) = \mathrm{Tr}(A^{p_0})$ for some $p_0 \in (1, \infty)$ (such an example exists among finite rank operators). Note that the norm in $\mathcal{L}_{p_0}$ is strictly monotone with respect to submajorization (see Proposition 2.1 in [3]). That is, if $K \in \mathcal{L}_{p_0}(H)$ and if $L \prec\prec K$, then either $\mu(L) = \mu(K)$ or $\|L\|_{p_0} < \|K\|_{p_0}$. Suppose that (i) holds, i.e. that $B \otimes C \prec\prec A \otimes C$ for some nonzero $C \in \mathcal{L}_1$ (that is, no closure is taken). We then have $\mathrm{Tr}((B \otimes C)^{p_0}) = \mathrm{Tr}((A \otimes C)^{p_0})$ and, by strict monotonicity, $\mu(k, B \otimes C) = \mu(k, A \otimes C)$ for all $k \geq 0$. Now, taking into account that the sequences $\mu(B \otimes C)$ and $\mu(A \otimes C)$ coincide with decreasing rearrangements of sequences $\mu(B) \otimes \mu(C)$ and $\mu(A) \otimes \mu(C)$ respectively, we infer that $\mu(A) = \mu(B)$.

As we shall see, the choice of the topology plays a crucial role. Prior to stating the precise question, we recall a few definitions and relevant facts.

There is a remarkable correspondence between sequence spaces and two-sided ideals in $\mathcal{L}(H)$ due to J.W. Calkin [2]. Recall that a linear subspace $\mathcal{J}$ in $\mathcal{L}(H)$ is a two-sided ideal if $X \in \mathcal{J}$ and $Y \in \mathcal{L}(H)$ imply $YX, XY \in \mathcal{J}$. Every nontrivial ideal necessarily consists of compact operators. A Calkin space J is a subspace of c_0 (the space of all vanishing sequences) such that $x \in J$ and $\mu(y) \leq \mu(x)$ imply $y \in J$, where $\mu(x)$ is the decreasing rearrangement of the sequence $|x|$. The Calkin correspondence may be explained as follows. If J is a Calkin space, then associate to it the subset $\mathcal{J}$ in $\mathcal{L}(H)$,

$$\mathcal{J} := \{X \in \mathcal{C}_0 : \mu(X) \in J\}.$$

Conversely, if $\mathcal{J}$ is a two-sided ideal, then associate to it the sequence space

$$J := \{x \in c_0 : \mu(x) = \mu(X) \text{ for some } X \in \mathcal{J}\}.$$

For the proof of the following theorem we refer to Calkin's original paper, [2], and to B. Simon's book, [13, Theorem 2.5].

[2]Here is an example of such a couple. Let p_0 be a rank 4 projection and let p_1 be a rank 1 projection orthogonal to p_0. Set $A = 2^{-\frac{1}{2}} p_0 + 2^{\frac{1}{2}} p_1$ and $B = p_0$. It is immediate that $\|A\|_p^p = 2^{2-\frac{p}{2}} + 2^{\frac{p}{2}}$ and $\|B\|_p^p = 4$ for every $p > 0$. Thus, $\|B\|_p \leq \|A\|_p$ for every $p > 0$ and $\|B\|_2 = \|A\|_2$.

Theorem 1 (Calkin correspondence)**.** *The correspondence $J \leftrightarrow \mathcal{J}$ is an inclusion lattice preserving bijection between Calkin spaces and two-sided ideals in $\mathcal{L}(H)$.*

In the recent papers [8], [14] this correspondence has been specialised to quasi-normed symmetrically-normed ideals and quasi-normed symmetric sequence spaces [10]. We use the notation $\|\cdot\|_\infty$ to denote the uniform norm on $\mathcal{L}(H)$.

Definition 2. (i) An ideal $\mathcal{E}$ in $\mathcal{L}(H)$ is said to be symmetrically (quasi)-normed if it is equipped with a Banach (quasi)-norm $\|\cdot\|_{\mathcal{E}}$ such that

$$\|XY\|_{\mathcal{E}}, \|YX\|_{\mathcal{E}} \leq \|X\|_{\mathcal{E}}\|Y\|_\infty, \quad X \in \mathcal{E}, Y \in \mathcal{L}(H).$$

(ii) A Calkin space E is a symmetric sequence space if it is equipped with a Banach (quasi)-norm $\|\cdot\|_E$ such that $\|y\|_E \leq \|x\|_E$ for every $x \in E$ and $y \in c_0$ such that $\mu(y) \leq \mu(x)$.

For convenience of the reader, we recall that a map $\|\cdot\|$ from a linear space X to $\mathbb{R}$ is a quasi-norm if for all $x, y \in X$ and scalars α the following properties hold:
(i) $\|x\| \geqslant 0$, and $\|x\| = 0 \Leftrightarrow x = 0$;
(ii) $\|\alpha x\| = |\alpha|\|x\|$;
(iii) $\|x + y\| \leqslant C(\|x\| + \|y\|)$ for some $C \geqslant 1$.
The couple $(X, \|\cdot\|)$ is a quasi-normed space and the least constant C satisfying inequality (iii) above is called the modulus of concavity of the quasi-norm $\|\cdot\|$ and denoted by C_X. A complete quasi-normed space is called quasi-Banach.

It easily follows from Definition 2 that if $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ is a quasi-Banach ideal and $X \in \mathcal{E}$ and $Y \in \mathcal{L}(H)$ are such that $\mu(Y) \leq \mu(X)$, then $Y \in \mathcal{E}$ and $\|Y\|_{\mathcal{E}} \leq \|X\|_{\mathcal{E}}$. In particular, it is easy to see that if E is a Calkin space corresponding to $\mathcal{E}$, then setting $\|x\|_E := \|X\|_{\mathcal{E}}$ (where $X \in \mathcal{E}$ is such that $\mu(x) = \mu(X)$) we obtain that $(E, \|\cdot\|_E)$ is a quasi-Banach symmetric sequence space. The converse implication is much harder and follows from Theorem 8.11 in [8] and Theorem 4 in [14].

With these preliminaries out of the way, we are now in a position to formulate the main question.

Question 3. Let $\mathcal{I}$ be a (quasi-)Banach ideal such that $\mathcal{I} \subset \bigcap_{p>1} \mathcal{L}_p$. Let $0 \leq A \in \mathcal{I}$. Consider the sets

$$\mathrm{PM}(A, \mathcal{I}) = \Big\{0 \leq B \in \mathcal{I} : \ \mathrm{Tr}(B^p) \leq \mathrm{Tr}(A^p) \ \forall p > 1\Big\},$$

$$\mathrm{Catal}(A, \mathcal{I}) = \Big\{0 \leq B \in \mathcal{I} : \ \exists 0 \leq C \in \mathcal{L}_1 : \ C \neq 0, \ B \otimes C \prec\prec A \otimes C\Big\}.$$

Let also $\overline{\mathrm{Catal}}(A, \mathcal{I})$ denote the closure of $\mathrm{Catal}(A, \mathcal{I})$ with respect to the quasi-norm of $\mathcal{I}$. Is it true that $\mathrm{PM}(A, \mathcal{I}) = \overline{\mathrm{Catal}}(A, \mathcal{I})$?

Note that $\mathrm{PM}(A, \mathcal{I})$ is a closed subset in $\mathcal{I}$. Indeed, let $B_n \in \mathrm{PM}(A, \mathcal{I})$ and let $B_n \to B$ in $\mathcal{I}$ as $n \to \infty$. Observe that it follows from Definition 2 that $\mathcal{I}$ is continuously embedded[3] into $\mathcal{L}(H)$, and therefore it follows from the Closed Graph Theorem that for every fixed $p > 1$, the identity embedding $\mathcal{I} \subset \mathcal{L}_p$ is continuous;

[3]We have to show that $\|A\|_\infty \leq \mathrm{const} \cdot \|A\|_{\mathcal{I}}$ for every $A \in \mathcal{I}$. Without loss of generality, $A \geq 0$. Set $p = E_A\{\|A\|_\infty\}$ (the spectral projection corresponding to the one-point set $\{\|A\|_\infty\}$ is nonzero since A is compact) and let $q \leq p$ be a rank one projection. Clearly, $qA = qpA = q \cdot \|A\|_\infty p = \|A\|_\infty q$ and, similarly, $Aq = \|A\|_\infty q$. Thus, A commutes with q and $A \geq qAq = \|A\|_\infty q$. Therefore, $\|A\|_{\mathcal{I}} \geq \|A\|_\infty \|q\|_{\mathcal{I}}$. Since all rank one projections are unitarily equivalent, it follows that they have the same norm. This proves the assertion.

in particular, there exists a constant $c(p,\mathcal{I})$ such that $\|C\|_p \le c(p,\mathcal{I})\|C\|_{\mathcal{I}}$, $C\in\mathcal{I}$. Thus,

$$\Big|\|B_n\|_p - \|B\|_p\Big| \le \|B-B_n\|_p \le c(p,\mathcal{I})\|B-B_n\|_{\mathcal{I}} \to 0.$$

Hence,

$$\mathrm{Tr}(B^p) = \lim_{n\to\infty}\mathrm{Tr}(B_n^p) \le \mathrm{Tr}(A^p), \quad p>1.$$

We also have that $\mathrm{Catal}(A,\mathcal{I}) \subset \mathrm{PM}(A,\mathcal{I})$. Indeed, if $B\otimes C \prec\prec A\otimes C$, then

$$\mathrm{Tr}(B^p) = \frac{\mathrm{Tr}((B\otimes C)^p)}{\mathrm{Tr}(C^p)} \le \frac{\mathrm{Tr}((A\otimes C)^p)}{\mathrm{Tr}(C^p)} = \mathrm{Tr}(A^p), \quad p>1.$$

Since $\mathrm{PM}(A,\mathcal{I})$ is closed, it follows that the inclusion $\overline{\mathrm{Catal}}(A,\mathcal{I}) \subset \mathrm{PM}(A,\mathcal{I})$ always holds.

In this paper, we show that the answer to Question 3 is positive when $\mathcal{I}=\mathcal{L}_1$ and negative when $\mathcal{I}=\mathcal{L}_{1,\infty}$. Recall that $\mathcal{L}_{1,\infty}$ is the principal ideal generated by the element $A_0 = \mathrm{diag}(\{1,\frac{1}{2},\frac{1}{3},\cdots\})$. Equivalently,

$$\mathcal{L}_{1,\infty} = \{A\in\mathcal{C}_0 : \sup_{k\ge0}(k+1)\mu(k,A) < +\infty\}.$$

It becomes a quasi-Banach space (see e.g. [8, 14]) when equipped with the quasi-norm

$$\|A\|_{1,\infty} = \sup_{k\ge0}(k+1)\mu(k,A), \quad A\in\mathcal{L}_{1,\infty}.$$

Here are our main results. We leave open the question of giving a complete description of the set $\overline{\mathrm{Catal}}(A,\mathcal{L}_{1,\infty})$.

Theorem 4. *For every $0\le A\in\mathcal{L}_1$, the sets $\mathrm{PM}(A,\mathcal{L}_1)$ and $\overline{\mathrm{Catal}}(A,\mathcal{L}_1)$ coincide.*

Theorem 5. *There exists $0\le A\in\mathcal{L}_{1,\infty}$ such that the set $\mathrm{PM}(A,\mathcal{L}_{1,\infty})$ strictly contains the set $\overline{\mathrm{Catal}}(A,\mathcal{L}_{1,\infty})$.*

It is actually simple to deduce Theorem 4 from the finite-dimensional considerations from [1], as we explain in Section 2. This is in sharp contrast with Theorem 5, whose proof is infinite-dimensional in its nature and uses crucially fine properties of Dixmier traces, which we introduce in Section 3. The heart of the argument behind Theorem 5 appears in Section 4, where we relegate some needed computations to Section 5.

2. The case of $\mathcal{L}_1$

We derive Theorem 4 from the following result which appears in [15] (see also Lemma 2 in [1]).

Lemma 6. *Let A,B be positive finite rank operators. Assume that for every $1\le p\le+\infty$, we have the strict inequality $\|B\|_p<\|A\|_p$. Then there exists a nonzero finite rank operator C such that $B\otimes C\prec\prec A\otimes C$.*

Proof of Theorem 4. Let us show the nontrivial inclusion, i.e. that every $B\in\mathrm{PM}(A,\mathcal{L}_1)$ belongs to $\overline{\mathrm{Catal}}(A,\mathcal{L}_1)$.

Let p_k, $k\ge0$, be a rank one eigenprojection of the operator A which corresponds to the eigenvalue $\mu(k,A)$. Similarly, let q_k, $k\ge0$, be a rank one eigenprojection of the operator B which corresponds to the eigenvalue $\mu(k,B)$. We have

$$A=\sum_{k=0}^{\infty}\mu(k,A)p_k, \quad B=\sum_{k=0}^{\infty}\mu(k,B)q_k.$$

Without loss of generality, $\mu(0,A)=1$. It follows that

$$(1-(1-\varepsilon)^p)\mathrm{Tr}(A^p) \geq (1-(1-\varepsilon)^p)\mu(0,A)^p = 1-(1-\varepsilon)^p \geq \varepsilon^p.$$

The latter readily implies

$$\mathrm{Tr}(A^p)-\varepsilon^p \geq (1-\varepsilon)^p\mathrm{Tr}(A^p), \quad p\geq 1, \quad \varepsilon\in(0,1).$$

Now, fix $\varepsilon\in(0,1)$ and select n such that

$$\sum_{k=n}^{\infty}\mu(k,A)<\varepsilon, \quad \sum_{k=n}^{\infty}\mu(k,B)<\varepsilon.$$

Set

$$A_n=\sum_{k=0}^{n-1}\mu(k,A)p_k, \quad B_n=\sum_{k=0}^{n-1}\mu(k,B)q_k.$$

It is clear that

$$\begin{aligned}\mathrm{Tr}(A_n^p)=\mathrm{Tr}(A^p)-\sum_{k=n}^{\infty}\mu(k,A)^p &\geq \mathrm{Tr}(A^p)-(\sum_{k=n}^{\infty}\mu(k,A))^p\\ &> \mathrm{Tr}(A^p)-\varepsilon^p \geq (1-\varepsilon)^p\mathrm{Tr}(A^p), \quad p\geq 1.\end{aligned}$$

Therefore,

$$(1-\varepsilon)^p\mathrm{Tr}(B_n^p) \leq (1-\varepsilon)^p\mathrm{Tr}(B^p) \leq (1-\varepsilon)^p\mathrm{Tr}(A^p) < \mathrm{Tr}(A_n^p), \quad p\geq 1.$$

Since both A_n and B_n are finite rank operators, it follows from Lemma 6 and the first footnote that there exists a finite rank operator C_n such that

$$(1-\varepsilon)B_n\otimes C_n \prec\prec A_n\otimes C_n \prec\prec A\otimes C_n.$$

In particular, we have that $(1-\varepsilon)B_n\in \mathrm{Catal}(A,\mathcal{L}_1)$. Observing that $\|B-B_n\|_1\leq 1$, we further obtain

$$\|B-(1-\varepsilon)B_n\|_1 \leq \varepsilon\|B\|_1+(1-\varepsilon)\|B-B_n\|_1 \leq \varepsilon(\|B\|_1+1).$$

Since ε is arbitrarily small, it follows that $B\in\overline{\mathrm{Catal}}(A,\mathcal{L}_1)$. □

3. Dixmier traces

The crucial ingredient in the proof is the notion of a Dixmier trace on $\mathcal{L}_{1,\infty}$. Let ℓ_∞ stand for the Banach space of all bounded sequences $x=(x_n)_{n\geq 0}$ equipped with the usual norm $\|x\|_\infty := \sup_{n\geq 0}|x_n|$. A *generalized limit* is any positive linear functional on ℓ_∞ which equals the ordinary limit on the subspace c of all convergent sequences.

Remark 7. Given a sequence $(x_n)_{n\geq 0}\in\ell_\infty$, there is a generalized limit ω such that $\omega((x_n))=\limsup_{n\to\infty}x_n$.

Proof. Fix $x=(x_n)\in\ell_\infty$ and let the sequence $(n_k)_{k\geq 0}$ be such that $\lim_{k\to\infty}x_{n_k}=\limsup_{n\to\infty}x_n$. Consider the set of functionals $(\varphi_{n_k})_{k\geq 0}$ on ℓ_∞ defined by $\varphi_{n_k}(y_n) := y_{n_k}$, $y=(y_n)\in\ell_\infty$, $k\geq 0$. The set $(\varphi_{n_k})_{k\geq 0}$ belongs to the unit ball B of the Banach dual ℓ_∞^*. The set B is compact in the weak* topology $\sigma(\ell_\infty^*,\ell_\infty)$, and therefore the set $(\varphi_{n_k})_{k\geq 0}$ possesses a cluster point $\omega\in\ell_\infty^*$ in that topology. The fact that ω is a generalized limit on ℓ_∞ such that $\omega((x_n))=\limsup_{n\to\infty}x_n$ follows immediately from the definition of the weak* topology. □

The Dixmier traces are defined as follows.

Theorem 8. *Let ω be a generalized limit. The mapping $\mathrm{Tr}_\omega : \mathcal{L}_{1,\infty}^+ \to \mathbb{R}^+$ defined for $0 \le A \in \mathcal{L}_{1,\infty}$ by setting*

$$\mathrm{Tr}_\omega(A) := \omega\left(\left\{\frac{1}{\log(N+2)}\sum_{k=0}^{N}\mu(k,A)\right\}_{N=0}^{\infty}\right)$$

is additive and, therefore, extends to a positive unitarily invariant linear functional on $\mathcal{L}_{1,\infty}$ called a Dixmier trace.

Note that the positivity of generalized limits implies that

$$|\mathrm{Tr}_\omega(A)| \le \|A\|_{1,\infty} \tag{1}$$

for every Dixmier trace Tr_ω and $A \in \mathcal{L}_{1,\infty}$.

Let us comment on how additivity is proved in Theorem 8. This is usually achieved under the extra assumption that ω is scale invariant (see Theorem 1.3.1 in [11]), i.e. that $\omega \circ \sigma_k = \omega$ for all positive integers k, where $\sigma_k : \ell_\infty \to \ell_\infty$ is defined as

$$\sigma_k(x_1, x_2, \dots, x_n, \dots) = (\underbrace{x_1, \dots, x_1}_{k \text{ times}}, \underbrace{x_2, \dots, x_2}_{k \text{ times}}, \dots, \underbrace{x_n, \dots, x_n}_{k \text{ times}}, \dots).$$

Under this extra assumption the map Tr_ω is actually additive on the larger ideal $\mathcal{M}_{1,\infty}$ (we refer to [11, Example 1.2.9] for the definition of the latter ideal and to [11, Section 6.8] for historical background). In the form presented here, Theorem 8 follows from Theorem 17 in [12]. For the reader's convenience we reproduce the argument here.

Proof of Theorem 8. Given $A \in \mathcal{L}_{1,\infty}$, consider the sequence $(x_N(A))_{N=0}^\infty$ defined by

$$x_N(A) = \frac{1}{\log(N+2)}\sum_{j=0}^{N}\mu(j,A).$$

It is not hard to check that for all positive integers k,

$$\lim_{N\to\infty} x_N(A) - x_{kN}(A) = 0. \tag{2}$$

Let $E \subset \ell_\infty$ be the subspace

$$E = \mathrm{span}\left\{\sigma_k(\{x_N(A)\}) \,:\, k \ge 1, A \in \mathcal{L}_{1,\infty}\right\}.$$

It follows from (2) that the equation $\omega \circ \sigma_k(x) = \omega(x)$ is satisfied for $x \in E$. By a version of the Hahn–Banach theorem (see [6], Theorem 3.3.1), the linear functional $\omega_{|E}$ can be extended to a generalized limit $\omega' : \ell_\infty \to \mathbf{R}$ which is scale-invariant. The usual argument ([11], Theorem 1.3.1) implies that $\mathrm{Tr}_{\omega'}$ (which coincides with Tr_ω on $\mathcal{L}_{1,\infty}$) is additive on $\mathcal{L}_{1,\infty}$. □

We also need a version of Fubini's theorem for Dixmier traces.

Theorem 9. *For every $A \in \mathcal{L}_{1,\infty}$ and for every $C \in \mathcal{L}_1$, we have $A \otimes C \in \mathcal{L}_{1,\infty}$ and*

$$\|A \otimes C\|_{1,\infty} \le \|A\|_{1,\infty}\|C\|_1. \tag{3}$$

Moreover, for every Dixmier trace Tr_ω on $\mathcal{L}_{1,\infty}$, we have

$$\mathrm{Tr}_\omega(A \otimes C) = \mathrm{Tr}_\omega(A)\mathrm{Tr}(C). \tag{4}$$

Proof. We may assume $\|A\|_{1,\infty} = 1$. Recall that $A_0 = \operatorname{diag}(\{1, \frac{1}{2}, \frac{1}{3}, \cdots\})$. We have for all $k \geq 0$,

$$\mu(k, A \otimes C) \leq \mu(k, A_0 \otimes C) \leq \frac{1}{k+1} \sum_{j=0}^{k} \mu(j, C) \leq \frac{\|C\|_1}{k+1},$$

where the second inequality follows from Proposition 3.14 in [5]. This proves (3).

Observe that both sides of (4) depend linearly on A and C (thanks to Theorem 8). Thus, we can assume without loss of generality that $A, C \geq 0$. When C is a rank one projection, (4) follows from Theorem 8 since in that case $\mu(k, A \otimes C) = \mu(k, A)$ for all $k \geq 0$. Again appealing to linearity of Dixmier traces, we infer the result for the finite rank operator C and when $A \in \mathcal{L}_{1,\infty}$ is arbitrary. Now consider a general $C \in \mathcal{L}_1$ and let (C_n) be a sequence of finite rank operators such that $\|C - C_n\|_1 \to 0$. We have

$$|\mathrm{Tr}_\omega(A \otimes C_n) - \mathrm{Tr}_\omega(A \otimes C)| \leq \|A \otimes (C - C_n)\|_{1,\infty} \leq \|A\|_{1,\infty} \|C - C_n\|_1,$$

and this quantity tends to 0 as n goes to infinity. Consequently,

$$\mathrm{Tr}_\omega(A \otimes C) = \lim_{n \to \infty} \mathrm{Tr}_\omega(A \otimes C_n) = \lim_{n \to \infty} \mathrm{Tr}_\omega(A)\mathrm{Tr}(C_n) = \mathrm{Tr}_\omega(A)\mathrm{Tr}(C). \quad \square$$

As a corollary, we obtain that Dixmier traces give necessary conditions for catalysis.

Corollary 10. *Let* $0 \leq A \in \mathcal{L}_{1,\infty}$ *and* $0 \leq B \in \overline{\mathrm{Catal}}(A, \mathcal{L}_{1,\infty})$. *Then for every Dixmier trace* Tr_ω, *one has*

$$\mathrm{Tr}_\omega(B) \leq \mathrm{Tr}_\omega(A). \tag{5}$$

Proof. We know from (1) that Dixmier traces are continuous on $\mathcal{L}_{1,\infty}$, and therefore we may assume that $B \in \mathrm{Catal}(A, \mathcal{L}_{1,\infty})$. By definition of the latter set (see Question 3), there exists a nonzero positive C in $\mathcal{L}_1$ with the property that $B \otimes C \prec\prec A \otimes C$. Combining the definition of Hardy-Littlewood submajorization $\prec\prec$ and the positivity from the definition of a Dixmier trace Tr_ω (see Theorem 8), we infer that the inequality $\mathrm{Tr}_\omega(B \otimes C) \leq \mathrm{Tr}_\omega(A \otimes C)$ holds for every Dixmier trace Tr_ω. Inequality (5) now follows from (4) and from the fact that $\mathrm{Tr}(C) > 0$. $\square$

4. The case of $\mathcal{L}_{1,\infty}$: The main argument

Here is the main technical result used in the proof of Theorem 5. In the lemma below, we tacitly identify a sequence in the space ℓ_∞ with the corresponding diagonal operator. For $I \subset \mathbb{N}$, we note by χ_I the sequence defined by $\chi_I(n) = 1$ if $n \in I$ and $\chi_I(n) = 0$ otherwise.

Lemma 11. *Let* I *be the subset of* $\mathbb{N}$ *defined as*

$$I = \bigcup_{n \geq 0} [2^{2n}, 2^{2n+1}).$$

Consider the operator[4]

$$B = \bigoplus_{m \in I} 2^{-m} \chi_{[0, 2^m)}.$$

[4]In the subsequent formulas, the symbol $\oplus$ stands for the direct sum of operators.

Then $B \in \mathcal{L}_{1,\infty}$. *Moreover,*

$$\limsup_{s\to 0+} s\mathrm{Tr}(B^{1+s}) \le \frac{5}{9\log 2} < \frac{2}{3\log 2} \le \limsup_{N\to\infty} \frac{1}{\log N}\sum_{k=0}^{N}\mu(k,B). \tag{6}$$

Let us postpone the proof of Lemma 11 and show how it implies the result stated in Theorem 5. Consider B as in Lemma 11 and fix a number α such that $\frac{5}{9\log 2} < \alpha < \frac{2}{3\log 2}$. Recall that $A_0 = \mathrm{diag}(\{1, \frac{1}{2}, \frac{1}{3}, \cdots\})$. Since

$$\lim_{s\to 0+} s\mathrm{Tr}((\alpha A_0)^{1+s}) = \lim_{s\to 0+} s\zeta(s)\alpha^{1+s} = \alpha,$$

it follows from (6) that there exists $\delta > 0$ such that the inequality

$$\mathrm{Tr}(B^{1+s}) \le \mathrm{Tr}((\alpha A_0)^{1+s}) \tag{7}$$

holds whenever $0 < s \le \delta$. Define the operator $A = \alpha A_0 \oplus \|B\|_{1+\delta} p$, where p is a rank one projection. We claim that $B \in \mathrm{PM}(A, \mathcal{L}_{1,\infty})$: indeed, for $s > \delta$ we may write

$$\mathrm{Tr}(B^{1+s}) = \mathrm{Tr}\left((B^{1+\delta})^{\frac{1+s}{1+\delta}}\right) \le \left(\mathrm{Tr}(B^{1+\delta})\right)^{\frac{1+s}{1+\delta}} = \|B\|_{1+\delta}^{1+s} \le \mathrm{Tr}(A^{1+s}),$$

while for $0 < s \le \delta$ the inequality $\mathrm{Tr}(B^{1+s}) \le \mathrm{Tr}(A^{1+s})$ follows immediately from (7).

We now assume by contradiction that B belongs to the set $\overline{\mathrm{Catal}}(A, \mathcal{L}_{1,\infty})$. We know from Corollary 10 that $\mathrm{Tr}_\omega(B) \le \mathrm{Tr}_\omega(A)$ for every Dixmier trace Tr_ω. Observing that any such trace vanishes on finite rank operators, we see that the value $\mathrm{Tr}_\omega(A)$ coincides with $\mathrm{Tr}_\omega(\alpha A_0)$ and hence is equal to α for for every Dixmier trace Tr_ω (see the definition given in Theorem 8). On the other hand, we may choose a generalized limit ω such that

$$\mathrm{Tr}_\omega(B) = \limsup_{N\to\infty} \frac{1}{\log N}\sum_{k=0}^{N}\mu(k,B)$$

and obtain from (6) that $\frac{2}{3\log 2} \le \alpha$, a contradiction.

We note that the Dixmier trace considered in the proof does not behave in a monotone way with respect to trace of powers: we have $\mathrm{Tr}(B^p) \le \mathrm{Tr}(A^p)$ for every $p > 1$, but $\mathrm{Tr}_\omega(B) > \mathrm{Tr}_\omega(A)$.

5. Proof of Lemma 11

Let I and B be as defined in Lemma 11, and denote by E_B the spectral measure of B. First, note that for every integer m,

$$\mathrm{Tr}(E_B(2^{-m}, \infty)) \le \sum_{l<m} 2^l \le 2^m. \tag{8}$$

Hence, for every positive integer n, writing $2^m \le n < 2^{m+1}$, we infer

$$\mathrm{Tr}(E_B(\frac{1}{n}, \infty)) \le \mathrm{Tr}(E_B(2^{-m-1}, \infty)) \le 2^{m+1} \le 2n. \tag{9}$$

Recall also (e.g., see [11, Chapter 2, Section 2.3]) that $\mu(k,B)$, $k \ge 0$, can be computed via the formula

$$\mu(k,B) = \inf\{s \ge 0 : \ \mathrm{Tr}(E_{|B|}(s,\infty)) \le k\}.$$

Hence, it follows from (9) that $\mu(k,B) \leq \frac{2}{k+1}$ for every $k \geq 0$ and, in particular, $B \in \mathcal{L}_{1,\infty}$. We now prove the right inequality in (6). For a given n, let $N = \mathrm{Tr}(E_B(2^{-2^{2n+1}},\infty))$. We know from (8) that $N \leq 2^{2^{2n+1}}$. Therefore,

$$\sum_{k=0}^{N-1} \mu(k,B) = \mathrm{Tr}(BE_B(2^{-2^{2n+1}},\infty)) = \mathrm{card}(I \cap [0, 2^{2n+1}]) = \frac{2}{3}\cdot 2^{2n+1} - \frac{1}{3}.$$

Hence, for N as above, we have

$$\frac{1}{\log(N)} \sum_{k=0}^{N-1} \mu(k,B) \geq \frac{1}{\log(2^{2^{2n+1}})} \cdot \Big(\frac{2}{3}\cdot 2^{2n+1} - \frac{1}{3}\Big) = \frac{2}{3\log(2)} + o(1),$$

as needed.

We now focus on the left inequality in (6) and use the following summation formula, whose proof we postpone. For a given sequence $(x_n) \in \ell_\infty$ and for a given $s > 0$, we have that

$$\sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} \sum_{l=0}^{k} x_l \right) 2^{-ms} = (1-2^{-s})^{-2} \sum_{l=0}^{\infty} x_l 2^{-ls}. \tag{10}$$

Note that $\mathrm{Tr}(B^{1+s}) = \sum_{m\in I} 2^{-ms} = \sum_{m\geq 0} \chi_I(m) 2^{-ms}$ (here, $\chi_I(0) = 0$). Applying (10) to $x = \chi_I$, we obtain, for every $M > 0$,

$$\begin{aligned}
\limsup_{s\to 0+} s \sum_{l\geq 0} \chi_I(l) 2^{-ls} &= \limsup_{s\to 0+} s(1-2^{-s})^2 \sum_{m\geq 0} (\sum_{k=0}^{m} \sum_{l=0}^{k} \chi_I(l)) 2^{-ms} \\
&= \limsup_{s\to 0+} s(1-2^{-s})^2 \sum_{m\geq M} \Big(\frac{1}{(m+1)^2} \sum_{k=0}^{m} \sum_{l=0}^{k} \chi_I(l) \Big) \cdot (m+1)^2 2^{-ms} \\
&\leq \Big(\sup_{m\geq M} \frac{1}{(m+1)^2} \sum_{k=0}^{m} \sum_{l=0}^{k} \chi_I(l) \Big) \\
&\quad \cdot \Big(\limsup_{s\to 0+} s(1-2^{-s})^2 \sum_{m\geq M} (m+1)^2 2^{-ms} \Big).
\end{aligned}$$

Passing $M \to \infty$, we infer that

$$\limsup_{s\to 0+} s \sum_{l\geq 0} \chi_I(l) 2^{-ls} \leq C \limsup_{s\to 0+} s(1-2^{-s})^2 \sum_{m\geq 0} (m+1)^2 2^{-ms},$$

where

$$C := \limsup_{m\to\infty} \frac{1}{(m+1)^2} \sum_{k=0}^{m} \sum_{l=0}^{k} \chi_I(l).$$

An elementary computation gives

$$\sum_{m=0}^{\infty} (m+1)^2 2^{-ms} = \frac{1+2^{-s}}{(1-2^{-s})^3}.$$

It follows that

$$\limsup_{s\to 0+} s\mathrm{Tr}(B^{1+s}) \leq \frac{2C}{\log 2}.$$

It remains to show that $C \leq 5/18$ (we actually show $C = 5/18$). To that end, we think of χ_I as an element of $L_\infty(0,\infty)$ and define $z \in L_\infty(0,\infty)$ by setting $z = \chi_{\bigcup_{n\in\mathbb{Z}}[2^{2n},2^{2n+1})}$. Observe that $\chi_I \leq z$. Therefore,

$$C \leq \limsup_{t\to\infty} \frac{1}{t^2}\int_0^t \int_0^s z(u)\,\mathrm{d}u\mathrm{d}s.$$

Since $z(4t) = z(t)$ for every $t > 0$, applying Fubini's theorem we have

$$C \leq \sup_{t\in(1,4)} \frac{1}{t^2}\int_0^t z(u)(t-u)\,\mathrm{d}u.$$

However,

$$\frac{1}{t^2}\int_0^t z(u)(t-u)\,\mathrm{d}u = \begin{cases} \frac{1}{2} - \frac{2}{3t} + \frac{2}{5t^2}, & 1 \leq t \leq 2, \\ \frac{4}{3t} - \frac{8}{5t^2}, & 2 \leq t \leq 4. \end{cases}$$

Hence, the latter supremum is, in fact, a maximum which is attained at $t = \frac{12}{5}$ and equal to $\frac{5}{18}$.

Proof of (10). Write

$$\begin{aligned}\sum_{m\geq 0}(\sum_{k=0}^{m}\sum_{l=0}^{k} x_l)2^{-ms} &= \sum_{m\geq k\geq 0}(\sum_{l=0}^{k} x_l)2^{-ms} = \sum_{k=0}^{\infty}(\sum_{l=0}^{k} x_l)\sum_{m=k}^{\infty} 2^{-ms} \\ &= (1-2^{-s})^{-1}\sum_{k=0}^{\infty}(\sum_{l=0}^{k} x_l)2^{-ks} = (1-2^{-s})^{-1}\sum_{k\geq l\geq 0} x_l 2^{-ks} \\ &= (1-2^{-s})^{-1}\sum_{l=0}^{\infty} x_l \sum_{k=l}^{\infty} 2^{-ks} = (1-2^{-s})^{-2}\sum_{l=0}^{\infty} x_l 2^{-ls}. \qquad \square\end{aligned}$$

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