New Calderón-Zygmund decomposition for Sobolev functions

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Abstract

We state a new Calderón-Zygmund decomposition for Sobolev spaces on a doubling Riemannian manifold. Our hypotheses are weaker than those of the already known decomposition which used classical Poincaré inequalities.

Key-words : Calderón-Zygmund decomposition, Sobolev spaces, Poincaré inequalities.
MSC : 42B20, 46E35.

Contents

1 Introduction

The purpose of this article is to weaken assumptions of the already known Calderón-Zygmund decomposition for Sobolev functions. This well-known tool was first stated by P. Auscher in [2]. It exactly corresponds to the Calderón-Zygmund decomposition in a context of Sobolev spaces.

Let us briefly recall the ideas of such decomposition. In [34], E. Stein stated this decomposition for Lebesgue spaces as following. Let (X, d, μ) be a space of homogeneous type and $p \geq 1$. Given a function $f \in L^p(X)$, the decomposition gives a precise way of partitioning X into two subsets: one where f is essentially small (bounded in L^∞ norm); the other a countable collection of cubes where f is essentially large, but where some control of the function is obtained in L^1 norm. This leads to the associated Calderón-Zygmund decomposition of f, where f is written as the sum of "good" and "bad" functions, using the above subsets.

This decomposition is a basic tool in Harmonic analysis and the study of singular integrals. One of the applications is the following : an L^2-bounded Calderón-Zygmund operator is of weak type $(1,1)$ and so L^p bounded for every $p \in (1, \infty)$.

In [2], P. Auscher extended these ideas for Sobolev spaces. His decomposition is the following:

Theorem 1.1 *Let $n \geq 1$, $p \in [1, \infty)$ and $f \in \mathcal{D}'(\mathbb{R}^n)$ be such that $\|\nabla f\|_{L^p} < \infty$. Let $\alpha > 0$. Then, one can find a collection of cubes $(Q_i)_i$, functions g and b_i such that*

$$f = g + \sum_i b_i$$

and the following properties hold:

$$\|\nabla g\|_{L^\infty} \leq C\alpha,$$

$$b_i \in W_0^{1,p}(Q_i) \text{ and } \int_{Q_i} |\nabla b_i|^p \leq C\alpha^p |Q_i|,$$

$$\sum_i |Q_i| \leq C\alpha^{-p} \int_{\mathbb{R}^n} |\nabla f|^p,$$

$$\sum_i \mathbf{1}_{Q_i} \leq N,$$

where C and N depend only on the dimension n and on p.

The important point in this decomposition is the fact that the functions b_i are supported in the corresponding balls, while the original Calderón-Zygmund decomposition applied to ∇f would not give this.

The proof relies on an appropriate use of Poincaré inequality and was then extended to a doubling manifold with Poincaré inequality by P. Auscher and T. Coulhon in [6].

This decomposition is used in many works and it appears in various forms and extensions. For example in [6] (same proof on manifolds), [8] (on $\mathbb{R}^n$ but with a doubling weight), B. Ben Ali's PhD thesis [16] and [5], [14] (the Sobolev space is modified to adapt to Schrödinger operators), N. Badr's PhD thesis [9] and [10, 11] (used toward interpolation of Sobolev spaces on manifolds and measured metric spaces) and in [13] (Sobolev spaces on graphs).

The aim of this article is to extend the proof using other kind of "Poincaré inequalities". This work can be integrated in several recent works, where the authors look for replacing the mean-value operators by other ones in the definition of Hardy spaces for example or in the definition of maximal operators (see [19, 20, 26, **?**, 32] ...). Mainly, Section 3 is devoted to the proof of Calderón-Zygmund decompositions for Sobolev functions (as in Theorem 1.1) in an abstract framework of a doubling Riemannian manifold under assumptions involving new kind of Poincaré inequalities. Then we give an application to the real interpolation of Sobolev spaces $W^{1,p}$. In Section 4, we focus on a particular case (using the heat semigroup) corresponding to the so-called pseudo-Poincaré inequalities. We specify that these new Poincaré inequalities are weaker than the classical ones and permit to insure the Calderón-Zygmund decomposition for Sobolev functions. We give some applications using this improvement.

2 Preliminaries

Throughout this paper we will denote by $\mathbf{1}_E$ the characteristic function of a set E and E^c the complement of E. If X is a metric space, Lip will be the set of real Lipschitz functions on X and Lip_0 the set of real, compactly supported Lipschitz functions on X. For a ball Q in a metric space, λQ denotes the ball co-centered with Q and with radius λ times that of Q. Finally, C will be a constant that may change from an inequality to another and we will use $u \lesssim v$ to say that there exists a constant C such that $u \leq Cv$ and $u \simeq v$ to say that $u \lesssim v$ and $v \lesssim u$.

In all this paper, M denotes a complete Riemannian manifold. We write μ for the Riemannian measure on M, ∇ for the Riemannian gradient, $|\cdot|$ for the length on the tangent space (forgetting the subscript x for simplicity) and $\|\cdot\|_{L^p}$ for the norm on $L^p := L^p(M, \mu)$, $1 \leq p \leq +\infty$. We denote by $Q(x, r)$ the open ball of center $x \in M$ and radius $r > 0$.
We will use the positive Laplace-Beltrami operator Δ defined by

$$\forall f, g \in C_0^\infty(M), \qquad \langle \Delta f, g \rangle = \langle \nabla f, \nabla g \rangle.$$

We deal with the Sobolev spaces of order 1 $W^{1,p} := W^{1,p}(M)$, where the norm is defined by:

$$\|f\|_{W^{1,p}(M)} := \|f\|_p + \| \, |\nabla f| \, \|_{L^p}.$$

2.1 The doubling property

Definition 2.1 (Doubling property) *Let M be a Riemannian manifold. One says that M satisfies the doubling property (D) if there exists a constant $C > 0$, such that for all $x \in M$, $r > 0$ we have*

$$\mu(Q(x,2r)) \leq C\mu(Q(x,r)). \tag{D}$$

Lemma 2.2 *Let M be a Riemannian manifold satisfying (D) and let $d = log_2 C$. Then for all x, $y \in M$ and $\theta \geq 1$*

$$\mu(Q(x,\theta R)) \leq C\theta^d \mu(Q(x,R)) \tag{1}$$

Observe that if M satisfies (D) then

$$diam(M) < \infty \Leftrightarrow \mu(M) < \infty \text{ (see [1]).}$$

Therefore if M is a complete Riemannian manifold satisfying (D) then $\mu(M) = \infty$.

Theorem 2.3 (Maximal theorem) *([22]) Let M be a Riemannian manifold satisfying (D). Denote by $\mathcal{M}$ the uncentered Hardy-Littlewood maximal function over open balls of M defined by*

$$\mathcal{M}f(x) := \sup_{\substack{Q \ ball \\ x \in Q}} |f|_Q$$

where $f_E := \fint_E f d\mu := \frac{1}{\mu(E)} \int_E f d\mu$. Then for every $p \in (1, \infty]$, $\mathcal{M}$ is L^p bounded and moreover of weak type $(1,1)$[1].
Consequently for $s \in (0, \infty)$, the operator $\mathcal{M}_s$ defined by

$$\mathcal{M}_s f(x) := [\mathcal{M}(|f|^s)(x)]^{1/s}$$

is of weak type (s,s) and L^p bounded for all $p \in (s, \infty]$.

2.2 Classical Poincaré inequality

Definition 2.4 (Classical Poincaré inequality on M) *We say that a complete Riemannian manifold M admits* ***a Poincaré inequality*** *(P_q) for some $q \in [1, \infty)$ if there exists a constant $C > 0$ such that, for every function $f \in \mathrm{Lip}_0(M)$*[2] *and every ball Q of M of radius $r > 0$, we have*

$$\left(\fint_Q |f - f_Q|^q d\mu \right)^{1/q} \leq Cr \left(\fint_Q |\nabla f|^q d\mu \right)^{1/q}. \tag{P_q}$$

Remark 2.5 *By density of $C_0^\infty(M)$ in $Lip_0(M)$, we can replace $\mathrm{Lip}_0(M)$ by $C_0^\infty(M)$.*

Let us recall some known facts about Poincaré inequalities with varying q.
It is known that (P_q) implies (P_p) when $p \geq q$ (see [29]). Thus, if the set of q such that (P_q) holds is not empty, then it is an interval unbounded on the right. A recent result of S. Keith and X. Zhong (see [30]) asserts that this interval is open in $[1, +\infty[$:

[1] An operator T is of weak type (p,p) if there is $C > 0$ such that for any $\alpha > 0$, $\mu(\{x; |Tf(x)| > \alpha\}) \leq \frac{C}{\alpha^p} \|f\|_p^p$.

[2] compaclty supported Lipshitz function defined on M.

Theorem 2.6 *Let (X,d,μ) be a complete metric-measure space with μ doubling and admitting a Poincaré inequality (P_q), for some $1<q<\infty$. Then there exists $\epsilon>0$ such that (X,d,μ) admits (P_p) for every $p>q-\epsilon$.*

2.3 Estimates for the heat kernel

We recall the following off-diagonal decays of the heat semigroup and the link between these decays and the boundedness of the Riesz transform, the doubling property and Poincaré inequality. We refer the reader to the work of P. Auscher, T. Coulhon, X. T. Duong and S. Hofmann [7] and [6] for more details about all these notions and how they are related. Let us consider the following two inequalities:

$$\| \nabla f \|_p \leq C(\|\Delta^{\frac{1}{2}} f\|_p + \|f\|_p) \tag{nhR_p}$$

and

$$(\|\Delta^{\frac{1}{2}} f\|_p + \|f\|_p) \leq C\| \nabla f \|_p. \tag{$nhRR_p$}$$

Theorem 2.7 *Let M be a complete doubling Riemannian manifold.*

- *The inequalities (nhR_2) and $(nhRR_2)$ are always satisfied.*
- *([23]) Assume that the heat kernel p_t of the semigroup $e^{-t\Delta}$ satisfies the following pointwise estimate:*
$$p_t(x,x) \lesssim \frac{1}{\mu(B(x,t^{1/2}))}. \tag{DUE}$$
Then for all $p\in(1,2]$, (nhR_p) and $(nhRR_{p'})$ hold [3].
- *([28], Theorem 1.1) Under (D), (DUE) self-improves into the following Gaussian upper-bound estimate of p_t*
$$p_t(x,y) \lesssim \frac{1}{\mu(B(y,t^{1/2}))} e^{-c\frac{d^2(x,y)}{t}}. \tag{UE}$$
Note that (UE) implies (L^1-L^∞) "off-diagonal" decays for $(e^{-t\Delta})_{t>0}$.
- *Under (UE), the collection $(\sqrt{t}\nabla e^{-t\Delta})_{t>0}$ satisfies "L^2-L^2 off-diagonal decays".*
- *Under (DUE) and by the analiticity of the heat semigroup, the following pointwise upper bound for the kernel of $\Delta e^{-t\Delta}$: $t\frac{\partial}{\partial t}p_t$ holds (see [25], Theorem 4 and [28], Corollary 3.3):*
$$t\left|\frac{\partial}{\partial t} p_t(x,y)\right| \lesssim \frac{1}{\mu(B(y,t^{1/2}))} e^{-c\frac{d^2(x,y)}{t}}. \tag{2}$$

Theorem 2.8 ([31, 33]) *The conjunction of (D) and Poincaré inequality (P_2) on M is equivalent to the following Li-Yau inequality*

$$\frac{1}{\mu(B(y,t^{1/2}))} e^{-c_1\frac{d^2(x,y)}{t}} \lesssim p_t(x,y) \lesssim \frac{1}{\mu(B(y,t^{1/2}))} e^{-c_2\frac{d^2(x,y)}{t}}, \tag{LY}$$

with some constants $c_1,c_2>0$.

[3] The assumptions in [23] are even weaker.

Theorem 2.9 ([7]) *The L^p-boundedness of the Riesz transform $\nabla(\Delta)^{-1/2}$ implies*

$$\left\| |\nabla e^{-t\Delta}| \right\|_{L^p \to L^p} \lesssim \frac{1}{\sqrt{t}}. \qquad (G_p)$$

Moreover, under (P_2) and (G_{p_0}) with $p_0 > 2$, the collection $(\sqrt{t}\nabla e^{-t\Delta})_{t>0}$ satisfies some $(L^p - L^p)$ "off-diagonal" decays for every $p \in [2, p_0)$.

Remark 2.10 *All these results are proved in their homogeneous version, with homogeneous properties (R_p) and (RR_p). It is essentially based on the well-known Calderón-Zygmund decomposition for Sobolev functions. This tool was extended for non-homogeneous Sobolev spaces (see [10]). Thus by exactly the same proof, we can obtain an analogous non-homogeneous version and then prove all these results.*

2.4 The K-method of real interpolation

We refer the reader to [17], [18] for details on the development of this theory. Here we only recall the essentials to be used in the sequel.

Let A_0, A_1 be two normed vector spaces embedded in a topological Hausdorff vector space V. For each $a \in A_0 + A_1$ and $t > 0$, we define the K-functional of interpolation by

$$K(a, t, A_0, A_1) = \inf_{a = a_0 + a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}).$$

For $0 < \theta < 1$, $1 \le q \le \infty$, we denote by $(A_0, A_1)_{\theta,q}$ the interpolation space between A_0 and A_1:

$$(A_0, A_1)_{\theta,q} = \left\{ a \in A_0 + A_1 : \|a\|_{\theta,q} = \left(\int_0^\infty (t^{-\theta} K(a, t, A_0, A_1))^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\}.$$

It is an exact interpolation space of exponent θ between A_0 and A_1 (see [18], Chapter II).

Definition 2.11 *Let f be a measurable function on a measure space (X, μ). The decreasing rearrangement of f is the function f^* defined for every $t \ge 0$ by*

$$f^*(t) = \inf \{\lambda : \mu(\{x : |f(x)| > \lambda\}) \le t\}.$$

The maximal decreasing rearrangement of f is the function f^{**} defined for every $t > 0$ by

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds.$$

Proposition 2.12 *From the properties of f^{**}, we mention:*

1. $(f + g)^{**} \le f^{**} + g^{**}$.
2. $(\mathcal{M}f)^* \sim f^{**}$.
3. $\mu(\{x; |f(x)| > f^*(t)\}) \le t$.
4. $\forall p \in (1, \infty]$, $\|f^{**}\|_p \sim \|f\|_p$.

We exactly know the functional K for Lebesgue spaces:

Proposition 2.13 *Take $0 < p_0 < p_1 \leq \infty$. We have:*

$$K(f, t, L^{p_0}, L^{p_1}) \simeq \left(\int_0^{t^\alpha} [f^*(s)]^{p_0}\, ds\right)^{1/p_0} + t\left(\int_{t^\alpha}^\infty [f^*(s)]^{p_1}\, ds\right)^{1/p_1},$$

where $\frac{1}{\alpha} = \frac{1}{p_0} - \frac{1}{p_1}$.

3 New "Calderón-Zygmund" decompositions for Sobolev functions.

In the introduction, we recalled the main use of "Calderón-Zygmund" decompositions for Sobolev functions. In the previously cited works, this decomposition relies on Poincaré inequalities and some "tricks" with the mean-value operators. We present here similar arguments with abstract operators, requiring new "Poincaré inequalities". Then, we give some applications to real interpolation of Sobolev spaces.

3.1 Decomposition using abstract "oscillation operators"

Let $\mathcal{A} := (A_Q)_Q$ be a collection of operators (acting from $W^{1,p}$ to $W^{1,p}_{loc}$) indexed by the balls of the manifold (A_Q can be thought to be similar to the mean operator over the ball Q).

Definition 3.1 *We define a new maximal operator associated to this collection: for $1 \leq s \leq p \leq \infty$ and all functions $f \in W^{1,p}$*

$$M_{\mathcal{A},s}(f)(x) := \sup_{Q;\, Q \ni x} \frac{1}{\mu(Q)^{1/s}} \left\| A_Q(f) \right\|_{W^{1,s}(Q)}.$$

Let us now define the assumptions that we need on the collection $\mathcal{A}$.

Definition 3.2 1) *We say that for $q \in [1, \infty]$ [4], the manifold M satisfies a Poincaré inequality (P_q) relatively to the collection $\mathcal{A}$ if there is a constant C such that for every ball Q (of radius r_Q) and for all functions $f \in W^{1,p}$; $p \geq q$:*

$$\left(\fint_Q |f - A_Q(f)|^q\, d\mu\right)^{1/q} \leq C r_Q \sup_{s \geq 1} \left(\fint_{sQ} (|f| + |\nabla f|)^q\, d\mu\right)^{1/q}.$$

2) *For $1 \leq q \leq r \leq \infty$, we say that the collection $\mathcal{A}$ satisfies "$L^q - L^r$ off-diagonal estimates" if*

a. there are constants $C' > 0$ and $N \in \mathbb{N}^*$ such that for all equivalent balls Q, Q' (i.e. $Q \subset Q' \subset NQ$) and all functions $f \in W^{1,p}$; $p \geq q$, we have

$$\frac{1}{\mu(Q)^{1/r}} \left\| A_Q(f) - A_{Q'}(f) \right\|_{L^r(NQ)} \leq C' r_Q \inf_{NQ} \mathcal{M}_q\left(|f| + |\nabla f|\right) \tag{3}$$

[4] we take the supremun instead of the L^q average when $q = \infty$.

b. and for every ball Q

$$\frac{1}{\mu(Q)^{1/r}}\left\|A_Q(f)\right\|_{W^{1,r}(Q)} \leq C' \inf_Q \mathcal{M}_q\left(|f|+|\nabla f|\right). \tag{4}$$

Here is our main result :

Theorem 3.3 *Let M be a complete Riemannian manifold satisfying (D) and of infinite measure. Consider a collection $\mathcal{A}=(A_Q)_Q$ of operators defined on M. Assume that M satisfies the Poincaré inequality (P_q) relatively to the collection $\mathcal{A}$ for some $q \in [1,\infty)$, and that $\mathcal{A}$ satisfies "$L^q - L^r$ off-diagonal estimates" for some $r \in (q,\infty]$.*
Let $q \leq p < r$, $f \in W^{1,p}$ and $\alpha > 0$. Then one can find a collection of balls (Q_i), functions $g \in W^{1,r}$ and $b_i \in W^{1,q}$ with the following properties

$$f = g + \sum_i b_i \tag{5}$$

$$\|g\|_{W^{1,r}} \lesssim \|f\|_{W^{1,p}}^{p/r}\alpha^{1-p/r}, \ \int_{\cup_i Q_i} (|g|^r + |\nabla g|^r) d\mu \lesssim \alpha^r \mu(\cup_i Q_i) \tag{6}$$

$$supp\,(b_i) \subset Q_i,\ \|b_i\|_{W^{1,q}} \lesssim \alpha \mu(Q_i)^{1/q} \tag{7}$$

$$\sum_i \mu(Q_i) \leq C\alpha^{-p} \int (|f|+|\nabla f|)^p d\mu \tag{8}$$

$$\sum_i \mathbf{1}_{Q_i} \leq N. \tag{9}$$

Remark 3.4 *From the assumed "$L^q - L^r$ off-diagonal estimates" for $\mathcal{A}$ and Theorem 2.3, we deduce that the maximal operator $M_{\mathcal{A},q}$ is continuous from $W^{1,q}$ to $L^{q,\infty}$ and from $W^{1,p}$ to L^p for $p \in (q,r]$.*

Proof : We follow the ideas of [10] where the result is proved for the particular case

$$A_Q(f) := \fint_Q f d\mu.$$

Let $f \in W^{1,p}$ and $\alpha > 0$. Consider the set

$$\Omega := \left\{x \in M;\ \mathcal{M}_q(|f|+|\nabla f|)(x) + M_{\mathcal{A},q}(f)(x) > \alpha\right\}.$$

We can assume that this set is non empty (otherwise the result is obvious taking $g = f$). With this assumption, the different maximal operators are of "weak type (p,p)" so

$$\begin{aligned}\mu(\Omega) &\leq C\alpha^{-p}\left(\int |f|^p d\mu + \int |\nabla f|^p d\mu\right) \\ &< +\infty.\end{aligned} \tag{10}$$

In particular $\Omega \neq M$ as $\mu(M)=\infty$. Let F be the complement of Ω. Since Ω is an open set distinct of M, we can take $(\underline{Q_i})$ a Whitney decomposition of Ω. That is the balls $\underline{Q_i}$ are pairwise disjoint and there exist two constants $C_2 > C_1 > 1$, depending only on the metric, such that

1. $\Omega = \cup_i Q_i$ with $Q_i = C_1 \underline{Q_i}$ and the balls Q_i have the bounded overlap property;

2. $r_i = r(Q_i) = \frac{1}{2} d(x_i, F)$ and x_i is the center of Q_i;

3. each ball $C_2 \underline{Q_i}$ intersects F ($C_2 = 4C_1$ works) and we define $\overline{Q_i} = 2C_2 \underline{Q_i}$.

For $x \in \Omega$, denote $I_x = \{i : x \in Q_i\}$. By the bounded overlap property of the balls Q_i, we have that $\sharp I_x \leq N$ with a numerical integer N. Fixing $j \in I_x$ and using the properties of the Q_i's, we easily see that $\frac{1}{3} r_i \leq r_j \leq 3r_i$ for all $i \in I_x$. In particular, $Q_i \subset 7Q_j$ for all $i \in I_x$.

Condition (9) is nothing but the bounded overlap property of the Q_i's and (8) follows from (9) and (10).

Observe that the doubling property and the fact that $\overline{Q_i} \cap F \neq \emptyset$ yield

$$\begin{aligned}
\int_{Q_i} (|f|^q + |\nabla f|^q + \left|A_{\overline{Q_i}} f\right|^q + \left|\nabla A_{\overline{Q_i}} f\right|^q) d\mu &\leq \int_{\overline{Q_i}} (|f|^q + |\nabla f|^q + \left|A_{\overline{Q_i}} f\right|^q + \left|\nabla A_{\overline{Q_i}} f\right|^q) d\mu \\
&\leq \inf_{\overline{Q_i}} \left[\mathcal{M}_q(|f| + |\nabla f|) + M_{\mathcal{A},q}(f)\right]^q \mu(\overline{Q_i}) \\
&\leq \alpha^q \mu(\overline{Q_i}) \\
&\lesssim \alpha^q \mu(Q_i). \qquad (11)
\end{aligned}$$

We now define the functions b_i. Let $(\chi_i)_i$ be a partition of unity of Ω associated to the covering $(\underline{Q_i})$, such that for all i, χ_i is a Lipschitz function supported in Q_i with $\| \, |\nabla \chi_i| \, \|_\infty \lesssim r_i^{-1}$. Set

$$b_i := (f - A_{\overline{Q_i}} f) \chi_i.$$

It is clear that $\mathrm{supp}(b_i) \subset Q_i$. Let us estimate $\|b_i\|_{W^{1,q}(Q_i)}$. We have

$$\begin{aligned}
\int_{Q_i} |b_i|^q d\mu &= \int_{Q_i} \left|(f - A_{\overline{Q_i}}(f)\right|^q d\mu \\
&\lesssim \int_{Q_i} |f|^q d\mu + \int_{Q_i} \left|A_{\overline{Q_i}}(f)\right|^q d\mu \\
&\lesssim \alpha^q \mu(Q_i).
\end{aligned}$$

We applied (11) in the last inequality. Since

$$\nabla \Big((f - A_{\overline{Q_i}} f) \chi_i \Big) = \chi_i \left(\nabla f - \nabla A_{\overline{Q_i}} f \right) + \left(f - A_{\overline{Q_i}} f \right) \nabla \chi_i,$$

we have

$$\int_{Q_i} |\nabla b_i|^q d\mu \lesssim \int_{Q_i} \left|\nabla f - \nabla A_{\overline{Q_i}}(f)\right|^q d\mu + \frac{1}{r_i^q} \int_{Q_i} \left|f - A_{\overline{Q_i}} f\right|^q d\mu.$$

The first term is estimated as above for b_i. Thus

$$\int_{Q_i} \left|\nabla f - \nabla A_{\overline{Q_i}}(f)\right|^q d\mu \lesssim \alpha^q \mu(Q_i).$$

For the second term, the Poincaré inequality (P_q) (relatively to the collection $\mathcal{A}$) shows that

$$\begin{aligned} \frac{1}{r_i^q}\int_{Q_i} \left|f - A_{\overline{Q_i}}(f)\right|^q d\mu &\lesssim \sup_{s\geq 1} \frac{\mu(Q_i)}{\mu(sQ_i)} \int_{sQ_i} (|f|^q + |\nabla f|^q) d\mu \\ &\lesssim \alpha^q \mu(Q_i). \end{aligned}$$

We used that for all $s \geq 1$, $s\overline{Q_i}$ meets F and (11) for sQ_i instead of Q_i. Therefore (7) is proved.

Set $g = f - \sum_i b_i$, then it remains to prove (6). Since the sum is locally finite on Ω, g is defined almost everywhere on M and $g = f$ on F. Observe that g is a locally integrable function on M. This follows from the fact that $b = f - g \in L^q$ here (for the homogeneous case, one can easily prove that $b \in L^1_{loc}$). Note that $\sum_i \chi_i = \mathbf{1}_\Omega$ and $\sum_i \nabla\chi_i = \nabla\mathbf{1}_\Omega$. We then have

$$\begin{aligned} \nabla g &= \nabla f - \sum_i \nabla b_i \\ &= \nabla f - \left(\sum_i \chi_i \left[\nabla f - \nabla A_{\overline{Q_i}} f\right]\right) - \sum_i (f - A_{\overline{Q_i}}(f))\nabla\chi_i \\ &= \mathbf{1}_F(\nabla f) + \sum_i \chi_i \nabla A_{\overline{Q_i}} f - \sum_i A_{\overline{Q_i}}(f)\nabla\chi_i - f\nabla\mathbf{1}_\Omega. \end{aligned} \tag{12}$$

The definition of F and the Lebesgue differentiation theorem yield $\mathbf{1}_F(|f| + |\nabla f|) \leq \alpha\ \mu-$a.e. We deduce that (with an interpolation inequality) for $\frac{1}{r} = \frac{\theta}{p}$:

$$\begin{aligned} \|\mathbf{1}_F(|f| + |\nabla f|)\|_{L^r} &\lesssim \|\mathbf{1}_F(|f| + |\nabla f|)\|_{L^p}^{\theta} \|\mathbf{1}_F(|f| + |\nabla f|)\|_{L^\infty}^{(1-\theta)} \\ &\lesssim \|f\|_{W^{1,p}}^{p/r} \alpha^{1-p/r}. \end{aligned}$$

We control the second term in (12) using the "off-diagonal" decays of $\mathcal{A}$: (4). We recall that $\overline{Q_i} = 2C_2\underline{Q_i}$. We deduce that

$$\begin{aligned} \left\| |\nabla A_{\overline{Q_i}} f| \right\|_{L^r(Q_i)} &\lesssim \mu(Q_i)^{1/r} \inf_{\overline{Q_i}} \mathcal{M}_q\left(|f| + |\nabla f|\right) \\ &\lesssim \alpha\mu(Q_i)^{1/r}. \end{aligned} \tag{13}$$

The last inequality is due to the fact that $\overline{Q_i} \cap F \neq \emptyset$. Then the bounded overlap property of the covering $(Q_i)_i$ gives us

$$\begin{aligned} \left\|\sum_i \chi_i(x)|\nabla A_{\overline{Q_i}} f|\right\|_{L^r} &\lesssim \left(\sum_i \left\| |\nabla A_{\overline{Q_i}} f| \right\|_{L^r(Q_i)}^r\right)^{1/r} \\ &\lesssim \left(\alpha^r \sum_i \mu(Q_i)\right)^{1/r} \\ &\lesssim \alpha(\mu(\Omega))^{1/r}. \end{aligned}$$

We claim that a similar estimate holds for $h = \sum_i \left[A_{\overline{Q_i}}(f) - f\right] \nabla\chi_i$: we have $\|h\|_{L^r} \lesssim \alpha(\mu(\Omega))^{1/r}$.
To prove this, we fix a point $x \in \Omega$ and let Q_j be a Whitney ball containing x. For all $i \in I_x$ as $r_{Q_i} \simeq r_{Q_j}$, we have

$$\left\|A_{\overline{Q_i}}(f) - A_{\overline{Q_j}}(f)\right\|_{L^r(Q_i)} \lesssim r_j \mu(Q_j)^{1/r}\alpha. \tag{14}$$

Indeed, since $Q_i \subset 7Q_j$, this is a direct consequence of the assumed "off-diagonal" decays and the fact that $10\overline{Q_i} \cap F \neq \emptyset$. Using $\sum_i \nabla\chi_i(x) = 0$, we deduce that

$$\|h\|_{L^r(Q_j)} \lesssim \sum_{i\in I_x} \left\|A_{\overline{Q_i}}(f) - A_{\overline{Q_j}}(f)\right\|_{L^r(Q_j)} r_j^{-1} \lesssim N\alpha\mu(Q_j)^{1/r} \lesssim \alpha\mu(Q_j)^{1/r}. \tag{15}$$

Using again the bounded overlap property of the $(Q_i)_i$'s, it follows that

$$\|h\|_{L^r} \lesssim \alpha(\mu(\Omega))^{1/r}.$$

Hence

$$\| \, |\nabla g| \, \|_{L^r(\Omega)} \lesssim \alpha(\mu(\Omega))^{1/r}.$$

Then (8) and the L^r estimate of $|\nabla g|$ on F yield $\|\nabla g\|_{L^r} \lesssim \|f\|_{W^{1,p}}^{p/r}\alpha^{1-p/r}$. Let us now estimate $\|g\|_{L^r}$. We have $g = f\mathbf{1}_F + \sum_i A_{\overline{Q_i}}(f)\chi_i$. Since $|f|\mathbf{1}_F \leq \alpha$, still need to estimate $\|\sum_i A_{\overline{Q_i}}(f)\chi_i\|_{L^r}$. Note that as in (13), we similarly have for every i

$$\left\|A_{\overline{Q_i}}(f)\right\|_{L^r(Q_i)} \lesssim \alpha\mu(Q_i)^{1/r}. \tag{16}$$

As above, this last inequality yields (thanks to the bounded overlap property of the $(Q_i)_i$)

$$\|g\|_{L^r(\Omega)} \lesssim \alpha(\mu(\Omega))^{1/r}.$$

Finally, (8) and the L^r estimate of g on F yield $\|g\|_{L^r} \lesssim \|f\|_{W^{1,p}}^{p/r}\alpha^{1-p/r}$. Therefore we proved that g belongs to $W^{1,r}$ with the desired boundedness. $\square$

Remark 3.5 *Note that in this decomposition, $\nabla\mathbf{1}_\Omega$ corresponds to a singular distribution, supported in $\partial\Omega$. In the previous proof, we considered that the distribution $\nabla\mathbf{1}_\Omega$ corresponds to a function, vanishing almost everywhere. The estimate (15) shows that h (considered as an L^1_{loc}-function) satisfies the good property. We also have to check that h can be considered as an L^1_{loc}-function. This is due to the following fact*

$$\sum_{i,j} \left[A_{\overline{Q_j}}(f)\chi_j - f\right] \nabla\chi_i = 0$$

in the distributional sense. This equality shows that when we are close to supp$(\sum \nabla\chi_i) = \partial\Omega$, the corresponding operator $A_{\overline{Q_j}}$ tends to the identity operator, due to Poincaré inequality. We do not detail this technical problem and refer to [4].

Remark 3.6 *In the case where the operator A_Q is the mean-operator over the ball Q, the assumption "$M_{\mathcal{A},q} = \mathcal{M}_q$ is continuous from $W^{1,p}$ to $L^{p,\infty}$" is always satisfied. The Poincaré inequality (P_q) corresponds to the "classical one" (in fact it is weaker since that in the classical one it appears only the $L^q(Q)$ norm of the gradient of the function) . Moreover "$L^q - L^\infty$ off-diagonal estimates" hold obviously. Thus, we regain the well-known Calderón-Zygmund decomposition in Sobolev spaces.*

3.2 Application to real Interpolation of Sobolev spaces.

As described in [11], such a "Calderón-Zygmund" decomposition in Sobolev spaces is sufficient to obtain a real interpolation result for Sobolev spaces.

Theorem 3.7 *Let M be a complete Riemannian manifold of infinite measure satisfying (D) and admitting a Poincaré inequality (P_q) for some $q \in [1, \infty)$ relatively to the collection $\mathcal{A}$. Assume that $\mathcal{A}$ satisfies "$L^q - L^r$ off-diagonal estimates" for an $r \in (q, \infty]$. Then for $1 \leq s \leq p < r \leq \infty$ with $p > q$, the space $W^{1,p}$ is a real interpolation space between $W^{1,s}$ and $W^{1,r}$. More precisely*

$$W^{1,p} = (W^{1,s}, W^{1,r})_{\theta,p}$$

where $\theta \in (0,1)$ such that

$$\frac{1}{p} := \frac{1-\theta}{s} + \frac{\theta}{r} < \frac{1}{q}.$$

We do not detail the proof and refer the reader to [11] for the link between such a "Calderón-Zygmund" decomposition and interpolation results. We briefly explain the main steps of the proof.

Proof : It is sufficient to prove that there exists $C > 0$ such that for every $f \in W^{1,p}$ and $t > 0$,

$$\begin{aligned} & K(f, t, W^{1,s}, W^{1,r}) \\ & \lesssim \left(t^{\frac{r}{r-s}} \left[|f|^{q**} + |\nabla f|^{q**} \right]^{1/q} (t^{\frac{rs}{r-s}}) + t \left[\int_{t^{\frac{rs}{r-s}}}^{\infty} (\mathcal{M}(|f| + |\nabla f|)^q)^{*r/q} (u) du \right]^{1/r} \right). \end{aligned} \quad (17)$$

We consider the previous Calderón-Zygmund decomposition for f with

$$\alpha = \alpha(t) = [\mathcal{M}_q(|f| + |\nabla f|) + M_{A,q}(f)]^{q* \frac{1}{q}} (t^{\frac{rs}{r-s}}).$$

We write $f = \sum_i b_i + g = b + g$ where $(b_i)_i$, g satisfy the properties of Theorem 3.3. From the bounded overlap property of the B_i's, it follows that

$$\begin{aligned} \|b\|_{W^{1,s}}^s &\leq N \sum_i \|b_i\|_{W^{1,s}}^s \\ &\lesssim \alpha^s(t) \sum_i \mu(B_i) \\ &\lesssim \alpha^s(t) \mu(\Omega_t), \end{aligned}$$

with $\Omega_t = \cup_i B_i$. For g, we have as in [11], proof of Theorem 4.2, p.15

$$\begin{aligned} \int_{F_t} (|g|^r + |\nabla g|^r) \, d\mu = & \int_{F_t} (|f|^r + |\nabla f|^r) \, d\mu \\ & \lesssim \int_{t^{\frac{rs}{r-s}}}^{\infty} (\mathcal{M}(|f| + |\nabla f|)^q)^{* \frac{r}{q}} (u) du \\ & + t^{\frac{rs}{r-s}} (|f|^{q**} + |\nabla f|^{q**})^{\frac{r}{q}} (t^{\frac{rs}{r-s}}) \end{aligned}$$

where F_t is the complement of Ω_t. For the Sobolev norm of g in Ω, we use the estimate of the Calderón-Zygmund decomposition. Moreover, since $(\mathcal{M}f)^* \sim f^{**}$ and $(f+g)^{**} \leq f^{**} + g^{**}$ (c.f [17],[18]) and thanks to the "$(L^q - L^r)$ off-diagonal" assumption on $\mathcal{A}$, we have

$$\alpha(t) \lesssim \left(|f|^{q**\frac{1}{q}}(t^{\frac{rs}{r-s}}) + |\nabla f|^{q**\frac{1}{q}}(t^{\frac{rs}{r-s}}) \right).$$

The choice of $\alpha(t)$ implies $\mu(\Omega_t) \leq t^{\frac{rs}{r-s}}$ (c.f [17],[18]). Finally (17) follows from the fact that

$$K(f,t,W^{1,s},W^{1,r}) \leq \|b\|_{W^{1,s}} + t\|g\|_{W^{1,r}}$$

and the good estimates of $\|b\|_{W^{1,s}}$ and $\|g\|_{W^{1,r}}$. □

Remark 3.8 *As explained in [10, 11], to interpolate the non-homogeneous Sobolev spaces, it is sufficient to assume local doubling (D_{loc}) and local Poincaré inequality (P_{qloc}) relatively to $\mathcal{A}$. In these assumptions, we restrict to balls Q of radius sufficiently small.*

We now give an homogeneous version of all these results and then give applications.

3.3 Homogeneous version

We begin recalling the definition of homogeneous Sobolev spaces on a manifold.

Let M be a C^∞ Riemannian manifold of dimension n. For $1 \leq p \leq \infty$, we define $\dot{E}^{1,p}$ to be the vector space of distributions φ with $|\nabla\varphi| \in L^p$, where $\nabla\varphi$ is the distributional gradient of φ. We equip $\dot{E}^{1,p}$ with the semi-norm

$$\|\varphi\|_{\dot{E}^{1,p}} = \|\,|\nabla\varphi|\,\|_{L^p}.$$

The homogeneous Sobolev space $\dot{W}^{1,p}$ is then the quotient space $\dot{E}^1_p/\mathbb{R}$.

Remark 3.9 *1.For all $\varphi \in \dot{E}^{1,p}$, $\|\overline{\varphi}\|_{\dot{W}^{1,p}} = \|\,|\nabla\varphi|\,\|_{L^p}$, where $\overline{\varphi}$ denotes the class of φ.*
2. The space $\dot{W}^{1,p}$ is a Banach space (see [27]).

We then have all the homogeneous version of our results. We only state them, their proofs being the same as in the non-homogeneous case with few modifications due to the homogeneous norm.

Let $\mathcal{A} := (A_Q)_Q$ be a collection of operators (acting from $\dot{W}^{1,p}$ to $\dot{W}^{1,p}_{loc}$) indexed by the balls of the manifold. We define analogously new homogeneous maximal operator associated to this collection: for $1 \leq s \leq p \leq \infty$ and all functions $f \in \dot{W}^{1,p}$

$$\dot{M}_{\mathcal{A},s}(f)(x) := \sup_{Q;\,Q\ni x} \frac{1}{\mu(Q)^{1/s}} \|\,|\nabla A_Q(f)|\,\|_{L^s(Q)}.$$

The assumptions that we need on the collection $\mathcal{A}$ are then the following:

Definition 3.10 1) *We say that for $q \in [1,\infty]$, the manifold M satisfies an homogeneous Poincaré inequality $(\dot{P}_q)$ relatively to the collection $\mathcal{A}$ if there is a constant C such that for every ball Q (of radius r_Q) and for all functions $f \in \dot{W}^{1,p}$; $p \geq q$:*

$$\left(\fint_Q |f - A_Q(f)|^q \, d\mu\right)^{1/q} \leq C r_Q \sup_{s\geq 1} \left(\fint_{sQ} |\nabla f|^q d\mu\right)^{1/q}.$$

2) *We say that the collection $\mathcal{A}$ satisfies "$L^q - L^r$ homogeneous off-diagonal estimates" if*

a. *there are constants $C' > 0$ and $N \in \mathbb{N}^*$ such that for all equivalent balls Q, Q' (i.e. $Q \subset Q' \subset NQ$; $N \in \mathbb{N}^*$) and all functions $f \in \dot{W}^{1,p}$; $p \geq q$, we have*

$$\frac{1}{\mu(Q)^{1/r}} \left\| A_Q(f) - A_{Q'}(f) \right\|_{L^r(NQ)} \leq C' r_Q \inf_{NQ} \mathcal{M}_q \left(|\nabla f|\right)$$

b. *and for every ball Q*

$$\frac{1}{\mu(Q)^{1/r}} \left\| \, |\nabla A_Q(f)| \, \right\|_{L^r(Q)} \leq C' \inf_{Q} \mathcal{M}_q \left(|\nabla f|\right). \tag{18}$$

Then, we get the homogeneous version of the Calderón-Zygmund decomposition:

Theorem 3.11 *Let M be a complete Riemannian manifold satisfying (D) and of infinite measure. Consider a collection $\mathcal{A} = (A_Q)_Q$ of operators defined on M. Assume that M satisfies the Poincaré inequality $(\dot{P}_q)$ relatively to the collection $\mathcal{A}$ for some $q \in [1,\infty)$ and that $\mathcal{A}$ satisfies $L^q - L^r$ " homogeneous off-diagonal estimates" for an $r \in (q,\infty]$. Let $f \in \dot{W}^{1,p}$ and $\alpha > 0$. Then one can find a collection of balls (Q_i), functions $g \in \dot{W}^{1,r}$ and $b_i \in \dot{W}^{1,q}$ with the following properties*

$$f = g + \sum_i b_i \tag{19}$$

$$\|g\|_{\dot{W}^{1,r}} \lesssim \|f\|_{\dot{W}^{1,p}}^{p/r} \alpha^{1-p/r}, \int_{\cup_i Q_i} |\nabla g|^r d\mu \lesssim \alpha^r \mu(\cup_i Q_i) \tag{20}$$

$$supp\,(b_i) \subset Q_i, \; \|b_i\|_{\dot{W}^{1,q}} \lesssim \alpha \mu(Q_i)^{1/q} \tag{21}$$

$$\sum_i \mu(Q_i) \leq C\alpha^{-p} \int |\nabla f|^p d\mu \tag{22}$$

$$\sum_i \mathbf{1}_{Q_i} \leq N. \tag{23}$$

This decomposition will give us the following homogeneous interpolation result:

Theorem 3.12 *Let M be a complete Riemannian manifold of infinite measure satisfying (D) and admitting a Poincaré inequality $(\dot{P}_q)$ for some $q \in [1,\infty)$ relatively to the collection $\mathcal{A}$. Assume that $\mathcal{A}$ satisfies $L^q - L^r$ " homogeneous off-diagonal estimates" for an $r \in (q,\infty]$.*
Then for $1 \leq s \leq p < r \leq \infty$ with $p > q$, the space $\dot{W}^{1,p}$ is a real interpolation space between $\dot{W}^{1,s}$ and $\dot{W}^{1,r}$. More precisely

$$\dot{W}^{1,p} = (\dot{W}^{1,s}, \dot{W}^{1,r})_{\theta,p}$$

where $\theta \in (0,1)$ such that

$$\frac{1}{p} := \frac{1-\theta}{s} + \frac{\theta}{r} < \frac{1}{q}.$$

4 Pseudo-Poincaré inequalities and Applications

4.1 The particular case of "Pseudo-Poincaré Inequalities"

Thanks to [2, 3], we know that under (D), a Poincaré inequality (P_q) guarantees the assumptions of Theorem 3.3 when A_Q is the mean-operator over the ball Q. Thus it permits to prove a Calderón-Zygmund decomposition for Sobolev functions.
The aim of this subsection is to show, using a particular choice of operators A_Q, that our assumptions are weaker than the classical Poincaré inequality used in the already known decomposition.
Let Δ be the positive Laplace-Beltrami operator and let us set $A_Q := e^{-r_Q^2\Delta}$ for each ball Q of radius r_Q. In all this section, we work with these operators. In order to obtain a Calderón-Zygmund decomposition as in Theorem 3.3, we need to put some assumptions on $(A_Q)_Q$ as those in Section 3.

According to this choice of operators, we define what are "Pseudo-Poincaré inequalities".

Definition 4.1 (Pseudo-Poincaré inequality on M) *We say that a complete Riemannian manifold M admits* ***a pseudo-Poincaré inequality*** $(\widetilde{P_q})$ *for some $q \in [1,\infty)$ if there exists a constant $C > 0$ such that, for every function $f \in C_0^\infty$ and every ball Q of M of radius $r > 0$, we have*

$$\left(\fint_Q |f - e^{-r^2\Delta} f|^q d\mu\right)^{1/q} \leq Cr \sup_{s\geq 1} \left(\fint_{sQ} |\nabla f|^q d\mu\right)^{1/q}. \qquad (\tilde{P}_q)$$

Pseudo-Poincaré inequalities corresponds to what we called Poincaré inequality relatively to this collection $\mathcal{A}$ (the homogeneous version, we can also consider the non-homogeneous one).

We begin showing that pseudo-Poincaré inequalities are implied by the classical Poincaré inequalities. We denote

$$q_0 := \inf\{q \in [1,\infty);\ (P_q) \text{ holds }\}. \qquad (q_0)$$

Proposition 4.2 *Let M be a complete manifold satisfying (D) and admitting a Poincaré inequality (P_q) for some $1 \leq q < \infty$.*

1. *If $q_0 < 2$ then the pseudo-Poincaré inequality $(\widetilde{P_q})$ holds.*
2. *If $q_0 \geq 2$, we moreover assume (DUE). Then $(\widetilde{P_q})$ also holds.*

Before proving this proposition, we give the following covering Lemma.

Lemma 4.3 *Let M be a complete manifold satisfying (D). Let Q a ball of radius r_Q. Then there exists a bounded covering $(Q_j)_j$ of Q with balls of radius $t^{1/2}$ for $0 < t \leq r_Q^2$. Moreover, for $s \geq 1$, the collection $(sQ_j)_j$ is a s-covering of sQ, that is :*

$$\sup_{x\in sQ} \sharp\{j,\ x \in sQ_j\} \lesssim s^d,$$

where d is the homogeneous dimension of the manifold.

Proof : We choose $\left(Q(x_j, t^{1/2}/3)\right)_j$ a maximal collection of disjoint balls in Q. Then we set $Q_j = Q(x_j, t^{1/2})$, which is a covering of Q.
Fix $x \in sQ$ and denote $J_x := \{j,\ x \in sQ_j\}$. Take $j_0 \in J_x$ (if $J_x \neq \emptyset$ otherwise, there is nothing to prove). By (D), we have

$$\begin{aligned}
(\sharp J_x)\, \mu\left(sQ_{j_0}\right) &\lesssim (\sharp J_x)\, s^d \mu\left(\frac{1}{3} Q_{j_0}\right) \\
&\lesssim s^d \sum_{j \in J_x} \mu\left(\frac{1}{3} Q_j\right) \\
&\lesssim s^d \mu\left(\cup_{j \in J_x} \frac{1}{3} Q_j\right) \\
&\lesssim s^d \mu\left(Q(x, 2st^{1/2})\right) \\
&\lesssim s^d \mu\left(sQ_{j_0}\right),
\end{aligned}$$

where we used the fact that the balls $\frac{1}{3}Q_j$ are disjoint and have equivalent measure when the index $j \in J_x$. $\square$

Proof of Proposition 4.2 Consider a ball Q of radius $r > 0$. We deal with the semigroup and write the oscillation as follows

$$f - e^{-r^2\Delta} f = -\int_0^{r^2} \frac{d}{dt} e^{-t\Delta} f dt = \int_0^{r^2} \Delta e^{-t\Delta} f dt.$$

Now we apply arguments used in [7], Lemma 3.2. Using the completeness of the manifold, we have

$$\begin{aligned}
\left(\frac{1}{\mu(Q)} \int_Q \left| \int_0^{r^2} \Delta e^{-t\Delta} f dt \right|^q d\mu\right)^{1/q} &\lesssim \int_0^{r^2} \left(\frac{1}{\mu(Q)} \int_Q \left|\Delta e^{-t\Delta} f\right|^q d\mu\right)^{1/q} dt \\
&\lesssim \int_0^{r^2} \left(\frac{1}{\mu(Q)} \sum_j \int_{Q_j} \left|\Delta e^{-t\Delta}(f - f_{Q_j})\right|^q d\mu\right)^{1/q} dt,
\end{aligned}$$

where $(Q_j)_j$ is a bounded covering of Q with balls of radius $t^{1/2}$ as in Lemma 4.3.
Fix $t \in (0, r^2)$ and denote by $C_k(Q_j) := 2^{k+1}Q_j \setminus 2^k Q_j$ for $k \geq 1$ and $C_0(Q_j) = 2Q_j$.

Then, arguing as in Lemma 3.2 in [7]

$$
\begin{aligned}
&\sum_j \int_{Q_j} \left|\Delta e^{-t\Delta}(f-f_{Q_j})\right|^q d\mu \\
&\quad\lesssim \sum_j \int_{Q_j} t^{-q} \left|\int_M \frac{e^{-cd^2(x,y)/t}}{\mu(Q(y,\sqrt{t}))}(f(y)-f_{Q_j})d\mu(y)\right|^q d\mu(x) \\
&\quad\lesssim \sum_{j,k;k\geq 0} \int_{Q_j} t^{-q}(\mu(2^{k+1}Q_j))^{q-1} \int_{C_k(Q_j)} \frac{e^{-cqd^2(x,y)/t}}{\mu(Q(y,\sqrt{t}))^q}|f(y)-f_{Q_j}|^q d\mu(y)d\mu(x) \\
&\quad\lesssim \sum_{j,k;k\geq 1} t^{-q}(\mu(2^{k+1}Q_j))^{q-1} \int_{C_k(Q_j)} \left(\int_{\{x;\, d(x,y)\geq 2^{k-1}\sqrt{t}\}} e^{-cqd^2(x,y)/t}d\mu(x)\right) \frac{|f(y)-f_{Q_j}|^q}{\mu(Q(y,\sqrt{t}))^q}d\mu(y) \\
&\quad+\sum_j t^{-q}\frac{1}{\mu(Q_j)^q}(\mu(2Q_j))^{q-1}\int_{2Q_j}\left(\int_{Q_j} d\mu(x)\right)\left|f(y)-f_{Q_j}\right|^q d\mu(y) \\
&\quad\lesssim \sum_j t^{-q}\sum_{k\geq 1} e^{-cq4^k}2^{kdq}\int_{C_k(Q_j)}\left|f(y)-f_{Q_j}\right|^q d\mu(y) \\
&\quad+\sum_j t^{-q}\int_{2Q_j}\left|f(y)-f_{Q_j}\right|^q d\mu(y) \\
&\quad\lesssim \sum_j t^{-q}\sum_{k\geq 1} e^{-cq4^k}2^{kdq}\int_{2^{k+1}Q_j}\left|f(y)-f_{2^{k+1}Q_j}\right|^q d\mu(y)+\sum_{l=1}^{k+1}\frac{\mu(2^{k+1}Q_j)}{\mu(2^lQ_j)}|f_{2^lQ_j}-f_{2^{l-1}Q_j}| \\
&\quad+\sum_j t^{-q}\int_{2Q_j}\left|f(y)-f_{Q_j}\right|^q d\mu(y) \\
&\quad\lesssim \sum_j t^{-q}\sum_{k\geq 1} e^{-cq4^k}2^{Mk}t^{q/2}\sum_{l=1}^{k+1}\int_{2^lQ_j}|\nabla f|^q d\mu+\sum_j t^{-q}t^{q/2}\int_{2Q_j}|\nabla f|^q d\mu.
\end{aligned}
$$

We used (2), (P_q), that for $y\in 2Q_j$, $\mu(Q(y,\sqrt{t}))\sim\mu(Q_j)$ and for $y\in C_k(Q_j)$, $k\geq 1$, $\frac{1}{\mu(Q(y,\sqrt{t}))}\leq C\frac{2^{kd}}{\mu(2^{k+1}Q_j)}$. We also used that for s, $t>0$,

$$\int_{\{x;\, d(x,y)\geq\sqrt{t}\}} e^{-cd^2(x,y)/s}d\mu(x)\leq Ce^{-ct/s}\mu(Q(y,\sqrt{s}))$$

thanks to (D) (see Lemma 2.1 in [24]).

Using that $(2^lQ_j)_j$ is a 2^l-bounded covering of 2^lQ, we deduce that

$$\sum_j\int_{2^lQ_j}|\nabla f|^q d\mu\lesssim 2^{ld}\int_{2^lQ}|\nabla f|^q d\mu\leq 4^{ld}\mu(Q)\sup_{s\geq 1}\fint_{sQ}|\nabla f|^q d\mu,$$

where d is the homogeneous dimension of the doubling manifold. Thus, it follows that

$$\left(\frac{1}{\mu(Q)}\int_Q\left|\int_0^{r^2}\Delta e^{-t\Delta}(f)dt\right|^q d\mu\right)^{1/q}\lesssim\left[\int_0^{r^2}t^{-1/2}dt\right]\sup_{s\geq 1}\left(\fint_{sQ}(|\nabla f|^q)d\mu\right)^{1/q},$$

which ends the proof. □

Before we prove off-diagonal estimates under the "classical" Poincaré inequality, let us recall the following result :

Proposition 4.4 *([6]) Let M be a complete Riemannian manifold satisfying (D) and (P_2). Then there exists $p_0 > 2$ such that the Riesz transform $\mathcal{R} := \nabla(-\Delta)^{-\frac{1}{2}}$ is L^p bounded for $1 < p < p_0$.*

We now let

$$p_0 := \sup\left\{p \in (2,\infty);\ \nabla(-\Delta)^{-\frac{1}{2}} \text{ is } L^p \text{ bounded}\right\} \tag{p_0}$$

and

$$s_0 := \sup\left\{s \in (1,\infty];\ (G_s) \text{ holds}\right\}. \tag{s_0}$$

Remark 4.5 *Note that the doubling property (D) and (DUE) imply for $p \in (1,2]$, the L^p boundedness of $\nabla\Delta^{-\frac{1}{2}}$ which implies (G_p) (see Subsection 2.3) and that $s_0 \geq p_0 > 2$.*

For the second off-diagonal condition (4), we obtain :

Proposition 4.6 *Let M be a complete manifold. Assume that M satisfies (D) and admits a classical Poincaré inequality (P_q) for some $q \in [1,\infty)$ as in Definition 2.4. Consider the following estimate*

$$M_{\mathcal{A},r}(f) \lesssim \mathcal{M}_q(|f| + |\nabla f|). \tag{24}$$

1. *If $q_0 < 2$, then (24) holds for all $r \in (q, s_0)$.*
2. *If $q_0 \geq 2$, assume moreover (DUE) and that $s_0 > q$. Then (24) holds for all $r \in (q, s_0)$.*

Consequently, (4) holds for all $r \in (q, s_0)$.

Proof : It is sufficient to prove the following inequalities

$$\left(\fint_Q |e^{-r^2\Delta} f|^r d\mu\right)^{1/r} \leq C\mathcal{M}_q(|f|)(x) \tag{25}$$

and

$$\left(\fint_Q |\nabla e^{-r^2\Delta} f|^r d\mu\right)^{1/r} \leq C\mathcal{M}_q(|\nabla f|)(x) \tag{26}$$

for every $x \in M$ and every ball Q containing x. We do not detail the proof as it uses analogous argument as in [7], subsection 3.1, Lemma 3.2 and the end of this subsection. For example, (26) is essentially inequality (3.12) in section 3 of [7] where $q_0 = 2$. We just mention that for (25), we use the L^r contractivity of the heat semigroup, (D) and (DUE). For (26), we moreover need the following L^r-Gaffney estimates for $\nabla e^{-t\Delta}$ with $r \in (q_0, s_0)$. We say that $(\nabla e^{-t\Delta})_{t>0}$ satisfies the L^p Gaffney estimate if there exists C, $\alpha > 0$ such that for all $t > 0$, E, F closed subsets of M and f supported in E

$$\|\sqrt{t}|\nabla e^{-t\Delta} f|\|_{L^p(F)} \leq Ce^{-\alpha d(E,F)^2/t}\|f\|_{L^p(E)}. \tag{Ga_p}$$

In the case where $q_0 \geq 2$, interpolating the already known (Ga_2) with (G_s) for every $2 < s < s_0$, we get the (Ga_p) for $2 < p < s_0$. When $q_0 < 2$, since in this case (G_s) holds for all $1 < s < 2$ and $2 < s < s_0$, interpolating again (G_s) and (Ga_2), we obtain the (Ga_p) for all $1 < p < s_0$. □

It remains to check (3).

Proposition 4.7 *Let M be a complete manifold satisfying (D) and admitting a classical Poincaré inequality (P_q) for some $1 \leq q < \infty$. Then*

1. *If $q_0 < 2$, for $r > q$, the collection $\mathcal{A}$ satisfies "$(L^q - L^r)$ off-diagonal" estimates (3).*

2. *If $q_0 \geq 2$, the same result holds under the additional assumption (DUE).*

Proof : Take Q_0, Q_1 two equivalent balls, let us say $Q_0 \subset Q_1 \subset 10Q_0$ with radius r_0 (resp. r_1). We choosed a numerical factor 10 just for convenience. We have to prove that

$$\left(\frac{1}{\mu(Q_0)} \int_{10Q_0} \left| e^{-r_0^2 \Delta} f - e^{-r_1^2 \Delta} f \right|^r d\mu \right)^{1/r} \lesssim r_0 \inf_{10Q_0} \mathcal{M}_q(|f| + |\nabla f|). \tag{27}$$

This is a consequence of

$$\left(\frac{1}{\mu(Q_0)} \int_{10Q_0} \left| e^{-r_0^2 \Delta} f - e^{-400 r_0^2 \Delta} f \right|^r d\mu \right)^{1/r} \lesssim r_0 \inf_{10Q_0} \mathcal{M}_q(|f| + |\nabla f|) \tag{28}$$

and

$$\left(\frac{1}{\mu(Q_0)} \int_{10Q_0} \left| e^{-400 r_0^2 \Delta} f - e^{-r_1^2 \Delta} f \right|^r d\mu \right)^{1/r} \lesssim r_0 \inf_{10Q_0} \mathcal{M}_q(|f| + |\nabla f|). \tag{29}$$

We use that

$$e^{-r_0^2 \Delta} f - e^{-400 r_0^2 \Delta} f = e^{-r_0^2 \Delta} \left[1 - e^{-399 r_0^2 \Delta} \right] (f)$$

and

$$e^{-400 r_0^2 \Delta} f - e^{-r_1^2 \Delta} f = -e^{-r_1^2 \Delta} \left[1 - e^{-(20 r_0)^2 - r_1^2) \Delta} \right] (f).$$

We only deal with (28), we do the same for (29). From (D) and (DUE), we know that (UE) holds and so we have very fast decays $(L^1 - L^\infty)$ for the semigroup, which permits to gain integrability from L^q to L^r. It follows

$$\begin{aligned} &\left(\frac{1}{\mu(Q_0)} \int_{10Q_0} \left| e^{-r_0^2 \Delta} f - e^{-400 r_0^2 \Delta} f \right|^r d\mu \right)^{1/r} \\ &\qquad \lesssim \sum_{j \geq 0} e^{-\gamma 4^j} \left(\frac{1}{\mu(Q_0)} \int_{C_j(Q_0)} \left| f - e^{-399 r_0^2 \Delta} f \right|^q d\mu \right)^{1/q}, \end{aligned}$$

where we make appear the dyadic coronas $C_j(Q_0)$ (see again [7], Lemma 3.2 and the end of subsection 3.1). Then we use (D) and (P_q). For each j, we choose a bounded covering

$(Q_i^j)_i$ of $2^{j+1}Q_0$ with balls of radius $\sqrt{399}r_0$ and obtain

$$\begin{aligned}
\frac{1}{\mu(Q_0)}\int_{C_j(Q_0)}\left|f-e^{-399r_0^2\Delta}f\right|^q d\mu &\lesssim \frac{1}{\mu(Q_0)}\sum_i\int_{Q_i^j}\left|f-e^{-399r_0^2\Delta}f\right|^q d\mu \\
&\lesssim \frac{1}{\mu(Q_0)}\sum_i\int_{Q_i^j}\left|f-e^{-399r_0^2\Delta}f\right|^q d\mu \\
&\lesssim \frac{1}{\mu(Q_0)}\sum_i r_0^q\mu(Q_i^j)\sup_{s\geq 1}\left(\fint_{sQ_i^j}|\nabla f|^q d\mu\right) \\
&\lesssim \frac{1}{\mu(Q_0)}\sum_i r_0^q\mu(Q_i^j)\sup_{s\geq 1}2^{dj}\left(\fint_{s2^{j+1}Q_0}|\nabla f|^q d\mu\right) \\
&\lesssim \frac{1}{\mu(Q_0)}\sum_i r_0^q\mu(Q_i^j)2^{dj}\inf_{Q_0}\mathcal{M}\left(|\nabla f|^q\right) \\
&\lesssim r_0^q 2^{dj}\frac{\mu(2^{j+1}Q_0)}{\mu(Q_0)}\left(\inf_{Q_0}\mathcal{M}_q(|\nabla f|)\right)^q \\
&\lesssim r_0^q 2^{2dj}\left(\inf_{Q_0}\mathcal{M}_q(|\nabla f|)\right)^q .
\end{aligned}$$

We applied (P_q) in the third inequality. In the fourth inequality, we used that $sQ_i^j \subset 2^{j+1}sQ_0$ and thanks to (D), $\mu(2^{j+1}sQ_0) \lesssim \mu(sQ_i^j)2^{jd}$. Then we applied the bounded overlap property in the sixth one.
Summing in j, we show the desired inequality (28). Similarly we prove (29), which completes the proof of (27). □

We get the following corollary:

Corollary 4.8 *Assume that M is complete, satisfies (D) and admits a classical Poincaré inequality (P_q) for some $q \in [1,\infty)$. In the case where $q_0 \geq 2$, we moreover assume (DUE) and $s_0 > q$. Then the assumptions of Theorem 3.3 and 3.7 hold. We have pseudo-Poincaré inequality $(\widetilde{P}_q)$ and $\mathcal{A}$ satisfies "$L^q - L^r$ off-diagonal estimates" for $r \in (q, s_0)$.*

Conclusion : When $q < 2$, the assumptions of Theorem 3.3 (according to this particular choice of $\mathcal{A}$) are weaker than the Poincaré inequality and are sufficient to get the Calderón-Zygmund decomposition.

We also have the homogeneous version:

Corollary 4.9 *Assume that M is complete, satisfies (D) and admits a classical Poincaré inequality (P_q) for some $1 \leq q < \infty$. In the case where $q_0 \geq 2$, we moreover assume (DUE).*
Let $\mathcal{A} := (A_Q)_Q$ with $A_Q := e^{-r_Q^2\Delta}$. Then the assumptions of Theorems 3.11 and 3.12 holds. We have pseudo-Poincaré inequality $(\widetilde{P}_q)$, $\mathcal{A}$ satisfies "homogeneous $L^q - L^r$ off-diagonal estimates" for $r \in (q, s_0)$.

4.2 Application to Reverse Riesz transform inequalities.

We refer the reader to [6, 7] for the study of the so-called (RR_p) inequalities :

$$\|\Delta^{1/2} f\|_{L^p} \lesssim \||\nabla f|\|_{L^p}. \tag{RR_p}$$

We know that (RR_2) is always satisfied and that (D) and (DUE) implies (RR_p) for all $p \in (2, \infty)$. For the exponents lower than 2, P. Auscher and T. Coulhon obtained the following result ([6]) :

Theorem 4.10 *Let M be a complete non-compact doubling Riemannian manifold. Moreover assume that the classical Poincaré inequality (P_q) holds for some $q \in (1, 2)$. Then for all $p \in (q, 2)$, (RR_p) is satisfied.*

This result is based on a Calderón-Zygmund decomposition for Sobolev functions. Using our new assumptions, we also obtain the following improvement :

Theorem 4.11 *Assume that M is complete, satisfies (D) and admits a pseudo-Poincaré inequality $(\widetilde{P}_q)$ for some $q \in (1, 2)$. If in addition, the collection $\mathcal{A}$ satisfies $L^q - L^2$ "off-diagonal estimates", then (RR_p) holds for all $p \in (q, 2)$.*

Remark 4.12 *Corollary 4.8 shows that these new assumptions are weaker than the Poincaré inequality (P_q).*

We do not prove this result and refer the reader to [6]. The proof is exactly the same as it relies on the Calderón-Zygmund decomposition.

Remark 4.13 *We refer the reader to other works of the authors [21, 15]. In [21], the assumption (RR_p) plays an important role in order to prove some maximal inequalities in dual Sobolev spaces $W^{-1,p}$, which do not require Poincaré inequalities. So it might be important to know how to prove (RR_p) without Poincaré inequality.*

4.3 Application to Gagliardo-Nirenberg inequalities.

We devote this subsection to the study of Gagliardo-Nirenberg inequalities. We refer the reader to [12] for a recent work on this subject.

Definition 4.14 *We introduce the Besov space. For $\alpha < 0$, we set $B^{\alpha}_{\infty,\infty}$ the set of all measurable functions f such that*

$$\|f\|_{B^{\alpha}_{\infty,\infty}} := \sup_{t>0} t^{-\frac{\alpha}{2}} \|e^{-t\Delta} f\|_{L^\infty} < \infty.$$

We have the following equivalence (Lemma 2.1 in [12]) :

$$\|f\|_{B^{\alpha}_{\infty,\infty}} \sim \sup_{t>0} t^{-\frac{\alpha}{2}} \|e^{-t\Delta}(f - e^{-t\Delta} f)\|_{L^\infty}.$$

Then, the so-called Gagliardo-Nirenberg inequalities are :

$$\|f\|_l \lesssim \| \, |\nabla f| \, \|_p^\theta \|f\|^{1-\theta}_{B^{\frac{\theta}{\theta-1}}_{\infty,\infty}} \tag{30}$$

where $\theta = \frac{p}{l}$ for some $p, l \in [1, \infty)$.

We first recall one of the main results of [12]:

Theorem 4.15 *Let M be a complete non-compact Riemannian manifold satisfying (D) and (P_q) for some $1 \leq q < \infty$. Moreover, assume that M satisfies the global pseudo-Poincaré inequalities (P'_q) and (P'_∞). Then (30) holds for all $q \leq p < l < \infty$.*

Here, the **global** pseudo-Poincaré inequality (P'_q) for some $q \in [1, \infty]$ corresponds to

$$\|f - e^{-t\Delta} f\|_{L^q} \leq C t^{\frac{1}{2}} \| \, |\nabla f| \, \|_{L^q}. \tag{P'_q}$$

This result requires global pseudo-Poincaré inequalities and some Poincaré inequalities with respect to balls. These two kinds of inequalities are quite different as they deal with oscillations with respect to the semigroup (for the pseudo-Poincaré inequalities) and to the mean value operators (for the Poincaré inequalities). We saw in the previous subsection, that Poincaré inequality implies pseudo-Poincaré inequality. That is why, we are looking for assumptions requiring only the Poincaré inequality, getting around the assumed global pseudo-Poincaré inequalities.

We begin first showing that pseudo-Poincaré inequalities related to balls yield global pseudo-Poincaré inequalities.

Proposition 4.16 *Let M be a complete Riemannian manifold satisfying (D) and admitting a pseudo-Poincaré inequality $(\widetilde{P}_q)$ for some $1 \leq q < \infty$. Then the global pseudo-Poincaré inequality (P'_q) holds.*

Proof : Let $t > 0$. Pick a countable set $\{x_j\}_{j \in J} \subset M$, such that $M = \bigcup_{j \in J} Q(x_j, \sqrt{t}) := \bigcup_{j \in J} Q_j$ and for all $x \in M$, x does not belong to more than N_1 balls Q_j. Then

$$\begin{aligned}
\|f - e^{-t\Delta} f\|_q^q &\leq \sum_j \int_{Q_j} |f - e^{-t\Delta} f|^q d\mu \\
&\lesssim \sum_j t^{\frac{q}{2}} \int_{Q_j} |\nabla f|^q d\mu \\
&\lesssim N_1 t^{\frac{q}{2}} \int_M |\nabla f|^q d\mu.
\end{aligned}$$

$\square$

Remark 4.17 *It is easy to see that the global pseudo-Poincaré inequality (P'_∞) is satisfied under (D) and (DUE) (see for instance [12], p.499).*

Using Propositions 4.16, 4.2 and Theorem 4.15, we get the following improvement version of Theorem 1.2 in [12] :

Theorem 4.18 *Let M be a complete Riemannian manifold satisfying (D) and admitting a Poincaré inequality (P_q) for some $1 \leq q < \infty$. If $q_0 \geq 2$, we moreover assume (DUE). Then (30) holds for all $q \leq p < l < \infty$.*

Using our new assumptions, we get also the following Gagliardo-Nirenberg theorem:

Theorem 4.19 *Assume that M satisfies the hypotheses of Theorem 3.12 with $A_Q = e^{-r_Q^2\Delta}$ and that $r = \infty$. Moreover, we assume (DUE). Then (30) holds for all $q \leq p < l < \infty$.*

Proof : The proof is analogous to that of Theorems 1.1 and 1.2 in [12]. We use our homogeneous interpolation result of Theorem 3.12. Also we need our non-homogeneous interpolation result of Theorem 3.7. It holds thanks to (25) which is true under (D) and (DUE). Moreover, (P'_q) is satisfied and (P'_∞) holds thanks to (D) and (DUE). □

As a Corollary, we obtain

Theorem 4.20 *Consider a complete Riemannian manifold M satisfying (D), (P_q) for some $1 \leq q < \infty$ and assume that there exists $C > 0$ such that for every $x, y \in M$ and $t > 0$*

$$|\nabla_x p_t(x, y)| \leq \frac{C}{\sqrt{t}\mu(B(y, \sqrt{t}))}. \tag{G}$$

((G) is equivalent to the assumption (G_∞).) In the case where $q_0 > 2$, we moreover assume (DUE). Then inequality (30) holds for all $q \leq p < l < \infty$.

Proof : In the case where $q \leq 2$, this result is already in [12]. For $q_0 \geq 2$, we are under the hypotheses of Theorem 4.19 thanks to subsection 4.1 and since (G) implies that $r = \infty$. □

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