REAL INTERPOLATION OF SOBOLEV SPACES ASSOCIATED TO A WEIGHT

NADINE BADR

Abstract. We hereby study the interpolation property of Sobolev spaces of order 1 denoted by $W^1_{p,V}$, arising from Schrödinger operators with positive potential. We show that for $1 \leq p_1 < p < p_2 < q_0$ with $p > s_0$, $W^1_{p,V}$ is a real interpolation space between $W^1_{p_1,V}$ and $W^1_{p_2,V}$ on some classes of manifolds and Lie groups. The constants s_0, q_0 depend on our hypotheses.

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1. Introduction

In [3], the Schrödinger operator $-\Delta + V$ on $\mathbb{R}^n$ with $V \in A_\infty$, the Muckenhoupt class (see [18]), is studied and the question whether the spaces defined by the norm

$$(*) \qquad \|f\|_p + \| \, |\nabla f| \, \|_p + \|V^{\frac{1}{2}} f\|_p \text{ or } \| \, |\nabla f| \, \|_p + \|V^{\frac{1}{2}} f\|_p$$

interpolate is posed. In fact, it is shown that

$$\| \, |\nabla f| \, \|_p + \|V^{\frac{1}{2}} f\|_p \sim \|(-\Delta + V)^{\frac{1}{2}} f\|_p$$

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whenever $1 < p < \infty$ and $p \leq 2q$, $f \in C_0^\infty(\mathbb{R}^n)$, where $q > 1$ is a reverse Hölder exponent of V. Hence the question of interpolation can be solved a posteriori using functional calculus and interpolation of L_p spaces. However, it is reasonable to expect a direct proof of the interpolation properties of the norms in $(*)$ that is not only valid on $\mathbb{R}^n$ but also on other geometric settings.

Here we provide such an argument with p lying in an interval depending on the Reverse Hölder exponent of V by estimating the K-functional of real interpolation. The particular case $V = 1$ is treated in [7] (also $V = 0$), the general case requires involved use of properties of Muckenhoupt weights. The method is actually valid on some Lie groups and even some Riemannian manifolds in which we place ourselves. Let us consider the following statements:

Definition 1.1. *Let M be a Riemannian manifold, $V \in A_\infty$. Consider for $1 \leq p < \infty$, the vector space $E^1_{p,V}$ of C^∞ functions f on M such that f, $|\nabla f|$ and $Vf \in L_p(M)$. We define the Sobolev space $W^1_{p,V}(M) = W^1_{p,V}$ as the completion of $E^1_{p,V}$ for the norm*

$$\|f\|_{W^1_{p,V}} = \|f\|_p + \| \, |\nabla f| \, \|_p + \|Vf\|_p.$$

Definition 1.2. *We denote by $W^1_{\infty,V}(M) = W^1_{\infty,V}$ the space of all bounded Lipschitz functions f on M with $\|Vf\|_\infty < \infty$.*

We have the following interpolation theorem for the non-homogeneous Sobolev spaces $W^1_{p,V}$:

Theorem 1.3. *Let M be a complete Riemannian manifold satisfying a local doubling property (D_{loc}). Let $V \in RH_{qloc}$ for some $1 < q \leq \infty$. Assume that M admits a local Poincaré inequality (P_{sloc}) for some $1 \leq s < q$. Then for $1 \leq r \leq s < p < q$, $W^1_{p,V}$ is a real interpolation space between $W^1_{r,V}$ and $W^1_{q,V}$.*

Definition 1.4. *Let M be a Riemannian manifold, $V \in A_\infty$. Consider for $1 \leq p < \infty$, the vector space $\dot{W}^1_{p,V}$ of distributions f such that $|\nabla f|$ and $Vf \in L_p(M)$. It is well known that the elements of $\dot{W}^1_{p,V}$ are in $L_{p,loc}$. We equip $\dot{W}^1_{p,V}$ with the semi-norm*

$$\|f\|_{\dot{W}^1_{p,V}} = \| \, |\nabla f| \, \|_p + \|Vf\|_p.$$

In fact, this expression is a norm since $V \in A_\infty$ yields $V > 0 \; \mu - a.e.$

Definition 1.5. *We denote $\dot{W}^1_{\infty,V}(M) = \dot{W}^1_{\infty,V}$ the space of all Lipschitz functions f on M with $\|Vf\|_\infty < \infty$.*

For the homogeneous Sobolev spaces $\dot{W}^1_{p,V}$, we have

Theorem 1.6. *Let M be a complete Riemannian manifold satisfying (D). Let $V \in RH_q$ for some $1 < q \leq \infty$ and assume that M admits a Poincaré inequality (P_s) for some $1 \leq s < q$. Then, for $1 \leq r \leq s < p < q$, $\dot{W}^1_{p,V}$ is a real interpolation space between $\dot{W}^1_{r,V}$ and $\dot{W}^1_{q,V}$.*

It is known that if $V \in RH_q$ then $V + 1 \in RH_q$ with comparable constants. Hence part of Theorem 1.3 can be seen as a corollary of Theorem 1.6. But the fact that $V + 1$ is bounded away from 0 also allows local assumptions in Theorem 1.3, which is why we distinguish in this way the non-homogeneous and the homogeneous case.

The proof of Theorem 1.3 and Theorem 1.6 is done by estimating the K-functional of interpolation. We were not able to obtain a characterization of the K-functional.

However, this suffices for our needs. When $q = \infty$ (for example if V is a positive polynomial on $\mathbb{R}^n$) and $r = s$, then there is a characterization. The key tools to estimate the K-functional will be a Calderón-Zygmund decomposition for Sobolev functions and the Fefferman-Phong inequality (see section 3).

Some comments on Theorems 1.3 and 1.6. One can not improve the condition $s < p$ in these theorems even in the case $V = 1$ or $V = 0$ (see [2]).
These interpolation theorems are new even in the Euclidian case. On $\mathbb{R}^n$, this result was only known in the particular case $V = 1$ (see [17], [9], [12]).
We apply them in our forthcoming paper with B. Ben Ali [8] devoted to the study of the L_p boundedness of the Riesz tranform associated to Schrödinger operators and its inverse on Riemannian manifolds and Lie groups. Although the two papers have some similar techniques, the objective is different in each.

We end this introduction with a plan of the paper. In section 2, we review the notions of doubling property, Poincaré inequality, Reverse Hölder classes as well as the real K interpolation method. At the end of this section, we summarize some properties for the Sobolev spaces defined above under some additional hypotheses on M and V. Section 3 is devoted to give the main tools: the Fefferman-Phong inequality and a Calderón-Zygmund decomposition adapted to our Sobolev spaces. In section 4, we estimate the K-functional of real interpolation for non-homogeneous Sobolev spaces in two steps: first of all for the global case and secondly for the local case. We interpolate and get Theorem 1.3 in section 5. Section 6 concerns the proof of Theorem 1.6. Finally, in section 7, we apply our interpolation result on Lie groups with an appropriate definition of $W^1_{p,V}$.

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2. Preliminaries

Throughout this paper we write $\mathbb{1}_E$ for the characteristic function of a set E and E^c for the complement of E. For a ball B in a metric space, λB denotes the ball co-centered with B and with radius λ times that of B. Finally, C will be a constant that may change from an inequality to another and we will use $u \sim v$ to say that there exist two constants C_1, $C_2 > 0$ such that $C_1 u \leq v \leq C_2 u$. Let M denote a complete non-compact Riemannian manifold. We write μ for the Riemannian measure on M, ∇ for the Riemannian gradient, $|\cdot|$ for the length on the tangent space (forgetting the subscript x for simplicity) and $\|\cdot\|_p$ for the norm on $L_p(M,\mu)$, $1 \leq p \leq +\infty$.

2.1. The doubling property and Poincaré inequality.

Definition 2.1. *Let (M, d, μ) be a Riemannian manifold. Denote by $B(x, r)$ the open ball of center $x \in M$ and radius $r > 0$. One says that M satisfies the local doubling property (D_{loc}) if there exist constants $r_0 > 0$, $0 < C = C(r_0) < \infty$, such that for all $x \in M$, $0 < r < r_0$ we have*

$$(D_{loc}) \qquad \mu(B(x, 2r)) \leq C\mu(B(x, r)).$$

Furthermore, M satisfies a global doubling property or simply doubling property (D) if one can take $r_0 = \infty$. We also say that μ is a locally (resp. globally) doubling Borel measure.

Observe that if M satisfies (D) then

$$diam(M) < \infty \Leftrightarrow \mu(M) < \infty \text{ ([1])}.$$

Theorem 2.2 (Maximal theorem)**.** *([13]) Let M be a Riemannian manifold satisfying (D). Denote by $\mathcal{M}$ the uncentered Hardy-Littlewood maximal function over open balls of M defined by*

$$\mathcal{M}f(x) = \sup_{B: x\in B} |f|_B$$

where $f_E := \fint_E f d\mu := \frac{1}{\mu(E)} \int_E f d\mu$. Then

1. $\mu(\{x : \mathcal{M}f(x) > \lambda\}) \leq \frac{C}{\lambda} \int_M |f| d\mu$ *for every* $\lambda > 0$;
2. $\|\mathcal{M}f\|_p \leq C_p \|f\|_p$, *for* $1 < p \leq \infty$.

2.2. **Poincaré inequality.**

Definition 2.3 (Poincaré inequality on M)**.** *Let M be a complete Riemannian manifold, $1 \leq s < \infty$. We say that M admits **a local Poincaré inequality** (P_{sloc}) if there exist constants $r_1 > 0$, $C = C(r_1) > 0$ such that, for every function $f \in C_0^\infty$, and every ball B of M of radius $0 < r < r_1$, we have*

$$(P_{sloc}) \qquad \fint_B |f - f_B|^s d\mu \leq C r^s \fint_B |\nabla f|^s d\mu.$$

M admits a global Poincaré inequality (P_s) if we can take $r_1 = \infty$ in this definition.

Remark 2.4. *By density of C_0^∞ in W_s^1, if (P_{sloc}) holds for every function $f \in C_0^\infty$, then it holds for every $f \in W_s^1$.*

Let us recall some known facts about Poincaré inequality with varying s. It is known that (P_{sloc}) implies (P_{ploc}) when $p \geq s$ (see [21]). Thus, if the set of s such that (P_{sloc}) holds is not empty, then it is an interval unbounded on the right. A recent result from Keith-Zhong [24] asserts that this interval is open in $[1, +\infty[$ in the following sense:

Theorem 2.5. *Let (X, d, μ) be a complete metric-measure space with μ locally doubling and admitting a local Poincaré inequality (P_{sloc}), for some $1 < s < \infty$. Then there exists $\epsilon > 0$ such that (X, d, μ) admits (P_{ploc}) for every $p > s - \epsilon$ (see [24] and [7], section 4).*

2.3. **Reverse Hölder classes.**

Definition 2.6. *Let M be a Riemannian manifold. A weight w is a non-negative locally integrable function on M. We say that $w \in A_p$ for $1 < p < \infty$ if there is a constant C such that for every ball $B \subset M$*

$$\left(\fint_B w d\mu\right)\left(\fint_B w^{\frac{1}{1-p}} d\mu\right)^{p-1} \leq C.$$

For $p = 1$, $w \in A_1$ if there is a constant C such that for every ball $B \subset M$

$$\fint_B w d\mu \leq C w(y) \quad \text{for } \mu - a.e.\ y \in B.$$

We pose $A_\infty = \bigcup_{1\le p<\infty} A_p$.

Definition 2.7. *Let M be a Riemannian manifold. The reverse Hölder classes are defined in the following way: a weight $w \in RH_q$, $1<q<\infty$, if*

1. *$wd\mu$ is a doubling measure;*
2. *there exists a constant C such that for every ball $B \subset M$*

$$\left(\fint_B w^q d\mu\right)^{\frac{1}{q}} \le C \fint_B w d\mu. \tag{2.1}$$

The endpoint $q=\infty$ is given by the condition: $w \in RH_\infty$ whenever, $wd\mu$ is doubling and for any ball B,

$$w(x) \le C \fint_B w \quad \text{for } \mu-a.e.\ x \in B. \tag{2.2}$$

We say that $w \in RH_{qloc}$ for some $1<q<\infty$ (resp. $q=\infty$) if $wd\mu$ is locally doubling and there exists $r_2>0$ such that (2.1) (resp. (2.2)) holds for all balls B of radius $0<r<r_2$.
The smallest C is called the RH_q (resp. RH_{qloc}) constant of w.

On $\mathbb{R}^n$, the condition $wd\mu$ doubling is superfluous. It could be the same on a Riemannian manifold.

Proposition 2.8. *(*[28]*,* [18]*)*

1. *$RH_\infty \subset RH_q \subset RH_p$ for $1<p\le q\le\infty$.*
2. *If $w\in RH_q$, $1<q<\infty$, then there exists $q<p<\infty$ such that $w \in RH_p$.*
3. *$A_\infty = \bigcup_{1<q\le\infty} RH_q$.*

Proof. These properties are standard, see for instance [28], [18]. □

Proposition 2.9. *(see section 11 in* [3]*,* [23]*) Let V be a non-negative measurable function. Then the following properties are equivalent:*

1. *$V \in A_\infty$.*
2. *For all $r \in]0,1[$, $V^r \in RH_{\frac{1}{r}}$.*
3. *There exists $r\in]0,1[$, $V^r \in RH_{\frac{1}{r}}$.*

Remark 2.10. *Propositions 2.8 and 2.9 still hold in the local case, that is, when the weights are considered in a local reverse Hölder class RH_{qloc} for some $1<q\le\infty$.*

2.4. **The K method of real interpolation.** The reader is referred to [9], [10] for details on the development of this theory. Here we only recall the essentials to be used in the sequel.

Let A_0, A_1 be two normed vector spaces embedded in a topological Hausdorff vector space V, and define for $a\in A_0+A_1$ and $t>0$,

$$K(a,t,A_0,A_1) = \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}).$$

For $0<\theta<1$, $1\le q\le\infty$, we denote by $(A_0,A_1)_{\theta,q}$ the interpolation space between A_0 and A_1:

$$(A_0,A_1)_{\theta,q} = \left\{ a\in A_0+A_1 : \|a\|_{\theta,q} = \left(\int_0^\infty (t^{-\theta}K(a,t,A_0,A_1))^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty \right\}.$$

It is an exact interpolation space of exponent θ between A_0 and A_1, see [10] Chapter II.

Definition 2.11. *Let f be a measurable function on a measure space (X,μ). We denote by f^* its decreasing rearrangement function: for every $t>0$,*

$$f^*(t) = \inf\{\lambda : \mu(\{x : |f(x)| > \lambda\}) \le t\}.$$

We denote by f^{**} the maximal decreasing rearrangement of f: for every $t>0$,

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds.$$

It is known that $(\mathcal{M}f)^* \sim f^{**}$ and $\mu(\{x : |f(x)| > f^*(t)\}) \le t$ for all $t>0$. We refer to [9], [10], [11] for other properties of f^* and f^{**}.

To end with this subsection let us quote the following theorem ([22]):

Theorem 2.12. *Let (X,μ) be a measure space where μ is a non-atomic positive measure. Take $0<p_0<p_1<\infty$. Then*

$$K(f,t,L_{p_0},L_{p_1}) \sim \left(\int_0^{t^\alpha} (f^*(u))^{p_0}du\right)^{\frac{1}{p_0}} + t\left(\int_{t^\alpha}^{\infty} (f^*(u))^{p_1}du\right)^{\frac{1}{p_1}},$$

where $\frac{1}{\alpha} = \frac{1}{p_0} - \frac{1}{p_1}$.

2.5. **Sobolev spaces associated to a weight V.** For the definition of the non-homogeneous Sobolev spaces $W^1_{p,V}$ and the homogeneous one $\dot{W}^1_{p,V}$ see the introduction. We begin showing that $W^1_{\infty,V}$ and $\dot{W}^1_{p,V}$ are Banach spaces.

Proposition 2.13. *$W^1_{\infty,V}$ equipped with the norm*

$$\|f\|_{W^1_{\infty,V}} = \|f\|_\infty + \|\,|\nabla f|\,\|_\infty + \|Vf\|_\infty$$

is a Banach space.

Proof. Let $(f_n)_n$ be a Cauchy sequence in $W^1_{\infty,V}$. Then it is a Cauchy sequence in W^1_∞ and converges to f in W^1_∞. Hence $Vf_n \to Vf$ $\mu - a.e.$ On the other hand, $Vf_n \to g$ in L_∞, then $\mu - a.e.$ The uniqueness of the limit gives us $g = Vf$. □

Proposition 2.14. *Let M be a complete Riemannian manifold satisfying (D) and admitting a Poincaré inequality (P_s) for some $1 \le s < \infty$ and that $V \in A_\infty$. Then, for $s \le p \le \infty$, $\dot{W}^1_{p,V}$ equipped with the norm*

$$\|f\|_{\dot{W}^1_{p,V}} = \|\,|\nabla f|\,\|_p + \|Vf\|_p$$

is a Banach space.

Proof. Let $(f_n)_n$ be a Cauchy sequence in $\dot{W}^1_{p,V}$. There exist a sequence of functions $(g_n)_n$ and a sequence of scalar $(c_n)_n$ with $g_n = f_n - c_n$ converging to a function g in $L_{p,loc}$ and ∇g_n converging to ∇g in L_p (see [19]). Moreover, since $(Vf_n)_n$ is a Cauchy sequence in L_p, it converges to a function h $\mu - a.e.$ Lemma 3.1 in section 3 below yields

$$\int_B (|\nabla(f_n - f_m)|^s + |V(f_n - f_m)|^s)\, d\mu \geq C(B,V) \int_B |f_n - f_m|^s d\mu$$

for all ball B of M. Thus, $(f_n)_n$ is a Cauchy sequence in $L_{s,loc}$. Since $(f_n - c_n)$ is also Cauchy in $L_{s,loc}$, the sequence of constants $(c_n)_n$ is Cauchy in $L_{s,loc}$ and therefore converges to a locally constant function c. By connexity of M, c is actually constant on M. Take $f := g + c$. We have $g_n + c = f_n - c_n + c \to f$ in $L_{p,loc}$. It follows that $f_n \to f$ in $L_{p,loc}$ and so $Vf_n \to Vf$ $\mu - a.e.$ The uniqueness of the limit gives us $h = Vf$. Hence, we conclude that $f \in \dot{W}^1_{p,V}$ and $f_n \to f$ in $\dot{W}^1_{p,V}$ which finishes the proof. □

In the following proposition we characterize the $W^1_{p,V}$. We have

Proposition 2.15. *Let M be a complete Riemannian manifold and let $V \in RH_{qloc}$ for some $1 < q < \infty$. Consider, for $1 \leq p \leq q$,*

$$H^1_{p,V}(M) = H^1_{p,V} = \{f \in L_p :\ |\nabla f| \ \text{ and } Vf \in L_p\}$$

and equip it with the same norm as $W^1_{p,V}$. Then C_0^∞ is dense in $H^1_{p,V}$ and hence $W^1_{p,V} = H^1_{p,V}$.

Proof. See the Appendix. □

Therefore, under the hypotheses of Proposition 2.15, $W^1_{p,V}$ is the set of distributions $f \in L_p$ such that $|\nabla f|$ and Vf belong to L_p.

3. Principal tools

We shall use the following form of Fefferman-Phong inequality. The proof is completely analogous to the one in $\mathbb{R}^n$ (see [26], [3]):

Lemma 3.1. *(Fefferman-Phong inequality). Let M be a complete Riemannian manifold satisfying (D). Let $w \in A_\infty$ and $1 \leq p < \infty$. We assume that M admits also a Poincaré inequality (P_p). Then there is a constant $C > 0$ depending only on the A_∞ constant of w, p and the constants in (D), (P_p), such that for all ball B of radius $R > 0$ and $u \in W^1_{p,loc}$*

$$\int_B (|\nabla u|^p + w|u|^p) d\mu \geq C \min(R^{-p}, w_B) \int_B |u|^p d\mu.$$

Proof. Since M admits a (P_p) Poincaré inequality, we have

$$\int_B |\nabla u|^p d\mu \geq \frac{C}{R^p \mu(B)} \int_B \int_B |u(x) - u(y)|^p d\mu(x) d\mu(y).$$

This and

$$\int_B w|u|^p d\mu = \frac{1}{\mu(B)} \int_B \int_B w(x)|u(x)|^p d\mu(x) d\mu(y)$$

lead easily to

$$\int_B (|\nabla u|^p + w|u|^p) d\mu \geq [\min(CR^{-p}, w)]_B \int_B |u|^p d\mu.$$

Now we use that $w \in A_\infty$: there exists $\varepsilon > 0$, independent of B, such that $E = \{x \in B : w(x) > \varepsilon w_B\}$ satisfies $\mu(E) > \frac{1}{2}\mu(B)$. Indeed, since $w \in A_\infty$ then there exists $1 \le p < \infty$ such that $w \in A_p$. Therefore,

$$\frac{\mu(B \setminus E)}{\mu(B)} \le C \left(\frac{w(B \setminus E)}{w(B)}\right)^{\frac{1}{p}} \le C\epsilon^{\frac{1}{p}}.$$

We take $\epsilon > 0$ such that $C\epsilon^{\frac{1}{p}} < \frac{1}{2}$. We obtain then

$$[\min(CR^{-p}, w)]_B \ge \frac{1}{2}\min(CR^{-p}, \varepsilon w_B) \ge C' \min(R^{-p}, w_B).$$

This proves the desired inequality and finishes the proof. □

We proceed to establish two versions of a Calderón-Zygmund decomposition. Denote $T_r f = |f|^r + |\nabla f|^r + |Vf|^r$ for $1 \le r < \infty$.

Proposition 3.2. *Let M be a complete non-compact Riemannian manifold satisfying (D). Let $V \in RH_q$, for some $1 < q < \infty$ and assume that M admits a Poincaré inequality (P_s) for some $1 \le s < q$. Let $f \in W^1_{p,V}$, $s \le p < q$, and $\alpha > 0$. Then one can find a collection of balls (B_i), functions $g \in W^1_{q,V}$ and $b_i \in W^1_{s,V}$ with the following properties*

$$f = g + \sum_i b_i \tag{3.1}$$

$$\int_{\bigcup_i B_i} T_q g \, d\mu \le C\alpha^q \mu(\bigcup_i B_i) \tag{3.2}$$

$$\text{supp } b_i \subset B_i, \ \int_{B_i} T_s b_i \, d\mu \le C\alpha^s \mu(B_i) \tag{3.3}$$

$$\sum_i \mu(B_i) \le \frac{C}{\alpha^p} \int_M T_p f \, d\mu \tag{3.4}$$

$$\sum_i \mathbb{1}_{B_i} \le N \tag{3.5}$$

where N, C depend only on the constants in (D), (P_s), p and the RH_q constant of V.

Proof. Let $f \in W^1_{p,V}$, $\alpha > 0$. Consider $\Omega = \{x \in M : \mathcal{M}T_s f(x) > \alpha^s\}$. If $\Omega = \emptyset$, then set

$$g = f, \ b_i = 0 \text{ for all } i$$

so that (3.2) is satisfied thanks to the Lebesgue differentiation theorem. Otherwise, the maximal theorem –Theorem 2.2– and $p \ge s$ yield

$$\mu(\Omega) \le \frac{C}{\alpha^p} \int_M T_p f \, d\mu < \infty. \tag{3.6}$$

In particular $\Omega \ne M$ as $\mu(M) = \infty$. Let F be the complement of Ω. Since Ω is an open set distinct from M, let $(\underline{B_i})$ be a Whitney decomposition of Ω ([14]). That is, the balls $\underline{B_i}$ are pairwise disjoint and there exist two constants $C_2 > C_1 > 1$, depending only on the metric, such that

1. $\Omega = \bigcup_i B_i$ with $B_i = C_1 \underline{B_i}$ and the balls B_i have the bounded overlap property;

2. $r_i = r(B_i) = \frac{1}{2}d(x_i, F)$ and x_i is the center of B_i;
3. each ball $\overline{B_i} = C_2\underline{B_i}$ intersects F ($C_2 = 4C_1$ works).

For $x \in \Omega$, denote $I_x = \{i : x \in B_i\}$. By the bounded overlap property of the balls B_i, we have that $\sharp I_x \leq N$. Fixing $j \in I_x$ and using the properties of the B_i's, we easily see that $\frac{1}{3}r_i \leq r_j \leq 3r_i$ for all $i \in I_x$. In particular, $B_i \subset 7B_j$ for all $i \in I_x$.

Condition (3.5) is nothing but the bounded overlap property of the B_i's and (3.4) follows from (3.5) and (3.6). Note that $V \in RH_q$ implies $V^q \in A_\infty$ because there exists $\epsilon > 0$ such that $V \in RH_{q+\epsilon}$ and hence $V^q \in RH_{1+\frac{\epsilon}{q}}$. Proposition 2.9 shows then that $V^s \in RH_{\frac{q}{s}}$. Applying Lemma 3.1, one gets

$$\int_{B_i} (|\nabla f|^s + |Vf|^s) d\mu \geq C \min(V_{B_i}^s, r_i^{-s}) \int_{B_i} |f|^s d\mu. \tag{3.7}$$

We declare B_i of type 1 if $V_{B_i}^s \geq r_i^{-s}$ and of type 2 if $V_{B_i}^s < r_i^{-s}$. One should read $V_{B_i}^s$ as $(V^s)_{B_i}$ but this is also equivalent to $(V_{B_i})^s$ since $V \in RH_q \subset RH_s$.

Let us now define the functions b_i. For this, we construct $(\chi_i)_i$ a partition of unity of Ω subordinated to the covering (B_i). Each χ_i is a Lipschitz function supported in B_i with $\| \, |\nabla \chi_i| \, \|_\infty \leq \dfrac{C}{r_i}$, for instance $\chi_i(x) = \psi(\dfrac{C_1 d(x_i, x)}{r_i})\Big(\sum_k \psi(\dfrac{C_1 d(x_k, x)}{r_k})\Big)^{-1}$, where ψ is a smooth function, $\psi = 1$ on $[0,1]$, $\psi = 0$ on $[\frac{1+C_1}{2}, +\infty[$ and $0 \leq \psi \leq 1$. Set

$$b_i = \begin{cases} f\chi_i & \text{if } B_i \text{ of type 1,} \\ (f - f_{B_i})\chi_i & \text{if } B_i \text{ of type 2.} \end{cases}$$

Let us estimate $\int_{B_i} T_s b_i \, d\mu$. We distinguish two cases:

1. If B_i is of type 2, then

$$\begin{aligned} \int_{B_i} |b_i|^s d\mu &= \int_{B_i} |(f - f_{B_i})\chi_i|^s d\mu \\ &\leq C \left(\int_{B_i} |f|^s d\mu + \int_{B_i} |f_{B_i}|^s d\mu \right) \\ &\leq C \int_{B_i} |f|^s d\mu \\ &\leq C \int_{\overline{B_i}} |f|^s d\mu \\ &\leq C\alpha^s \mu(\overline{B_i}) \\ &\leq C\alpha^s \mu(B_i) \end{aligned}$$

where we used that $\overline{B_i} \cap F \neq \emptyset$ and the property (D). The Poincaré inequality (P_s) gives us

$$\begin{aligned} \int_{B_i} |\nabla b_i|^s d\mu &\leq C \int_{B_i} |\nabla f|^s d\mu \\ &\leq C\mathcal{M}T_s f(y) \mu(B_i) \\ &\leq C\alpha^s \mu(B_i) \end{aligned}$$

as y can be chosen in $F \cap \overline{B_i}$. Finally,

$$\begin{aligned}\int_{B_i} |Vb_i|^s d\mu &= \int_{B_i} |V(f - f_{B_i})\chi_i|^s d\mu \\ &\leq C \int_{B_i} |Vf|^s d\mu + C \int_{B_i} |Vf_{B_i}|^s d\mu \\ &\leq C(|Vf|^s)_{B_i}\mu(B_i) + C(V^s)_{B_i}(|f|^s)_{B_i}\mu(B_i) \\ &\leq C\alpha^s \mu(B_i) + C\left(|\nabla f|^s + |Vf|^s\right)_{B_i}\mu(B_i) \\ &\leq C\alpha^s\mu(B_i).\end{aligned}$$

We used that $\overline{B_i} \cap F \neq \emptyset$, Jensen's inequality and (3.7), noting that B_i is of type 2.

2. If B_i is of type 1, then

$$\begin{aligned}\int_{B_i} T_s b_i \, d\mu &\leq C \int_{B_i} T_s f \, d\mu + C r_i^{-s} \int_{B_i} |f|^s d\mu \\ &\leq C \int_{B_i} T_s f \, d\mu \\ &\leq C\alpha^s \mu(B_i)\end{aligned}$$

where we used that $\overline{B_i} \cap F \neq \emptyset$ and that B_i is of type 1.

Set now $g = f - \sum_i b_i$, where the sum is over balls of both types and is locally finite by (3.5). The function g is defined almost everywhere on M, $g = f$ on F and $g = \sum^2 f_{B_i}\chi_i$ on Ω where $\sum^k$ means that we are summing over balls of type k. Observe that g is a locally integrable function on M. Indeed, let $\varphi \in L_\infty$ with compact support. Since $d(x, F) \geq r_i$ for $x \in \operatorname{supp}\ b_i$, we obtain

$$\int \sum_i |b_i|\, |\varphi| \, d\mu \leq \Big(\int \sum_i \frac{|b_i|}{r_i} \, d\mu\Big) \sup_{x \in M} \Big(d(x, F)|\varphi(x)|\Big).$$

If B_i is of type 2, then

$$\begin{aligned}\int \frac{|b_i|}{r_i} d\mu &= \int_{B_i} \frac{|f - f_{B_i}|}{r_i} \chi_i \, d\mu \\ &\leq \Big(\mu(B_i)\Big)^{\frac{1}{s'}} \Big(\int_{B_i} |\nabla f|^s d\mu\Big)^{\frac{1}{s}} \\ &\leq C\alpha\mu(B_i).\end{aligned}$$

We used the Hölder inequality, (P_s) and that $\overline{B_i} \cap F \neq \emptyset$, s' being the conjugate of s. If B_i is of type 1, then

$$\begin{aligned}\int \frac{|b_i|}{r_i} d\mu &\leq \mu(B_i)^{\frac{1}{s'}} \left(\int_{B_i} \frac{|b_i|^s}{r_i^s} d\mu\right)^{\frac{1}{s}} \\ &\leq \mu(B_i)^{\frac{1}{s'}} r_i^{-s} \left(\int_{B_i} |f|^s d\mu\right)^{\frac{1}{s}} \\ &\leq C\mu(B_i)^{\frac{1}{s'}} \left(\int_{B_i} (|\nabla f|^s + |Vf|^s) d\mu\right)^{\frac{1}{s}}\end{aligned}$$

$$\leq C\alpha\mu(B_i).$$

Hence $\displaystyle\int \sum_i |b_i||\varphi| d\mu \leq C\alpha\mu(\Omega) \sup_{x\in M}\Big(d(x,F)|\varphi(x)|\Big)$. Since $f \in L_{1,loc}$, we conclude that $g \in L_{1,loc}$. (Note that since $b \in L_1$ in our case, we can say directly that $g \in L_{1,loc}$. However, for the homogeneous case –section 5– we need this observation to conclude that $g \in L_{1,loc}$.)

It remains to prove (3.2). Note that $\displaystyle\sum_i \chi_i(x) = 1$ for all $x \in \Omega$. A computation of the sum $\sum_i \nabla b_i$ leads us to

$$\nabla g = 1\!\!1_F(\nabla f) - \sum{}^1 f\nabla\chi_i - \sum{}^2 \left(f - f_{B_i}\right)\nabla\chi_i.$$

Set $h_k = \sum^k \left(f - f_{B_i}\right)\nabla\chi_i$ and $h = h_1 + h_2$. Then

$$\nabla g = (\nabla f)1\!\!1_F - \sum{}^1 f\nabla\chi_i - (h - h_1) = (\nabla f)1\!\!1_F - \sum{}^1 f_{B_i}\nabla\chi_i - h.$$

By definition of F and the differentiation theorem, $|\nabla g|$ is bounded by α almost everywhere on F. By already seen arguments for type 1 balls, $|f_{B_i}| \leq C\alpha r_i$. Therefore, $|\sum^1 f_{B_i}\nabla\chi_i| \leq C\sum^1 1\!\!1_{B_i}\alpha \leq CN\alpha$. It remains to control $\|h\|_\infty$. For this, note first that h vanishes on F and the sum defining h is locally finite on Ω. Then fix $x \in \Omega$. Observe that $\sum_i \nabla\chi_i(x) = 0$ and by definition of I_x, the sum reduces $i \in I_x$. Hence, for all $j \in I_x$,

$$\sum_i (f(x) - f_{B_i})\nabla\chi_i(x) = \sum_{i\in I_x}(f(x) - f_{B_i})\nabla\chi_i(x) = \sum_{i\in I_x}(f_{B_j} - f_{B_i})\nabla\chi_i(x).$$

We claim that $|f_{B_i} - f_{B_j}| \leq cr_j\alpha$. Indeed, since $B_i \subset 7B_j$, we get

$$\begin{aligned}
|f_{B_i} - f_{7B_j}| &\leq \frac{1}{\mu(B_i)}\int_{B_i}|f - f_{7B_j}|d\mu \\
&\leq \frac{C}{\mu(B_j)}\int_{7B_j}|f - f_{7B_j}|d\mu \\
&\leq Cr_j\Big(\fint_{7B_j}|\nabla f|^q d\mu\Big)^{\frac{1}{q}} \\
&\leq Cr_j\alpha \qquad (3.8)
\end{aligned}$$

where we used Hölder inequality, (D), (P_q) and that $\overline{B_i} \cap F \neq \emptyset$. Analogously, $|f_{7B_j} - f_{B_j}| \leq Cr_j\alpha$. Thus

$$\begin{aligned}
|h(x)| &= |\sum_{i\in I_x}(f_{B_j} - f_{B_i})\nabla\chi_i(x)| \\
&\leq C\sum_{i\in I_x}|f_{B_j} - f_{B_i}|r_i^{-1} \\
&\leq CN\alpha.
\end{aligned}$$

Let us now estimate $\int_\Omega T_q g\, d\mu$. We have

$$\begin{aligned}
\int_\Omega |g|^q d\mu &= \int_M |(\sum{}^2 f_{B_i}\chi_i)|^q d\mu \\
&\leq C\sum{}^2 |f_{B_i}|^q\mu(B_i)
\end{aligned}$$

$$\leq CN\alpha^q\mu(\Omega).$$

We used the estimate

$$(|f|_{B_i})^s \leq (|f|^s)_{B_i} \leq (\mathcal{M}T_s f)(y) \leq \alpha^s$$

as y can be chosen in $F \cap \overline{B_i}$. For $|\nabla g|$, we have

$$\begin{aligned}\int_\Omega |\nabla g|^q d\mu &= \int_\Omega |h_2|^q d\mu \\ &\leq C\alpha^q\mu(\Omega).\end{aligned}$$

Finally, since by Proposition 2.9 $V^s \in RH_{\frac{q}{s}}$, it comes that

$$\begin{aligned}\int_\Omega V^q|g|^q d\mu &\leq \sum{}^2 \int_{B_i} V^q |f_{B_i}|^q d\mu \\ &\leq C\sum{}^2 (V^s_{B_i}|f_{B_i}|^s)^{\frac{q}{s}}\mu(B_i).\end{aligned}$$

By construction of the type 2 balls and by (3.7) we have $V^s_{B_i}|f_{B_i}|^s \leq V^s_{B_i}(|f|^s)_{B_i} \leq C(|\nabla f|^s + |Vf|^s)_{B_i} \leq C\alpha^s$. Therefore, $\int_\Omega V^q|g|^q d\mu \leq C\sum{}^2\alpha^q\mu(B_i) \leq NC\alpha^q\mu(\Omega)$.

To finish the proof, we have to verify that $g \in W^1_{q,V}$. For that, we just have to control $\int_F T_q g\, d\mu$. As $g = f$ on F, this readily follows from

$$\begin{aligned}\int_F T_q f d\mu &= \int_F (|f|^q + |\nabla f|^q + |Vf|^q) d\mu \\ &\leq \int_F (|f|^p|f|^{q-p} + |\nabla f|^p|\nabla f|^{q-p} + |Vf|^p|Vf|^{q-p}) d\mu \\ &\leq \alpha^{q-p}\|f\|^p_{W^1_{p,V}}.\end{aligned}$$

□

Remark 3.3. *1-It is a straightforward consequence from (3.3) that $b_i \in W^1_{r,V}$ for all $1 \leq r \leq s$ with $\|b_i\|_{W^1_{r,V}} \leq C\alpha\mu(B_i)^{\frac{1}{r}}$.*
2-The estimate $\int_F T_q g\, d\mu$ above is too crude to be used in the interpolation argument. Note that (3.2) only involves control of $T_q g$ on $\Omega = \bigcup_i B_i$. Compare with (3.10) in the next argument when $q = \infty$.

Proposition 3.4. *Let M be a complete non-compact Riemannian manifold satisfying (D). Let $V \in RH_\infty$ and assume that M admits a Poincaré inequality (P_s) for some $1 \leq s < \infty$. Let $f \in W^1_{p,V}$, $s \leq p < \infty$, and $\alpha > 0$. Then one can find a collection of balls (B_i), functions b_i and a Lipschitz function g such that the following properties hold:*

$$f = g + \sum_i b_i \tag{3.9}$$

$$\|g\|_{W^1_{\infty,V}} \leq C\alpha \tag{3.10}$$

$$\text{supp } b_i \subset B_i,\ \forall\, 1 \leq r \leq s \int_{B_i} T_r b_i\, d\mu \leq C\alpha^r\mu(B_i) \tag{3.11}$$

$$(3.12)\qquad \sum_i \mu(B_i) \leq \frac{C}{\alpha^p}\int T_p f\, d\mu$$

$$(3.13)\qquad \sum_i \chi_{B_i} \leq N$$

where C and N only depend on the constants in (D), (P_s), p and the RH_∞ constant of V.

Proof. The only difference between the proof of this proposition and that of Proposition 3.2 is the estimate (3.10). Indeed, as we have seen in the proof of Proposition 3.2, we have $|\nabla g| \leq C\alpha$ almost everywhere. By definition of F and the differentiation theorem, $(|g| + |Vg|)$ is bounded by α almost everywhere on F. We have also seen that for all i, $|f|_{B_i} \leq \alpha$.

Fix $x \in \Omega$, then

$$\begin{aligned} |g(x)| &= |\sum_{i\in I_x} f_{B_i}| \\ &\leq \sum_{i\in I_x} |f_{B_i}| \\ &\leq N\alpha. \end{aligned}$$

It remains to estimate $|Vg|(x)$. We have

$$\begin{aligned} |Vg|(x) &\leq \sum{}_{i:x\in B_i}^{2} V(x)|f_{B_i}| \\ &\leq C\sum{}_{i:x\in B_i}^{2} (V_{B_i})|f_{B_i}| \\ &\leq C\sum{}_{i:x\in B_i}^{2} ((V^s)_{B_i}(|f|^s)_{B_i})^{\frac{1}{s}} \\ &\leq C\sum{}_{i:x\in B_i}^{2} (|\nabla f|^s + |Vf|^s)_{B_i}^{\frac{1}{s}} \\ &\leq NC\alpha \end{aligned}$$

where we used the definition of RH_∞ and Jensen's inequality as $s \geq 1$. We also used (3.7) and the bounded overlap property of the B_i's. □

4. Estimation of the K-functional in the non-homogeneous case

Denote for $1 \leq r < \infty$, $T_r f = |f|^r + |\nabla f|^r + |Vf|^r$, $T_{r*}f = |f|^{r*} + |\nabla f|^{r*} + |Vf|^{r*}$, $T_{r**}f = |f|^{r**} + |\nabla f|^{r**} + |Vf|^{r**}$. We have $tT_{r**}f(t) = \int_0^t T_{r*}f(u)du$ for all $t > 0$.

Theorem 4.1. *Under the same hypotheses as in Theorem 1.3, with $V \in RH_{\infty loc}$ and $1 \leq r \leq s < \infty$:*

1. *there exists $C_1 > 0$ such that for every $f \in W^1_{r,V} + W^1_{\infty,V}$ and $t > 0$*

$$K(f, t^{\frac{1}{r}}, W^1_{r,V}, W^1_{\infty,V}) \geq C_1 \left(\int_0^t T_{r*}f(u)du\right)^{\frac{1}{r}} \sim (tT_{r**}f(t))^{\frac{1}{r}};$$

2. *for $s \leq p < \infty$, there is $C_2 > 0$ such that for every $f \in W^1_{p,V}$ and $t > 0$*

$$K(f, t^{\frac{1}{r}}, W^1_{r,V}, W^1_{\infty,V}) \leq C_2 t^{\frac{1}{r}} (T_{s**}f(t))^{\frac{1}{s}}.$$

In the particular case when $r = s$, we obtain the upper bound of K for every $f \in W^1_{s,V} + W^1_{\infty,V}$ and therefore get a true characterization of K.

Proof. We refer to [7] for an analogous proof. $\square$

Theorem 4.2. *We consider the same hypotheses as in Theorem 1.3 with $V \in RH_{qloc}$ for some $1 < q < \infty$. Then*

1. *there exists C_1 such that for every $f \in W^1_{r,V} + W^1_{q,V}$ and $t > 0$*

$$K(f,t,W^1_{r,V},W^1_{q,V}) \geq C_1 \left(t^{\frac{q}{q-r}} (T_{r**}f)^{\frac{1}{r}} (t^{\frac{qr}{q-r}}) + t \left(\int_{t^{\frac{qr}{q-r}}}^{\infty} T_{r*}f(u)du \right)^{\frac{1}{r}} \right);$$

2. *for $s \leq p < q$, there is C_2 such that for every $f \in W^1_{p,V}$ and $t > 0$*

$$K(f,t,W^1_{r,V},W^1_{q,V}) \leq C_2 \left(t^{\frac{q}{q-r}} (T_{s**}f)^{\frac{1}{s}} (t^{\frac{qr}{q-r}}) + t \left(\int_{t^{\frac{qr}{q-r}}}^{\infty} (\mathcal{M}T_s f)^{*\frac{q}{s}} (u)du \right)^{\frac{1}{q}} \right).$$

Proof. In a first step, we prove this theorem in the global case. This will help to understand the proof of the more general local case.

4.1. **The global case.** Let M be a complete Riemannian manifold satisfying (D). Let $V \in RH_q$ for some $1 < q < \infty$ and assume that M admits a Poincaré inequality (P_s) for some $1 \leq s < q$. The principal tool to prove Theorem 4.2 in this case will be the Calderón-Zygmund decomposition of Proposition 3.2.

We prove the left inequality by applying Theorem 2.12 with $p_0 = r$ and $p_1 = q$ which gives for all $f \in L_r + L_q$:

$$K(f,t,L_r,L_q) \sim \left(\int_0^{t^{\frac{qr}{q-r}}} f^{*r}(u)du \right)^{\frac{1}{r}} + t \left(\int_{t^{\frac{qr}{q-r}}}^{\infty} f^{*q}(u)du \right)^{\frac{1}{q}}.$$

Moreover, we have

$$K(f,t,W^1_{r,V},W^1_{q,V}) \geq K(f,t,L_r,L_q) + K(|\nabla f|,t,L_r,L_q) + K(Vf,t,L_r,L_q)$$

since the operator

$$(I,\, \nabla,\, V) : W^1_{l,V} \to L_l(M;\mathbb{C} \times TM \times \mathbb{C})$$

is bounded for every $1 \leq l \leq \infty$.
Hence, we conclude that

$$K(f,t,W^1_{r,V},W^1_{q,V}) \geq C \left(\int_0^{t^{\frac{qr}{q-r}}} T_{r*}f(u)du \right)^{\frac{1}{r}} + Ct \left(\int_{t^{\frac{qr}{q-r}}}^{\infty} T_{q*}f(u)du \right)^{\frac{1}{q}}.$$

We now prove item 2. Let $f \in W^1_{p,V}$, $s \leq p < q$ and $t > 0$. We consider the Calderón-Zygmund decomposition of f given by Proposition 3.2 with $\alpha = \alpha(t) = (\mathcal{M}T_s f)^{*\frac{1}{s}}(t^{\frac{qr}{q-r}})$. Thus f can be written as $f = b + g$ with $b = \sum_i b_i$ where $(b_i)_i$, g satisfy the properties of the proposition. For the L_r norm of b, we have

$$\|b\|_r^r \leq \int_M (\sum_i |b_i|)^r d\mu$$

$$
\begin{aligned}
&\le N\sum_i \int_{B_i} |b_i|^r d\mu\\
&\le C\alpha^r(t)\sum_i \mu(B_i)\\
&\le NC\alpha^r(t)\mu(\Omega_t).
\end{aligned}
$$

This follows from the fact that $\displaystyle\sum_i \chi_{B_i} \le N$ and $\Omega_t = \Omega = \bigcup_i B_i$. Similarly, we obtain $\| \, |\nabla b| \, \|_r^r \le C\alpha^r(t)\mu(\Omega_t)$ and $\|Vb\|_r^r \le C\alpha^r(t)\mu(\Omega_t)$. For g, we have $\|g\|_{W^1_{q,V}} \le C\alpha(t)\mu(\Omega_t)^{\frac{1}{q}} + \left(\int_{F_t} T_q f d\mu\right)^{\frac{1}{q}}$, where $F_t = F$ in the Proposition 3.2 with this choice of α.

Moreover, since $(\mathcal{M}f)^* \sim f^{**}$ and $(f+g)^{**} \le f^{**} + g^{**}$, we obtain

$$\alpha(t) = (\mathcal{M}T_s f)^{*\frac{1}{s}}(t^{\frac{qr}{q-r}}) \le C(T_{s**}f)^{\frac{1}{s}}(t^{\frac{qr}{q-r}}).$$

Notice that for every $t > 0$, $\mu(\Omega_t) \le t^{\frac{qr}{q-r}}$. It comes that

$$K(f,t,W^1_{r,V},W^1_{q,V}) \le Ct^{\frac{q}{q-r}}(T_{s**}f)^{\frac{1}{s}}(t^{\frac{qr}{q-r}}) + Ct\left(\int_{F_t} T_q f d\mu\right)^{\frac{1}{q}}. \tag{4.1}$$

Let us estimate $\int_{F_t} T_q f d\mu$. Consider E_t a measurable set such that

$$\Omega_t \subset E_t \subset \left\{x : \mathcal{M}T_s f(x) \ge (\mathcal{M}T_s f)^*(t^{\frac{qr}{q-r}})\right\}$$

and $\mu(E_t) = t^{\frac{qr}{q-r}}$. Remark that $\int_{E_t}(\mathcal{M}T_s f)^l d\mu = \int_0^{t^{\frac{qr}{q-r}}} (\mathcal{M}T_s f)^{*l}(u)du$ for $l \ge 1$ (see [27], Chapter V, Lemma 3.17). Denote $G_t := E_t \setminus \Omega_t$. Then

$$
\begin{aligned}
\int_{F_t} T_q f d\mu &= \int_{E_t^c} T_q f d\mu + \int_{G_t} T_q f d\mu\\
&\le C\int_{t^{\frac{qr}{q-r}}}^\infty (\mathcal{M}T_s f)^{*\frac{q}{s}}(u)du + C\mu(G_t)(T_{s**}f)^{\frac{q}{s}}(t^{\frac{qr}{q-r}})\\
&\le C\int_{t^{\frac{qr}{q-r}}}^\infty (\mathcal{M}T_s f)^{*\frac{q}{s}}(u)du + C\mu(E_t)(T_{s**}f)^{\frac{q}{s}}(t^{\frac{qr}{q-r}})\\
&= C\int_{t^{\frac{qr}{q-r}}}^\infty (\mathcal{M}T_s f)^{*\frac{q}{s}}(u)du + Ct^{\frac{qr}{q-r}}(T_{s**}f)^{\frac{q}{s}}(t^{\frac{qr}{q-r}}). \qquad (4.2)
\end{aligned}
$$

Combining (4.1) and (4.2), we deduce that

$$K(f,t,W^1_{r,V},W^1_{q,V}) \le Ct^{\frac{q}{q-r}}(T_{s**}f)^{\frac{1}{s}}(t^{\frac{qr}{q-r}}) + Ct\left(\int_{t^{\frac{qr}{q-r}}}^\infty (\mathcal{M}T_s f)^{*\frac{q}{s}}(u)du\right)^{\frac{1}{q}}$$

which finishes the proof in that case.

4.2. **The local case.** Let M be a complete non-compact Riemannian manifold satisfying a local doubling property (D_{loc}). Consider $V \in RH_{qloc}$ for some $1 < q < \infty$ and assume that M admits a local Poincaré inequality (P_{sloc}) for some $1 \le s < q$.

Denote by $\mathcal{M}_E$ the Hardy-Littlewood maximal operator relative to a measurable subset E of M, that is, for $x \in E$ and every f locally integrable function on M:

$$\mathcal{M}_E f(x) = \sup_{B:\, x \in B} \frac{1}{\mu(B \cap E)} \int_{B \cap E} |f| d\mu$$

where B ranges over all open balls of M containing x and centered in E. We say that a measurable subset E of M has the relative doubling property if there exists a constant C_E such that for all $x \in E$ and $r > 0$ we have

$$\mu(B(x, 2r) \cap E) \leq C_E \mu(B(x, r) \cap E).$$

This is equivalent to saying that the metric measure space $(E, d/E, \mu/E)$ has the doubling property. On such a set $\mathcal{M}_E$ is of weak type $(1, 1)$ and bounded on $L^p(E, \mu)$, $1 < p \leq \infty$.

We now prove Theorem 4.2 in the local case. To fix ideas, we assume $r_0 = 5$, $r_1 = 8$, $r_2 = 2$. The lower bound of K in item 1. is trivial (same proof as for the global case). It remains to prove the upper bound. For all $t > 0$, take $\alpha = \alpha(t) = (\mathcal{M}T_s f)^{*\frac{1}{s}}(t^{\frac{qr}{q-r}})$.

Consider

$$\Omega = \{x \in M : \mathcal{M}T_s f(x) > \alpha^s(t)\}.$$

We have $\mu(\Omega) \leq t^{\frac{qr}{q-r}}$. If $\Omega = M$, then

$$\begin{aligned} \int_M T_r f \, d\mu &= \int_\Omega T_r f \, d\mu \\ &\leq C \int_0^{\mu(\Omega)} T_{r*} f(l) dl \\ &\leq C \int_0^{t^{\frac{qr}{q-r}}} T_{r*} f(l) dl \\ &\leq C t^{\frac{qr}{q-r}} T_{r**} f(t^{\frac{qr}{q-r}}) \end{aligned}$$

Therefore, since $r \leq s$

$$K(f, t, W^1_{r,V}, W^1_{q,V}) \leq C t^{\frac{q}{q-r}} (T_{s**} f)^{\frac{1}{s}} (t^{\frac{qr}{q-r}}).$$

We thus obtain item 2. in this case.

Now assume $\Omega \neq M$. Pick a countable set $\{x_j\}_{j \in J} \subset M$, such that $M = \bigcup_{j \in J} B(x_j, \frac{1}{2})$ and for all $x \in M$, x does not belong to more than N_1 balls $B^j := B(x_j, 1)$. Consider a C^∞ partition of unity $(\varphi_j)_{j \in J}$ subordinated to the balls $\frac{1}{2}B^j$ such that $0 \leq \varphi_j \leq 1$, *supp* $\varphi_j \subset B^j$ and $\| |\nabla \varphi_j| \|_\infty \leq C$ uniformly with respect to j. Consider $f \in W^1_{p,V}$, $s \leq p < q$. Let $f_j = f \varphi_j$ so that $f = \sum_{j \in J} f_j$. We have for $j \in J$, $f_j, V f_j \in L_p$ and $\nabla f_j = f \nabla \varphi_j + \nabla f \varphi_j \in L_p$. Hence $f_j \in W^1_p(B^j)$. The balls B^j satisfy the relative doubling property with the constant independent of the balls B^j. This follows from the next lemma quoted from [4], p.947.

Lemma 4.3. *Let M be a complete Riemannian manifold satisfying (D_{loc}). Then the balls B^j above, equipped with the induced distance and measure, satisfy the relative doubling property (D), with the doubling constant that may be chosen independently of j. More precisely, there exists $C \geq 0$ such that for all $j \in J$*

$$\mu(B(x, 2R) \cap B^j) \leq C \, \mu(B(x, R) \cap B^j) \quad \forall x \in B^j, \; R > 0, \tag{4.3}$$

and

$$\mu(B(x,R)) \leq C\mu(B(x,R)\cap B^j) \quad \forall x \in B^j,\, 0 < R \leq 2. \tag{4.4}$$

Let us return to the proof of the theorem. For any $x \in B^j$ we have

$$\begin{aligned}
\mathcal{M}_{B^j} T_s f_j(x) &= \sup_{B:\, x\in B,\, R(B)\leq 2} \frac{1}{\mu(B^j\cap B)} \int_{B^j\cap B} T_s f_j d\mu \\
&\leq \sup_{B:\, x\in B,\, R(B)\leq 2} C\frac{\mu(B)}{\mu(B^j\cap B)} \frac{1}{\mu(B)} \int_B T_s f d\mu \\
&\leq C\mathcal{M}T_s f(x). \qquad (4.5)
\end{aligned}$$

where we used (4.4) of Lemma 4.3. Consider now

$$\Omega_j = \left\{x \in B^j : \mathcal{M}_{B^j} T_s f_j(x) > C\alpha^s(t)\right\}$$

where C is the constant in (4.5). The set Ω_j is an open subset of B^j, then of M, and $\Omega_j \subset \Omega$ for all $j \in J$. For the f_j's, and for all $t > 0$, we have a Calderón-Zygmund decomposition similar to the one done in Proposition 3.2: there exist $b_{jk},\ g_j$ supported in B^j, and balls $(B_{jk})_k$ of M, contained in Ω_j, such that

$$f_j = g_j + \sum_k b_{jk} \tag{4.6}$$

$$\int_{\Omega_j} T_q g_j \, d\mu \leq C\alpha^q(t)\mu(\Omega_j) \tag{4.7}$$

$$supp\ b_{jk} \subset B_{jk},\, \forall 1 \leq r \leq s \int_{B_{jk}} T_r b_{jk}\, d\mu \leq C\alpha^r(t)\mu(B_{jk}) \tag{4.8}$$

$$\sum_k \mu(B_{jk}) \leq C\alpha^{-p}(t) \int_{B^j} T_p f_j \, d\mu \tag{4.9}$$

$$\sum_k \chi_{B_{jk}} \leq N \tag{4.10}$$

with C and N depending only on q, p and the constant $C(r_0), C(r_1), C(r_2)$ in (D_{loc}) and (P_{sloc}) and the RH_{qloc} condition of V, which is independent of B^j.
The proof of this decomposition is the same as in the Proposition 3.2. We take for all $j \in J$, a Whitney decomposition $(B_{jk})_k$ of $\Omega_j \neq M$. We use the doubling property for balls whose radii do not exceed $3 < r_0$ and the Poincaré inequality for balls whose radii do not exceed $7 < r_1$ and the RH_{qloc} property of V for balls whose radii do not exceed $1 < r_2$. By the above decomposition, we can write $f = \sum_{j\in J}\sum_k b_{jk} + \sum_{j\in J} g_j = b + g$. Let us now estimate $\|b\|_{W^1_{r,V}}$ and $\|g\|_{W^1_{q,V}}$.

$$\begin{aligned}
\|b\|_r^r &\leq N_1 N \sum_j \sum_k \|b_{jk}\|_r^r \\
&\leq C\alpha^r(t) \sum_j \sum_k (\mu(B_{jk}))
\end{aligned}$$

$$
\begin{aligned}
&\leq NC\alpha^r(t)\Big(\sum_j \mu(\Omega_j)\Big)\\
&\leq N_1 C\alpha^r(t)\mu(\Omega).
\end{aligned}
$$

We used the bounded overlap property of the $(\Omega_j)_{j\in J}$'s and that of the $(B_{jk})_k$'s for all $j \in J$. It follows that $\|b\|_r \leq C\alpha(t)\mu(\Omega)^{\frac{1}{r}}$. Similarly we get $\| \, |\nabla b| \, \|_r \leq C\alpha(t)\mu(\Omega)^{\frac{1}{r}}$ and $\|Vb\|_r \leq C\alpha(t)\mu(\Omega)^{\frac{1}{r}}$.

For g we have

$$
\begin{aligned}
\int_\Omega |g|^q d\mu &\leq N\sum_j \int_{\Omega_j} |g_j|^q d\mu\\
&\leq NC\alpha^q(t)\sum_j \mu(\Omega_j)\\
&\leq N_1 NC\alpha^q(t)\mu(\Omega).
\end{aligned}
$$

Analogously, $\int_\Omega |\nabla g|^q d\mu \leq C\alpha^q(t)\mu(\Omega)$ and $\int_\Omega |Vg|^q d\mu \leq C\alpha^q(t)\mu(\Omega)$. Noting that $g \in W^1_{q,V}$ –same argument as in the proof of the global case–, it follows that

$$
\begin{aligned}
K(f,t,W^1_{r,V},W^1_{q,V}) &\leq \|b\|_{W^1_{r,V}} + t\|g\|_{W^1_{q,V}}\\
&\leq C\alpha(t)\mu(\Omega)^{\frac{1}{r}} + Ct\alpha(t)\mu(\Omega)^{\frac{1}{q}} + t\left(\int_{F_t} T_q f d\mu\right)^{\frac{1}{q}}\\
&\leq Ct^{\frac{q}{q-r}}(T_{s**}f)^{\frac{1}{s}}(t^{\frac{qr}{q-r}}) + t\left(\int_{t^{\frac{qr}{q-r}}}^\infty (\mathcal{M}T_s f)^{*\frac{q}{s}}(u)du\right)^{\frac{1}{q}}.
\end{aligned}
$$

Thus, we get the desired estimation for $f \in W^1_{p,V}$. □

5. Interpolation of non-homogeneous Sobolev spaces

Proof of Theorem 1.3. The proof of the case when $V \in RH_{\infty loc}$ is the same as the one in section 4 in [7]. Consider now $V \in RH_{qloc}$ for some $1 < q < \infty$. For $1 \leq r \leq s < p < q$, we define the interpolation space $W^1_{p,r,q,V}(M) = W^1_{p,r,q,V}$ between $W^1_{r,V}$ and $W^1_{q,V}$ by

$$
W^1_{p,r,q,V} = (W^1_{r,V}, W^1_{q,V})_{\frac{q(p-r)}{p(q-r)},p}.
$$

We claim that $W^1_{p,r,q,V} = W^1_{p,V}$ with equivalent norms. Indeed, let $f \in W^1_{p,r,q,V}$. We have

$$
\begin{aligned}
\|f\|_{\frac{q(p-r)}{p(q-r)},p} &= \left\{\int_0^\infty \left(t^{\frac{q(r-p)}{p(q-r)}}K(f,t,W^1_{r,V},W^1_{q,V})\right)^p \frac{dt}{t}\right\}^{\frac{1}{p}}\\
&\geq \left\{\int_0^\infty \left(t^{\frac{q(r-p)}{p(q-r)}}t^{\frac{q}{q-r}}(T_{r**}f)^{\frac{1}{r}}(t^{\frac{qr}{q-r}})\right)^p \frac{dt}{t}\right\}^{\frac{1}{p}}\\
&= \left\{\int_0^\infty t^{\frac{qr}{q-r}-1}(T_{r**}f)^{\frac{p}{r}}(t^{\frac{qr}{q-r}})dt\right\}^{\frac{1}{p}}\\
&= \left\{\int_0^\infty (T_{r**}f)^{\frac{p}{r}}(t)dt\right\}^{\frac{1}{p}}
\end{aligned}
$$

$$
\begin{aligned}
&\geq \|f^{r**}\|_{\frac{p}{r}}^{\frac{1}{r}} + \| \, |\nabla f|^{r**}\|_{\frac{p}{r}}^{\frac{1}{r}} + \| \, |Vf|^{r**}\|_{\frac{p}{r}}^{\frac{1}{r}} \\
&\sim \|f^{r}\|_{\frac{p}{r}}^{\frac{1}{r}} + \| \, |\nabla f|^{r}\|_{\frac{p}{r}}^{\frac{1}{r}} + \| \, |Vf|^{r}\|_{\frac{p}{r}}^{\frac{1}{r}} \\
&= \|f\|_{W^1_{p,V}}
\end{aligned}
$$

where we used that for $l > 1$, $\|f^{**}\|_l \sim \|f\|_l$. Therefore $W^1_{p,r,q,V} \subset W^1_{p,V}$, with $\|f\|_{\frac{q(p-r)}{p(q-r)},p} \geq C\|f\|_{W^1_{p,V}}$.

On the other hand, let $f \in W^1_{p,V}$. By the Calderón-Zygmund decomposition of Proposition 3.2, $f \in W^1_{r,V} + W^1_{q,V}$. Next,

$$
\begin{aligned}
\|f\|_{\frac{q(p-r)}{p(q-r)},p} &\leq C\left\{\int_0^\infty \left(t^{\frac{q(r-p)}{p(q-r)}} t^{\frac{q}{q-r}} (T_{s**}f)^{\frac{1}{s}}(t^{\frac{qr}{q-r}})\right)^p \frac{dt}{t}\right\}^{\frac{1}{p}} \\
&\quad + C\left\{\int_0^\infty \left(t^{\frac{q(r-p)}{p(q-r)}} t \left(\int_{t^{\frac{qr}{q-r}}}^\infty (\mathcal{M}T_s f)^{*\frac{q}{s}}(u)du\right)^{\frac{1}{q}}\right)^p \frac{dt}{t}\right\}^{\frac{1}{p}} \\
&= I + II.
\end{aligned}
$$

Using the same computation as above, we conclude that

$$
\begin{aligned}
I &\leq C\left\{\int_0^\infty (T_{s**}f)^{\frac{p}{s}}(t)dt\right\}^{\frac{1}{p}} \\
&\leq C\|f\|_{W^1_{p,V}}.
\end{aligned}
$$

It remains to estimate II. We have

$$
\begin{aligned}
II &\leq C\left\{\int_0^\infty t^{\frac{q(r-p)}{q-r}} t^p \left(\int_{t^{\frac{qr}{q-r}}}^\infty (\mathcal{M}T_s f)^{*\frac{q}{s}}(u)du\right)^{\frac{p}{q}} \frac{dt}{t}\right\}^{\frac{1}{p}} \\
&\leq C\left\{\int_0^\infty t^{-\frac{p}{q}} \left(\int_t^\infty \left(u(\mathcal{M}T_s f)^{*\frac{q}{s}}(u)\right) \frac{du}{u}\right)^{\frac{p}{q}} dt\right\}^{\frac{1}{p}} \\
&\leq C\left\{\int_0^\infty t^{-\frac{p}{q}} \left(\int_t^\infty \left(u(\mathcal{M}T_s f)^{*\frac{q}{s}}(u)\right)^{\frac{p}{q}} \frac{du}{u}\right) dt\right\}^{\frac{1}{p}} \\
&\leq \frac{C}{1-\frac{p}{q}}\left\{\int_0^\infty t^{-\frac{p}{q}} (t(t^{\frac{p}{q}-1}(\mathcal{M}T_s f)^{*\frac{p}{s}}(t)))dt\right\}^{\frac{1}{p}} \\
&= C\|(\mathcal{M}T_s f)^*\|_{\frac{p}{s}}^{\frac{1}{s}} \\
&\leq C\|\mathcal{M}T_s f\|_{\frac{p}{s}}^{\frac{1}{s}} \\
&\leq C\|T_s f\|_{\frac{p}{s}}^{\frac{1}{s}} \\
&\leq C\|f\|_{W^1_{p,V}}.
\end{aligned}
$$

We used the monotonicity of $(\mathcal{M}T_s f)^*$ together with $\frac{p}{q} < 1$, the following Hardy inequality

$$
\int_0^\infty \left[\int_t^\infty g(u)du\right] t^{l-1}dt \leq \left(\frac{1}{l}\right) \int_0^\infty [ug(u)]\, u^{l-1}du
$$

for $l=1-\frac{p}{q}>0$, the fact that $\|g^*\|_l\sim\|g\|_l$ for all $l\geq 1$ and Theorem 2.2. Therefore, $W^1_{p,V}\subset W^1_{p,r,q,V}$ with $\|f\|_{\frac{q(p-r)}{p(q-r)},p}\leq C\|f\|_{W^1_{p,V}}$. □

Let $A_V=\{q\in]1,\infty]: V\in RH_{qloc}\}$ and $q_0=\sup A_V$, $B_M=\{s\in[1,q_0[: (P_{sloc}) \text{ holds }\}$ and $s_0=\inf B_M$.

Corollary 5.1. *For all p, p_1, p_2 such that $1\leq p_1<p<p_2<q_0$ with $p>s_0$, $W^1_{p,V}$ is a real interpolation space between $W^1_{p_1,V}$ and $W^1_{p_2,V}$.*

Proof. Since $p_2<q_0$, item 1. of Proposition 2.8 gives us that $V\in RH_{p_2loc}$. Therefore, Theorem 1.3 yields the corollary. (We could prove this corollary also using the reiteration theorem.)

□

6. Interpolation of homogeneous Sobolev spaces

Denote for $1\leq r<\infty$, $\dot{T}_rf=|\nabla f|^r+|Vf|^r$, $\dot{T}_{r*}f=|\nabla f|^{r*}+|Vf|^{r*}$ and $\dot{T}_{r**}f=|\nabla f|^{r**}+|Vf|^{r**}$. For the estimation of the functional K for homogeneous Sobolev spaces we have the corresponding results:

Theorem 6.1. *Under the hypotheses of Theorem 1.6 with $q<\infty$:*

1. *there exists C_1 such that for every $f\in\dot{W}^1_{r,V}+\dot{W}^1_{q,V}$ and $t>0$*

$$K(f,t,\dot{W}^1_{r,V},\dot{W}^1_{q,V})\geq C_1\left\{\left(\int_0^{t^{\frac{qr}{q-r}}}\dot{T}_{r*}f(u)du\right)^{\frac{1}{r}}+t\left(\int_{t^{\frac{qr}{q-r}}}^\infty\dot{T}_{q*}f(u)du\right)^{\frac{1}{q}}\right\};$$

2. *for $s\leq p<q$, there exists C_2 such that for every $f\in\dot{W}^1_{p,V}$ and $t>0$*

$$K(f,t,\dot{W}^1_{r,V},\dot{W}^1_{q,V})\leq C_2\left\{\left(\int_0^{t^{\frac{qr}{q-r}}}\dot{T}_{s*}f(u)du\right)^{\frac{1}{s}}+t\left(\int_{t^{\frac{qr}{q-r}}}^\infty\left(\mathcal{M}\dot{T}_sf\right)^{*\frac{q}{s}}(u)du\right)^{\frac{1}{q}}\right\}.$$

Theorem 6.2. *Under the hypotheses of Theorem 1.6 with $V\in RH_\infty$:*

1. *there exists C_1 such that for every $f\in\dot{W}^1_{r,V}+\dot{W}^1_{\infty,V}$ and $t>0$*

$$K(f,t^{\frac{1}{r}},\dot{W}^1_{r,V},\dot{W}^1_{\infty,V})\geq C_1t^{\frac{1}{r}}(\dot{T}_{r**}f)^{\frac{1}{r}}(t);$$

2. *for $s\leq p<\infty$, there exists C_2 such that for every $f\in\dot{W}^1_{p,V}$ and every $t>0$*

$$K(f,t^{\frac{1}{r}},\dot{W}^1_{r,V},\dot{W}^1_{\infty,V})\leq C_2t^{\frac{1}{r}}(\dot{T}_{s**}f)^{\frac{1}{s}}(t).$$

Before we prove Theorems 6.1, 6.2 and 1.6, we give two versions of a Calderón-Zygmund decomposition.

Proposition 6.3. *Let M be a complete non-compact Riemannian manifold satisfying (D). Let $1\leq q<\infty$ and $V\in RH_q$. Assume that M admits a Poincaré inequality (P_s) for some $1\leq s<q$. Let $s\leq p<q$ and consider $f\in\dot{W}^1_{p,V}$ and $\alpha>0$. Then there exist a collection of balls $(B_i)_i$, functions $b_i\in\dot{W}^1_{r,V}$ for $1\leq r\leq s$ and a function $g\in\dot{W}^1_{q,V}$ such that the following properties hold:*

$$f=g+\sum_i b_i \tag{6.1}$$

$$\int_{\bigcup_i B_i} \dot{T}_q g \, d\mu \leq C \, \alpha^q \mu(\bigcup_i B_i) \tag{6.2}$$

$$supp \ b_i \subset B_i \ \ and \ \ \forall 1 \leq r \leq s \int_{B_i} \dot{T}_r b_i \, d\mu \leq C\alpha^r \mu(B_i) \tag{6.3}$$

$$\sum_i \mu(B_i) \leq C\alpha^{-p} \int \dot{T}_p f \, d\mu \tag{6.4}$$

$$\sum_i \chi_{B_i} \leq N \tag{6.5}$$

with C and N depending only on q, s and the constants in (D), (P_s) and the RH_q condition.

Proposition 6.4. *Let M be a complete non-compact Riemannian manifold satisfying (D). Consider $V \in RH_\infty$. Assume that M admits a Poincaré inequality (P_s) for some $1 \leq s < \infty$. Let $s \leq p < \infty$, $f \in \dot{W}^1_{p,V}$ and $\alpha > 0$. Then there exist a collection of balls $(B_i)_i$, functions b_i and a function g such that the following properties hold :*

$$f = g + \sum_i b_i \tag{6.6}$$

$$\dot{T}_1 g \leq C\alpha \quad \mu - a.e. \tag{6.7}$$

$$supp \ b_i \subset B_i \ \ and \ \ \forall 1 \leq r \leq s \int_{B_i} \dot{T}_r b_i d\mu \leq C\alpha^r \mu(B_i) \tag{6.8}$$

$$\sum_i \mu(B_i) \leq C\alpha^{-p} \int \dot{T}_p f \, d\mu \tag{6.9}$$

$$\sum_i \chi_{B_i} \leq N \tag{6.10}$$

with C and N depending only on q, p and the constant in (D), (P_s) and the RH_∞ condition.

The proof of these two decompositions goes as in the case of non-homogeneous Sobolev spaces, but taking $\Omega = \left\{ x \in M : \mathcal{M}\dot{T}_s f(x) > \alpha^s \right\}$ as $\|f\|_p$ is not under control. We note that in the non-homogeneous case, we used that $f \in L_p$ only to control $b \in L_r$ and $g \in L_\infty$ when $V \in RH_\infty$ and $\int_\Omega |g|^q d\mu$ when $V \in RH_q$ and $q < \infty$.

Proof of Theorem 6.1 and 6.2. We refer to [7] for the proof of Theorem 6.2. The proof of item 1. of Theorem 6.1 is the same as in the non-homogeneous case. Let us turn to inequality 2. Consider $f \in \dot{W}^1_{p,V}$, $t > 0$ and $\alpha(t) = (\mathcal{M}\dot{T}_s f)^{*\frac{1}{s}}(t^{\frac{qr}{q-r}})$. By the Calderón-Zygmund decomposition with $\alpha = \alpha(t)$, f can be written $f = b + g$ with $\|b\|_{\dot{W}^1_{r,V}} \leq C\alpha(t)\mu(\Omega)^{\frac{1}{r}}$ and $\int_\Omega \dot{T}_q g d\mu \leq C\alpha^q(t)\mu(\Omega)$. Since we have $\mu(\Omega) \leq t^{\frac{qr}{q-r}}$, we get then as in the non-homogeneous case

$$K(f, t, \dot{W}^1_{r,V}, \dot{W}^1_{q,V}) \leq Ct^{\frac{q}{q-r}} (\dot{T}_{s**} f)^{\frac{1}{s}} (t^{\frac{qr}{q-r}}) + Ct \left(\int_{t^{\frac{qr}{q-r}}}^\infty (\mathcal{M}\dot{T}_s f)^{*\frac{q}{s}}(u) du \right)^{\frac{1}{q}}.$$

□

Proof of Theorem 1.6. We refer to [7] when $q=\infty$. When $q<\infty$, the proof follows directly from Theorem 6.1. Indeed, item 1. of Theorem 6.1 gives us that

$$(\dot{W}^1_{r,V},\dot{W}^1_{q,V})_{\frac{q(p-r)}{p(q-r)},p}\subset \dot{W}^1_{p,V}$$

with $\|f\|_{\dot{W}^1_{p,V}}\leq C\|f\|_{\frac{q(p-r)}{p(q-r)},p}$, while item 2. gives us as in section 5 for non-homogeneous Sobolev spaces, that

$$\dot{W}^1_{p,V}\subset(\dot{W}^1_{r,V},\dot{W}^1_{q,V})_{\frac{q(p-r)}{p(q-r)},p}$$

with $\|f\|_{\frac{q(p-r)}{p(q-r)},p}\leq C\|f\|_{\dot{W}^1_{p,V}}$. □

Let $A_V=\{q\in]1,\infty]: V\in RH_q\}$ and $q_0=\sup A_V$, $B_M=\{s\in[1,q_0[: (P_s) \text{ holds }\}$ and $s_0=\inf B_M$.

Corollary 6.5. *For all p, p_1, p_2 such that $1\leq p_1<p<p_2<q_0$ with $p>s_0$, $\dot{W}^1_{p,V}$ is a real interpolation space between $\dot{W}^1_{p_1,V}$ and $\dot{W}^1_{p_2,V}$.*

7. Interpolation of Sobolev spaces on Lie Groups

Consider G a connected Lie group. Assume that G is unimodular and let $d\mu$ be a fixed Haar measure on G. Let $X_1,...,X_k$ be a family of left invariant vector fields such that the X_i's satisfy a Hörmander condition. In this case, the Carnot-Carathéodory metric ρ is a distance, and G equipped with the distance ρ is complete and defines the same topology as the topology of G as manifold (see [15] page 1148). It is known that G has an exponential growth or polynomial growth. In the first case, G satisfies the local doubling property (D_{loc}) and admits a local Poincaré inequality (P_{1loc}). In the second case, it admits the global doubling property (D) and a global Poincaré inequality (P_1) (see [15], [20], [25], [29] for more details).

Definition 7.1 (Sobolev spaces $W^1_{p,V}$)**.** *For $1\leq p<\infty$ and for a weight $V\in A_\infty$, we define the Sobolev space $W^1_{p,V}$ as the completion of C^∞ functions for the norm:*

$$\|u\|_{W^1_{p,V}}=\|f\|_p+\|\,|Xf|\,\|_p+\|Vf\|_p$$

where $|Xf|=\left(\sum_{i=1}^k|X_if|^2\right)^{\frac{1}{2}}$.

Definition 7.2. *We denote by $W^1_{\infty,V}$ the space of all bounded Lipschitz functions f on G such that $\|Vf\|_\infty<\infty$ which is a Banach space.*

Proposition 7.3. *Let $V\in RH_{qloc}$ for some $1\leq q<\infty$. Consider, for $1\leq p<q$,*

$$H^1_{p,V}=\{f\in L_p:\ |\nabla f| \text{ and } Vf\in L_p\}$$

and equip it with the same norm as $W^1_{p,V}$. Then as in Proposition 2.15 in the case of Riemannian manifolds, C_0^∞ is dense in $H^1_{p,V}$ and hence $W^1_{p,V}=H^1_{p,V}$.

Interpolation of $W^1_{p,V}$: Let $V \in RH_{qloc}$ for some $1 < q \leq \infty$. To interpolate the $W^1_{p_i,V}$, we distinguish between the polynomial and the exponential growth cases. If G has polynomial growth and $V \in RH_q$, then we are in the global case. Otherwise, we are in the local case. In the two cases, we obtain the following theorem:

Theorem 7.4. *Let G be a connected Lie group as in the beginning of this section and assume that $V \in RH_{qloc}$ with $1 < q \leq \infty$. Denote $T_1 f = |f| + |Xf| + |Vf|$, $T_{r*}f = |f|^{r*} + |Xf|^{r*} + |Vf|^{r*}$ for $1 \leq r < \infty$.*

a. *If $q < \infty$, then*

1. *there exists $C_1 > 0$ such that for every $f \in W^1_{1,V} + W^1_{q,V}$ and $t > 0$*

$$K(f,t,W^1_{1,V},W^1_{q,V}) \geq C_1 \left\{ \left(\int_0^{t^{\frac{q}{q-1}}} T_{1*}f(u)du \right)^{\frac{1}{s}} + t \left(\int_{t^{\frac{q}{q-1}}}^{\infty} T_{q*}f(u)du \right)^{\frac{1}{q}} \right\};$$

2. *for $1 \leq p < \infty$, there exists $C_2 > 0$ such that for every $f \in W^1_{p,V}$ and $t > 0$,*

$$K(f,t,W^1_{1,V},W^1_{q,V}) \leq C_2 \left\{ \int_0^{t^{\frac{q}{q-1}}} T_{1*}f(u)du + t \left(\int_{t^{\frac{q}{q-1}}}^{\infty} (\mathcal{M}T_1 f)^{*q}(u)du \right)^{\frac{1}{q}} \right\}.$$

b. *If $q = \infty$, then for every $f \in W^1_{1,V} + W^1_{\infty,V}$ and $t > 0$*

$$K(f,t,W^1_{1,V},W^1_{\infty,V}) \sim \int_0^t T_{1*}f(u)du.$$

Theorem 7.5. *Let G be as above, $V \in RH_{qloc}$, for some $1 < q \leq \infty$. Then, for $1 \leq p_1 < p < p_2 < q_0$, $W^1_{p,V}$ is a real interpolation space between $W^1_{p_1,V}$ and $W^1_{p_2,V}$ where $q_0 = \sup\{q \in]1,\infty] : V \in RH_{qloc}\}$.*

Proof. Combine Theorem 7.4 and the reiteration theorem. □

Remark 7.6. *For $V \in A_\infty$, define the homogeneous Sobolev spaces $\dot{W}^1_{p,V}$ as the vector space of distributions f such that Xf and $Vf \in L_p$ and equip this space with the norm*

$$\|f\|_{\dot{W}^1_{p,V}} = \| \, |Xf| \, \|_p + \|Vf\|_p$$

and $\dot{W}^1_{\infty,V}$ as the space of all Lipschitz functions f on G with $\|Vf\|_\infty < \infty$. Theses spaces are Banach spaces. If G has polynomial growth, we obtain interpolation results analog to those of section 6.

Examples: For examples of spaces on which our interpolation result applies see section 11 of [7].

Examples of RH_q weights in $\mathbb{R}^n$ for $q < \infty$ are the power weights $|x|^{-\alpha}$ with $-\infty < \alpha < \frac{n}{q}$ and positive polynomials for $q = \infty$. We give another example of RH_q weights on a Riemannian manifold M: consider f, $g \in L_1(M)$, $1 \leq r < \infty$ and $1 < s \leq \infty$, then $V(x) = (\mathcal{M}f(x))^{-(r-1)} \in RH_\infty$ and $W(x) = (\mathcal{M}g(x))^{\frac{1}{s}} \in RH_q$ for all $q < s$ ($q = s$ if $s = \infty$) and hence $V + W \in RH_q$ for all $q < s$ ($q = s$ if $s = \infty$) (see [5], [6] for details).

8. Appendix

Proof of Proposition 2.15: We follow the method of Davies [16]. Let $L(f) = L_0(f) + L_1(f) + L_2(f) := \int_M |f|^p d\mu + \int_M |\nabla f|^p d\mu + \int_M |Vf|^p d\mu$. We will prove the proposition in three steps:

1. Let $f \in H^1_{p,V}$. Fix $p_0 \in M$ and let $\varphi \in C_0^\infty(\mathbb{R})$ satisfies $\varphi \geq 0$, $\varphi(\alpha) = 1$ if $\alpha < 1$ and $\varphi(\alpha) = 0$ if $\alpha > 2$. Then put $f_n(x) = f(x)\varphi(\frac{d(x,p_0)}{n})$. Elementary calculations establish that f_n lies in $H^1_{p,V}$. Moreover,

$$\begin{aligned}
L(f - f_n) &= \int_M |f(x)\{1 - \varphi(\frac{d(x,p_0)}{n})\}|^p d\mu(x) \\
&+ \int_M |\nabla f(x)\{1 - \varphi(\frac{d(x,p_0)}{n})\} - n^{-1} f(x)\varphi'(\frac{d(x,p_0)}{n})\nabla(d(x,p_0))|^p d\mu(x) \\
&+ \int_M |V(x)f(x)(1 - \varphi(\frac{d(x,p_0)}{n}))|^p d\mu(x) \\
&\leq \int_M |f(x)\{1 - \varphi(\frac{d(x,p_0)}{n})\}|^p d\mu(x) \\
&+ 2^{p-1}\int_M |\nabla f(x)\{1 - \varphi(\frac{d(x,p_0)}{n})\}|^p d\mu(x) + 2^{p-1}n^{-p}\int_M |f(x)|^p |\varphi'(\frac{d(x,p_0)}{n})|^p d\mu(x) \\
&+ \int_M V^p(x)|f(x)|^p |1 - \varphi(\frac{d(x,p_0)}{n})|^p d\mu(x).
\end{aligned}$$

   This converges to zero as $n \to \infty$ by the dominated convergence theorem. Thus, the set of functions $f \in H^1_{p,V}$ with compact support is dense in $H^1_{p,V}$.

2. Let $f \in H^1_{p,V}$ with compact support. Let $n > 0$ and $F_n : \mathbb{R} \to \mathbb{R}$ be a smooth increasing function such that

$$F_n(s) = \begin{cases} s \quad \text{if } -n \leq s \leq n, \\ n+1 \quad \text{if } s \geq n+2, \\ -n-1 \quad \text{if } s \leq -n-2 \end{cases}$$

   and $0 \leq F_n'(s) \leq 1$ for all $s \in \mathbb{R}$. If we put $f_n(x) := F_n(f(x))$ then $|f_n(x)| \leq |f(x)|$ and $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in M$. The dominated convergence theorem yields

$$\lim_{n\to\infty} L_0(f - f_n) = \lim_{n\to\infty} \int_M |f - f_n|^p d\mu = 0$$

   and

$$\lim_{n\to\infty} L_2(f - f_n) = \lim_{n\to\infty} \int_M V^p |f - f_n|^p d\mu = 0.$$

   Also

$$\begin{aligned}
\lim_{n\to\infty} L_1(f - f_n) &= \lim_{n\to\infty} \int_M |\nabla f - F_n'(f(x))\nabla f|^p d\mu(x) \\
&= \lim_{n\to\infty} \int_M |1 - F_n'(f(x))|^p |\nabla f(x)|^p d\mu(x) \\
&= 0.
\end{aligned}$$

Therefore, the set of bounded functions $f \in H^1_{p,V}$ with compact support is dense in $H^1_{p,V}$.

3. Let now $f \in H^1_{p,V}$ be bounded and with compact support. Consider locally finite coverings of M, $(U_k)_k$, $(V_k)_k$ with $\overline{U_k} \subset V_k$, V_k being endowed with a real coordinate chart ψ_k. Let $(\varphi_k)_k$ be a partition of unity subordinated to the covering $(U_k)_k$, that is, for all k, φ_k is a C^∞ function supported in U_k, $0 \leq \varphi_k \leq 1$ and $\sum_{k=1}^\infty \varphi_k = 1$. There exists a finite subset I of $\mathbb{N}$ such that $f = \sum_{k\in I} f\varphi_k := \sum_{k\in I} f_k$. Take $\epsilon > 0$. The functions $g_k = f_k \circ \psi_k^{-1}$ –which belong to $W^1_p(\mathbb{R}^n)$ since f and $|\nabla f| \in L_{ploc}$– can be approximated by smooth functions w_k with compact support (standard approximation by convolution). The w_k are defined as $w_k = g_k * \alpha_k$ where $\alpha_k \in C_0^\infty(\mathbb{R}^n)$ is a standard mollifier, $\operatorname{supp} w_k \subset \psi_k(V_k)$ and $\|g_k - w_k\|_{W^1_p} \leq \frac{\epsilon}{2^k}$. Define
$$h_k(x) = \begin{cases} w_k \circ \psi_k(x) & \text{if } x \in V_k, \\ 0 & \text{otherwise.} \end{cases}$$
Thus $\operatorname{supp} h_k \subset V_k$ and
$$\|f_k - h_k\|_p = \left(\int_{V_k} |f_k - h_k|^p d\mu\right)^{\frac{1}{p}} = \|g_k - w_k\|_p \leq \frac{\epsilon}{2^k}.$$
$$\|\,|\nabla(f_k - h_k)|\|_p = \left(\int_{V_k} |\nabla(f_k - h_k)|^p d\mu\right)^{\frac{1}{p}} = \|\,|\nabla(g_k - w_k)|\,\|_p \leq \frac{\epsilon}{2^k}.$$
Hence, the series $\sum_{k\in I}(f_k - h_k)$ is convergent in W^1_p. Moreover, $\sum_{k\in I}(f_k - h_k) = f - h_\epsilon$ where $h_\epsilon = \sum_{k\in I} h_k$, and $\|f - h_\epsilon\|_{W^1_p} \leq \sum_{k\in I} \|f_k - h_k\|_{W^1_p} \leq \epsilon$. If $l_\epsilon := |f - h_\epsilon|^p$ then $\lim_{\epsilon\to 0} \|l_\epsilon\|_1 = 0$ and there exists a compact set K which contains the support of every l_ϵ. We have $\|h_\epsilon\|_\infty \leq \sharp I \|f\|_\infty$ for all $\epsilon > 0$. Indeed,
$$\begin{aligned}
\sum_{k\in I} |h_k(x)| &= \sum_{k\in I} \int_{\mathbb{R}^n} |g_k(y)|\, \alpha_k(\psi_k(x) - y) dy \\
&= \int_{\mathbb{R}^n} \sum_{k\in I} |f\varphi_k(\psi_k^{-1}(y))|\, \alpha_k(\psi_k(x) - y) dy \\
&\leq \|f\|_\infty \int_{\mathbb{R}^n} \sum_{k\in I} \varphi_k(\psi_k^{-1}(y))\, \alpha_k(\psi_k(x) - y) dy \\
&\leq \|f\|_\infty \sum_{k\in I} \int_{\psi_k(U_k)} \varphi_k(\psi_k^{-1}(y))\, \alpha_k(\psi_k(x) - y) dy \\
&\leq \|f\|_\infty \sum_{k\in I} \int_{\mathbb{R}^n} \alpha_k(z) dz \\
&\leq \sharp I \|f\|_\infty.
\end{aligned}$$
It follows that $\|l_\epsilon\|_\infty \leq 2^{p-1}(1 + \sharp I)\|f\|_\infty^p = C\|f\|_\infty^p$ (C being independent of ϵ it depends just on f) for all $\epsilon > 0$. We claim that these facts suffice to deduce that $\lim_{\epsilon\to 0} \int_M l_\epsilon V^p d\mu = 0$, that is
$$\lim_{\epsilon\to 0} L_2(f - l_\epsilon) = 0.$$

Therefore C_0^∞ is dense in $H^1_{p,V}$.

4. It remains to prove the above claim. Thanks to Proposition 2.8, $V \in RH_{ploc}$ and there exists $r > p$ such that $V \in RH_{rloc}$. Therefore $V^p \in L_{t,loc}$ where $t = \frac{r}{p} > 1$. Hence, by Hölder inequality we get

$$\begin{aligned} 0 \leq \int_M l_\epsilon V^p d\mu &= \int_K l_\epsilon V^p d\mu \\ &\leq \|l_\epsilon\|_{L_{t'}(K)} \|V^p\|_{L_t(K)} \\ &\leq C\|f\|_\infty^{\frac{p}{r}} \epsilon^{\frac{1}{t'}} \end{aligned}$$

for all $\epsilon > 0$, t' being the conjugate exponent of t. The proof of Proposition 2.15 is therefore complete.

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N.Badr, Institut Camille Jordan, Université Claude Bernard Lyon 1, UMR du CNRS 5208, 43 boulevard du 11 novembre 1918, F-69622 Villeurbanne cedex.
E-mail address: badr@math.univ-lyon1.fr