An atomic decomposition of the Hajłasz Sobolev space M_1^1 on manifolds *

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Abstract

Several possible notions of Hardy-Sobolev spaces on a Riemannian manifold with a doubling measure are considered. Under the assumption of a Poincaré inequality, the space M_1^1, defined by Hajłasz, is identified with a Hardy-Sobolev space defined in terms of atoms. Decomposition results are proved for both the homogeneous and the nonhomogeneous spaces.

Key words: Hardy-Sobolev spaces, atomic decomposition, metric measure spaces, Hajłasz-Sobolev spaces

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1 Introduction

The aim of this paper is to compare different definitions of Hardy-Sobolev spaces on manifolds. In particular, we consider characterizations of these spaces in terms of maximal functions, atomic decompositions, and gradients, some of which have been shown in the Euclidean setting, and apply them to the L_1 Sobolev space defined by Hajłasz.

In the Euclidean setting, specifically on a domain $\Omega \subset \mathbb{R}^n$, Miyachi [28] shows that for a locally integrable function f to have partial derivatives $\partial^\alpha f$ (taken in the sense of distributions) belonging to the real Hardy space $H_p(\Omega)$, is equivalent to a certain maximal function of f being in $L_p(\Omega)$. Earlier work by Gatto, Jiménez and Segovia [14] on Hardy-Sobolev spaces, defined via powers of the Laplacian, used a maximal function introduced by Calderón [6] in characterizing Sobolev spaces for $p > 1$ to extend his results to $p \leq 1$. Calderón's maximal function was subsequently studied by Devore and Sharpley [12], who showed that it is pointwise equivalent to the following variant of the sharp function. For simplicity we only give the definition in the special case corresponding to one derivative in L_1, which is what this article is concerned with. We will call this function the *Sobolev sharp maximal function* (it is also called a "fractional sharp maximal function" in [21]):

Definition 1.1. *For $f \in L_{1,\mathrm{loc}}$, define Nf by*

$$Nf(x) = \sup_{B:\, x\in B} \frac{1}{r(B)} \fint_B |f - f_B| d\mu,$$

where B denotes a ball, $r(B)$ its radius and f_B the average of f over B.

Another definition of Hardy-Sobolev spaces on $\mathbb{R}^n$, using second differences, is given by Strichartz [30], who also obtains an atomic decomposition. Further characterizations of Hardy-Sobolev spaces on $\mathbb{R}^n$ by means of atoms are given in [8] and [25]. For related work see [20].

Several recent results provide a connection between Hardy-Sobolev spaces and the $p = 1$ case of Hajłasz's definition of L_p Sobolev spaces on a metric measure space (X, d, μ):

Definition 1.2 (Hajłasz)**.** *Let $1 \leq p \leq \infty$. The (homogeneous) Sobolev space $\dot{M}^1_p$ is the set of all functions $u \in L_{1,\mathrm{loc}}$ such that there exists a measurable function $g \geq 0$, $g \in L_p$, satisfying*

$$|u(x) - u(y)| \leq d(x, y)(g(x) + g(y)) \ \mu - a.e. \tag{1}$$

We equip $\dot{M}^1_p$ with the semi-norm

$$\|u\|_{\dot{M}^1_p} = \inf_{g \text{ satisfies}(1)} \|g\|_p.$$

In the Euclidean setting, Hajłasz [15] showed the equivalence of this definition with the usual one for $1 < p \leq \infty$. For $p \in (n/n+1, 1]$, Koskela and Saksman [22] proved that $\dot{M}^1_p(\mathbb{R}^n)$ coincides with the homogeneous Hardy-Sobolev space $\dot{H}^1_p(\mathbb{R}^n)$ defined by requiring all first-order partial derivatives of f to lie in the real Hardy space H_p (the same space defined by Miyachi [28]). In recent work [23], the Hajłasz Sobolev spaces $\dot{M}^s_p$, for $0 < s \leq 1$ and $\frac{n}{n+s} < p < \infty$, are characterized as homogeneous grand Triebel-Lizorkin spaces.

In the more general setting of a metric space with a doubling measure, Kinnunen and Tuominen [21] show that Hajłasz's condition is equivalent to Miyachi's maximal function characterization, extending to $p = 1$ a previous result of Hajłasz and Kinnunen [17] for $p > 1$:

Theorem 1.3 ([17],[21]). *For* $1 \leq p < \infty$

$$\dot{M}^1_p = \{f \in L_{1,\mathrm{loc}} : Nf \in L_p\}$$

with

$$\|f\|_{\dot{M}^1_p} \sim \|Nf\|_p.$$

Moreover, if $f \in L_{1,\mathrm{loc}}$ *and* $Nf \in L_1$*, then* f *satisfies*

$$|f(x) - f(y)| \leq Cd(x,y)(Nf(x) + Nf(y)) \tag{2}$$

for $\mu - a.e.\, x$, y.

We now restrict the discussion to a complete Riemannian manifold M satisfying a doubling condition and a Poincaré inequality (see below for definitions). In this setting, Badr and Bernicot [5] defined a family of homogeneous atomic Hardy-Sobolev spaces $\dot{HS}^1_{t,\mathrm{ato}}$ and proved the following comparison between these spaces:

Theorem 1.4. ([5]) *Let* M *be a complete Riemannian manifold satisfying a doubling condition and a Poincaré inequality* (P_q) *for some* $q > 1$. *Then* $\dot{HS}^1_{t,\mathrm{ato}} \subset \dot{HS}^1_{\infty,\mathrm{ato}}$ *for every* $t \geq q$ *and therefore* $\dot{HS}^1_{t_1,\mathrm{ato}} = \dot{HS}^1_{t_2,\mathrm{ato}}$ *for every* $q \leq t_1, t_2 \leq \infty$.

In particular, under the assumption of the Poincaré inequality (P_1), for every $t > 1$ we can take $1 < q \leq t$ for which (P_q) holds, so all the atomic Hardy-Sobolev spaces $\dot{HS}^1_{t,\mathrm{ato}}$ coincide and can be denoted by $\dot{HS}^1_{\mathrm{ato}}$.

The main result of this paper is to identify this atomic Hardy-Sobolev space with Hajłasz's Sobolev space for $p = 1$:

Theorem 1.5. *Let* M *be a complete Riemannian manifold satisfying a doubling condition and the Poincaré inequality* (P_1). *Then*

$$\dot{M}^1_1 = \dot{HS}^1_{\mathrm{ato}}.$$

The definition of the atomic Hardy-Sobolev spaces, as well as the doubling condition, the Poincaré inequality, and other preliminaries, can be found in Section 2. The proof of Theorem 1.5, based on the characterization given by Theorem 1.3 and a Calderón-Zygmund decomposition, follows in Section 3. In Section 4, a nonhomogeneous version of Theorem 1.5 is obtained. Finally, in Section 5, we characterize our Hardy-Sobolev spaces in terms of derivatives. In particular, we show that the space of differentials df of our Hardy-Sobolev functions coincides with the molecular Hardy space of differential one-forms defined by Auscher, McIntosh and Russ [3] (and by Lou and McIntosh [24] in the Euclidean setting).

2 Preliminaries

In all of this paper M denotes a complete non-compact Riemannian manifold. We write T_xM for the tangent space at the point $x \in M$, $\langle \cdot, \cdot \rangle_x$ for the Riemannian metric at x, and μ for the Riemannian measure (volume) on M. The Riemannian metric induces a distance function ρ which makes (M, ρ) into a metric space, and $B(x, r)$ will denote the ball of radius r centered at x in this space.

Let T_x^*M be the cotangent space at x, ΛT_x^*M the complex exterior algebra, and d the exterior derivative acting on $C_0^\infty(\Lambda T^*M)$. We will work only with functions (0-forms) and hence for a smooth function f, df will be a 1-form. In fact, in most of the paper we will deal instead with the gradient ∇f, defined as the image of df under the isomorphism between T_x^*M and T_xM (see [32], Section 4.10). Since this isomorphism preserves the inner product, we have

$$\langle df, df \rangle_x = \langle \nabla f, \nabla f \rangle_x \tag{3}$$

Letting $L_p := L_p(M, \mu)$, $1 \leq p \leq \infty$, and denoting by $|\cdot|$ the length induced by the Riemannian metric on the tangent space (forgetting the subscript x for simplicity), we can define $\|\nabla f\|_p := \||\nabla f|\|_{L_p(M,\mu)}$ and, in view of (3), $\|df\|_p = \|\nabla f\|_p$. If d^* denotes the adjoint of d on $L_2(\Lambda T_x^*M)$, then the Laplace-Beltrami operator Δ is defined by $dd^* + d^*d$. However since d^* is null on 0-forms, this simplifies to $\Delta f = d^*df$ on functions and we have, for $f, g \in C_0^\infty(M)$, using (3),

$$\langle \Delta f, g \rangle_{L_2(M)} = \int_M \langle \Delta f, g \rangle_x d\mu = \int_M \langle df, dg \rangle_x d\mu = \langle \nabla f, \nabla g \rangle_{L_2(M)}.$$

We will use $\mathrm{Lip}(M)$ to denote the space of Lipschitz functions, i.e. functions f satisfying, for some $C < \infty$, the global Lipschitz condition

$$|f(x) - f(y)| \leq C\rho(x, y) \quad \forall\, x, y \in M.$$

The smallest such constant C will be denoted by $\|f\|_{\mathrm{Lip}}$. By $\mathrm{Lip}_0(M)$ we will denote the space of compactly supported Lipschitz functions. For such functions the gradient

∇f can be defined μ-almost everywhere and is in $L_\infty(M)$, with $\|\nabla f\|_\infty \approx \|f\|_{\mathrm{Lip}}$ (see [7] for Rademacher's theorem on metric measure spaces and also the discussion of upper gradients in [18], Section 10.2).

2.1 The doubling property

Definition 2.1. *Let M be a Riemannian manifold. One says that M satisfies the (global) doubling property (D) if there exists a constant $C > 0$, such that for all $x \in M$, $r > 0$ we have*

$$\mu(B(x,2r)) \leq C\mu(B(x,r)). \tag{D}$$

Observe that if M satisfies (D) then

$$\mathrm{diam}(M) < \infty \Leftrightarrow \mu(M) < \infty$$

(see [1]). Therefore if M is a complete non-compact Riemannian manifold satisfying (D) then $\mu(M) = \infty$.

Lemma 2.2. *Let M be a Riemannian manifold satisfying (D) and let $s = \log_2 C_{(\mathrm{D})}$. Then for all x, $y \in M$ and $\theta \geq 1$*

$$\mu(B(x,\theta R)) \leq C\theta^s \mu(B(x,R)). \tag{4}$$

Theorem 2.3 (Maximal theorem, [9])**.** *Let M be a Riemannian manifold satisfying (D). Denote by $\mathcal{M}$ the non-centered Hardy-Littlewood maximal function over open balls of M, defined by*

$$\mathcal{M}f(x) := \sup_{\substack{B \ ball \\ x \in B}} |f|_B,$$

where $f_E := \fint_E f d\mu := \dfrac{1}{\mu(E)} \displaystyle\int_E f d\mu$. Then for every $1 < p \leq \infty$, $\mathcal{M}$ is L_p bounded and moreover it is of weak type $(1,1)$. Consequently, for $r \in (0,\infty)$, the operator $\mathcal{M}_r$ defined by

$$\mathcal{M}_r f(x) := [\mathcal{M}(|f|^r)(x)]^{1/r}$$

is of weak type (r,r) and L_p bounded for all $r < p \leq \infty$.

Recall that an operator T is of weak type (p,p) if there is $C > 0$ such that for any $\alpha > 0$, $\mu(\{x : |Tf(x)| > \alpha\}) \leq \frac{C}{\alpha^p}\|f\|_p^p$.

2.2 Poincaré inequality

Definition 2.4 (Poincaré inequality on M)**.** *We say that a complete Riemannian manifold M admits **a Poincaré inequality** (P_q) for some $q \in [1,\infty)$ if there exists* *a constant $C > 0$ such that, for every function $f \in \mathrm{Lip}_0(M)$ and every ball B of M of radius $r > 0$, we have*

$$\left(\fint_B |f - f_B|^q d\mu\right)^{1/q} \leq Cr \left(\fint_B |\nabla f|^q d\mu\right)^{1/q}. \tag{P_q}$$

We also recall the following result

Theorem 2.5. ([16], Theorem 8.7) *Let $u \in \dot{M}_1^1$ and $g \in L_1$ such that (u, g) satisfies (1). Take $\frac{s}{s+1} \leq q < 1$ and $\lambda > 1$. Then (u, g) satisfies the following Sobolev-Poincaré inequality: there is a constant $C > 0$ depending on (D) and λ, independent of (u, g) such that for all balls B of radius $r > 0$,*

$$\left(\fint_B |u - u_B|^{q^*} d\mu\right)^{1/q^*} \leq Cr \left(\fint_{\lambda B} g^q d\mu\right)^{1/q}, \tag{5}$$

where $q^* = \frac{sq}{s-q}$.

Applying this together with Theorem 1.3, for $u \in \dot{M}_1^1$ we have

$$\left(\fint_B |u - u_B|^{q^*} d\mu\right)^{1/q^*} \leq Cr \left(\fint_{\lambda B} (Nu)^q d\mu\right)^{1/q} \tag{6}$$

for all balls B.

2.3 Comparison between Nf and $|\nabla f|$

The following Proposition shows that the maximal function Nf controls the gradient of f in the pointwise almost-everywhere sense. In the Euclidean setting this result was demonstrated by Calderón (see [6], Theorem 4) for his maximal function $N(f, x)$ (denoted by $f^\star$ in Section 4.2 below), which was shown to be pointwise equivalent to our Nf by Devore and Sharpley (see also the stronger inequality (5.5) in [28], which bounds the maximal function of the partial derivatives).

Recall that if $u \in C_0^\infty(M)$, given any smooth vector field $\mathbf{\Phi}$ with compact support, we can write, based on (3) and the definition of d^*,

$$\int_M \langle \nabla u, \mathbf{\Phi} \rangle_x d\mu := \int_M \langle du, \omega_{\mathbf{\Phi}} \rangle_x d\mu = \int_M u(d^* \omega_{\mathbf{\Phi}}) d\mu,$$

where $\omega_{\mathbf{\Phi}}$ is the 1-form corresponding to $\mathbf{\Phi}$ under the isomorphism between the tangent space $T_x M$ and the co-tangent space $T_x^* M$ (see [32], Section 4.10). Denoting $d^* \omega_{\mathbf{\Phi}}$ by div $\mathbf{\Phi}$, we can define, for $u \in L_{1,\mathrm{loc}}$, the gradient ∇u in the sense of distributions by

$$\langle \nabla u, \mathbf{\Phi} \rangle := -\int_M u(\mathrm{div}\ \mathbf{\Phi}) d\mu \tag{7}$$

for all smooth vector fields $\mathbf{\Phi}$ with compact support (see [27]). When M is orientable, div $\mathbf{\Phi}$ is given by $*d*\omega_{\mathbf{\Phi}}$ with $*$ the Hodge star operator (see [32]), and in the Euclidean case this corresponds to the usual notion of divergence of a vector field.

Proposition 2.6. *Assume that M satisfies (D), and suppose $u \in L_{1,\mathrm{loc}}$ with $Nu \in L_1$. Then ∇u, initially defined by (7), is given by an L_1 vector field and satisfies*

$$|\nabla u| \leq CNu \qquad \mu-\text{a.e.}$$

Proof. Fix $r > 0$. We begin with a covering of M by balls $B_i = B(x_i, r)$, $i = 1, 2...$ such that

1. $M \subset \cup_i B_i$,
2. $\sum_i 1\!\!1_{6B_i} \leq K$.

Note that the constant K can be taken independent of r. Then we take $\{\varphi_i\}_i$ a partition of unity related to the covering $\{B_i\}_i$ such that $0 \leq \varphi_i \leq 1$, $\varphi_i = 0$ on $(6B_i)^c$, $\varphi_i \geq c$ on $3B_i$ and $\sum_i \varphi_i = 1$. The φ_i's are C/r Lipschitz. For details concerning this covering we refer to [13], [21], [19], [10]. Now let (see [13], p. 1908 and [21], Section 3.1)

$$u_r(x) = \sum_j \varphi_j(x) u_{3B_j}. \tag{8}$$

The sum is locally finite and defines a Lipschitz function so we can take its gradient and we have, for μ-almost every x,

$$\begin{aligned} |\nabla u_r(x)| &= |\sum_j \nabla\varphi_j(x) u_{3B_j}| \\ &= |\sum_{\{j: x\in 6B_j\}} \nabla\varphi_j(x)(u_{3B_j} - u_{B(x,9r)})| \\ &\leq CK\frac{1}{r} \fint_{B(x,9r)} |u - u_{B(x,9r)}| d\mu \\ &\leq CKNu(x). \end{aligned} \tag{9}$$

We used the fact that $\sum \nabla\phi_j = 0$ and that for $x \in 6B_j$, $3B_j \subset B(x, 9r)$.

To see that $u_r \to u$ $\mu - a.e.$ and moreover in L_1 when $r \to 0$ (see also [13],p. 1908), write, for x a Lebesgue point of μ,

$$|u_r(x) - u(x)| \leq \sum_j |\varphi_j(x)||u(x) - u_{3B_j}| \leq \sum_{\{j: x\in 6B_j\}} |u(x) - u_{3B_j}| \leq CKr\mathcal{M}_q(Nu)(x)$$

where $\frac{s}{s+1} \leq q < 1$. The last inequality follows from estimates of $|u(x) - u_{B(x,9r)}|$ and $|u_{3B_j} - u_{B(x,9r)}|$, $x \in 6B_j$, which are the same as estimates (12)-(14) in the proof of Lemma 1 in [21], using the doubling property and (6).

Now let $\mathbf{\Phi}$ be a smooth vector field with compact support. Using the convergence in L_1, the fact that div $\mathbf{\Phi} \in C_0^\infty(M)$, and the estimate on $|\nabla u_r|$ above, we have

$$
\begin{aligned}
|\int_M \langle \nabla u, \mathbf{\Phi} \rangle_x d\mu| &= |\int_M u(\text{div } \mathbf{\Phi}) d\mu| \\
&= |\lim_{r \to 0} \int_M u_r(\text{div } \mathbf{\Phi}) d\mu| \\
&\leq \limsup_{r \to 0} \int_M |\nabla u_r||\mathbf{\Phi}| d\mu \leq CK \int |Nu||\mathbf{\Phi}| d\mu.
\end{aligned}
$$

Taking the supremum of the left-hand-side over all such $\mathbf{\Phi}$ with $|\mathbf{\Phi}| \leq 1$, we get that the total variation of u is bounded (see [27], (1.4), p. 104), i.e.

$$|Du|(M) \leq C\|Nu\|_{L_1(M)} < \infty,$$

hence u is a function of bounded variation on M, and $|Du|$ defines a finite measure on M. We can write the distributional gradient as

$$\langle \nabla u, \mathbf{\Phi} \rangle = \int_M \langle X_u, \mathbf{\Phi} \rangle_x d|Du|$$

for some vector field X_u with $|X_u| = 1$ a.e. (see again [27], p. 104 where this is expressed in terms of the corresponding 1-form σ_u). Moreover, from the above estimates and the fact that $Nu \in L_1$, we further deduce that the measure $|Du|$ is absolutely continuous with respect to the Riemannian measure μ, so there is an L_1 function g such that we can write $\nabla u = gX_u$, and $|\nabla u| \leq CNu$, $\mu - a.e.$ □

Corollary 2.7. *Assume that M satisfies (D). Then*

$$\dot{M}_1^1 \subset \dot{W}_1^1.$$

Proof. The result follows from Proposition 2.6 and Theorem 1.3. □

2.4 Hardy spaces

We begin by introducing the maximal function characterization of the real Hardy space H_1.

Definition 2.8. *Let $f \in L_{1,\text{loc}}(M)$. We define its grand maximal function, denoted by f^+, as follows:*

$$f^+(x) := \sup_{\varphi \in \mathcal{T}_1(x)} \left| \int f \varphi d\mu \right| \tag{10}$$

where $\mathcal{T}_1(x)$ is the set of all test functions $\psi \in \mathrm{Lip}_0(M)$ such that for some ball $B := B(x,r)$ containing the support of ψ,

$$\|\psi\|_\infty \le \frac{1}{\mu(B)}, \qquad \|\nabla\psi\|_\infty \le \frac{1}{r\mu(B)}. \tag{11}$$

Set $H_{1,\max}(M) = \{f \in L_{1,\mathrm{loc}}(M) : f^+ \in L_1(M)\}$.

While this definition assumes f to be only locally integrable, by taking an appropriate sequence $\varphi_\epsilon \in \mathcal{T}_1(x)$, the Lebesgue differentiation theorem implies that

$$|f(x)| = \lim_{\epsilon \to 0} \left| \int f\varphi_\epsilon \right| \le f^+(x) \ \text{ for } \mu\text{-a.e. } x, \tag{12}$$

so $H_{1,\max}(M) \subset L_1(M)$.

Another characterization is given in terms of atoms (see [10]).

Definition 2.9. *Fix $1 < t \le \infty$, $\frac{1}{t} + \frac{1}{t'} = 1$. We say that a function a is an H_1-atom if*

1. *a is supported in a ball B,*
2. *$\|a\|_t \le \mu(B)^{-\frac{1}{t'}}$, and*
3. *$\int a d\mu = 0$.*

We say f lies in the atomic Hardy space $H_{1,\mathrm{ato}}$ if f can be represented, in $L_1(M)$, by

$$f = \sum \lambda_j a_j \tag{13}$$

for sequences of H_1-atoms $\{a_j\}$ and scalars $\{\lambda_j\} \in \ell^1$. Note that this representation is not unique and we define

$$\|f\|_{H_{1,\mathrm{ato}}} := \inf \sum |\lambda_j|,$$

where the infimum is taken over all atomic decompositions (13).

A priori this definition depends on the choice of t. However, we claim

Proposition 2.10. *Let M be a complete Riemannian manifold satisfying (D). Then*

$$H_{1,\mathrm{ato}}(M) = H_{1,\max}(M)$$

with equivalent norms

$$\|f\|_{H_{1,\mathrm{ato}}} \approx \|f^+\|_1$$

(where the constants of proportionality depend on the choice of t).

In the case of a space of homogeneous type (X, d, μ), this was shown in [26] (Theorem 4.13) for a normal space of order α and in [31] (Theorem C) under the assumption of the existence of a family of Lipschitz kernels (see also the remarks following Theorem (4.5) in [10]). For the manifold M this will follow as a corollary of the atomic decomposition for the Hardy-Sobolev space below. We first prove the inclusion

$$H_{1,\mathrm{ato}}(M) \subset H_{1,\max}(M). \tag{14}$$

Proof. We show that if $f \in H_{1,\mathrm{ato}}$ then $f^+ \in L_1$. Let $t > 1$ and a be an atom supported in a ball $B_0 = B(x_0, r_0)$. We want to prove that $a^+ \in L_1$. First take $x \in 2B_0$. We have $a^+(x) = \sup_{\varphi \in \mathcal{T}_1(x)} \left|\int_B a\varphi d\mu\right| \leq C\mathcal{M}(a)(x)$. Then by the L_t-boundedness of the Hardy-Littlewood maximal function for $t > 1$ (Theorem 2.3) and the size condition on a,

$$\begin{aligned}\int_{2B_0} |a^+(x)| d\mu &\leq \mu(B_0)^{1/t'} \left(\int_{2B_0} |a^+|^t d\mu\right)^{1/t} \leq C\mu(B_0)^{1/t'} \|\mathcal{M}a\|_t \\ &\leq C_t \mu(B_0)^{1/t'} \|a\|_t \leq C_t. \end{aligned} \tag{15}$$

Note that the constant depends on t due to the dependence of the constant in the boundedness of the Hardy-Littlewood maximal function, which blows up as $t \to 1^+$.

Now if $x \in M \setminus 2B_0$, there exists $k \in \mathbb{N}^*$ such that $x \in C_k(B_0) := 2^{k+1}B_0 \setminus 2^k B_0$. Let $\varphi \in \mathcal{T}_1(x)$ and take a ball $B = B(x, r)$ such that φ is supported in and satisfies (11) with respect to B. Using the moment condition for a and the Lipschitz bound on φ, we get

$$\begin{aligned}\left|\int_B a\varphi d\mu\right| &= \left|\int_{B\cap B_0} a(y)(\varphi(y) - \varphi(x_0)) d\mu(y)\right| \\ &\leq C \int_{B\cap B_0} |a(y)| \frac{d(y, x_0)}{r\mu(B)} d\mu(y) \\ &\leq C \frac{r_0}{r\mu(B)} \|a\|_1. \end{aligned}$$

Note that for the integral not to vanish we must have $B \cap B_0 \neq \emptyset$. We claim that this implies

$$r > 2^{k-1} r_0 \text{ and } 2^{k+1} B_0 \subset 8B. \tag{16}$$

To see this, let $y \in B \cap B_0$. Then $r > d(y, x) \geq d(x, x_0) - d(y, x_0) \geq 2^k r_0 - r_0 \geq 2^{k-1} r_0$. Thus if $d(z, x_0) < 2^{k+1} r_0$ then $d(z, x) \leq d(z, x_0) + d(x, x_0) < 2^{k+1} r_0 + 2^{k+1} r_0 < 8r$ and we deduce that $2^{k+1} B_0 \subset 8B$. We then have

$$\mu(2^{k+1} B_0) \leq C 8^s \mu(B)$$

by (4). Using this estimate and the fact that $\|a\|_1 \leq 1$, we have

$$\begin{aligned}
\int_{x \notin 2B_0} |a^+|(x) d\mu &= \sum_{k \geq 1} \int_{C_k(B_0)} |a^+|(x) d\mu \\
&\leq C\|a\|_1 \sum_{k \geq 1} \frac{8^s 2^{1-k}}{\mu(2^{k+1}B_0)} \mu(C_k(B_0)) \\
&\leq C 8^s \sum_{k \geq 1} 2^{1-k} \\
&\leq C.
\end{aligned}$$

Thus $a^+ \in L_1$ with $\|a^+\|_1 \leq C_t$.

Now for $f \in H_{1,\mathrm{ato}}$, take an atomic decomposition of f as in (13). By the convergence of the series in L_1, we have, for each x and each $\varphi \in \mathcal{T}_1(x)$,

$$\left| \int f \varphi d\mu \right| \leq \sum |\lambda_j| \left| \int a_j \varphi d\mu \right| \leq \sum |\lambda_j| a_j^+(x)$$

so f^+ is pointwise dominated by $\sum |\lambda_j| a_j^+$, giving

$$\|f^+\|_1 \leq \sum_j |\lambda_j| \|a_j^+\|_1 \leq C_t \sum_j |\lambda_j|.$$

Taking the infimum over all the atomic decompositions of f yields $\|f^+\|_1 \leq C_t \|f\|_{H_1}$. □

The proof of the converse, namely that if $f^+ \in L_1$ then $f \in H_{1,\mathrm{ato}}$, relies on an atomic decomposition and will follow from the proof of Proposition 3.4 below.

2.5 Atomic Hardy-Sobolev spaces

In [5], the authors defined atomic Hardy-Sobolev spaces. Let us recall their definition of homogeneous Hardy-Sobolev atoms. These are similar to H_1 atoms but instead of the usual L_t size condition they are bounded in the Sobolev space $\dot{W}_t^1$.

Definition 2.11 ([5])**.** *For $1 < t \leq \infty$, $\frac{1}{t} + \frac{1}{t'} = 1$, we say that a function a is a homogeneous Hardy-Sobolev $(1,t)$-atom if*

1. *a is supported in a ball B,*
2. *$\|a\|_{\dot{W}_t^1} := \|\nabla a\|_t \leq \mu(B)^{-\frac{1}{t'}}$, and*
3. *$\int a d\mu = 0$.*

They then define, for every $1 < t \leq \infty$, the homogeneous Hardy-Sobolev space $\dot{HS}^1_{t,\mathrm{ato}}$ as follows: $f \in \dot{HS}^1_{t,\mathrm{ato}}$ if there exists a sequence of homogeneous Hardy-Sobolev $(1,t)$-atoms $\{a_j\}_j$ such that

$$f = \sum_j \lambda_j a_j \tag{17}$$

with $\sum_j |\lambda_j| < \infty$. This space is equipped with the semi-norm

$$\|f\|_{\dot{HS}^1_{t,\mathrm{ato}}} = \inf \sum_j |\lambda_j|,$$

where the infimum is taken over all possible decompositions (17).

Remarks 2.12. *1. Since condition 2 implies that the homogeneous Sobolev $\dot{W}^1_1$ semi-norm of the atoms is bounded by a constant, the sum in (17) converges in $\dot{W}^1_1$ and therefore we can consider $\dot{HS}^1_{t,\mathrm{ato}}$ as its subspace.*

2. *Since we are working with homogeneous spaces, we can modify functions by constants so the cancellation conditions are, in a sense, irrelevant. As we will see below, and when comparing to other definitions in the literature (see, for example, [25]), condition 3 can be replaced by one of the following:*

   $3'$. $\|a\|_1 \leq r(B)$, *or*

   $3''$. $\|a\|_t \leq r(B)\mu(B)^{-\frac{1}{t'}}$,

   where $r(B)$ is the radius of the ball B. Clearly condition $3''$ implies $3'$, and conditions 2 and 3 imply $3'$ (respectively $3''$) if we assume the Poincaré inequality (P_1) (respectively (P_t)). It is most common to consider the case $t = 2$ under the assumption (P_2).

3. *As mentioned in the introduction, from Theorem 1.4 we have that under (P_1) all the spaces $\dot{HS}^1_{t,\mathrm{ato}}$ can be identified as one space $\dot{HS}^1_{ato}$. As we will see, in this case the atomic decomposition can be taken with condition $3'$ instead of 3.*

3 Atomic decomposition of $\dot{M}^1_1$ and comparison with $\dot{HS}^1_{t,\mathrm{ato}}$

We begin by proving that under the Poincaré inequality (P_1), $\dot{HS}^1_{ato} \subset \dot{M}^1_1$. While under this assumption the space $\dot{HS}^1_{ato}$ is equivalent to any one of the spaces $\dot{HS}^1_{t,\mathrm{ato}}$ defined above, if we want to consider the norms we need to fix some $t > 1$.

Proposition 3.1. *Let M be a complete Riemannian manifold satisfying (D) and (P_1). Let $1 < t \leq \infty$ and a be a homogeneous Hardy-Sobolev $(1,t)$-atom. Then $a \in \dot{M}^1_1$ with $\|a\|_{\dot{M}^1_1} \leq C_t$, the constant C depending only on t, the doubling constant and the constant appearing in (P_1), and independent of a.*

Consequently $\dot{HS}^1_{t,\mathrm{ato}} \subset \dot{M}^1_1$ with

$$\|f\|_{\dot{M}^1_1} \leq C_t \|f\|_{\dot{HS}^1_{t,\mathrm{ato}}}.$$

Proof. Let a be an $(1,t)$-atom supported in a ball $B_0 = B(x_0, r_0)$. We want to prove that $Na \in L_1$. For $x \in 2B_0$ we have, using (P_1),

$$Na(x) = \sup_{B\colon x \in B} \frac{1}{r(B)} \fint_B |a - a_B| d\mu \leq C \sup_{B\colon x \in B} \fint_B |\nabla a| d\mu = C\mathcal{M}(|\nabla a|)(x).$$

Then, exactly as in (15), by the L_t boundedness of $\mathcal{M}$ for $t > 1$ (with a constant depending on t), and properties 1 and 2 of $(1,t)$-Hardy-Sobolev atoms,

$$\int_{2B_0} |Na(x)| d\mu \leq C\mu(B_0)^{1/t'} \left(\int_{2B_0} (\mathcal{M}(|\nabla a|))^t d\mu \right)^{1/t} \leq C_t \mu(B_0)^{1/t'} \|\nabla a\|_t \leq C_t.$$

Now if $x \notin 2B_0$, then there exists $k \in \mathbb{N}^*$ such that $x \in C_k(B_0) := 2^{k+1}B_0 \setminus 2^k B_0$. Let $B = B(y, r(B))$ be a ball containing x. Then

$$\begin{aligned} \frac{1}{r(B)} \fint_B |a - a_B| d\mu &= \frac{1}{r(B)} \frac{1}{\mu(B)} \left(\int_{B \cap B_0} |a - a_B| d\mu + \int_{B \cap B_0^c} |a_B| d\mu \right) \\ &\leq \frac{3}{r(B)} \frac{1}{\mu(B)} \int_{B \cap B_0} |a| d\mu. \end{aligned} \tag{18}$$

From (16) we have that $B \cap B_0 \neq \emptyset$ implies $r(B) > 2^{k-1} r_0$ and $\mu(2^{k+1}B_0) \leq C8^s \mu(B)$. This, together with the doubling and Poincaré assumptions (D) and (P_1), the cancellation condition 3 for a and the size condition 2 for ∇a, yield

$$Na(x) \leq \frac{3}{2^{k-1} r_0} \frac{8^s}{\mu(2^{k+1}B_0)} \int_{B_0} |a| d\mu \leq \frac{3}{2^{k-1}} \frac{8^s}{\mu(2^{k+1}B_0)} \int_{B_0} |\nabla a| d\mu \leq 3 \frac{2^{-k+1} 8^s}{\mu(2^{k+1}B_0)}.$$

Note that at this point we could have used condition $3'$ (see Remarks 2.12) instead of conditions $2, 3, (D)$ and (P_1).

Therefore

$$\begin{aligned} \int_{x \notin 2B_0} |Na|(x) d\mu &= \sum_{k \geq 1} \int_{C_k(B_0)} |Na|(x) d\mu \leq C8^s \sum_{k \geq 1} 2^{-k+1} \int_{C_k(B_0)} \frac{1}{\mu(2^{k+1}B_0)} d\mu(x) \\ &\leq C8^s \sum_{k \geq 1} 2^{-k+1} = C_s. \end{aligned}$$

Thus $Na \in L_1$ with $\|Na\|_1 \le C_{s,t}$.

Now if $f \in \dot{HS}^1_{t,\text{ato}}$, take an atomic decomposition of f: $f = \sum_j \lambda_j a_j$ with a_j $(1,t)$-atoms and $\sum_j |\lambda_j| < \infty$. Then the sum $\sum_j \lambda_j N a_j$ converges absolutely in L_1 so by Theorem 1.3 the sequence of functions $f_k = \sum_{j=1}^k \lambda_j a_j$ has a limit, g, in the Banach space $\dot{M}^1_1$. By Proposition 2.6, this implies convergence in $\dot{W}^1_1$. Since (as pointed out in Remarks 2.12) the convergence of the decomposition $f = \sum_j \lambda_j a_j$ also takes place in $\dot{W}^1_1$, we get that $f = g$ in $\dot{W}^1_1$. This allows us to consider f as a (locally integrable) element of $\dot{M}^1_1$, take Nf and estimate

$$\|Nf\|_1 \le \sum_j |\lambda_j| \|Na_j\|_1 \le C_t \sum_j |\lambda_j|.$$

Taking the infimum over all the atomic decompositions of f yields $\|Nf\|_1 \le C_t \|f\|_{\dot{HS}^1_{t,\text{ato}}}$.
□

Remark 3.2. *As pointed out in the proof, Proposition 3.1 remains valid if we take, for the definition of a* $(1,t)$*-atom, instead of condition* 3 *of Definition 2.11, condition* $3'$ *or* $3''$ *of Remarks 2.12.*

Now for the converse, that is, to prove that $\dot{M}^1_1 \subset \dot{HS}^1_{t,\text{ato}}$, we establish an atomic decomposition for functions $f \in \dot{M}^1_1$. To attain this goal, we need a Calderón-Zygmund decomposition for such functions. We refer to [2] for the original proof of the Calderón-Zygmund decomposition for Sobolev spaces on Riemannian manifolds.

Proposition 3.3 (Calderón-Zygmund decomposition)**.** *Let M be a complete Riemannian manifold satisfying (D). Let $f \in \dot{M}^1_1$, $\frac{s}{s+1} < q < 1$ and $\alpha > 0$. Then one can find a collection of balls $\{B_i\}_i$, functions $b_i \in \dot{W}^1_1$ and a Lipschitz function g such that the following properties hold:*

$$f = g + \sum_i b_i,$$

$$|\nabla g(x)| \le C\alpha \quad \text{for } \mu - a.e.\ x \in M, \tag{19}$$

$$\operatorname{supp} b_i \subset B_i,\ \|b_i\|_1 \le C\alpha\mu(B_i) r_i,\ \|\nabla b_i\|_q \le C\alpha\mu(B_i)^{\frac{1}{q}},$$

$$\sum_i \mu(B_i) \le \frac{C}{\alpha} \int N f d\mu, \tag{20}$$

and

$$\sum_i \chi_{B_i} \le K. \tag{21}$$

The constants C and K only depend on the constant in (D).

Proof. Let $f \in \dot{M}^1_1$, $\frac{s}{s+1} < q < 1$ and $\alpha > 0$. Consider the open set

$$\Omega = \{x : \mathcal{M}_q(Nf)(x) > \alpha\}.$$

If $\Omega = \emptyset$, then set

$$g = f \ , \quad b_i = 0 \text{ for all } i$$

so that (19) is satisfied according to the Lebesgue differentiation theorem. Otherwise

$$\begin{aligned}\mu(\Omega) &\leq \frac{C}{\alpha}\int_M \mathcal{M}_q(Nf)d\mu \\ &\leq \frac{C}{\alpha}\int_M (\mathcal{M}(Nf)^q)^{1/q}\,d\mu \\ &\leq \frac{C}{\alpha}\int_M Nf d\mu < \infty. \end{aligned} \tag{22}$$

We used the fact the $\mathcal{M}$ is $L_{1/q}$ bounded since $1/q > 1$ and Theorem 1.3. In particular $\Omega \neq M$ as $\mu(M) = +\infty$.

Let F be the complement of Ω. Since Ω is an open set distinct from M, let $\{\underline{B_i}\}_i$ be a Whitney decomposition of Ω (see [10]). That is, the $\underline{B_i}$ are pairwise disjoint, and there exist two constants $C_2 > C_1 > 1$, depending only on the metric, such that

1. $\Omega = \cup_i B_i$ with $B_i = C_1\underline{B_i}$, and the balls B_i have the bounded overlap property;
2. $r_i = r(B_i) = \frac{1}{2}d(x_i, F)$ and x_i is the center of B_i;
3. each ball $\overline{B_i} = C_2 B_i$ intersects F ($C_2 = 4C_1$ works).

For $x \in \Omega$, denote $I_x = \{i : x \in B_i\}$. By the bounded overlap property of the balls B_i, we have that $\sharp I_x \leq K$, and moreover, fixing $k \in I_x$, $\frac{1}{r_i} \leq r_k \leq 3r_i$ and $B_i \subset 7B_k$ for all $i \in I_x$.

Condition (21) is nothing but the bounded overlap property of the B_i's and (20) follows from (21) and (22). Note also that using the doubling property, we have

$$\int_{B_i} |Nf|^q d\mu \leq C\mu(B_i) \fint_{\overline{B_i}} |Nf|^q d\mu \leq \mu(B_i)\mathcal{M}_q^q(Nf)(y) \leq C\alpha^q\mu(B_i) \tag{23}$$

for some $y \in \overline{B_i} \cap F$, whose existence is guaranteed by property 3 of the Whitney decomposition.

Let us now define the functions b_i. For this, we construct a partition of unity $\{\chi_i\}_i$ of Ω subordinate to the covering $\{B_i\}_i$. Each χ_i is a Lipschitz function supported in B_i with $0 \leq \chi_i \leq 1$ and $\|\nabla\chi_i\|_\infty \leq \dfrac{C}{r_i}$ (see for example [13], p. 1908).

We set $b_i = (f - c_i)\chi_i$ where $c_i := \frac{1}{\chi_i(B_i)} \int_{B_i} f\chi_i d\mu$ and $\chi_i(B_i)$ means $\int_{B_i} \chi_i d\mu$, which is comparable to $\mu(B_i)$. Note that by the properties of the χ_i we have the trivial estimate

$$\|b_i\|_1 \leq \int_{B_i} |f - c_i| d\mu \leq \int_{B_i} |f| d\mu + \frac{\mu(B_i)}{\chi_i(B_i)} \int_{B_i} |f| d\mu \leq C \int \mathbb{1}_{B_i} |f| d\mu, \tag{24}$$

but we need a better estimate, as follows:

$$\begin{aligned}
\|b_i\|_1 &\leq \frac{1}{\chi_i(B_i)} \int_{B_i} \left| \int_{B_i} (f(x) - f(y))\chi_i(y) d\mu(y) \right| d\mu(x) \\
&\leq \frac{1}{\chi_i(B_i)} \int_{B_i} \int_{B_i} |f(x) - f(y)| d\mu(y) d\mu(x) \\
&\leq 2 \frac{\mu(B_i)}{\chi_i(B_i)} \int_{B_i} |f(x) - f_{B_i}| d\mu(x) \\
&\leq C r_i \left(\fint_{\overline{B_i}} |Nf|^q d\mu \right)^{1/q} \mu(B_i) \\
&\leq C r_i \mathcal{M}_q(Nf)(y) \mu(B_i) \\
&\leq C r_i \alpha \mu(B_i),
\end{aligned} \tag{25}$$

as in (23). Here we have used the Sobolev-Poincaré inequality (6) with $\lambda = 4$ and the fact that $q^* > 1$.

Together with the estimate on $\|b_i\|_1$, we use the fact that $|\nabla f|$ is in L_1 (see Proposition 2.6) to bound $\|\nabla b_i\|_1$ and conclude that $b_i \in W_1^1$:

$$\begin{aligned}
\|\nabla b_i\|_1 &\leq \int_{B_i} |f - c_i| |\nabla \chi_i| d\mu + \int_{B_i} |\nabla f| d\mu \\
&\leq C \frac{1}{r_i} r_i \mu(B_i) \left(\fint_{4B_i} |Nf|^q d\mu \right)^{1/q} + \int_{B_i} |\nabla f| d\mu \\
&\leq C\alpha\mu(B_i) + \int_{B_i} |\nabla f| d\mu < \infty.
\end{aligned} \tag{26}$$

Similarly, we can estimate b_i in the Sobolev space $\dot{W}_q^1$; note again that by Proposition 2.6, $|\nabla f|$ is in L_1 and can be bounded pointwise μ-a.e. by Nf:

$$\begin{aligned}
\|\nabla b_i\|_q &\leq \| \, |(f - c_i)\nabla\chi_i| \, \|_q + \| \, |\nabla f| \chi_i \|_q \\
&\leq \frac{\mu(B_i)^{\frac{1}{q}-1}}{\chi_i(B_i)} \int_{B_i} \int_{B_i} |f(x) - f(y)| \chi_i(y) |\nabla \chi_i(x)| d\mu(y) d\mu(x) + \left(\int_{B_i} |\nabla f|^q d\mu \right)^{1/q} \\
&\leq C \left(\fint_{\overline{B_i}} |Nf|^q d\mu \right)^{1/q} \mu(B_i)^{1/q} + \left(\int_{\overline{B_i}} |Nf|^q d\mu \right)^{1/q}
\end{aligned}$$

$$\leq C\alpha\mu(B_i)^{1/q} \tag{27}$$

by (23).

Set now $g = f - \sum_i b_i$. Since the sum is locally finite on Ω, g is defined almost everywhere on M and $g = f$ on F. Observe that g is a locally integrable function on M. Indeed, let $\varphi \in L_\infty$ with compact support. Since $d(x,F) \geq r_i$ for $x \in \text{supp } b_i$, we obtain

$$\int \sum_i |b_i|\,|\varphi|\,d\mu \leq \Big(\int \sum_i \frac{|b_i|}{r_i}\,d\mu\Big) \sup_{x\in M}\Big(d(x,F)|\varphi(x)|\Big)$$

Hence by (25) and the bounded overlap property,

$$\int \sum_i |b_i||\varphi|d\mu \leq C\alpha \sum_i \mu(B_i) \sup_{x\in M}\Big(d(x,F)|\varphi(x)|\Big) \leq CK\alpha\mu(\Omega) \sup_{x\in M}\Big(d(x,F)|\varphi(x)|\Big).$$

Since $f \in L_{1,loc}$, we conclude that $g \in L_{1,\text{loc}}$.

It remains to prove (19). Indeed, using the fact that on Ω we have $\sum \chi_i = 1$ and $\sum \nabla\chi = 0$, we get

$$\begin{aligned} \nabla g &= \nabla f - \sum_i \nabla b_i \\ &= \nabla f - (\sum_i \chi_i)\nabla f - \sum_i (f - c_i)\nabla\chi_i \\ &= 1\!\!1_F \nabla f - \sum_i (f - c_i)\nabla\chi_i. \end{aligned} \tag{28}$$

From Proposition 2.6, the definition of F and the Lebesgue differentiation theorem, we have that $1\!\!1_F|\nabla f| \leq 1\!\!1_F Nf \leq \alpha$, $\mu-$a.e. We claim that a similar estimate holds for

$$h = \sum_i (f - c_i)\nabla\chi_i,$$

i.e. $|h(x)| \leq C\alpha$ for all $x \in M$. For this, note first that by the properties of the balls B_i and the partition of unity, h vanishes on F and the sum defining h is locally finite on Ω. Then fix $x \in \Omega$ and let B_k be some Whitney ball containing x. Again using the fact that $\sum\limits_i \nabla\chi_i(x) = 0$, we can replace $f(x)$ by any constant in the sum above, so we can write

$$h(x) = \sum_{i\in I_x} \Big(\fint_{7B_k} f d\mu - c_i\Big) \nabla\chi_i(x).$$

For all $i,k \in I_x$, by the construction of the Whitney collection, the balls B_i and B_k have equivalent radii and $B_i \subset 7B_k$. Thus

$$\Big|c_i - \fint_{7B_k} f d\mu\Big| \leq \frac{1}{\chi_i(B_i)} \int_{B_i} \Big|f - \fint_{7B_k} f d\mu\Big| \chi_i d\mu$$

$$\lesssim \fint_{7B_k} |f - f_{7B_k}| d\mu$$
$$\lesssim r_k \left(\fint_{7\lambda B_k} |Nf|^q d\mu \right)^{1/q}$$
$$\lesssim \alpha r_k. \tag{29}$$

We used (D), (6) , $\chi_i(B_i) \simeq \mu(B_i)$ and (23) for $7B_k$. Hence

$$|h(x)| \lesssim \sum_{i \in I_x} \alpha r_k (r_i)^{-1} \leq CK\alpha. \tag{30}$$

□

Proposition 3.4. *Let M be a complete Riemannian manifold satisfying (D). Let $f \in \dot{M}_1^1$. Then for all $\frac{s}{s+1} < q < 1$, $q^* = \frac{sq}{s-q}$, there is a sequence of homogeneous $(1, q^*)$ Hardy-Sobolev atoms $\{a_j\}_j$, and a sequence of scalars $\{\lambda_j\}_j$, such that*

$$f = \sum_j \lambda_j a_j \quad \text{in } \dot{W}_1^1, \text{ and } \quad \sum |\lambda_j| \leq C_q \|f\|_{\dot{M}_1^1}.$$

Consequently, $\dot{M}_1^1 \subset \dot{HS}^1_{q^*,ato}$ with $\|f\|_{\dot{HS}^1_{q^*,\mathrm{ato}}} \leq C_q \|f\|_{\dot{M}_1^1}$.

Remark 3.5. *Note that for the inclusion $\dot{M}_1^1 \subset \dot{HS}^1_{q^*,ato}$, we do not need to assume any additional hypothesis, such as a Poincaré inequality, on the doubling manifold.*

Proof of Proposition 3.4. Let $f \in \dot{M}_1^1$. We follow the general scheme of the atomic decomposition for Hardy spaces, found in [29], Section III.2.3. For every $j \in \mathbb{Z}^*$, we take the Calderón-Zygmund decomposition, Proposition 3.3, for f with $\alpha = 2^j$. Then

$$f = g^j + \sum_i b_i^j$$

with b_i^j, g^j satisfying the properties of Proposition 3.3.

We want to write

$$f = \sum_{-\infty}^{\infty} (g^{j+1} - g^j) \tag{31}$$

in $\dot{W}_1^1$. First let us see that $g^j \to f$ in as $j \to \infty$. Indeed, since the sum is locally finite we can write

$$\|\nabla(g^j - f)\|_1 = \|\nabla(\sum_i b_i^j)\|_1 \leq \sum_i \|\nabla b_i^j\|_1.$$

By (26),

$$\sum_i \|\nabla b_i^j\|_1 \le CK2^j\mu(\Omega_j) + K\int_{\Omega_j} |\nabla f| d\mu$$
$$= I_j + II_j. \tag{32}$$

When $j \to \infty$, $I_j \to 0$ since $\sum_j 2^j \mu(\Omega_j) \approx \int \mathcal{M}_q(Nf)d\mu < \infty$. This also implies $\mathcal{M}_q(Nf)$ is finite μ-a.e., hence $\bigcap \Omega_j = \emptyset$ so $II_j \to 0$, since $|\nabla f| \in L_1$.

When $j \to -\infty$, we want to show $\|\nabla g_j\|_1 \to 0$. Breaking ∇g up as in (28), we know that

$$\int_{F^j} |\nabla g^j| = \int \mathbb{1}_{F^j}|\nabla f| \le \int_{\{Nf\le 2^j\}} Nf \to 0, \tag{33}$$

since $Nf \in L_1$. For the other part we have, by (30),

$$\int_{\Omega^j} |\nabla g^j| = \int |h(x)| \le CK2^j\mu(\Omega^j) \to 0 \tag{34}$$

from the convergence of $\sum 2^j\mu(\Omega^j)$, as above.

Denoting $g^{j+1} - g^j$ by ℓ^j, we have $\operatorname{supp} \ell^j \subset \Omega_j$ so using the partition of unity $\{\chi_k^j\}$ corresponding to the Whitney decomposition for Ω_j, we can write $f = \sum_{j,k} \ell^j\chi_k^j$ in $\dot{W}_1^1$. Let us compute $\|\ell^j\chi_k^j\|_{\dot{W}^1_{q^*}}$. We have

$$\nabla(\ell^j\chi_k^j) = (\nabla\ell^j)\chi_k^j + \ell^j\nabla\chi_k^j.$$

From the estimate $\|\nabla g^j\|_\infty \le C2^j$ it follows that $\left(\fint_{B_k^j} |\nabla\ell^j|^{q^*} d\mu\right)^{1/q^*} \le C2^j$, while

$$\ell^j\nabla\chi_k^j = \left(\sum_{i:B_k^j\cap B_i^j\neq\emptyset} (f - c_i^j)\chi_i^j - \sum_{l:B_k^j\cap B_l^{j+1}\neq\emptyset} (f - c_l^{j+1})\chi_l^{j+1}\right)\nabla\chi_k^j. \tag{35}$$

Observe that since $\Omega_{j+1} \subset \Omega_j$, for a fixed k, the balls B_l^{j+1} with $B_k^j \cap B_l^{j+1} \neq \emptyset$ must have radii $r_l^{j+1} \le cr_k^j$ for some constant c. Therefore $B_l^{j+1} \subset (B_k^j)' := (1 + 2c)B_k^j$. Moreover, by the properties of the Whitney balls, given $\lambda > 1$ we can take c sufficiently large so that $(B_k^j)'$ contains λB_i^j for all B_i^j intersecting B_k^j. Using this fact as well as (6) and (23), and proceeding in the same way as in the derivations of (25) and (29), we get

$$(r_k^j)^{q^*} \int_{B_k^j} |\ell^j\nabla\chi_k^j|^{q^*} d\mu \le K^{q^*-1}\int_{B_k^j} \Big(\sum_i \mathbb{1}_{B_i^j}|f - c_i^j|^{q^*} + \sum_l \mathbb{1}_{B_l^{j+1}}|f - c_l^{j+1}|^{q^*}\Big)d\mu$$
$$\le K^{q^*-1}\sum_{i:B_k^j\cap B_i^j\neq\emptyset} \int_{B_i^j} |f - c_i^j|^{q^*} d\mu$$

$$+ K^{q^*-1}\int_{(B_k^j)'}\sum_l \mathbb{1}_{B_l^{j+1}}|f-f_{(B_k^j)'}+f_{(B_k^j)'}-c_l^{j+1}|^{q^*}d\mu \tag{36}$$
$$\lesssim K^{q^*-1}\sum_{i:B_k^j\cap B_i^j\neq\emptyset}(r_i^j 2^j)^{q^*}\mu(B_i^j)+K^{q^*}(r_k^j 2^j)^{q^*}\mu((B_k^j)')$$
$$\lesssim K^{q^*}(r_k^j 2^j)^{q^*}\mu((B_k^j)').$$

Therefore

$$\left(\fint_{(B_k^j)'}|\ell^j\nabla\chi_k^j|^{q^*}d\mu\right)^{\frac{1}{q^*}}\le CK2^j. \tag{37}$$

The $\ell^j\chi_k^j$'s seem to be a good choice for our atoms but unfortunately they do not satisfy the cancellation condition. If we wanted to get atoms with property $3'$ (see Remarks 2.12) instead of the vanishing moment condition 3, we could use (25) to bound the L_1 norm of $\ell^j\chi_k^j$, then normalize as below. However, if we want to obtain the vanishing moment condition, we need to consider instead the following decomposition of the ℓ^j's: $\ell^j=\sum_k \ell_k^j$ with

$$\ell_k^j=(f-c_k^j)\chi_k^j-\sum_l(f-c_l^{j+1})\chi_l^{j+1}\chi_k^j+\sum_l c_{k,l}\chi_l^{j+1}, \tag{38}$$

where

$$c_{k,l}:=\frac{1}{\chi_l^{j+1}(B_l^{j+1})}\int_{B_l^{j+1}}(f-c_j^{l+1})\chi_l^{j+1}\chi_k^j d\mu.$$

First, this decomposition holds since $\sum_k\chi_k^j=1$ on the support of χ_l^{j+1} and $\sum_k c_{k,l}=0$. Furthermore, the cancellation condition

$$\int_M \ell_k^j d\mu=0$$

follows from the fact that $\int_M(f-c_k^j)\chi_k^j d\mu=0$ and the definition of $c_{k,l}$, which immediately gives $\int\left((f-c_l^{j+1})\chi_l^{j+1}\chi_k^j-c_{k,l}\chi_l^{j+1}\right)d\mu=0$.

Noting that ℓ_k^j is supported in the ball $(B_k^j)'$ (see above), let us estimate $\|\nabla\ell_k^j\|_{L_{q^*}((B_k^j)')}$. Write

$$\begin{aligned}\nabla\ell_k^j&=(\nabla f)\chi_k^j+(f-c_k^j)\nabla\chi_k^j-\sum_l(f-c_l^{j+1})\nabla\chi_l^{j+1}\chi_k^j\\&-\sum_l(f-c_l^{j+1})\chi_l^{j+1}\nabla\chi_k^j-(\nabla f)\mathbb{1}_{\Omega_{j+1}}\chi_k^j+\sum_l c_{k,l}\nabla\chi_l^{j+1}\\&=\nabla f(1-\mathbb{1}_{\Omega_{j+1}})\chi_k^j+((f-c_k^j)-\sum_l(f-c_l^{j+1})\chi_l^{j+1})\nabla\chi_k^j\\&-\sum_l(f-c_l^{j+1})\nabla\chi_l^{j+1}\chi_k^j+\sum_l c_{k,l}\nabla\chi_l^{j+1}.\end{aligned}$$

Since the first term, concerning the gradient of f, is supported in $B_k^j \cap F_{j+1}$, we can use Proposition 2.6, the definition of F_{j+1} and the Lebesgue differentiation theorem to bound it, namely

$$\int_{B_k^j} |\nabla f|^{q^*} d\mu \leq 2^{(j+1)q^*} \mu(B_k^j).$$

Recalling (35), we see that the estimate of the L_{q^*} norm of the second term is given by (37). The third term can be handled by the pointwise estimate (30):

$$\| \sum_l (f - c_l^{j+1}) \nabla \chi_l^{j+1} \chi_k^j \|_{q^*} \leq CK 2^{j+1} \mu(B_k^j)^{1/q^*}.$$

For $\sum_l c_{k,l} \nabla \chi_l^{j+1}$, note first that $c_{k,l} = 0$ when $B_k^j \cap B_l^{j+1} = \emptyset$ and $|c_{k,l}| \leq C 2^j r_l^{j+1}$ thanks to (25). By the properties of the partition of unity, this gives $|c_{k,l} \nabla \chi_l^{j+1}| \leq C 2^j$ for every l, and as the sum has at most K terms at each point we get the pointwise bound

$$| \sum_l c_{k,l} \nabla \chi_l^{j+1} | \leq CK 2^j,$$

from which it follows that

$$\| \sum_l c_{k,l} \nabla \chi_l^{j+1} \|_{q^*} \leq CK 2^j \mu((B_k^j)')^{1/q^*}.$$

Thus

$$\| \nabla \ell_k^j \|_{q^*} \leq \gamma 2^j \mu((B_k^j)')^{1/q^*}. \tag{39}$$

We now set $a_k^j = \gamma^{-1} 2^{-j} \mu((B_k^j)')^{-1} \ell_k^j$ and $\lambda_{j,k} = \gamma 2^j \mu((B_k^j)')$. Then $f = \sum_{j,k} \lambda_{j,k} a_k^j$, with a_k^j being $(1, q^*)$ homogeneous Hardy-Sobolev atoms and

$$\begin{aligned}
\sum_{j,k} |\lambda_{j,k}| &= \gamma \sum_{j,k} 2^j \mu((B_k^j)') \\
&\leq \gamma' \sum_{j,k} 2^j \mu(\underline{B_k^j}) \\
&\leq \gamma' \sum_j 2^j \mu(\{x : \mathcal{M}_q(Nf)(x) > 2^j\}) \\
&\leq C \int \mathcal{M}_q(Nf) d\mu \\
&\leq C_q \|Nf\|_1 \sim \|f\|_{\dot{M}_1^1}.
\end{aligned}$$

We used that $\mu((B_k^j)') \sim \mu(\underline{B_k^j})$ thanks to (D), and the fact that the $\underline{B_k^j}$ are disjoint.

$\square$

Remark 3.6. *As pointed out in the proof following (37), we can get an atomic decomposition as in Proposition 3.4, but replacing the vanishing moment condition* 3 *of the atoms from Definition 2.11 by condition* $3'$ *in Remarks 2.12. This does not assume a Poincaré inequality.*

Conclusion: Let M be a complete Riemannian manifold satisfying (D). Then

1. for all $\frac{s}{s+1} < q < 1$,
$$\dot{M}_1^1 \subset H\dot{S}^1_{q^*,\mathrm{ato}}.$$
2. (Theorem 1.5) If moreover we assume (P_1), then
$$\dot{M}_1^1 = H\dot{S}^1_{t,\mathrm{ato}}$$
for all $t > 1$.

4 The nonhomogeneous case

We begin by recalling the definitions of the nonhomogeneous versions of the spaces considered above.

Definition 4.1. ([16]) *Let* $1 \leq p \leq \infty$. *The Sobolev space* M_p^1 *is the set of all functions* $u \in L_p$ *such that there exists a measurable function* $g \geq 0$, $g \in L_p$, *satisfying*

$$|u(x) - u(y)| \leq d(x,y)(g(x) + g(y)) \ \mu - a.e. \tag{40}$$

That is, $M_p^1 = L_p \cap \dot{M}_p^1$. *We equip* M_p^1 *with the norm*

$$\|u\|_{M_p^1} = \|u\|_p + \inf_{g \textit{ satisfies } (40)} \|g\|_p.$$

From Theorem 1.3, we deduce that for $1 \leq p \leq \infty$,

$$M_p^1 = \{f \in L_p : \ Nf \in L_p\}$$

with equivalent norm

$$\|f\|_{M_p^1} = \|f\|_p + \|Nf\|_p.$$

Definition 4.2. *We define the Hardy-Sobolev space* $\widetilde{M_1^1}$ *as the set of all functions* $u \in H_{1,\max}$ *such that there exists a measurable function* $g \geq 0$, $g \in L_1$, *satisfying*

$$|u(x) - u(y)| \leq d(x,y)(g(x) + g(y)) \ \mu - a.e. \tag{41}$$

We equip $\widetilde{M_1^1}$ *with the norm*

$$\|u\|_{\widetilde{M_1^1}} = \|u^+\|_1 + \inf_{g \textit{ satisfies } (41)} \|g\|_1.$$

We have $\widetilde{M_1^1} = H_{1,\max} \cap \dot{M}_1^1$.

Again by Theorem 1.3,

$$\widetilde{M_1^1} = \{f \in H_{1,\max} : \ Nf \in L_1\},$$

with equivalent norm

$$\|f\|_{\widetilde{M_1^1}} = \|f^+\|_1 + \|Nf\|_1.$$

By (12) and Corollary 2.7, we have

$$\widetilde{M_1^1} \subset M_1^1 \subset W_1^1.$$

In [5], the authors also defined the nonhomogeneous atomic Hardy-Sobolev spaces. Let us recall their definition.

Definition 4.3 ([5])**.** *For $1 < t \leq \infty$, we say that a function a is a nonhomogeneous Hardy-Sobolev $(1,t)$-atom if*

1. *a is supported in a ball B,*
2. *$\|a\|_{W_t^1} := \|a\|_t + \|\nabla a\|_t \leq \mu(B)^{-\frac{1}{t'}}$,*
3. *$\int a d\mu = 0$.*

They then define, for every $1 < t \leq \infty$, the nonhomogeneous Hardy-Sobolev space $HS^1_{t,\mathrm{ato}}$ as follows: $f \in HS^1_{t,\mathrm{ato}}$ if there exists a sequence of nonhomogeneous Hardy-Sobolev $(1,t)$-atoms $\{a_j\}_j$ such that $f = \sum_j \lambda_j a_j$ with $\sum_j |\lambda_j| < \infty$. This space is equipped with the norm

$$\|f\|_{HS^1_{t,\mathrm{ato}}} := \inf \sum_j |\lambda_j|,$$

where the infimum is taken over all such decompositions.

We also recall the following comparison between these atomic Hardy-Sobolev spaces.

Theorem 4.4. ([5]) *Let M be a complete Riemannian manifold satisfying (D) and a Poincaré inequality (P_q) for some $q > 1$. Then $HS^1_{t,\mathrm{ato}} \subset HS^1_{\infty,\mathrm{ato}}$ for every $t \geq q$ and therefore $HS^1_{t_1,\mathrm{ato}} = HS^1_{t_2,\mathrm{ato}}$ for every $q \leq t_1, t_2 \leq \infty$.*

4.1 Atomic decomposition of $\widetilde{M_1^1}$ and comparison with $HS^1_{t,\mathrm{ato}}$

As in the homogeneous case, under the Poincaré inequality (P_1), $HS^1_{t,\mathrm{ato}} \subset \widetilde{M_1^1}$:

Proposition 4.5. *Let M be a complete Riemannian manifold satisfying (D) and (P_1). Let $1 < t \leq \infty$ and a be a nonhomogeneous Hardy-Sobolev $(1,t)$-atom. Then $a \in \widetilde{M_1^1}$ with $\|a\|_{\widetilde{M_1^1}} \leq C_t$, the constant depending only on t, the doubling constant and the constant appearing in (P_1) , but not on a. Consequently $HS^1_{t,\mathrm{ato}} \subset \widetilde{M_1^1}$ with*

$$\|f\|_{\widetilde{M_1^1}} \leq C_t \|f\|_{HS^1_{t,\mathrm{ato}}}.$$

Proof. The proof follows analogously to that of Proposition 3.1, noting that in the nonhomogeneous case every Hardy-Sobolev $(1,t)$-atom a is an H_1 atom and so by (14) is in $H_{1,\max}$ with norm bounded by a constant. $\square$

Now for the converse, that is, to prove that $\widetilde{M_1^1} \subset HS^1_{t,\mathrm{ato}}$, we establish, as in the homogeneous case, an atomic decomposition for functions $f \in \widetilde{M_1^1}$ using a Calderón-Zygmund decomposition for such functions.

Proposition 4.6 (Calderón-Zygmund decomposition). *Let M be a complete Riemannian manifold satisfying (D). Let $f \in \widetilde{M_1^1}$, $\frac{s}{s+1} < q < 1$ and $\alpha > 0$. Then one can find a collection of balls $\{B_i\}_i$, functions $b_i \in W_1^1$ and a Lipschitz function g such that the following properties hold:*

$$f = g + \sum_i b_i,$$

$$|g(x)| + |\nabla g(x)| \leq C\alpha \quad \textit{for } \mu - a.e\ x \in M,$$

$$\operatorname{supp} b_i \subset B_i,\ \|b_i\|_1 \leq C\alpha\mu(B_i)r_i,\ \|\, b_i + |\nabla b_i|\,\|_q \leq C\alpha\mu(B_i)^{\frac{1}{q}},$$

$$\sum_i \mu(B_i) \leq \frac{C}{\alpha}\int (f^+ + Nf)d\mu,$$

$$\textit{and} \quad \sum_i \chi_{B_i} \leq K.$$

The constants C and K only depend on the constant in (D).

Proof. The proof follows the same steps as that of Proposition 3.3. We will only mention the changes that occur due to the nonhomogeneous norm. Let $f \in \widetilde{M_1^1}$, $\frac{s}{s+1} < q < 1$ and $\alpha > 0$. The first change is that we consider the open set

$$\Omega = \{x : \mathcal{M}_q(f^+ + Nf)(x) > \alpha\}.$$

We define, as in the homogeneous case, the partition of unity χ_i corresponding to the Whitney decomposition $\{\underline{B_i}\}_i$ of Ω, the functions $b_i = (f - c_i)\chi_i$ with $c_i := \frac{1}{\chi_i(B_i)}\int_{B_i} f\chi_i d\mu$, and $g = f - \sum b_i$. In addition to the previous estimates (25) - (27) for b_i and ∇b_i, we need here to estimate $\|b_i\|_q$.

We begin by showing that for $x \in \Omega$,

$$|c_i| \leq C\alpha \tag{42}$$

for every $i \in I_x$. Set $\varphi_i = \gamma\frac{\chi_i}{\chi_i(B_i)}$. From the properties of χ_i, in particular since $\chi_i(B_i) \approx \mu(B_i)$, we see that we can choose γ (independent of i) so that $\varphi_i \in \mathcal{T}_1(y)$ and thus

$$|c_i| \leq \gamma^{-1}f^+(y) \text{ for all } y \in B_i.$$

Recall that the ball $\overline{B_i} = C_2 B_i$ has nonempty intersection with F. Taking $y_0 \in F \cap \overline{B_i}$, we get, by integrating the inequality above,

$$|c_i| \leq \gamma^{-1} \left(\fint_{B_i} (f^+)^q d\mu \right)^{\frac{1}{q}} \leq C \left(\fint_{\overline{B_i}} (f^+)^q d\mu \right)^{\frac{1}{q}} \leq C\mathcal{M}_q(f^+)(y_0) \leq C\alpha.$$

Combining this with (12), we have

$$\|b_i\|_q \leq \left(\int_{B_i} |f - c_i|^q \right)^{\frac{1}{q}} \leq \left(\fint_{\overline{B_i}} |f^+|^q d\mu \right)^{\frac{1}{q}} \mu(B_i)^{\frac{1}{q}} + |c_i| \mu(B_i)^{\frac{1}{q}} \leq C\alpha\mu(B_i)^{\frac{1}{q}}.$$

For g, we need to prove that $\|g\|_\infty \leq C\alpha$. We have

$$g = f\mathbb{1}_F + \sum_i c_i \chi_i. \tag{43}$$

For the first term we have $|f| \leq f^+ \leq \mathcal{M}_q(f^+)$ at all Lebesgue points and thus $|f\mathbb{1}_F| \leq \alpha$ μ-a.e. For the second term, thanks to the bounded overlap property and (42), we get the desired estimate. □

Proposition 4.7. *Let M be a complete Riemannian manifold satisfying (D). Let $f \in \widetilde{M}_1^1$. Then for all $\frac{s}{s+1} < q < 1$, there is a sequence of $(1, q^*)$ $(q^* = \frac{sq}{s-q})$ nonhomogeneous atoms $\{a_j\}_j$, and a sequence of scalars $\{\lambda_j\}_j$, such that*

$$f = \sum_j \lambda_j a_j \quad \text{in } W_1^1, \text{ and} \quad \sum |\lambda_j| \leq C_q \|f\|_{\widetilde{M}_1^1}.$$

Consequently, $\widetilde{M}_1^1 \subset HS^1_{q^*,ato}$ with $\|f\|_{HS^1_{q^*,\text{ato}}} \leq C_q \|f\|_{\widetilde{M}_1^1}$.

Proof. Again, we will only mention the additional properties that one should verify in comparison with the proof of Proposition 3.4.

First let us see that (31) holds in the nonhomogeneous Sobolev space W_1^1. We already showed convergence in the homogeneous $\dot{W}_1^1$ norm so we only need to verify convergence in L_1. By (24)

$$\|g^j - f\|_1 \leq \sum_i \|b_i^j\|_1 \leq C \sum_i \int \mathbb{1}_{B_i^j} |f| d\mu \leq CK \int_{\Omega_j} |f| d\mu \to 0, \tag{44}$$

as $j \to \infty$. Here we've used the properties of the χ_i^j, the bounded overlap property of the B_i^j, the fact that $f \in L_1$ and that $\bigcap \Omega_j = \emptyset$ since $\mathcal{M}_q(f^+ + Nf)$ is finite μ-a.e.

Taking now $j \to -\infty$, we write, by (43), (42), and the bounded overlap property

$$\int |g^j| \leq \int_{F^j} |f| + \int \sum_i |c_i^j| \chi_i^j \leq \int_{\{\mathcal{M}_q(f^+) \leq 2^j\}} \mathcal{M}_q(f^+) + CK 2^j |\Omega^j| \to 0. \tag{45}$$

For the functions $\ell^j = g^{j+1} - g^j$, we have

$$\|\ell^j \chi_k^j\|_{q^*} \leq C2^j \mu(B_k^j)^{\frac{1}{q^*}}$$

since by Proposition 4.6, $\|g^j\|_\infty \leq C2^j$. This estimate also applies when we replace $\ell^j \chi_k^j$ by the moment-free "pre-atoms"

$$\begin{aligned}
\ell_k^j &:= (f - c_k^j)\chi_k^j - \sum_l (f - c_l^{j+1})\chi_l^{j+1}\chi_k^j + \sum_l c_{k,l}\chi_l^{j+1} \\
&= f(1 - \sum_l \chi_l^{j+1})\chi_k^j + c_k^j\chi_k^j + \sum_l c_l^{j+1}\chi_l^{j+1}\chi_k^j + \sum_l c_{k,l}\chi_l^{j+1}.
\end{aligned}$$

The first term, involving f, is $f\mathbb{1}_{F_{j+1}}\chi_k^j$ which is bounded by 2^{j+1} since $|f| \leq f^+ \leq \mathcal{M}_q(f^+)$ μ-a.e. For the second and third terms, we use (42) and the bounded overlap property of the B_l^{j+1}. Finally, that

$$|c_{k,l}| = \left| \frac{1}{\chi_l^{j+1}(B_l^{j+1})} \int_{B_l^{j+1}} (f - c_j^{l+1})\chi_l^{j+1}\chi_k^j d\mu \right| \leq c2^j$$

follows by arguing as in the proof of (42), since $\frac{\chi_l^{j+1}\chi_k^j}{\chi_l^{j+1}(B_l^{j+1})}$ can be considered as a multiple of some $\varphi \in \mathcal{T}_1(x)$ for every $x \in \overline{B_l^{j+1}}$, due to the fact that $|\nabla\chi_k^j| \lesssim (r_k^j)^{-1} \lesssim (r_l^{j+1})^{-1}$ when $B_l^{j+1} \cap B_k^j \neq \emptyset$.

Thus we obtain the stronger L_∞ estimate

$$\|\ell_k^j\|_\infty \leq C2^j \tag{46}$$

from which we conclude, as ℓ_k^j is supported in the ball $(B_k^j)' = (1 + 2c)B_k^j$, that $\|\ell_k^j\|_{q^*} \leq C2^j\mu(B_k^j)^{\frac{1}{q^*}}$.

The rest of the proof is exactly the same as that of Proposition 3.4. $\square$

Now we can state the converse inclusion from Theorem 2.10:

Corollary 4.8. *Let M be a complete Riemannian manifold satisfying (D). Then*

$$H_{1,\max}(M) \subset H_{1,\text{ato}}(M)$$

with

$$\|f\|_{H_{1,\text{ato}}} \lesssim \|f^+\|_1,$$

for any choice of t in the definition of H_1 atoms, $1 < t \leq \infty$, with a constant independent of t.

Proof. Assuming $f^+ \in L_1$ and letting

$$\Omega_j = \{x : \mathcal{M}_q(f^+)(x) > 2^j\},$$

we follow the steps outlined in the proofs of Propositions 4.6 and 4.7, which use only the maximal function f^+, while ignoring the estimates on the gradients from the proofs of Proposition 3.3 and 3.4, which are the only ones involving Nf. From the L_∞ bound (46) we are able to obtain atoms satisfying the conditions of Definition 2.9 with $t = \infty$, hence for every other t with uniform bounds. $\square$

Conclusion: Let M be a complete Riemannian manifold satisfying (D). Then

1. for all $\frac{s}{s+1} < q < 1$,
$$\widetilde{M}_1^1 \subset HS^1_{q^*,\text{ato}}.$$

2. If we moreover assume (P_1), then
$$\widetilde{M}_1^1 = HS^1_{t,\text{ato}}$$
for all $t > 1$.

4.2 Atomic decomposition for the Sobolev space M_1^1

For this we need to define new nonhomogeneous atomic spaces $LS^1_{t,\text{ato}}$, where the L is used to indicate that the atoms will now be in L_1 but not necessarily in H_1. Let us define our atoms.

Definition 4.9. *For $1 < t \leq \infty$, we say that a function a is an $LS^1_{t,\text{ato}}$-atom if*

1. *a is supported in a ball B;*
2. *$\|\nabla a\|_t \leq \mu(B)^{-\frac{1}{t'}}$; and*
3. *$\|a\|_1 \leq \min(1, r(B))$.*

We then say that f belongs to $LS^1_{t,\text{ato}}$ if there exists a sequence of $LS^1_{t,\text{ato}}$-atoms $\{a_j\}_j$ such that $f = \sum_j \lambda_j a_j$ in W_1^1, with $\sum_j |\lambda_j| < \infty$. This space is equipped with the norm

$$\|f\|_{LS^1_{t,\text{ato}}} = \inf \sum_j |\lambda_j|,$$

where the infimum is taken over all such decompositions.

Remark 4.10. *As discussed previously, condition* 3 *in Definition 4.9 is a substitute for the cancellation condition* 3 *in Definition 2.11. Assuming a Poincaré inequality* (P_t)*,* $LS^1_{t,\mathrm{ato}}$*-atoms corresponding to small balls (with* $r(B)$ *bounded above) can be shown (see [11], Appendix B) to be elements of Goldberg's local Hardy space (defined by restricting the supports of the test functions in Definition 2.8 to balls of radii* $r < R$ *for some fixed* R *- see [29], Section III.5.17), so that* $LS^1_{t,\mathrm{ato}}$ *is a subset of the "localized" space* $H_{1,\mathrm{loc}}$*.*

As in the homogeneous case, under the Poincaré inequality (P_1), $LS^1_{t,\mathrm{ato}} \subset M^1_1$:

Proposition 4.11. *Let* M *be a complete Riemannian manifold satisfying* (D) *and* (P_1)*. Let* $1 < t \leq \infty$ *and* a *be an* $LS^1_{t,\mathrm{ato}}$*-atom. Then* $a \in M^1_1$ *with* $\|a\|_{M^1_1} \leq C_t$*, the constant* C *depending only on* t*, the doubling constant and the constant appearing in* (P_1)*, and independent of* a*.*

Consequently $LS^1_{t,\mathrm{ato}} \subset M^1_1$ *with*

$$\|f\|_{M^1_1} \leq C_t \|f\|_{LS^1_{t,\mathrm{ato}}}.$$

Proof. The proof follows analogously to that of Proposition 3.1, noting that we can use Remark 3.2 thanks to property 3 in Definition 4.9, and that this property also implies every atom a is in L_1. □

Now for the converse, that is, to prove that $M^1_1 \subset LS^1_{t,\mathrm{ato}}$, we again establish an atomic decomposition for functions $f \in M^1_1$. In order to do that we must introduce an equivalent maximal function $f^\star$, which is a variant of the one originally defined by Calderón [6] and denoted by $N(f,x)$ (here we are only defining it in the special case $q = 1$ and $m = 1$, where for x a Lebesgue point of f, the constant $P(x,y)$ in Calderón's definition is equal to $f(x)$, and we are allowing for the balls not to be centered at x):

Definition 4.12. *Let* $f \in L_{1,\mathrm{loc}}(M)$*. Suppose* x *is a Lebesgue point of* f*, i.e.*

$$\lim_{r \to 0} \fint_{B(x,r)} |f(y) - f(x)| d\mu(y) = 0.$$

We define

$$f^\star(x) := \sup_{B:\, x \in B} \frac{1}{r(B)} \fint_B |f(y) - f(x)| d\mu(y).$$

Then $f^\star$ *is defined* μ*-almost everywhere.*

We now show the equivalence of $f^\star$ and Nf. As discussed in the Introduction, the following Proposition was proved in [12] (see also [28]) in the Euclidean case:

Proposition 4.13. *Let M be a complete Riemannian manifold satisfying (D). Then, there exist constants C_1, C_2 such that for all $f \in L_{1,\mathrm{loc}}(M)$*

$$C_1 Nf \leq f^\star \leq C_2 Nf$$

pointwise μ-almost everywhere.

Proof. Let $f \in L_{1,\mathrm{loc}}$ and x be a Lebesgue point of f, so that there exists a sequence of balls $B_n = B(x, r_n)$ with $r_n \to 0$ and $f_{B_n} \to f(x)$. Given a ball B containing x, take n sufficiently large so that $B_n \subset B$. Since $x \in B$, there is a smallest $k \geq 1$ such that $2^k B_n = B(x, 2^k r_n) \supset B$, and for this k we have $2^k r_n \leq 4r(B)$, so

$$\begin{aligned}
|f_B - f_{B_n}(x)| &\leq \fint_B |f - f_{2^k B_n}| d\mu + \sum_{j=1}^{k} |f_{2^j B_n} - f_{2^{j-1} B_n}| \\
&\leq \frac{\mu(2^k B_n)}{\mu(B)} \fint_{2^k Bn} |f - f_{2^k B_n}| d\mu + \sum_{j=1}^{k} \frac{\mu(2^j B_n)}{\mu(2^{j-1} B_n)} \fint_{2^j B_n} |f - f_{2^j B_n}| d\mu \\
&\leq 2C_{(\mathrm{D})}^2 \sum_{j=1}^{k} 2^j r_n Nf(x) \\
&\leq 16 C_{(\mathrm{D})}^2 r(B) Nf(x).
\end{aligned}$$

Taking the limit as $n \to \infty$, we see that $|f_B - f(x)| \leq Cr(B)Nf(x)$ so that

$$\fint_B |f(y) - f(x)| d\mu(y) \leq \fint_B |f(y) - f_B| d\mu(y) + |f_B - f(x)| \leq Cr(B)Nf(x).$$

Dividing by $r(B)$ and taking the supremum over all balls B containing x, we conclude that $f^\star(x) \leq CNf(x)$.

For the converse, again take any Lebesgue point x and let B be a ball containing x. Writing $|f(y) - f_B| \leq |f(y) - f(x)| + |\fint_B f - f(x)|$, we have

$$\fint_B |f(y) - f_B| d\mu(y) \leq 2 \fint_B |f(y) - f(x)| d\mu(y) \leq 2r(B) f^\star(x).$$

Taking the supremum over all balls B containing x, we deduce that $Nf(x) \leq 2f^\star(x)$. $\square$

Proposition 4.14 (Calderón-Zygmund decomposition)**.** *Let M be a complete Riemannian manifold satisfying (D). Let $f \in M_1^1$, $\frac{s}{s+1} < q < 1$ and $\alpha > 0$. Then one can find a collection of balls $\{B_i\}_i$, functions $b_i \in W_1^1$ and a Lipschitz function g such that the following properties hold:*

$$f = g + \sum_i b_i,$$

$$|g(x)| + |\nabla g(x)| \leq C\alpha \quad for\, \mu - a.e\; x \in M, \tag{47}$$

$$\operatorname{supp} b_i \subset B_i,\ \|b_i\|_1 \leq C\alpha\mu(B_i)r_i,\ \|\, b_i + |\nabla b_i|\, \|_q \leq C\alpha\mu(B_i)^{\frac{1}{q}}, \tag{48}$$

$$\sum_i \mu(B_i) \leq \frac{C_q}{\alpha} \int (|f| + Nf) d\mu, \tag{49}$$

$$and \quad \sum_i \chi_{B_i} \leq K. \tag{50}$$

The constants C and K only depend on the constant in (D).

Proof. The proof follows the same steps as that of Propositions 3.3 and 4.6. Again we will only mention the changes that occur. Let $f \in M_1^1$, $\frac{s}{s+1} < q < 1$ and $\alpha > 0$. By Proposition 4.13, we have $f^\star \in L_1$ with norm equivalent to $\|Nf\|_1$. Thus if we consider the open set

$$\Omega = \{x :\, \mathcal{M}_q(|f| + f^\star)(x) > \alpha\},$$

its Whitney decomposition $\{B_i\}_i$, and the corresponding partition of unity $\{\chi_i\}_i$, we get immediately (50) and (49) by the bounded overlap property and the boundedness of the maximal function in $L_{1/q}$.

We again define $b_i = (f - c_i)\chi_i$ but this time we set $c_i = f(x_i)$ for some $x_i \in \overline{B_i}$ chosen as follows. Recall that $\overline{B_i} = 4B_i$ contains some point y of $F = M \setminus \Omega$ so that

$$\fint_{\overline{B_i}} |f|^q \leq \mathcal{M}_q(f)^q(y) \leq \alpha^q \tag{51}$$

as well as

$$\fint_{\overline{B_i}} (f^\star)^q \leq \mathcal{M}_q(f^\star)^q(y) \leq \alpha^q. \tag{52}$$

Let

$$E_i = \{x \in \overline{B_i} : x \text{ is a Lebesgue point of } f \text{ and } |f|^q, \text{ and } |f(x)| \leq 2\alpha\}.$$

We claim that

$$\mu(E_i) \geq (1 - 2^{-q})\mu(\overline{B_i}).$$

Otherwise we would have $\mu(\overline{B_i} \setminus E_i) > 2^{-q}\mu(\overline{B_i})$ and so, since f and $|f|^q$ are locally integrable and the set of points which are not their Lebesgue points has measure zero,

$$\int_{\overline{B_i}\setminus E_i} |f|^q \geq (2\alpha)^q \mu(\overline{B_i} \setminus E_i) > \alpha^q \mu(\overline{B_i}),$$

contradicting (51).

Now we claim that for an appropriate constant c_q (to be chosen independent of i and α), there exists a point $x_i \in E_i$ with

$$f^\star(x_i) \leq c_q \alpha. \tag{53}$$

Again, suppose not. Then we have, by (52),

$$(c_q\alpha)^q\mu(E_i) \le \int_{E_i} (f^\star)^q d\mu \le \alpha^q \mu(\overline{B_i}),$$

implying that $\mu(E_i) \le c_q^{-q}\mu(\overline{B_i})$. Taking $c_q > (1-2^{-q})^{-1/q}$, we get a contradiction.

Thanks to our choice of x_i, we now have

$$|c_i| = |f(x_i)| \le 2\alpha$$

and

$$\|b_i\|_1 \le C\int_{B_i} |f(y)-f(x_i)|d\mu(y) \le C\mu(B_i)r_i f^\star(x_i) \le Cc_q r_i \alpha\mu(B_i).$$

Moreover for $\|b_i\|_q$, one has, by (51),

$$\|b_i\|_q \le C\left(\int_{B_i} |f-c_i|^q d\mu\right)^{\frac{1}{q}} \le C\left(\int_{B_i} |f|^q d\mu\right)^{\frac{1}{q}} + C2\alpha\mu(B_i)^{\frac{1}{q}} \le C\alpha\mu(B_i)^{\frac{1}{q}}.$$

Finally, for ∇b_i, we can estimate the L_1 norm by

$$\begin{aligned}
\|\nabla b_i\|_1 &\le \|(f-c_i)\nabla\chi_i\|\|_1 + \|(\nabla f)\chi_i\|_1 \\
&\le \int_{B_i} |f(x)-f(x_i)||\nabla\chi_i(x)|d\mu(x) + \int_{B_i} |\nabla f| d\mu \\
&\le C\mu(B_i) f^\star(x_i) + \int_{B_i} |\nabla f| d\mu \\
&\le Cc_q\alpha\mu(B_i) + \int_{B_i} |\nabla f| d\mu, \qquad (54)
\end{aligned}$$

showing (since $|\nabla f|$ in L_1 by Proposition 2.6) that $b_i \in W_1^1$, and the L_q norm by

$$\begin{aligned}
\|\nabla b_i\|_q^q &\le \|(f-c_i)\nabla\chi_i\|\|_q^q + \|(\nabla f)\chi_i\|_q^q \\
&\le \mu(B_i)^{1-q}\left(\int_{B_i} |f(x)-f(x_i)||\nabla\chi_i(x)|d\mu(x)\right)^q + \int_{B_i} |\nabla f|^q d\mu \\
&\le C\mu(B_i) f^\star(x_i)^q + \int_{\overline{B_i}} |Nf|^q d\mu \\
&\le C(c_q\alpha)^q\mu(B_i) + \int_{\overline{B_i}} |f^\star|^q d\mu \\
&\le C\alpha^q\mu(B_i),
\end{aligned}$$

where we used Propositions 2.6 and 4.13, and (52). Taking the $1/q$-th power on both sides, we get (48).

It remains to prove (47). First note that $\|g\|_\infty \le C\alpha$ since

$$g = f\mathbb{1}_F + \sum_i c_i\chi_i$$

and for the first term, by the Lebesgue differentiation theorem, we have $|f\mathbb{1}_F| \le \mathcal{M}_q(f)\mathbb{1}_F \le \alpha$ μ-a.e., while for the second term, thanks to the bounded overlap property and $|c_i| \le 2\alpha$, we get the desired estimate.

Now for the gradient, we write, as in (28),

$$\nabla g = \mathbb{1}_F(\nabla f) - \sum_i (f - f(x_i))\nabla\chi_i.$$

Again we have, by Propositions 2.6 and 4.13, that $\mathbb{1}_F(|\nabla f|) \le C\mathbb{1}_F(Nf) \le C\mathbb{1}_F(f^\star) \le C\alpha$ $\mu-$a.e. Let

$$h = \sum_i (f - f(x_i))\nabla\chi_i.$$

We will show $|h(x)| \le C\alpha$ for all $x \in M$. Note first that the sum defining h is locally finite on Ω and vanishes on F. Then take $x \in \Omega$ and a Whitney ball B_k containing x. As before, since $\sum_i \nabla\chi_i(x) = 0$, we can replace $f(x)$ in the sum by any constant so

$$h(x) = \sum_{i\in I_x} \left(f(x_k) - f(x_i)\right) \nabla\chi_i(x).$$

Recall that for all $i, k \in I_x$, by the construction of the Whitney collection, the balls B_i and B_k have equivalent radii and $B_i \subset 7B_k$. Thus

$$\begin{aligned}|f(x_k) - f(x_i)| &\le |f_{7B_k} - f(x_k)| + |f_{7B_k} - f(x_i)| \\ &\le \fint_{7B_k} |f - f(x_k)|d\mu + \fint_{7B_k} |f - f(x_i)|d\mu \\ &\le 7r_k(f^\star(x_k) + f^\star(x_i)) \le 14r_k c_q\alpha,\end{aligned} \tag{55}$$

by (53). Therefore we again get the estimate (30).

$\square$

Proposition 4.15. *Let M be a complete Riemannian manifold satisfying (D). Let $f \in M_1^1$. Then for all $\frac{s}{s+1} < q < 1$, there is a sequence of $LS^1_{q^*,\mathrm{ato}}$-atoms $\{a_j\}_j$ ($q^* = \frac{sq}{s-q}$), as in Definition 4.9, and a sequence of scalars $\{\lambda_j\}_j$, such that*

$$f = \sum_j \lambda_j a_j \quad \textit{in } W_1^1, \textit{ and } \quad \sum |\lambda_j| \le C_q\|f\|_{M_1^1}.$$

Consequently, $M_1^1 \subset HS^1_{q^*,ato}$ with $\|f\|_{LS^1_{q^*,\mathrm{ato}}} \le C_q\|f\|_{M_1^1}$.

Proof. Here as well we will only mention the additional properties that one should verify in comparison with Proposition 3.4 and 4.7. We use the Calderón-Zygmund decomposition (Proposition 4.14) above with Ω^j corresponding to $\alpha = 2^j$, and denote the resulting functions by g^j and b_i^j, recalling that for the definition of the constant c_i^j we have $c_i^j = f(x_i^j)$ for a specially chosen point $x_i^j \in \overline{B_i^j}$.

First let us see that $g^j \to f$ in W_1^1. For the convergence in L_1 we just repeat (44) and (45) from the nonhomogeneous case, replacing f^+ by $|f|$. For the convergence in $\dot{W}_1^1$, we can estimate $\sum_i \|\nabla b_i^j\|_1$ exactly as in (32), using (54) instead of (26), and replacing Nf by $f^\star$ and $\mathcal{M}_q(Nf)$ by $\mathcal{M}_q(|f| + f^\star)$. This gives $\nabla g^j \to \nabla f$ in L_1 as $j \to \infty$. For the convergence of ∇g^j to 0 as $j \to -\infty$, we imitate (33) and (34), using (28) and (30) with $f^\star$ and our new choice of c_i^j.

We define the functions $\ell^j = g^{j+1} - g^j$ as in Proposition 3.4 but this time we just use

$$\ell_k^j := \ell^j \chi_k^j$$

for the "pre-atoms", since we no longer need to have the moment condition $\int \ell_k^j = 0$ (see Remark 3.6). From the L_∞ bounds (47) on g^j and ∇g^j in Proposition 4.14, we immediately get

$$\|\ell_k^j\|_1 \le C 2^j \mu(B_k^j)$$

and $\| \, |\nabla \ell^j| \chi_k^j\|_{q^*} \le C 2^j \mu(B_k^j)^{1/q^*}$. We need a similar estimate on $\|\ell^j |\nabla \chi_k^j|\|_{q^*}$ in order to bound $\|\nabla \ell_k^j\|_{q^*}$. As in (36), write

$$r_k^j \left(\fint_{B_k^j} |\ell^j \nabla \chi_k^j|^{q^*} d\mu \right)^{1/q^*} \le C \left(\fint_{B_k^j} \Big(\sum_i \mathbb{1}_{B_i^j} |f - c_i^j| + \sum_l \mathbb{1}_{B_l^{j+1}} |f - c_l^{j+1}| \Big)^{q^*} d\mu \right)^{1/q^*}$$

Expanding $|f - c_i^j| = |f - f_{B_k^j} + f_{B_k^j} - c_k^j + c_k^j - c_i^j|$ and using the bounded overlap property of the balls, the Sobolev-Poincaré inequality (6), Proposition 4.13, and properties (53) and (55) of the constants $c_i^j = f(x_i^j)$, we have for the integral of the first sum on the right-hand-side:

$$\begin{aligned}
\left(\fint_{B_k^j} \Big(\sum_i \mathbb{1}_{B_i^j} |f - c_i^j| \Big)^{q^*} d\mu \right)^{1/q^*} &\le K \left(\fint_{B_k^j} |f - f_{B_k^j}|^{q^*} d\mu \right)^{1/q^*} + K |f_{B_k^j} - c_k^j| \\
&\quad + \left(\fint_{B_k^j} \Big(\sum_{B_i^j \cap B_k^j \neq \emptyset} \mathbb{1}_{B_i^j} |c_k^j - c_i^j| \Big)^{q^*} d\mu \right)^{1/q^*} \\
&\le C K r_k^j \left(\fint_{\overline{B_k^j}} (Nf)^q \right)^{1/q} + K r_k^j f^\star(x_k^j) + C K r_k^j 2^j \\
&\le C K r_k^j 2^j.
\end{aligned}$$

The analogous estimate holds for the integral of the second sum, in l, since as pointed out previously, when $B_l^{j+1} \cap B_k^j \neq \emptyset$ we have that $r_l^{j+1} \leq c r_k^j$. This gives

$$\|\nabla \ell_k^j\|_{q^*} \leq \gamma 2^j \mu((B_k^j)')^{1/q^*},$$

as desired. The rest of the proof follows in the same way as that of Propositions 3.4 and 4.7. □

Conclusion: Let M be a complete Riemannian manifold satisfying (D). Then

1. for all $\frac{s}{s+1} < q < 1$,
$$M_1^1 \subset LS_{q^*,\mathrm{ato}}^1.$$
2. If moreover we assume (P_1), then
$$M_1^1 = LS_{t,\mathrm{ato}}^1$$
for all $t > 1$.

5 Comparison between $\dot{M}_1^1$ and Hardy-Sobolev spaces defined in terms of derivatives

5.1 Using a maximal function definition

In the Euclidean case, the homogeneous Hardy-Sobolev space $\dot{HS}^1$ consists of all locally integrable functions f such that $\nabla f \in H_1(\mathbb{R}^n)$ (i.e. the weak partial derivatives $D_j f = \frac{\partial f}{\partial x_j}$ belong to the real Hardy space $H_1(\mathbb{R}^n)$). In [28], it was proved that this space is nothing else than $\{f \in L_{1,\mathrm{loc}}(\mathbb{R}^n) : Nf \in L_1\}$, which also coincides with the Sobolev space $\dot{M}_1^1$ ([22]).

Does this theory extends to the case of Riemannian manifolds? If this is the case, which hypotheses should one assume on the geometry of the manifold? We proved an atomic characterization of $\dot{M}_1^1$ but we would like to clarify the relation with Hardy-Sobolev spaces defined using maximal functions.

Definition 5.1. *We define the (maximal) homogeneous Hardy-Sobolev space $\dot{HS}_{\max}^1$ as follows:*

$$\dot{HS}_{\max}^1 := \left\{ f \in L_{1,\mathrm{loc}}(M) : (\nabla f)^+ \in L_1 \right\}$$

where ∇f is the distributional gradient, as defined in (7)*, and the corresponding maximal function is defined, analogously to* (10)*, by*

$$(\nabla f)^+(x) := \sup \left| \int f \left(\langle \nabla \varphi, \mathbf{\Phi} \rangle + \varphi \mathrm{div}\, \mathbf{\Phi} \right) d\mu \right|,$$

where the supremum is taken over all pairs $\varphi \in \mathcal{T}_1(x)$, $\mathbf{\Phi} \in C_0^1(M, TM)$ *such that*

$$\|\mathbf{\Phi}\|_\infty \leq 1 \quad \text{and } \|\text{div } \mathbf{\Phi}\|_\infty \leq \frac{1}{r}$$

for the radius r *of the same ball* B *containing* x *for which* φ *satisfies (11). We equip this space with the semi-norm*

$$\|f\|_{\dot{HS}^1_{\max}} = \|(\nabla f)^+\|_1.$$

Note that in case both φ and $\mathbf{\Phi}$ are smooth, the quantity $\langle \nabla\varphi, \mathbf{\Phi}\rangle + \varphi \text{div } \mathbf{\Phi}$ represents the divergence of the product $\varphi\mathbf{\Phi}$, so the definition coincides with that of the maximal function $M^{(1)}f$ given in [4] for the case of domains in $\mathbb{R}^n$, but here we want to allow for the case of Lipschitz φ.

Proposition 5.2. *Let* $f \in \dot{HS}^1_{\max}$. *Then* ∇f, *initially defined by (7), is given by an* L_1 *function and satisfies*

$$|\nabla f| \leq C(\nabla f)^+ \qquad \mu - \text{a.e.}$$

Consequently,

$$\dot{HS}^1_{\max} \subset \dot{W}^1_1$$

with

$$\|f\|_{\dot{W}^1_1} \leq C\|f\|_{\dot{HS}^1_{\max}}.$$

Proof. We follow the ideas in the proof of Proposition 2.6. Let Ω be any open subset of M and consider the total variation of u on Ω, defined by

$$|Df|(\Omega) := \sup |\langle \nabla f, \mathbf{\Phi}\rangle| ,$$

where the supremum is taken over all vector fields $\mathbf{\Phi} \in C_0^1(\Omega, TM)$ with $\|\mathbf{\Phi}\|_\infty \leq 1$. For such a vector field $\mathbf{\Phi}$, take $r > 0$ sufficiently small so that $\|\text{div } \mathbf{\Phi}\|_\infty \leq r^{-1}$ and $\text{dist}(\text{supp}(\mathbf{\Phi}), M \setminus \Omega) > 12r$. As in the proof of Proposition 2.6, take a collection of balls $B_i = B(x_i, r)$ with $6B_i$ having bounded overlap (with a constant K independent of r), covering M, and a Lipschitz partition of unity $\{\varphi_i\}_i$ subordinate to $\{6B_i\}_i$, with $0 \leq \varphi_i \leq 1$ and $|\nabla\varphi_i| \leq r^{-1}$. Then for all $x \in B_i$, $\varphi_i/\mu(B_i) \in \mathcal{T}_1(x)$, so

$$\left|\int f[\langle \nabla\varphi_i, \mathbf{\Phi}\rangle + \varphi_i \text{div } \mathbf{\Phi}]d\mu\right| \leq (\nabla f)^+(x)\mu(B_i).$$

Hence

$$\left|\int f[\langle \nabla\varphi_i, \mathbf{\Phi}\rangle + \varphi_i \text{div } \mathbf{\Phi}]d\mu\right| \leq \int_{B_i} (\nabla f)^+(x)d\mu.$$

Summing up over i such that $6B_i \subset \Omega$, by the choice of r we still get $\sum \varphi_i = 1$ on the support of $\mathbf{\Phi}$, hence $\sum \nabla\varphi_i = 0$, so using the bounded overlap of the balls we have

$$\left|\int f \text{ div } \mathbf{\Phi}\, d\mu\right| \leq \sum_{\{i:6B_i\subset\Omega\}} \int_{B_i} (\nabla f)^+ d\mu \leq K \int_\Omega (\nabla f)^+ d\mu \leq K\|(\nabla f)^+\|_1 < \infty.$$

The rest of the proof proceeds as in the proof of Proposition 2.6, replacing Nf by $(\nabla f)^+$. □

Proposition 5.3. *Let $f \in L_{1,\mathrm{loc}}$. Then at every point of M,*

$$(\nabla f)^+ \leq Nf.$$

Consequently,

$$\dot{M}_1^1 \subset \dot{HS}^1_{\max}$$

with

$$\|f\|_{\dot{HS}^1_{\max}} \leq C\|f\|_{\dot{M}_1^1}.$$

Proof. Let $f \in L_{1,\mathrm{loc}}$ and $x \in M$. Take $\varphi \in \mathcal{T}_1(x)$, $\mathbf{\Phi} \in C_0^1(M, TM)$ as in Definition 5.1. Then

$$\int (\langle\nabla\varphi, \mathbf{\Phi}\rangle + \varphi\text{div } \mathbf{\Phi})d\mu = 0$$

so we can write

$$\begin{aligned}\left|\int f(\langle\nabla\varphi, \mathbf{\Phi}\rangle + \varphi\text{div } \mathbf{\Phi})d\mu\right| &= \left|\int (f - f_B)(\langle\nabla\varphi, \mathbf{\Phi}\rangle + \varphi\text{div } \mathbf{\Phi})d\mu\right| \\ &\leq \frac{1}{r\mu(B)} \int |f - f_B| d\mu \\ &\leq Nf(x).\end{aligned}$$

□

We would like to prove the reverse inclusion. However, this would require some tools such as Lemma 6 in [22] or Lemma 10 in [4] (solving div $\Psi = \phi$ with Ψ having compact support) which are particular to $\mathbb{R}^n$.

Another possible maximal function we can use, following the ideas in [21] (see Section 4.1), is given by

Definition 5.4.

$$\mathcal{M}^*(\nabla f)(x) := \sup_j |\nabla f_{r_j}|$$

with the "discrete convolution" f_{r_j} defined as in (8), corresponding to an enumeration of the positive rationals $\{r_j\}_j$, where for each j we have a covering of M by balls $\{B_i^j\}_i$ of radius r_j, and a partition of unity φ_i^j subordinate to this covering.

We have already shown in the proof of Proposition 2.6 (see (9)) that

Lemma 5.5. *Let $f \in L_{1,\mathrm{loc}}$. Then at μ-almost every point of M,*

$$\mathcal{M}^*(\nabla f) \leq Nf.$$

5.2 Derivatives of molecular Hardy spaces

As noted in the previous section, on a manifold, obtaining a decomposition with atoms of compact support from a maximal function definition is not obvious. In [3], the authors considered instead Hardy spaces generated by molecules. We begin by recalling their definition of $H_{\mathrm{mol},1}(\wedge^1 T^*M)$ (a special case with $N = 1$ of $H^1_{\mathrm{mol},N}(\wedge T^*M)$ in Definition 6.1 of [3], where we have dropped the superscript 1 for convenience). If in addition the heat kernel on M satisfies Gaussian upper bounds, this space coincides with the space $H^1(\wedge T^*M)$, which also has a maximal function characterization (see [3], Theorem 8.4).

A sequence of non-negative Lipschitz functions $\{\chi_k\}_k$ is said to be (a partition of unity) adapted to a ball B of radius r if $\operatorname{supp}\chi_0 \subset 4B$, $\operatorname{supp}\chi_k \subset 2^{k+2}B \setminus 2^{k-1}B$ for all $k \geq 1$,

$$\|\nabla\chi_k\|_\infty \leq C2^{-k}r^{-1} \tag{56}$$

and

$$\sum_k \chi_k = 1 \text{ on } M.$$

A 1-form $a \in L^2(\wedge^1 T^*M)$ is called a 1-molecule if $a = db$ for some $b \in L_2(M)$ and there exists a ball B with radius r, and a partition of unity $\{\chi_k\}_k$ adapted to B, such that for all $k \geq 0$

$$\|\chi_k a\|_{L^2(\wedge^1 T^*M)} \leq 2^{-k}(\mu(2^kB))^{-1/2} \tag{57}$$

and

$$\|\chi_k b\|_2 \leq 2^{-k}r(\mu(2^kB))^{-1/2}.$$

Summing in k, this implies that $\|a\|_{L^2(\wedge^1 T^*M)} \leq 2(\mu(B))^{-1/2}$ and $\|b\|_{L^2} \leq 2r(\mu(B))^{-1/2}$. Moreover, there exists a constant C', depending only on the doubling constant in (D), such that

$$\|\mathbb{1}_{2^{k+2}B\setminus 2^{k-1}B}b\|_2 \leq \left\|\sum_{l=k-3}^{k+3}\chi_l b\right\|_2 \leq C'r2^{-k}(\mu(2^{k+2}B))^{-1/2}. \tag{58}$$

Definition 5.6 ([3])**.** *We say that $f \in H_{\mathrm{mol},1}(\wedge^1 T^*M)$ if there is a sequence $\{\lambda_j\}_j \in \ell^1$ and a sequence of 1-molecules $\{a_j\}_j$ such that*

$$f = \sum_j \lambda_j a_j$$

in $L_1(\wedge^1 T^*M)$, with the norm defined by

$$\|f\|_{H_{\mathrm{mol},1}(\wedge^1 T^*M)} = \inf \sum_j |\lambda_j|.$$

Here the infimum is taken over all such decompositions. The space $H_{\mathrm{mol},1}(\wedge^1 T^*M)$ is a Banach space.

Proposition 5.7. *Let M be a complete Riemannian manifold satisfying (D) and (P_1). We then have*

$$H_{\mathrm{mol},1}(\wedge^1 T^*M) = d(\dot{H}S^1_{2,\mathrm{ato}}(M)). \tag{59}$$

Moreover

$$\|g\|_{H_{\mathrm{mol},1}(\wedge^1 T^*M)} \sim \inf_{df=g} \|f\|_{\dot{H}S^1_{2,\mathrm{ato}}(M)}.$$

Consequently, in this case we have an atomic decomposition for $H_{\mathrm{mol},1}(\wedge^1 T^*M)$ (this was already proved in [3], after Theorem 8.4).

Remark 5.8. *As pointed out in Remarks 3.2 and 3.6, we can define the atomic Hardy-Sobolev space $\dot{H}S^1_{2,\mathrm{ato}}(M)$ by using $(1,2)$-atoms satisfying condition $3''$ of Remarks 2.12 instead of condition 3 of Definition 2.11. As will be seen from the proof below, if we restrict ourselves to this kind of atoms we do not require the hypothesis (P_1) for (59). Under the assumption (P_1), we actually get the stronger conclusion*

$$H_{\mathrm{mol},1}(\wedge^1 T^*M) = d(\dot{H}S^1_{2,\mathrm{ato}}) = d(\dot{H}S^1_{t,\mathrm{ato}}) = d(\dot{M}^1_1)$$

for all $t > 1$.

Proof. Take $f \in \dot{H}S^1_{2,\mathrm{ato}}$. There exists a sequence $\{\lambda_j\}_j \in \ell^1$ and $(1,2)-$atoms b_j such that $f = \sum_j \lambda_j b_j$ in $\dot{W}^1_1$. This means $\sum_j \lambda_j \nabla b_j$ converges in L_1 to ∇f, and by the isometry between the vector fields and the 1-forms, we have $df = \sum_j \lambda_j db_j$ in $L_1(\wedge^1 T^*M)$.

We claim that $a_j = db_j$ are 1-molecules. Indeed, fix j, take B_j to be the ball containing the support of b_j and let $\{\chi_j^k\}_k$ be a partition of unity adapted to B_j. Then

$$\|\chi_j^0 a_j\|_2 \le \|db_j\|_2 = \|\nabla b_j\|_2 \le \frac{1}{\mu(B_j)^{\frac{1}{2}}}$$

and by condition $3''$ of Remarks 2.12 (alternatively condition 3 of Definition 2.11 and (P_1)) we get

$$\|\chi_j^0 b_j\|_2 \le \|b_j\|_2 \le r_j \frac{1}{\mu(B_j)^{\frac{1}{2}}}.$$

For $k \ge 1$, there is nothing to do since $\operatorname{supp} b_j \subset B_j$ and $\operatorname{supp} \chi_j^k \subset 2^{k+2}B_j \setminus 2^{k-1}B_j \subset (B_j)^c$. Consequently, $df \in H_{\mathrm{mol},1}(\wedge^1 T^*M)$ with $\|df\|_{H_{\mathrm{mol},1}(\wedge^1 T^*M)} \le \sum_j |\lambda_j|$. Taking the infimum over all such decompositions, we get $\|df\|_{H_{\mathrm{mol},1}(\wedge^1 T^*M)} \le \|f\|_{\dot{H}S^1_{2,\mathrm{ato}}}$.

Now for the converse, let $g \in H_{\mathrm{mol},1}(\wedge^1 T^*M)$. Write

$$g = \sum_j \lambda_j a_j := \sum_j \lambda_j db_j$$

where $\sum_j |\lambda_j| < \infty$, for every j, a_j is a 1-molecule associated to a ball B_j, and the convergence is in L_1. Let $\{\chi_j^k\}_k$ be the partition of unity adapted to B_j. Then

$$g = \sum_j \lambda_j \sum_k db_j \chi_j^k = \sum_j \lambda_j d(\sum_k b_j \chi_j^k) = \sum_j \lambda_j \sum_k d(b_j \chi_j^k)$$

since the sum is locally finite and $\sum_k \chi_j^k = 1$.

We claim that for every j, k, $\beta_j^k := 2^{k-1}\gamma b_j \chi_j^k$, with γ a constant to be determined, satisfies properties $1, 2$ and $3''$ (see Definition 2.11 and Remarks 2.12) of a $(1,2)$-homogeneous Hardy-Sobolev atom. Indeed, β_j^k is supported in the ball $2^{k+2}B_j$ with

$$\|\beta_j^k\|_2 \leq 2^{k-1}\gamma \frac{2^{-k} r_j}{\mu(2^k B_j)^{\frac{1}{2}}} \leq \frac{2^{k+2} r_j}{\mu(2^{k+2}B_j)^{\frac{1}{2}}}$$

for an appropriate choice of γ depending only on the doubling constant in (D). Furthermore, by (57), (56), and (58),

$$\begin{aligned}
\|\nabla \beta_j^k\|_2 &= 2^{k-1}\gamma \|d(b_j\chi_j^k)\|_2 \\
&\leq 2^{k-1}\gamma(\|a_j \chi_j^k\|_2 + \|b_j d\chi_j^k\|_2) \\
&\leq 2^{k-1}\gamma(2^{-k}(\mu(2^k B_j))^{-1/2} + C2^{-k} r_i^{-1} \|\mathbb{1}_{2^{k+2}B_j \setminus 2^{k-1}B_j} b_j\|_2) \\
&\leq \mu(2^{k+2}B_j)^{-1/2}.
\end{aligned}$$

Here we again chose γ conveniently, depending only on the doubling constant, and used the fact that $k \geq 0$.

Since $\sum_{j,k} |\lambda_j| \gamma^{-1} 2^{1-k} \leq 4\gamma^{-1} \sum_j |\lambda_j| < \infty$, the sum $f := \sum_j \lambda_j \sum_k \gamma^{-1} 2^{1-k} \beta_j^k$ defines an element of $\dot{H}S^1_{2,\mathrm{ato}}$, with the convergence being in $\dot{W}^1_1$. This means that in L_1 we have

$$df = d\left(\sum_{j,k} \lambda_j (b_j \chi_j^k)\right) = \sum_j \lambda_j \sum_k d(b_j \chi_j^k) = g.$$

Therefore $g = df = d\left(\sum_{j,k} \lambda_j (b_j \chi_j^k)\right)$, with $\|f\|_{\dot{H}S^1_{2,\mathrm{ato}}} \leq 4\gamma^{-1} \sum_j |\lambda_j|$. Taking the infimum over all such decompositions of g, we see that

$$\inf_{df=g} \|f\|_{\dot{H}S^1_{2,\mathrm{ato}}} \leq 4\gamma^{-1} \|g\|_{H_{\mathrm{mol},1}(\wedge^1 T^*M)}.$$

□

Corollary 5.9. *In the Euclidean case, we then obtain*

$$H_{\mathrm{mol},1}(\mathbb{R}^n, \wedge^1) = \mathcal{H}^1_d(\mathbb{R}^n, \wedge^1) = d(\dot{M}^1_1) = d(\dot{H}S^1_{t,\mathrm{ato}})$$

for all $t > 1$. (For details on $\mathcal{H}^1_d(\mathbb{R}^n, \wedge^1)$, see [24]).

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