Hölder regularity for approximations of second order elliptic operators by Galerkin schemes

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Abstract.   We prove Hölder regularity estimates for approximate solutions of second order elliptic equations obtained by Galerkin methods. The proofs rely on interpolation results for Sobolev spaces on graphs. Estimates for second order elliptic operators on rather general graphs are also obtained.

Keywords: Elliptic operators, elliptic equations, interpolation, Hölder regularity, Sobolev spaces, graphs, Galerkin methods.

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Contents

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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded C^2 domain. Let $A : \Omega \to \mathcal{M}_n(\mathbb{R})$ (where $\mathcal{M}_n(\mathbb{R})$ is the space of $n \times n$ matrices with real coefficients) be a bounded measurable function satisfying the following uniform ellipticity condition: there exists $c > 0$ such that, for almost every $x \in \Omega$ and all $\xi \in \mathbb{R}^n$,

$$A(x)\xi \cdot \xi \geq c \, |\xi|^2 \, , \tag{1.1}$$

where, for all $u, v \in \mathbb{R}^n$, $u \cdot v$ is the standard scalar product of u and v and $|\xi|$ stands for the Euclidean norm of ξ. Given $f \in L^2(\Omega)$, the Lax-Milgram theorem shows that there exists a unique function $u \in W_0^{1,2}(\Omega)$ such that

$$\mathrm{div}(A\nabla u) = f, \tag{1.2}$$

in the sense that, for all function $\varphi \in \mathcal{D}(\Omega)$,

$$\int_\Omega A(x)\nabla u(x) \cdot \nabla \varphi(x) dx = - \int_\Omega f(x)\varphi(x) dx. \tag{1.3}$$

A classical result of N. Meyers ([17]) asserts that there exists $\varepsilon > 0$, only depending on Ω and A, such that, for all $p \in (2, 2 + \varepsilon)$, $u \in W_0^{1,p}(\Omega)$ and

$$\|u\|_{W^{1,p}(\Omega)} \leq C_p \|f\|_{W^{-1,p}(\Omega)}$$

where $C_p > 0$ only depends on Ω, A and p. This results still holds when Ω is Lipschitz ([11]). Let us now focus on the case when $n = 2$ and $\Omega \subset \mathbb{R}^2$ is a polygonal domain. Let A as before. We shall be interested in approximate solutions of (1.2) obtained by Galerkin approximations based on spaces of piecewise polynomial functions. Let us briefly present these approximation schemes. By triangulation of Ω, we mean a family $\mathcal{T}$ of triangles such that

$$\Omega = \bigcup_{T \in \mathcal{T}} T.$$

Say that $\mathcal{T}$ is admissible if and only if, for all $T, T' \in \mathcal{T}$, $T \cap T'$ is either empty, or a vertex, or a common side. If $\mathcal{T}$ is an admissible triangulation of Ω, for all $T \in \mathcal{T}$, denote by h_T

the diameter of T and by ρ_T the inner diameter of T (*i.e.* the diameter of the largest ball included in T). Define also

$$V^1(\mathcal{T}) := \{v \in C(\Omega);\ v|_T \in \mathcal{P}_1\ \forall T \in \mathcal{T}\}$$

where $\mathcal{P}_1$ denotes the space of polynomials with degree less or equal to 1. Define also $V_0^1(\mathcal{T})$ as the subspace of $V^1(\mathcal{T})$ made of functions vanishing on $\partial\Omega$. It is plain to see that $V^1(\mathcal{T}) \subset W^{1,2}(\Omega)$, so that $V_0^1(\mathcal{T}) \subset W_0^{1,2}(\Omega)$. Another application of the Lax-Milgram theorem gives a unique function $u_{\mathcal{T}} \in V_0^1(\mathcal{T})$ such that

$$\int_\Omega A(x)\nabla u_{\mathcal{T}}(x) \cdot \nabla v(x)dx = - \int_\Omega f(x)v(x)dx\ \forall v \in V_0^1(\mathcal{T}) \tag{1.4}$$

(note that the bilinear form given by the left-hand side of (1.4) is still coercive on $V_0^1(\mathcal{T}) \times V_0^1(\mathcal{T})$). Assume now that, for all $h > 0$, we have an admissible triangulation $\mathcal{T}_h$ of Ω, such that

$$\sup_{T \in \mathcal{T}} h_T = h.$$

Say that the family of triangulations $(\mathcal{T}_h)_{h>0}$ is regular if and only if there exists $\sigma > 0$ such that

$$\frac{h_T}{\rho_T} \leq \sigma\ \forall T \in \mathcal{T}_h,\ \forall h > 0.$$

For all $h > 0$, write u_h instead of $u_{\mathcal{T}_h}$. Provided that the solution u of (1.2) is regular enough, the approximated solution u_h converges to u in the Sobolev sense when $h \to 0$. More precisely (see [5], Chapter 5):

Theorem 1.1 *Assume that the solution u of (1.2) in $W_0^{1,2}(\Omega)$ belongs to $W^{2,2}(\Omega)$. Then, for all $h > 0$,*

$$\|u - u_h\|_{W^{1,2}(\Omega)} \leq Ch \left\|D^2 u\right\|_{L^2(\Omega)},$$

where $C > 0$ is independent of u and h.

Note that the solution u of (1.2) need not be in $W^{2,2}(\Omega)$ in general. However, when the matrix A is of class $C^1(\Omega)$ and Ω is convex, one has $u \in W^{2,2}(\Omega)$ and $\|u\|_{W^{2,2}(\Omega)} \leq C \|f\|_{L^2(\Omega)}$, so that Theorem 1.1 yields

$$\|u - u_h\|_{W^{1,2}(\Omega)} \leq Ch \|f\|_{L^2(\Omega)}. \tag{1.5}$$

Many other convergence results in Sobolev spaces for approximated solutions of (1.2) by finite elements methods can be found in [5, 6, 12]. In the present work, we prove a convergence result analogous to Theorem 1.1 in *Hölder* spaces. Namely, we establish:

Theorem 1.2 *Assume that $\Omega \subset \mathbb{R}^2$ is a convex polygonal subset and that $A \in C^1(\Omega)$ is bounded and uniformly elliptic, in the sense that A satisfies (1.1). Then, there exist $C > 0$ and $\eta > 0$ with the following property: for all $f \in L^2(\Omega)$ and all $h > 0$, the solution $u_h \in V_0^1(\mathcal{T}_h)$ of (1.4) belongs to $C^{0,\eta}(\overline{\Omega})$ and*

$$\|u_h\|_{C^{0,\eta}(\overline{\Omega})} \leq C \|f\|_{L^2(\Omega)}. \tag{1.6}$$

Moreover, for all $\eta' \in (0, \eta)$, there exists $\theta \in (0, 1)$ such that

$$\|u_h - u\|_{C^{0,\eta'}(\overline{\Omega})} \leq Ch^\theta \|f\|_{L^2(\Omega)}. \tag{1.7}$$

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Theorem 1.2 will be an immediate consequence of:

Theorem 1.3 *Assume that $\Omega \subset \mathbb{R}^2$ is a convex polygonal subset and that $A \in C^1(\Omega)$ is bounded and uniformly elliptic, in the sense that A satisfies (1.1). Then, there exist $C > 0$ and $\varepsilon > 0$ with the following property: for all $f \in L^2(\Omega)$ and all $h > 0$, the solution $u_h \in V_0^1(\mathcal{T}_h)$ of (1.4) belongs to $W_0^{1,2+\varepsilon}(\Omega)$ and*

$$\|u_h\|_{W^{1,2+\varepsilon}(\Omega)} \le C \|f\|_{L^2(\Omega)}. \tag{1.8}$$

Moreover, for all $p \in (2, 2+\varepsilon)$, there exists $\theta \in (0,1)$ such that

$$\|u_h - u\|_{W^{1,p}(\Omega)} \le Ch^\theta \|f\|_{L^2(\Omega)}. \tag{1.9}$$

Let us give a few comments about these resuts.

Observe first that Theorem 1.2 can be seen as a De Giorgi type regularity result for approximate solutions of (1.2) obtained by the finite elements method. Under the extra assumption that A is symmetric, a version of Theorem 1.2 was proved in [19], Théorème 6.2. As in [19], the proof of Theorem 1.2 goes through the analysis of some second order elliptic operators on graphs. However, we simplify the arguments in a significant way and also get rid of the symmetry assumption on A. Moreover, we also prove the convergence result in (1.7), while only (1.6) was established in [19]. To our best knowledge, this is the first convergence result in Hölder spaces for approximations of second order elliptic equations by finite element methods.

To our best knowledge, Theorem 1.3 was never stated before. It can be seen as a discrete version of the results of [11, 17] stated at the very beginning of the paper. Since $\Omega \subset \mathbb{R}^2$, Theorem 1.3 yields Theorem 1.2 at once by means of the classical Sobolev embeddings.

Using ideas analogous to those involved in the proofs of Theorems 1.2 and 1.3, we also establish L^∞ and $C^{0,\eta}$ estimates for second order uniformly elliptic operators on quite general graphs. A lot of work was devoted to estimates for reversible Markov chains on infinite graphs (see for instance [7, 8, 10] and the references therein). In these references, the operators under consideration have positive coefficients. In the present work, we deal with more general operators, for which no positivity assumption is made on the coefficients but which are still elliptic (see Section 5 below). It seems therefore impossible to use techniques such as the Moser iteration or the maximum principle.

Let us describe our strategy for the proof of Theorem 1.3. Given a triangulation $\mathcal{T}_h$, we define a graph Γ_h, the vertices (resp. the edges) of which are the vertices (resp. the edges) of $\mathcal{T}_h$, and consider a suitable "second order elliptic" operator on Γ_h, denoted by L_h, which approximates L on Γ_h. We then reduce (1.8) to proving that the inverse of L_h is $L^2(\Gamma_h) - W^{1,2+\varepsilon}(\Gamma_h)$ bounded for some $\varepsilon > 0$.

The proof of this boundedness result for L_h^{-1} is inspired by an analogous perturbation argument for $2m$-th order operators in $\mathbb{R}^{2m}$ (see [1], Chapter 1). Namely, we first observe that L_h is bounded from $W^{1,p}(\Gamma_h)$ to $W^{-1,p}(\Gamma_h)$ for all $p \in (1, +\infty)$, and that it is an isomorphism from $W^{1,2}(\Gamma_h)$ into $W^{-1,2}(\Gamma_h)$. Then, using the fact that the $W^{1,p}(\Gamma_h)$ and the $W^{-1,p}(\Gamma_h)$ form an interpolation scale for the *complex* method, we use a general perturbation result to

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conclude that L_h is an isomorphism from $W^{1,p}(\Gamma_h)$ onto $W^{-1,p}(\Gamma_h)$ for $p \in (2 - \varepsilon, 2 + \varepsilon)$ for some $\varepsilon > 0$, which completes the proof.

We argue similarly to obtain L^∞ and $C^{0,\eta}$ estimates for second order elliptic operators on graphs Γ, under some geometric assumptions on Γ. In particular, the volume of a ball with radius $r > 0$ has to be controlled from below by r^2. These geometric assumptions imply that $L^2(\Gamma)$ embeds into $W^{-1,p}(\Gamma)$ and $W^{1,p}(\Gamma)$ embeds into $C^\eta(\Gamma)$ for $p > 2$, which concludes the proof.

Theorems 1.2 and 1.3 are stated for open polygonal subsets of $\mathbb{R}^2$. It is plausible that an adaptation of our method would yield an analogue of Theorem 1.3 for polyhedral subsets of $\mathbb{R}^3$. However, this would not yield an analogue of Theorem 1.2. Obtaining a De Giorgi type regularity result for approximate solutions of (1.2) by a finite elements method is an open challenging problem.

The plan of the paper is as follows. In section 2, we present the discrete setting. Section 3 is devoted to general interpolation results for Sobolev spaces on graphs for the real and the complex method. In Section 4, we show Theorem 1.3, while estimates for general second order elliptic operators on graphs are stated and established in Section 5.

Notation: if $A(f)$ and $B(f)$ are two quantities depending on a function f ranging in a set E, say that $A(f) \lesssim B(f)$ if and only if there exists $C > 0$ such that, for all $f \in E$,

$$A(f) \leq CB(f),$$

and that $A(f) \sim B(f)$ if and only if $A(f) \lesssim B(f)$ and $B(f) \lesssim A(f)$.

2 Presentation of the discrete framework

Let us give precise definitions of our framework. The following presentation is partly borrowed from [10]. In the sequel, Γ stands for a non-empty finite or infinite set. When Γ is finite, fix a non-empty strict subset of Γ, called the boundary of Γ and denoted by $\partial \Gamma$.

2.1 The metric on Γ

Let $h_{xy} = h_{yx} \geq 0$ a symmetric weight on $\Gamma \times \Gamma$. We assume that $h_{xx} = 0$ for all $x \in \Gamma$. If $x, y \in \Gamma$, say that $x \sim y$ if and only if $h_{xy} > 0$ or $x = y$. We define for all $x \in \Gamma$

$$h_x := \sup_{y \sim x} h_{xy}$$

and

$$h := \sup_{x,\, y \in \Gamma} h_{xy}.$$

Denote by E the set of edges in Γ, *i.e.*

$$E := \{(x,y) \in \Gamma \times \Gamma;\ x \sim y\},$$

and notice that, due to the symmetry of h, $(x,y) \in E$ if and only if $(y,x) \in E$. In the sequel, we always assume that Γ is locally uniformly finite, which means that there exists $N \in \mathbb{N}^*$ such that

$$\sup_{x \in \Gamma} \sharp\{y \in \Gamma;\ y \sim x\} \leq N \tag{2.10}$$

(here and after, $\sharp A$ denotes the cardinal of any subset A of Γ).

We also assume that h is locally uniformly controlled with a constant $C_W \geq 1$, which means that for all $x, y, z \in \Gamma$ such that $x \sim y$ with $x \neq y$ and $x \sim z$ with $x \neq z$, we have

$$\frac{1}{C_W} \leq \frac{h_{xy}}{h_{xz}} \leq C_W. \tag{2.11}$$

For $x, y \in \Gamma$, a path C joining x to y is a finite sequence of vertices $x_0 = x, ..., x_N = y$ such that, for all $0 \leq i \leq N-1$, $x_i \sim x_{i+1}$. By definition, the length of such a path is

$$l(C) := \sum_{k=0}^{N-1} h_{x_k x_{k+1}}.$$

Assume that Γ is connected, which means that, for all $x, y \in \Gamma$, there exists a path joining x to y. For all $x, y \in \Gamma$, the distance between x and y, denoted by $d(x,y)$, is defined as the infimum of the lengths of all paths joining x and y. For all $x \in \Gamma$ and all $r > 0$, let $B(x,r) := \{y \in \Gamma,\ d(y,x) < r\}$.

Remark 2.1 *If $x \in \Gamma$ and $r > 0$ are such that $r \leq \frac{h_x}{C_W}$, then $B(x,r) = \{x\}$. Indeed, let $y \in B(x,r)$, so that $d(y,x) < r \leq \frac{h_x}{C_W}$. If $y \neq x$, there exists $z \sim x$ such that*

$$h_{xz} \leq d(y,x) < \frac{h_x}{C_W} \leq h_{xz},$$

which is impossible.

If $B = B(x,r)$ is a ball, set $\alpha B := B(x, \alpha r)$ for all $\alpha > 0$.

2.2   A measure on the graph and the edges

We also consider measures on E and Γ. Assume that, for all $x \sim y \in \Gamma$, we are given a coefficient $\mu_{xy} > 0$ such that $\mu_{xy} = \mu_{yx}$. Assume also that there exists $C > 0$ such that, for all $x, y, z \in \Gamma$ satisfying $x \sim y$ and $x \sim z$,

$$\frac{\mu_{xy}}{\mu_{xz}} \leq C. \tag{2.12}$$

For all $x \in \Gamma$, set

$$m(x) := \sum_{y \sim x} \mu_{xy}. \tag{2.13}$$

If $A \subset \Gamma$, let

$$m(A) := \sum_{x \in A} m(x).$$

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For all $1 \leq p < +\infty$, say that a function f on Γ belongs to $L^p(\Gamma, m)$ (or $L^p(\Gamma)$) if and only if

$$\|f\|_p := \left(\sum_{x \in \Gamma} |f(x)|^p \, m(x) \right)^{1/p} < +\infty.$$

Say that $f \in L^\infty(\Gamma, m)$ (or $L^\infty(\Gamma)$) if and only if

$$\|f\|_\infty := \sup_{x \in \Gamma} |f(x)| < +\infty.$$

When Γ is finite and $1 \leq p \leq +\infty$, say that $f \in L_0^p(\Gamma)$ if and only if $f \in L^p(\Gamma)$ and $f(x) = 0$ for all $x \in \partial\Gamma$.

Remark 2.2 *It is an elementary observation that the space of functions on Γ with finite support is dense in $L^p(\Gamma, m)$ for all $1 \leq p < +\infty$.*

For all $1 \leq p < +\infty$, say that a function F on E belongs to $L^p(E, \mu)$ (or $L^p(E)$) if and only if F is antisymmetric (which means that $F(x, y) = -F(y, x)$ for all $(x, y) \in E$) and

$$\|F\|_p := \left(\sum_{(x,y) \in E} |F(x, y)|^p \, \mu_{xy} \right)^{1/p} < +\infty.$$

Say that $F \in L^\infty(E, \mu)$ (or $L^\infty(E)$) if and only F is antisymmetric and

$$\|F\|_\infty := \sup_{(x,y) \in E} |F(x, y)| < +\infty.$$

2.3 Differential and gradient

In the spirit of [3], for all function $f : \Gamma \to \mathbb{R}$ and all $(x, y) \in E$, define

$$df(x, y) = \begin{cases} \dfrac{f(y) - f(x)}{h_{xy}} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Say that d is the differential operator on Γ.

We also define the sublinear operator "length of the gradient" by

$$\nabla f(x) = \frac{1}{h_x} \left(\sum_{y \sim x} |f(y) - f(x)|^2 \right)^{1/2}$$

for all function f on Γ and all $x \in \Gamma$. It is a consequence of (2.10), (2.11), (2.12) and (2.13) that, for all function $f : \Gamma \to \mathbb{R}$ and all $p \in [1, +\infty]$,

$$\|df\|_{L^p(E,\mu)} \sim \|\nabla f\|_{L^p(\Gamma,m)}.$$

2.4 Sobolev and Hölder spaces

Let $1 \leq p \leq +\infty$. Say that a scalar-valued function f on Γ belongs to the (inhomogeneous) Sobolev space $W^{1,p}(\Gamma)$ (see also [3, 13, 18]) if and only if

$$\|f\|_{W^{1,p}(\Gamma)} := \|f\|_{L^p(\Gamma)} + \|\nabla f\|_{L^p(\Gamma)} < +\infty.$$

In the case when Γ is finite, denote by $W_0^{1,p}(\Gamma)$ the subspace of $W^{1,p}(\Gamma)$ made of functions f such that $f = 0$ on $\partial \Gamma$.

It is then routine to check that both $W^{1,p}(\Gamma)$ and $W_0^{1,p}(\Gamma)$ are Banach spaces.

Finally, if $1 < p < +\infty$, as in the Euclidean case, we define $W^{-1,p}(\Gamma)$ as the dual space of $W^{1,p'}(\Gamma)$ (when Γ is infinite) or of $W_0^{1,p'}(\Gamma)$ (when Γ is finite), equipped with its natural norm. Here and after, p' is defined by $\frac{1}{p} + \frac{1}{p'} = 1$.

Remark 2.3 *It is easy to check that, for all $1 \leq p < +\infty$, the space of functions with finite support is dense in $W^{1,p}(\Gamma)$.*

For all $\mu > 0$, we define the discrete homogeneous (resp. inhomogeneous) Hölder spaces of exponent μ as

$$\dot{C}^\mu(\Gamma) := \{u \in \mathbb{R}^\Gamma; \ |u|_{\dot{C}^\mu(\Gamma)} = \sup_{x \neq y} \frac{|u(x) - u(y)|}{d(x,y)^\mu} < +\infty\}$$

and

$$C^\mu(\Gamma) := \{u \in \mathbb{R}^\Gamma; \ \|u\|_{C^\mu(\Gamma)} = \|u\|_{L^\infty(\Gamma)} + |u|_{\dot{C}^\mu(\Gamma)} < +\infty\}.$$

2.5 Geometric assumptions on Γ

Let us now introduce some geometric assumptions on (Γ, m).

1. Say that (Γ, m) satisfies the doubling property if there exists $C > 0$ such that, for all $x \in \Gamma$ and all $r > 0$,
$$V(x, 2r) \leq CV(x, r). \tag{D}$$

   Note that this assumption implies that there exist $C, D > 0$ such that, for all $x \in \Gamma$, all $r > 0$ and all $\theta > 1$,
$$V(x, \theta r) \leq C\theta^D V(x, r). \tag{2.14}$$

   Remark 2.4 *Observe also that, when Γ is infinite, it is also unbounded (since it is locally uniformly finite) so that, if (D) holds, then $m(\Gamma) = +\infty$ (see [16]).*

2. Say that (Γ, m) satisfies (L_σ) if there exist $\sigma > 0$, $c > 0$ such that, for all $x \in \Gamma$ and all $r > 0$,
$$V(x, r) \geq cr^\sigma. \tag{L_σ}$$

3. If $1 \leq q < +\infty$, say that (Γ, m) satisfies a scaled L^q Poincaré inequality on balls (this inequality will be denoted by (P_q) in the sequel) if there exists $C > 0$ such that, for any $x \in \Gamma$, any $r > 0$ and any function f on Γ,

$$\sum_{y \in B(x,r)} |f(y) - f_B|^q \, m(y) \leq Cr^q \sum_{y \in B(x,r)} |\nabla f(y)|^q \, m(y), \qquad (P_q)$$

where

$$f_B = \frac{1}{V(B)} \sum_{x \in B} f(x) m(x)$$

is the mean value of f on B.

When Γ is finite, we will consider local versions of these properties. Let $r_0 > 0$. Say that Γ satisfies (D_{loc}) if there exists $C > 0$ such that, for all $0 < r < r_0$ and all $x \in \Gamma$,

$$V(x, 2r) \leq CV(x, r).$$

Say that Γ satisfies $(L_{\sigma, loc})$ if there exists $C > 0$ such that, for all $x \in \Gamma$ and $0 < r \leq r_0$,

$$V(x, r) \geq Cr^\sigma.$$

Let $1 \leq q < +\infty$. Say that Γ satisfies $(P_{q,loc})$ if there exists $C > 0$ such that, for all $x \in \Gamma$, all $0 < r < r_0$ and any function f on Γ,

$$\sum_{y \in B(x,r)} |f(y) - f_B|^q \, m(y) \leq Cr^q \sum_{y \in B(x,r)} |\nabla f(y)|^q \, m(y).$$

3 Interpolation and embeddings of Sobolev spaces

3.1 Real and complex interpolation of Sobolev spaces: the case of unbounded graphs

Let us first focus on the case when Γ is infinite, and therefore unbounded. We will make constant use of the following notation here and in the sequel:

Definition 3.1 *When (P_q) holds for some $1 \leq q < \infty$, we define $q_0 = \inf \{q \in [1, \infty) : (P_q) \text{ holds}\}$.*

Note that if (P_q) holds for some $q > 1$, then $q_0 < q$ (see [15]), and that (P_r) holds for all $q_0 < r < +\infty$.

Let us first state the following real interpolation theorem for Sobolev spaces:

Theorem 3.2 *Let $q \in [1, +\infty)$ and assume that (D) and (P_q) hold. Then, for all $1 \leq r \leq q < p < +\infty$, $W^{1,p}(\Gamma) = (W^{1,r}(\Gamma), W^{1,\infty}(\Gamma))_{1-\frac{r}{p}, p}$.*

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The corresponding result for Sobolev spaces on Riemannian manifolds was proved in [2], Theorem 1.1. We established in [3], Theorem 1.18, a statement very similar to Theorem 3.2 for homogeneous Sobolev spaces, and the proof, which we skip here, is analogous to the one of Theorem 1.18 in [3], even if the definition of the gradient is slightly different. The only important assumptions to be able to interpolate are (D) and (P_q), see Theorem 7.11 in [2].

As an immediate corollary, we obtain:

Corollary 3.3 (The reiteration theorem) *Assume that Γ satisfies (D), (P_q) for some $1 \leq q < +\infty$. Let $q_0 < p_1 < p < p_2 \leq +\infty$, with $\dfrac{1}{p} = \dfrac{1-\theta}{p_1} + \dfrac{\theta}{p_2}$, then*

$$W^{1,p}(\Gamma) = \left(W^{1,p_1}(\Gamma), W^{1,p_2}(\Gamma)\right)_{\theta,p}.$$

If $q_0 = 1$, then one can also take $1 = p_1 < p < p_2 \leq +\infty$.

Let us now turn to the complex interpolation of Sobolev spaces. We first recall the following complex reiteration theorem:

Theorem 3.4 *[4, 9] For any compatible couple of Banach spaces (A_1, A_2) we have*

$$[(A_1, A_2)_{\lambda_1,p_2}, (A_1, A_2)_{\lambda_2,p_2}]_\alpha = (A_1, A_2)_{\beta,p}$$

for all λ_1, λ_2 and α in $(0,1)$ and all p_1, p_2 in $[1, +\infty]$, except for the case $p_1 = p_2 = \infty$. Here β and p are given by $\beta = (1-\alpha)\lambda_1 + \alpha\lambda_2$ and $\frac{1}{p} = \frac{1-\alpha}{p_1} + \frac{\alpha}{p_2}$.

From this, we deduce a complex interpolation result for Sobolev spaces:

Corollary 3.5 *Assume that Γ satisfies (D) and (P_q) for some $1 \leq q < +\infty$. Let $q_0 < p_1 < p < p_2 < +\infty$ where q_0 is given by Definition 3.1. Then for $\alpha = \dfrac{\frac{1}{p_1} - \frac{1}{p}}{\frac{1}{p_1} - \frac{1}{p_2}} = \dfrac{p_1(p-p_1)}{p(p_2-p_1)}$ one has*

$$\left[W^{1,p_1}(\Gamma), W^{1,p_2}(\Gamma)\right]_\alpha = W^{1,p}(\Gamma).$$

Proof: We apply Theorem 3.4 with $A_1 = W^{1,1}(\Gamma)$, $A_2 = W^{1,\infty}(\Gamma)$, $\lambda_1 = 1 - \frac{1}{p_1}$, $\lambda_2 = 1 - \frac{1}{p_2}$ and $\beta = 1 - \frac{1}{p}$. We obtain

$$\left[W^{1,p_1}(\Gamma), W^{1,p_2}(\Gamma)\right]_\alpha = (W^{1,1}(\Gamma), W^{1,\infty}(\Gamma))_{1-\frac{1}{p},p}$$

and the result follows from Theorem 3.2 which yields

$$(W^{1,1}(\Gamma), W^{1,\infty}(\Gamma))_{1-\frac{1}{p},p} = W^{1,p}(\Gamma)$$

since $p > q_0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now for dual spaces, we recall the following theorem:

Theorem 3.6 *([4], Theorem 3.7.1) Let A_1, A_2 a compatible couple of Banach spaces. Then for all $\theta \in (0,1)$ and $1 \leq s \leq \infty$, we have*

$$[(A_1, A_2)_{\theta,s}]^* = (A_1^*, A_2^*)_{\theta,s'}.$$

Combining this Theorem with Theorem 3.2, we deduce:

Theorem 3.7 *Assume that Γ satisfies (D) and (P_q) for some $1 \leq q < +\infty$. Then, for all $1 \leq r_1 < r < r_2 \leq \infty$ with $r' > q_0$, we have*

$$\left(W^{-1,r_1}(\Gamma), W^{-1,r_2}(\Gamma)\right)_{\theta,r} = W^{-1,r}(\Gamma)$$

where $\theta \in (0,1)$ such that $\frac{1}{r} = \frac{1-\theta}{r_1} + \frac{\theta}{r_2}$.

Proof: We have $1 \leq r_2' < r' < r_1' \leq \infty$ with $r' > q_0$ and $\frac{1}{r'} = \frac{\theta}{r_2'} + \frac{1-\theta}{r_1'}$. Then

$$[(W^{1,r_2'}(\Gamma), W^{1,r_1'}(\Gamma))_{1-\theta,r'}]^* = (W^{1,r'}(\Gamma))^*$$
$$= W^{-1,r}(\Gamma) = (W^{-1,r_2}(\Gamma), W^{-1,r_1}(\Gamma))_{1-\theta,r} = (W^{-1,r_1}(\Gamma), W^{-1,r_2}(\Gamma))_{\theta,r}.$$

$\square$

Theorem 3.8 *Assume that Γ satisfies (D), (P_q) for some $1 \leq q < +\infty$. Let $1 < p_1 < p < p_2 < q_0'$, where q_0 is given by Definition 3.1. Then*

$$\left[W^{-1,p_1}(\Gamma), W^{-1,p_2}(\Gamma)\right]_\alpha = W^{-1,p}(\Gamma)$$

where $\beta = 1 - \frac{1}{p}$ and $\alpha = \frac{\frac{1}{p_1} - \frac{1}{p}}{\frac{1}{p_1} - \frac{1}{p_2}}$.

Proof: We have $p_1 < p < p_2 < q_0'$. Then Theorem 3.7 shows that

$$(W^{-1,1}(\Gamma), W^{-1,\infty}(\Gamma))_{1-\frac{1}{p_1},p_1} = W^{-1,p_1}(\Gamma), \quad (W^{-1,1}(\Gamma), W^{-1,\infty}(\Gamma))_{1-\frac{1}{p_2},p_2} = W^{-1,p_2}(\Gamma)$$

and

$$(W^{-1,1}(\Gamma), W^{-1,\infty}(\Gamma))_{1-\frac{1}{p},p} = W^{-1,p}(\Gamma).$$

This, with Theorem 3.4, yields

$$\left[W^{-1,p_1}(\Gamma), W^{-1,p_2}(\Gamma)\right]_\alpha = (W^{-1,1}(\Gamma), W^{-1,\infty}(\Gamma))_{\beta,p} = W^{-1,p}(\Gamma).$$

$\square$

3.2 Real and complex interpolation of Sobolev spaces: the case of bounded graphs

Assume now that the graph Γ is bounded.

Theorem 3.9 *Let $q \in [1,+\infty)$ and assume that (D_{loc}) and $(P_{q,loc})$ hold. Then, for all $1 \leq r \leq q < p < +\infty$, $W_0^{1,p}(\Gamma) = \left(W_0^{1,r}(\Gamma), W_0^{1,\infty}(\Gamma)\right)_{1-\frac{r}{p},p}$.*

Proof We refer to [2] and [3] for the proof. We just mention the differences between the proof in [2, 3] and the present situation. We have to estimate the functional K of real interpolation. Let

$$\Omega = \{x \in \Gamma; \mathcal{M}((|\nabla f| + |f|)^q)(x) > \alpha^q(t)\}$$

where $\mathcal{M}$ is the uncentered Hardy-Littlewood maximal function, that is

$$\mathcal{M}h(x) := \sup_{B \ni x} \frac{1}{V(B)} \sum_{y \in B} |h(y)|\, m(y)$$

and $\alpha(t) = \mathcal{M}((|\nabla f| + |f|)^q)^{*\frac{1}{q}}(t)$.We recall that for a function g on Γ, g^* denotes its decreasing rearrangement function.

If $\Omega = \Gamma$ (note that this may happen since $m(\Gamma) < +\infty$), this is the easy case, we argue as in [2], Section 3.2.2.

Otherwise, we write Γ as the union of balls B^i, of radius ρ small enough (namely $0 < \rho < r_0$) and having the bounded overlap property. We take $(\varphi^i)_i$ a partition of unity subordinated to the covering $(B^i)_i$ and write $f = \sum_i f\varphi^i = \sum_i f^i$. We estimate the functional K as in [2], Section 3.2.2, using the Calderón-Zygmund decomposition applied to f^i. Note that here the functions $\left(b_j^i\right)_j$ of the decomposition should belong to $W_0^{1,r}(\Gamma)$, that is $b_j^i = 0$ on $\partial\Gamma$. For this, we take $b_j^i = f^i\chi_j^i$ if $B_j^i \cap \partial\Gamma \neq \emptyset$ and, if not, we define as usual $b_j^i = (f^i - (f^i)_{B_j^i})\chi_j^i$. We characterize then K as in [2]. Integrating K, we get the interpolation result. $\qquad\square$

As in section 3.1, from this theorem, we deduce:

Corollary 3.10 *Assume that Γ satisfies (D_{loc}) and $(P_{q,loc})$ for some $1 \leq q < +\infty$. Let $q_0 < p_1 < p < p_2 < +\infty$ where q_0 is defined by (3.1). Then for $\alpha = \frac{\frac{1}{p_1} - \frac{1}{p}}{\frac{1}{p_1} - \frac{1}{p_2}} = \frac{p_1(p-p_1)}{p(p_2-p_1)}$ one has*

$$\left[W_0^{1,p_1}(\Gamma), W_0^{1,p_2}(\Gamma)\right]_\alpha = W_0^{1,p}(\Gamma).$$

As far as dual spaces are concerned, we have:

Theorem 3.11 *Assume that Γ satisfies (D_{loc}) and $(P_{q,loc})$ for some $1 \leq q < +\infty$. Then, for all $1 \leq r_1 < r < r_2 \leq \infty$ with $r' > q_0$, we have*

$$\left(W^{-1,r_1}(\Gamma), W^{-1,r_2}(\Gamma)\right)_{\theta,r} = W^{-1,r}(\Gamma)$$

where $\theta \in (0,1)$ such that $\frac{1}{r} = \frac{1-\theta}{r_1} + \frac{\theta}{r_2}$.

Finally,

Theorem 3.12 *Assume that Γ satisfies (D_{loc}), $(P_{q,loc})$ for some $1 \leq q < +\infty$. Let $1 < p_1 < p < p_2 < q_0'$. Then*

$$\left[W^{-1,p_1}(\Gamma), W^{-1,p_2}(\Gamma)\right]_\alpha = W^{-1,p}(\Gamma)$$

where $\beta = 1 - \frac{1}{p}$ and $\alpha = \frac{\frac{1}{p_1} - \frac{1}{p}}{\frac{1}{p_1} - \frac{1}{p_2}}$

3.3 Sobolev embeddings

Assuming (D), (P_q) and (L_σ) for some $\sigma > 0$, we obtain Sobolev embeddings analogous to the Euclidean situation. Namely:

Proposition 3.13 *Assume that Γ satisfies (D), (P_q) and (L_σ) for some $\sigma > 0$. Then*

$$\|f\|_{L^{p^*}(\Gamma)} \lesssim \|\nabla f\|_{L^p(\Gamma)}$$

for all $q_0 < p < \sigma$ where $p^* = \frac{\sigma p}{\sigma - p}$.

This result is proved in [20], Corollary 3.3.3, p.75.

Proposition 3.14 *Assume that Γ satisfies (D), (P_q) and (L_σ) for some $\sigma > 0$. Then, for all $p > \max(q_0, \sigma)$,*

$$W^{1,p}(\Gamma) \hookrightarrow C^\eta(\Gamma)$$

with $\eta := 1 - \frac{\sigma}{p}$, where the symbol $\hookrightarrow$ means that the embedding is continuous.

Proof: Let $u \in W^{1,p}(\Gamma)$, $x, y \in \Gamma$ with $x \neq y$ and set $r := d(x, y)$. Define $B := B(x, r)$, so that $B \subset B(x, 2r) \cap B(y, 2r)$. One has

$$|u(x) - u(y)| \leq |u(x) - u_B| + |u(y) - u_B|, \qquad (3.15)$$

and we estimate each term of the right-hand side of (3.15). Let us first focus on the second term. Using Remark 2.1, write

$$|u(y) - u_B| \leq \sum_{k=-\infty}^{0} |u_{B(y,2^k r)} - u_{B(y,2^{k+1} r)}| + |u_{B(y,2r)} - u_B|$$

$$\leq \sum_{k=-\infty}^{0} \frac{1}{V(y,2^k r)} \sum_{z \in B(y,2^k r)} |u(z) - u_{B(y,2^{k+1} r)}| m(z) + \frac{1}{V(B)} \sum_{z \in B} |u(z) - u_{B(y,2r)}| m(z)$$

$$\leq \sum_{k=-\infty}^{0} \frac{V(y,2^{k+1} r)}{V(y,2^k r)} \frac{1}{V(y,2^{k+1} r)} \sum_{z \in B(y,2^{k+1} r)} |u(z) - u_{B(y,2^{k+1} r)}| m(z)$$

$$+ \frac{C}{V(y,2r)} \sum_{z \in B(y,2r)} |u(z) - u_{B(y,2r)}| m(z)$$

$$\leq C \sum_{k=-\infty}^{0} 2^k r \left(\frac{1}{V(y,2^{k+1} r)} \sum_{z \in B(y,2^{k+1} r)} |\nabla u(z)|^p m(z) \right)^{\frac{1}{p}} + Cr \left(\frac{1}{V(y,2r)} \sum_{z \in B(y,2r)} |\nabla u(z)|^p \right)^{\frac{1}{p}}$$

$$\leq C \left(\sum_{k=-\infty}^{0} \left(2^k r \right)^{1-\frac{\sigma}{p}} \right) \|\nabla u\|_{L^p(\Gamma)} + Cr^{1-\frac{\sigma}{p}} \|\nabla u\|_{L^p(\Gamma)}$$

$$\leq Cr^{1-\frac{\sigma}{p}} \|\nabla u\|_{L^p(\Gamma)} = Cd(x,y)^{1-\frac{\sigma}{p}} \|\nabla u\|_{L^p(\Gamma)},$$

where we used (D) and (P_p) in the third and fourth inequalities and (L_σ) in the fifth one.

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Arguing similarly, one obtains

$$|u(x) - u_B| \leq C d(x,y)^{1-\frac{\sigma}{p}} \|\nabla u\|_{L^p(\Gamma)}. \tag{3.16}$$

Thus $u \in \dot{C}^\eta(\Gamma)$ and $|u|_{\dot{C}^\eta(\Gamma)} \leq C\|\nabla u\|_{L^p(\Gamma)}$.
It remains to prove that $u \in L^\infty(\Gamma)$. For all $x \in \Gamma$, write

$$|u(x)| \leq |u(x) - u_{B(x,1)}| + |u_{B(x,1)} - u(y)| + |u(y)| \tag{3.17}$$

for all $y \in B(x,1)$.
Applying (3.16) with $r = 1$, one obtains

$$|u(x) - u_{B(x,1)}| \leq C \|\nabla u\|_{L^p(B(x,1))}. \tag{3.18}$$

Then, taking the average over $B(x,1)$ in (3.17) yields

$$|u(x)| \leq |u(x) - u_{B(x,1)}| + \frac{1}{V(x,1)} \sum_{y \in B(x,1)} |u(y) - u_{B(x,1)}| m(y) + \frac{1}{V(x,1)} \sum_{y \in B(x,1)} |u(y)| m(y)$$

$$\leq C \left(1 + \frac{1}{V(x,1)^{\frac{1}{p}}} \right) \|\nabla u\|_{L^p(B(x,1))} + \frac{C}{V(x,1)^{\frac{1}{p}}} \|u\|_{L^p(B(x,1))}$$

$$\leq C\|u\|_{W^{1,p}(\Gamma)}.$$

We used (3.18), (P_p) and (L_σ) in the last inequality. Thus $\|u\|_{L^\infty(\Gamma)} \leq C\|u\|_{W^{1,p}(\Gamma)}$ and therefore $u \in C^\eta(\Gamma)$ with $\|u\|_{C^\eta(\Gamma)} \leq C\|u\|_{W^{1,p}(\Gamma)}$. $\qquad\square$

4 Hölder regularity for Galerkin schemes

This section is devoted to the proof of Theorem 1.3.

4.1 Definition of a graph and an operator

Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain and $(\mathcal{T}_h)_{h>0}$ be a regular family of triangulations of Ω as in the statement of Theorem 1.2. Let Γ_h be the set of vertices of $\mathcal{T}_h$. Define $\partial\Gamma_h := \Gamma_h \cap \partial\Omega$. For all $x, y \in \Gamma_h$, say that $x \sim y$ if and only if $x = y$ or there exists a triangle T in $\mathcal{T}_h$ such that the segment joining x and y is an edge of T. For all $x \sim y \in \Gamma_h$, set

$$h_{xy} = |x - y|,$$

where we recall that $|u|$ denotes the Euclidean norm of $u \in \mathbb{R}^2$. Note that, since the triangulation is admissible, (2.10) and (2.11) hold. For all $x \sim y$, define also

$$\mu_{xy} := h_{xy}^2,$$

so that (2.12) holds and

$$m(x) \sim h_x^2$$

for all $x \in \Gamma_h$.

Lemma 4.1 *The graph Γ_h, endowed with h_{xy} and μ_{xy} satisfies (D_{loc}), $(P_{2,loc})$ and $(L_{2,loc})$, with constants independent of h.*

Proof: The proof uses essentially the regularity of the graph Γ_h and Poincaré inequality $(P_{2,loc})$ on Ω. A detailed proof can be found in [19], Chapter 7, section 7.2. □

For all $x \in \Gamma_h \cap \Omega$, define φ_x as the unique function in $V_0^1(\mathcal{T}_h)$ such that

$$\varphi_x(y) = \delta_{xy}$$

for all $y \in \Gamma_h$. The functions $(\varphi_x)_{x \in \Gamma_h \cap \Omega}$ form a basis of $V_0^1(\mathcal{T}_h)$.
Let R_h be the "reconstruction" operator defined by

$$R_h \widetilde{u_h} := \sum_{x \in \Gamma_h \cap \Omega} \widetilde{u_h}(x) \varphi_x$$

for all function $\widetilde{u_h}$ on Γ_h vanishing on $\partial \Gamma_h$. In other words, $R_h \widetilde{u_h}$ is the function in $V_0^1(\mathcal{T}_h)$ which coincides with $\widetilde{u_h}$ at the vertices of $\mathcal{T}_h$. We claim that the norms of $\widetilde{u_h}$ and $R_h \widetilde{u_h}$ in the various functional spaces introduced before are equivalent:

Proposition 4.2 *Let $p \in (1, +\infty)$ and $\eta \in (0,1)$. For all function $\widetilde{u_h}$ on Γ_h vanishing on $\partial \Gamma_h$,*

1. $\|\widetilde{u_h}\|_{L^p(\Gamma_h)} \sim \|R_h \widetilde{u_h}\|_{L^p(\Omega)}$,

2. $\|\nabla \widetilde{u_h}\|_{L^p(\Gamma_h)} \sim \|\nabla R_h \widetilde{u_h}\|_{L^p(\Omega)}$,

3. $\|\widetilde{u_h}\|_{C^\eta(\Gamma_h)} \sim \|R_h \widetilde{u_h}\|_{C^{0,\eta}(\overline{\Omega})}$.

The implicit constants in these equivalences of norms are independent of h.

Proof: It is a straightforward consequence of the definitions of the norms, of R_h, the properties of the triangulation: namely the regularity of Γ_h and the fact that it is locally unifomly bounded, and finally the Poincaré inequality on a bounded open convex subset of $\mathbb{R}^2$. A detailed proof can be found in Appendix E of [19]. □

Let us now define a maximal accretive operator on $L_0^2(\Gamma_h)$ associated to the Galerkin scheme. For all $\widetilde{u_h}, \widetilde{v_h} \in W_0^{1,2}(\Gamma_h)$, define

$$Q_h\left(\widetilde{u_h}, \widetilde{v_h}\right) := \int_\Omega A(x) \nabla \left(R_h \widetilde{u_h}\right)(x) \cdot \nabla \left(R_h \widetilde{v_h}\right)(x) dx.$$

The operator $\nabla R_h : L_0^2(\Gamma_h) \to L^2(\Omega, \mathbb{C}^n)$ is defined on the whole $L_0^2(\Gamma_h)$ space, and the operator $F \mapsto AF$ is obviously bounded on $L^2(\Omega, \mathbb{R}^n)$. Moreover, if $\widetilde{u_h} \in W_0^{1,2}(\Gamma_h)$,

$$
\begin{aligned}
\int_\Omega A(x) \nabla \left(R_h \widetilde{u_h}\right)(x) \cdot \nabla \left(R_h \widetilde{u_h}\right)(x) dx &\geq \delta \left\|\nabla \left(R_h \widetilde{u_h}\right)\right\|_{L^2(\Omega)}^2 \\
&\gtrsim \|R_h \widetilde{u_h}\|_{L^2(\Omega)}^2 \\
&\gtrsim \|\widetilde{u_h}\|_{L_0^2(\Gamma_h)}^2.
\end{aligned}
$$

As a consequence, using a construction due to Kato ([14]), there exists a unique maximal accretive operator L_h on $L_0^2(\Gamma_h)$ such that, for all $\widetilde{u_h} \in \mathcal{D}(L_h)$ (here and after, $\mathcal{D}(L)$ stands for the domain of an operator L) and all $\widetilde{v_h} \in W_0^{1,2}(\Gamma_h)$,

$$\langle L_h \widetilde{u_h}, \widetilde{v_h} \rangle_{L_0^2(\Gamma_h)} = \int_\Omega A(x) \nabla \left(R_h \widetilde{u_h} \right)(x) \cdot \nabla \left(R_h \widetilde{v_h} \right)(x) dx.$$

The following facts are easily checked:

Proposition 4.3 *For all $p \in (1, +\infty)$, the operator L_h extends to a bounded operator from $W_0^{1,p}(\Gamma_h)$ to $W^{-1,p}(\Gamma_h)$. Moreover, L_h is an isomorphism from $W_0^{1,2}(\Gamma_h)$ into $W^{-1,2}(\Gamma_h)$. The norms of L_h and of its inverse (for $p = 2$) do not depend on h.*

Proof: Let $p \in (1, +\infty)$ and $p' \in (1, +\infty)$ such that $\frac{1}{p} + \frac{1}{p'} = 1$. Let $\widetilde{u_h} \in W_0^{1,p}(\Gamma_h)$. The map $\widetilde{v_h} \mapsto \int_\Omega A(x) \nabla \left(R_h \widetilde{u_h} \right)(x) \cdot \nabla \left(R_h \widetilde{v_h} \right)(x) dx$ is clearly a linear functional on $W_0^{1,p'}(\Gamma_h)$. Moreover, for all $\widetilde{v_h} \in W_0^{1,p'}(\Gamma_h)$,

$$\begin{aligned} |\langle L_h \widetilde{u_h}, \widetilde{v_h} \rangle| &\lesssim \left\| \nabla \left(R_h \widetilde{u_h} \right) \right\|_{L^p(\Omega)} \left\| \nabla \left(R_h \widetilde{v_h} \right) \right\|_{L^{p'}(\Omega)} \\ &\lesssim \left\| \nabla \widetilde{u_h} \right\|_{L^p(\Gamma_h)} \left\| \nabla \widetilde{v_h} \right\|_{L^{p'}(\Gamma_h)}. \end{aligned}$$

This ends the proof of the first part of the claim. Assume now that $p = 2$. For all bounded linear functional T on $W_0^{1,2}(\Gamma_h)$, the Lax-Milgram theorem yields a unique function $\widetilde{u_h} \in W_0^{1,2}(\Gamma_h)$ such that, for all $\widetilde{v_h} \in W_0^{1,2}(\Gamma_h)$, $\langle L_h \widetilde{u_h}, \widetilde{v_h} \rangle_{L_0^2(\Gamma_h)} = T(\widetilde{v_h})$, and one has $\|\widetilde{u_h}\|_{W_0^{1,2}(\Gamma_h)} \lesssim \|T\|_{W^{-1,2}(\Gamma_h)}$. This shows that L_h is an isomorphism from $W_0^{1,2}(\Gamma_h)$ into $W^{-1,2}(\Gamma_h)$. Moreover, the norms of L_h and L_h^{-1} do not depend on h because of Lemma 4.1 and Proposition 4.2. $\qquad\square$

4.2 Conclusion of the proof

Our next step is a perturbation argument, in order to show that L_h is still an isomorphism from $W_0^{1,p}(\Gamma_h)$ into $W^{-1,p}(\Gamma_h)$ for p close enough to 2. Indeed, since $(P_{2,loc})$ holds, there exists $q_0 < 2$ such that, for $q_0 < p_0 < p < p_1 < q_0'$, the $W^{1,p}(\Gamma)$ and $W^{-1,p}(\Gamma)$ form a complex interpolation scale (see Corollary 3.5 and Theorem 3.8). We then combine this fact, Proposition 4.3 and the following general result ([21]):

Lemma 4.4 *Let X^s, Y^s, $s \in [0, 1]$ be two scales of complex interpolation Banach spaces. If $T : X^s \to Y^s$ is bounded for each $s \in [0, 1]$, then the set of s for which there exists $C > 0$ such that $\|Tf\|_{Y^s} \geq C\|f\|_{X^s}$ holds for all $f \in X^s$ is open.*

Thus, there exists $\varepsilon > 0$ such that L_h is an isomorphism from $W_0^{1,p}(\Gamma_h)$ into $W^{-1,p}(\Gamma_h)$ for all $p \in (2 - \varepsilon, 2 + \varepsilon)$. Take $p \in (2, 2 + \varepsilon)$. Proposition 3.13 applied with $q = \sigma = 2$ yields

$$L^2(\Gamma_h) \hookrightarrow W^{-1,p}(\Gamma_h)$$

and the norm of the continuous embedding is independent of h, since it only depends on the geometric constants of Γ_h. Therefore:

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Proposition 4.5 *The operator L_h^{-1} maps $L_0^2(\Gamma_h)$ into $W_0^{1,p}(\Gamma_h)$ for $p \in (2, 2 + \varepsilon)$ with a norm independent of h.*

Let us now conclude the proof of Theorem 1.3. Take $f \in L^2(\Omega)$ and define, for all $x \in \Gamma_h$,

$$f_h(x) := \begin{cases} \dfrac{1}{m(x)} \displaystyle\int_\Omega f(y)\varphi_x(y)dy & \text{when } x \in \Gamma_h \cap \Omega, \\ 0 & \text{when } x \in \Gamma_h \cap \partial\Omega. \end{cases}$$

We claim that, if $u_h \in V_0^1(\mathcal{T}_h)$ solves (1.4) and $\widetilde{u_h} \in W_0^{1,2}(\Gamma_h)$ is defined by

$$\widetilde{u_h}(x) = \begin{cases} u_h(x) & \text{when } x \in \Gamma_h \cap \Omega, \\ 0 & \text{when } x \in \Gamma_h \cap \partial\Omega, \end{cases}$$

then

$$L_h\widetilde{u_h} = -f_h. \tag{4.19}$$

Indeed, $L_h\widetilde{u_h} \in W^{-1,2}(\Gamma_h)$, and (4.19) exactly means that, for all $x \in \Gamma_h \cap \Omega$,

$$-\langle L_h\widetilde{u_h}, \widetilde{\varphi_x} \rangle = \langle f_h, \widetilde{\varphi_x} \rangle,$$

where $\widetilde{\varphi_x}$ is the function in $W_0^{1,2}(\Gamma_h)$ defined by $\widetilde{\varphi_x}(y) = \delta_{xy}$ for all $y \in \Gamma_h$, so that $\varphi_x = R_h\widetilde{\varphi_x}$. But,

$$\langle L_h\widetilde{u_h}, \widetilde{\varphi_x} \rangle = m(x)L_h\widetilde{u_h}(x), \tag{4.20}$$

and

$$\begin{aligned} -\langle L_h\widetilde{u_h}, \widetilde{\varphi_x} \rangle &= -\int_\Omega A(y)\nabla\left(R_h\widetilde{u_h}\right)(y) \cdot \nabla\left(R_h\widetilde{\varphi_x}\right)(y)dy \\ &= -\int_\Omega A(y)\nabla u_h(y) \cdot \nabla\varphi_x(y) \\ &= \int_\Omega f(y)\varphi_x(y)dy \\ &= m(x)f_h(x) \\ &= \langle f_h, \widetilde{\varphi_x} \rangle, \end{aligned} \tag{4.21}$$

where we used (1.4) in the third equality and the definition of f_h in the fourth one. Gathering (4.20) and (4.21), we obtain (4.19). It therefore follows from (4.19) and Proposition 4.5 that, for some $\varepsilon > 0$,

$$\|\widetilde{u_h}\|_{W^{1,2+\varepsilon}(\Gamma_h)} \lesssim \|f_h\|_{L^2(\Gamma_h)}. \tag{4.22}$$

Proposition 4.2 shows that

$$\|u_h\|_{W^{1,2+\varepsilon}(\Omega)} \sim \|\widetilde{u_h}\|_{W^{1,2+\varepsilon}(\Gamma_h)}. \tag{4.23}$$

Moreover,

$$\left\|\widetilde{f_h}\right\|_{L^2(\Gamma_h)} \leq C\|f\|_{L^2(\Omega)}. \tag{4.24}$$

Indeed, by Cauchy-Schwarz and the fact that the measure of the support of φ_x is controlled by a constant times $m(x)$, one has for all $x \in \Gamma_h$,

$$\left| \widetilde{f_h}(x) \right|^2 \leq \frac{C}{m(x)} \int_{supp(\varphi_x)} |f(z)|^2 \, dz,$$

and (4.24) follows by summing up on $x \in \Gamma_h$ and using the fact that the supports of the functions φ_x have the bounded overlap property. Gathering (4.22), (4.23) and (4.24) yields (1.8).

Let us now prove (1.9). First, by the results of [11] stated in the introduction, and since $L^2(\Omega) \hookrightarrow W^{-1,2+\varepsilon}(\Omega)$, one has

$$\|u\|_{W^{1,2+\varepsilon}(\Omega)} \leq C \|f\|_{L^2(\Omega)}. \tag{4.25}$$

so that

$$\|u - u_h\|_{W^{1,2+\varepsilon}(\Omega)} \leq C \|f\|_{L^2(\Omega)}. \tag{4.26}$$

Interpolating between (4.26) and (1.5) gives (1.9) with

$$\theta := \frac{\frac{1}{p} - \frac{1}{2+\varepsilon}}{\frac{1}{2} - \frac{1}{2+\varepsilon}}. \square$$

5 Estimates for general second elliptic operators on graphs

In the present section, we prove estimates in L^∞ and Hölder spaces for general second order elliptic on graphs, assuming suitable geometric properties on the graph Γ, which is assumed to be unbounded.

5.1 Definition of the operators

Assume that, for all $x \sim y$, a coefficient $c_{xy} \in \mathbb{C}$ is given, and that there exist $C_\infty > 0$ such that

$$|c_{xy}| \leq C_\infty \; \forall x \sim y \tag{5.27}$$

and $\delta > 0$ such that, for all $u \in W^{1,2}(\Gamma)$,

$$\mathrm{Re} \sum_{x \sim y} c_{xy} |du(x,y)|^2 \mu_{xy} \geq \delta \sum_{x \sim y} |du(x,y)|^2 \mu_{xy}. \tag{5.28}$$

We associate to these coefficients an operator, which is the discrete version of second order uniformly elliptic operators in divergence form in $\mathbb{R}^n$. To that purpose, we use again a classical construction of maximal accretive operators due to Kato ([14]). The operator $d : L^2(\Gamma) \to L^2(E)$ is densely defined (see Remark 2.2) and closed, and the operator $F \mapsto cF$ is

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clearly $L^2(E)$ bounded by (5.27). Since (5.28) holds, there exists a unique maximal accretive operator L on $L^2(\Gamma)$ such that, for all $u \in \mathcal{D}(L)$ and all $v \in W^{1,2}(\Gamma)$,

$$\sum_{x \in \Gamma} Lu(x)\overline{v(x)}m(x) = \sum_{(x,y) \in E} c_{xy}du(x,y)\overline{dv(x,y)}\mu_{xy} \qquad (5.29)$$

(recall that $\mathcal{D}(L)$ stands for the domain of L).

Remark 5.1 *Note that assumption* (5.28) *is satisfied in particular when the coefficients c_{xy} are real-valued and satisfy $c_{xy} \geq \delta$ whenever $x \neq y$. However, we do* **not** *make this assumption in the sequel, and all that is needed is* (5.28).

We say that L is a second order uniformly elliptic operator on $L^2(\Gamma)$ with ellipticity constants C_∞ and δ. Recall that $\mathcal{D}(L)$ is dense in $L^2(\Gamma)$ and that L generates a holomorphic semigroup on $L^2(\Gamma)$. Note finally that, for all $u \in \mathcal{D}(L)$,

$$\operatorname{Re}\langle Lu, u \rangle \geq c\delta \left\|\nabla u\right\|_2^2. \qquad (5.30)$$

Remark 5.2 *Let $1 < p < +\infty$. Assume that $u \in \mathcal{D}(L) \cap W^{1,p}(\Gamma)$ and $v \in W^{1,2}(\Gamma) \cap W^{1,p'}(\Gamma)$. From the definition of L, we deduce that*

$$|\langle Lu, v \rangle| \leq C\|\nabla u\|_{L^p(\Gamma)}\|\nabla v\|_{L^{p'}(\Gamma)}.$$

This and Remark 2.3 show that Lu extends to a bounded linear form on $W^{1,p'}(\Gamma)$, with norm less or equal to $C\|u\|_{W^{1,p}(\Gamma)}$. Thus L is a bounded operator from $W^{1,p}(\Gamma)$ to $W^{-1,p}(\Gamma)$, with bound less or equal to C.

5.2   The estimates on L

Our first estimates deal with the resolvent of L. Let $\omega \in \left(0, \frac{\pi}{2}\right)$ be such that L is ω-accretive on $L^2(\Gamma)$, $\mu \in \left(\frac{\pi}{2}, \pi - \omega\right)$ and

$$\Sigma_\mu := \{z \in \mathbb{C};\ z = 0 \text{ or } |\arg z| < \mu\}.$$

Let $f \in L^2(\Gamma)$ and $\lambda \in \Sigma_\mu$. We are interested in weak solutions of the equation

$$Lu + \lambda u = f. \qquad (5.31)$$

By "weak solution", we mean a function $u \in W^{1,2}(\Gamma)$ such that, for all $v \in W^{1,2}(\Gamma)$,

$$\sum_{(x,y) \in E} c_{xy}du(x,y)\overline{dv(x,y)}\mu_{xy} + \lambda \sum_{x \in \Gamma} u(x)\overline{v(x)}m(x) = \sum_{x \in \Gamma} f(x)\overline{v(x)}m(x).$$

We establish that, if (D), (L_2) and (P_2) hold, given $f \in L^2(\Gamma)$ and $\lambda \in \Sigma_\mu$, (5.31) has a unique weak solution in $W^{1,2}(\Gamma)$, and that this solution actually belongs to $C^\eta(\Gamma)$ for some $\eta > 0$:

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Theorem 5.3 *Let Γ be a graph satisfying (D), (L_2) and (P_2). Let L be a second order uniformly elliptic operator on $L^2(\Gamma)$ with ellipticity constants C_∞ and δ. Let ω be such that L is ω-accretive on $L^2(\Gamma)$ and $\mu \in \left(\frac{\pi}{2}, \pi - \omega\right)$. Then, for all $f \in L^2(\Gamma)$ and all $\lambda \in \Sigma_\mu$, there exists a unique solution $u \in W^{1,2}(\Gamma)$ of (5.31). Moreover, there exist $C, \eta > 0$ only depending on the constants in (D), (L_2) and (P_2) and the constants of ellipticity of L such that*

$$\|u\|_{L^\infty(\Gamma)} \le C |\lambda|^{-\frac{1}{2}} \|f\|_{L^2(\Gamma)}$$

and

$$|u|_{\dot{C}^\eta(\Gamma)} \le C |\lambda|^{\frac{\eta}{2}} |\lambda|^{-\frac{1}{2}} \|f\|_{L^2(\Gamma)}.$$

Theorem 5.3 is a version of the De Giorgi regularity theorem for elliptic equations in this context.

We also obtain estimates on the kernel of the semigroup generated by L. Before stating the Theorem, we introduce the following notations: for all $x, y \in \Gamma$, $h^*_{x \to y}$ is equal to 0 if $x = y$ and is the supremum of the weights of edges where at least one of the two vertices belongs to the ball $B(x, d(x, y))$, when $x \ne y$. We then define

$$h^*_{xy} = \min(h^*_{x \to y}, h^*_{y \to x}).$$

Theorem 5.4 *Let Γ be a graph satisfying (D), (L_2) and (P_2). Let L be a second order uniformly elliptic operator on $L^2(\Gamma)$ with ellipticity constants C_∞ and δ. We then have:*

1. *The operator L generates a holomorphic semi-group $(e^{-tL})_{t>0}$ on $L^2(\Gamma)$ which has a kernel, denoted by $K_t(x, y)$ in the sequel. For all $u \in L^2(\Gamma)$ and all $x \in \Gamma$,*

$$(e^{-tL}u)(x) = \sum_{y \in \Gamma} K_t(x, y)u(y).$$

   Moreover, there exist $C, C', \beta > 0$ depending only of the geometric constants and that of ellipticity, such that for every $(x, y) \in \Gamma^2$ and $t > 0$ we have

   a. *if $t \le C'h^*_{xy}d(x, y)$,*

   $$|K_t(x, y)| \le \frac{C}{t} e^{-\beta \frac{d(x,y)}{h^*_{xy}}};$$

   b. *if $t \ge C'h^*_{xy}d(x, y)$,*

   $$|K_t(x, y)| \le \frac{C}{t} e^{-\beta \frac{d^2(x,y)}{t}}.$$

2. *There exist constants C'' and $\eta > 0$ depending only on the geometric constants and that of ellipticity such that for every $t > 0$ and for every $x, x', y \in \Gamma$, we have*

$$|K_t(x, y) - K_t(x', y)| \le \frac{C''}{t} \left(\frac{d(x, x')}{\sqrt{t}}\right)^\eta.$$

5.3 Proofs

We first focus on Theorem 5.3. We begin with the following observation:

Proposition 5.5 *Let Γ as above and L a second order elliptic operator on Γ, with ellipticity constants C_∞ and δ. Then, for all $\lambda \in \mathbb{C}$, $L + \lambda I$ is a continuous operator from $W^{1,2}(\Gamma)$ to $W^{-1,2}(\Gamma)$. Moreover, when $\lambda = 1$, $L + I$ is an isomorphism from $W^{1,2}(\Gamma)$ onto $W^{-1,2}(\Gamma)$ and the norm of its inverse is bounded by a constant only depending on δ.*

Proof: That $L + \lambda I$ is bounded from $W^{1,2}(\Gamma)$ to $W^{-1,2}(\Gamma)$ was already seen in Remark 5.2 above. The Lax-Milgram theorem, applied with the sesquilinear form

$$\mathcal{B}(u,v) := \sum_{(x,y)\in E} c_{x,y} du(x,y) \overline{dv(x,y)} \mu_{xy} + \sum_{x\in\Gamma} u(x)\overline{v(x)} m(x)$$

which is clearly continuous and coercive on $W^{1,2}(\Gamma)$ thanks to (5.30), yields the invertibility of $L + I$ and the bound on the norm of the inverse. $\qquad\square$

Relying on Lemma 4.4 again and arguing as in Proposition 4.5, we prove:

Proposition 5.6 *Assume that Γ satisfies (D) and (P_2) and L is an elliptic operator as above. Then there exists $\varepsilon > 0$ only depending on the geometric constants of Γ and the ellipticity constants of L such that $L+I$ is invertible from $W^{1,p}$ to $W^{-1,p}$ for $p \in (2-\varepsilon, 2+\varepsilon)$.*

5.4 Proof of the estimates on L

We prove here Theorems 5.3 and 5.4.

5.5 Proof of Theorem 5.3

Let $f \in L^2(\Gamma)$. Since $L + I$ is an isomorphism from $W^{1,2}(\Gamma)$ onto $W^{-1,2}(\Gamma)$, there exists a unique $u \in W^{1,2}(\Gamma)$ such that (5.31) is satisfied. Moreover, we have seen in Proposition 5.6 that, for some $\varepsilon > 0$ $L+I$ is an isomorphism from $W^{1,p}(\Gamma)$ onto $W^{-1,p}(\Gamma)$ for $p \in (2-\varepsilon, 2+\varepsilon)$. Let $p \in (2, 2+\varepsilon)$. Propositions 3.13 and 3.14 applied with $q = \sigma = 2$ yield

$$L^2(\Gamma) \hookrightarrow W^{-1,p}(\Gamma)$$

and

$$W^{1,p}(\Gamma) \hookrightarrow C^\eta(\Gamma).$$

Therefore $(L+I)^{-1}$ maps $L^2(\Gamma)$ into $C^\eta(\Gamma)$ with $\eta = 1 - \frac{2}{p}$.
With an analogous argument, $L + I$ can be replaced by $L + \lambda I$ for $\lambda \in \Sigma_\mu$ with $|\lambda| = 1$:

$$\|(L+\lambda)^{-1}u\|_\infty \leq C\|u\|_{L^2(\Gamma)} \tag{5.32}$$

and

$$\left|(L+\lambda)^{-1}u\right|_{\dot{C}^\eta} \leq C\|u\|_{L^2(\Gamma)}. \tag{5.33}$$

We now claim that there is a constant $C > 0$ such that, for all $\lambda \in \Sigma_\mu \setminus \{0\}$,

$$\|(L + \lambda)^{-1}u\|_\infty \le C|\lambda|^{-\frac{1}{2}}\|u\|_{L^2(\Gamma)}$$

and

$$|(L + \lambda)^{-1}u|_{\dot{C}^\eta} \le C|\lambda|^{\frac{\eta}{2}-\frac{1}{2}}\|u\|_{L^2(\Gamma)}.$$

We treat the case where $\lambda > 0$ using a scaling argument. A similar proof can be done for a general $\lambda \in \Sigma_\mu$.

The scaling argument is as follows. For all $\alpha > 0$, let Γ_α be the set Γ, endowed with the weights αh_{xy} and the measures $\alpha^2 \mu_{xy}$. Note that, if ∇_α stands for the gradient in Γ_α, one has $\nabla_\alpha f = \frac{1}{\alpha}\nabla f$. As a consequence, it is easily checked that, endowed with the distance d_α defined by the weights αh_{xy} and the measure $\alpha^2 m$, the graph Γ_α satisfies the assumptions (D), (P_2) and (L_2) with the same constants as Γ.

Consider now the operator L_α on Γ_α given by the coefficients c_{xy}. A straightforward computation yields that L_α has the same ellipticity constants on Γ_α as L on Γ.

Let $f \in L^2(\Gamma)$ and $\lambda > 0$ be given. Define $g := \frac{1}{\lambda}f$. Applying the conclusion of Theorem 5.3 with $\Gamma_{\sqrt{\lambda}}$ and the operator $L_{\sqrt{\lambda}}$, one obtains that there exists a unique function $u \in W^{1,2}(\Gamma_{\sqrt{\lambda}})$ such that

$$(L_{\sqrt{\lambda}} + I)u = g$$

on $\Gamma_{\sqrt{\lambda}}$, and

$$\|u\|_{C^\eta(\Gamma_{\sqrt{\lambda}})} \le C\|g\|_{L^2(\Gamma_{\sqrt{\lambda}})} = \frac{C}{\sqrt{\lambda}}\|f\|_{L^2(\Gamma)}.$$

Therefore, $(L + \lambda I)u = f$ on Γ. Moreover,

$$\|u\|_{L^\infty(\Gamma)} \le \frac{C}{\sqrt{\lambda}}\|f\|_{L^2(\Gamma)},$$

and

$$|u|_{\dot{C}^\eta(\Gamma)} \le C\lambda^{\frac{\eta-1}{2}}\|f\|_{L^2(\Gamma)},$$

which ends the proof of Theorem 5.3. $\qquad\square$

5.6 Proof of Theorem 5.4

The proof of Theorem 5.4 is analogous to that of Proposition 28, p.59 in [1] in the Euclidean case, and we only indicate the main differences.

Let us first recall that, for all $t > 0$,

$$e^{-tL} = \frac{1}{2\pi i}\int_\gamma e^{z\lambda}(L + \lambda)^{-1}d\lambda \tag{5.34}$$

where γ is made of two rays $\gamma_\pm := \left\{re^{\pm i\theta};\ r \ge \frac{1}{t}\right\}$ and of the arc $\gamma_0 := \left\{\frac{1}{t}e^{i\sigma};\ |\sigma| \le \theta\right\}$ where $\theta \in]\frac{\pi}{2}, \pi[$, and γ is described counterclockwise.

Combining (5.34) with Theorem 5.3, we obtain

$$\|e^{-tL}u\|_{L^\infty(\Gamma)} \le Ct^{-\frac{1}{2}}\|u\|_{L^2(\Gamma)}, \qquad t > 0$$

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and
$$|e^{-tL}u|_{\dot{C}^\eta} \le Ct^{-(\frac{1}{2}+\frac{\eta}{2})}\|u\|_2, \qquad t > 0.$$

We finish the proof as in [19], Chapter 5, section 5.2, see also [1], Chapter 1. $\qquad\square$

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