J. Symbolic Computation (2001) **32**, 643–662
doi:10.1006/jsco.2001.0487
Available online at http://www.idealibrary.com on IDEAL®

Algorithm for Computing Bernstein–Sato Ideals Associated with a Polynomial Mapping

ROUCHDI BAHLOUL†

Département de Mathématiques, Université d'Angers, 2 bd Lavoisier, 49045 Angers, France

Let $f_1, \ldots, f_p$ be polynomials in n variables with coefficients in a field $\mathbf{K}$. We associate with these polynomials a number of functional equations and related ideals $\mathcal{B}$, $\mathcal{B}_j$ and $\mathcal{B}_\Sigma$ of $\mathbf{K}[s_1, \ldots, s_p]$ called Bernstein–Sato ideals. Using standard basis techniques, our aim is to present an algorithm for computing generators of $\mathcal{B}_j$ and $\mathcal{B}_\Sigma$.

1. Introduction

Let n, p be two strictly positive integers, and let $f_1(x), \ldots, f_p(x) \in \mathbf{K}[x] := \mathbf{K}[x_1, \ldots, x_n]$ be p polynomials of n variables with coefficients in a field $\mathbf{K}$ of characteristic zero. Denote by $\mathbf{A}_n = \mathbf{K}[x_1, \ldots, x_n]\langle \partial_{x_1}, \ldots, \partial_{x_n} \rangle$ the Weyl algebra with n variables and let $s_1, \ldots, s_p$ be new variables.

Denote by $\mathcal{L} = \mathbf{K}[x][f_1^{-1}, \ldots, f_p^{-1}, s_1, \ldots, s_p] \cdot f^s$ the free module generated by the symbol f^s where f^s is a notation for $f_1^{s_1} \cdots f_p^{s_p}$. $\mathcal{L}$ has a natural $\mathbf{A}_n[s]$-module structure where $\mathbf{A}_n[s] := \mathbf{A}_n[s_1, \ldots, s_p]$. We have, for instance:

$$\partial_{x_i}(g(x,s)f^{s-m}) = \left(\frac{\partial g}{\partial x_i} f^{-m} + \sum_{j=1}^{p} g(x,s)(s_j - m_j) \frac{\partial f_j}{\partial x_i} f_j^{-1} f^{-m} \right) f^s$$

where $g(x, s) \in \mathbf{K}[x][s]$ and $m \in \mathbf{N}^p$.
Consider the following ideals of $\mathbf{K}[s]$, called Bernstein–Sato ideals:

- $\mathcal{B} = \{b(s) \in \mathbf{K}[s] \ / \ b(s)f^s \in \mathbf{A}_n[s]f^{s+1}\}$,
- $\mathcal{B}_j = \{b(s) \in \mathbf{K}[s] \ / \ b(s)f^s \in \mathbf{A}_n[s]f_j f^s\}$ for $j \in \{1, \ldots, p\}$,
- $\mathcal{B}_\Sigma = \{b(s) \in \mathbf{K}[s] \ / \ b(s)f^s \in \sum_{j=1}^{p} \mathbf{A}_n[s]f_j f^s\}$,

where $f^{s+1} := f_1^{s_1+1} \cdots f_p^{s_p+1}$.

Note that, in the local analytic case, these ideals have been studied by C. Sabbah (see Sabbah, 1987) who showed that they are not zero. In the algebraic case studied here, $\mathcal{B} \neq 0$ can be obtained by a method similar to that used when $p = 1$ (see Bernstein, 1972 and Björk, 1979). We take the notations of Maynadier (see Maynadier, 1996 and Maynadier, 1997).

†E-mail: rouchdi.bahloul@univ-angers.fr

Our aim, in this paper, is to present an explicit algorithm for computing the ideals $\mathcal{B}_j$ and $\mathcal{B}_\Sigma$ (i.e. an algorithm that gives generators of these ideals).
In Oaku and Takayama (1999), T. Oaku and N. Takayama solved the problem for $\mathcal{B}$ using the relation:

$$\mathcal{B} = (\mathrm{Ann} f^s + \mathbf{A}_n[s] f_1 \cdots f_p) \cap \mathbf{K}[s]$$

by a computation of $\mathrm{Ann} f^s$ the annihilator of f^s in $\mathbf{A}_n[s]$. The ideal $\mathrm{Ann} f^s$ is obtained as the intersection of the ideal I (the annihilator of f^s in $\mathbf{A}_{n+p} = \mathbf{A}_n[t_1, \dots, t_p] \langle \partial_{t_1}, \dots, \partial_{t_p} \rangle$, see Section 4 of this paper, for instance) and the subring $\mathbf{A}_n[-\partial_{t_1} t_1, \dots, -\partial_{t_p} t_p]$. This intersection is obtained by a Groebner basis computation in $2n + 4p$ variables.
The relations:

$$\mathcal{B}_j = (\mathrm{Ann} f^s + \mathbf{A}_n[s] f_j) \cap \mathbf{K}[s],$$
$$\mathcal{B}_\Sigma = (\mathrm{Ann} f^s + \mathbf{A}_n[s] f_1 + \cdots + \mathbf{A}_n[s] f_p) \cap \mathbf{K}[s]$$

show that our initial problem is completely resolved using this method.
On the other hand, in the case $p = 1$, those three kinds of ideals are equal and another algorithm to compute $\mathcal{B}$ has been proposed by Oaku (in Oaku, 1997). In this paper, Oaku works with a filtration, called the V-filtration, which gives what we call the V-order, and with these objects he computes a V-standard basis of the ideal I of $\mathbf{A}_{n+1} = \mathbf{A}_n[t]\langle \partial_t \rangle$, the annihilator of f^s in $\mathbf{A}_{n+1}$, after which he obtains generators of an ideal of $\mathbf{A}_n[s]$ denoted $\psi(I)$ and with an elimination algorithm, he finally obtains a system of generators of the ideal $\mathcal{B}$.

In this paper, we try to generalize this algorithm. In our case, where $p \geq 2$, we have to deal with p filtrations $V_1, \dots, V_p$ or equivalently with a multifiltration V. For each ideal $J = \mathcal{B}_j, \mathcal{B}_\Sigma$, we have the $\prec_J$-order which allows us to compute a $\prec_J$-standard basis of I in $\mathbf{A}_{n+p}$ but the difficulty in our case is the following: to compute $\psi_J(I)$ (the analogue of Oaku's $\psi(I)$) we have to find generators of I which are adapted to the ideal J (for instance, V-regular for $\mathcal{B}_\Sigma$, Bj-good for $\mathcal{B}_j$, etc.) and for this reason, we use an homogenization technique (see Section 6) which makes a number of undesirable divisions impossible. After this homogenization, the rest of the algorithm is similar to Oaku's.
Note that with this method, Groebner basis computations are done in (at most) $2n + 3p$ variables.

In Section 2, we recall how to compute a standard basis with respect to an order which is not a well-order. Section 3 introduces the V- multifiltration in $\mathbf{A}_{n+p}$ and all the objects which are linked with it, and all the usual elementary properties of these objects. In Section 4, following Malgrange (see Malgrange, 1975), we introduce I the annihilator of f^s in $\mathbf{A}_{n+p}$ and we establish the link between functional equations and this ideal. In Section 5, we introduce the three Bernstein–Sato ideals that we are interested in, and following Oaku, we also introduce the ideals $\psi_{\mathcal{B}}$, $\psi_{\mathcal{B}_j}$ and $\psi_{\mathcal{B}_\Sigma}$ of $\mathbf{A}_n[s]$ and we show the link between these ideals and the Bernstein–Sato ones. In Section 6, we give the algorithms for computing $\mathcal{B}_j$ and $\mathcal{B}_\Sigma$ (in fact $\psi_{\mathcal{B}_j}$ and $\psi_{\mathcal{B}_\Sigma}$), after having described and used the homogenization techniques which are specific to these algorithms. We end off this section with some examples. One has been calculated by hand and the rest using KAN (see Takayama, 1991).
In this paper, ideal means left ideal.

2. Standard Basis with Respect to a Non-well Order

In this section, we will recall the process of building a standard basis as in Castro and Narváez (1997). This is classical except for the fact that we add parameters $u_1, \ldots, u_q$, as developed later, in view of frequent applications, especially in Section 6.5. Note that in the case of a polynomial ring and a well-order, this process has been treated in Buchberger (1970).

We are going to work in $\mathbf{A}_n[u] = \mathbf{A}_n[u_1, \ldots, u_q]$, with $q \in \mathbf{N}$. It is a usual polynomial ring in which u_i commutes with the other variables. This section is, nevertheless, worthwhile even if $q = 0$, where we work in $\mathbf{A}_n$. Let $\prec$ be a total order of $\mathbf{N}^{2n+q}$ which is compatible with the structure of $\mathbf{A}_n$, i.e.:

$$\exp_\prec(1) \prec \exp_\prec(x_i \partial_{x_i}) = \exp_\prec(\partial_{x_i} x_i)$$

and we suppose this order to be compatible with sums while not necessarily being a well-order.

We recall that for $P \in \mathbf{A}_n[u]$ written:

$$P = \sum p_{\alpha,\beta,\gamma} x^\alpha \partial_x^\beta u^\gamma \text{ with } p_{\alpha,\beta,\gamma} \in \mathbf{K},\ (\alpha, \beta, \gamma) \in \mathbf{N}^{2n+q}.$$

We define:

- the privileged exponent of P:

$$\exp_\prec(P) = \{\max_\prec(\alpha, \beta, \gamma)/p_{\alpha,\beta,\gamma} \neq 0\}$$

- the privileged monomial of P:

$$\mathrm{mp}_\prec(P) = p_{\exp_\prec(P)}(x, \partial_x, u)^{\exp_\prec(P)}$$

- the Newton diagram of P:

$$\mathrm{ND}(P) = \{(\alpha, \beta, \gamma) \in \mathbf{N}^{2n+q}/p_{\alpha,\beta,\gamma} \neq 0\}.$$

Let I be an ideal of $\mathbf{A}_n[u]$. We suppose that we know a finite system of generators of I. The purpose of this section is to give an algorithm to compute a $\prec$-standard basis of I and, more precisely, we are going to prove:

Proposition 2.1. *There exists $P_1, \ldots, P_m \in I$ such that for each $P \in I$ there exists $W_1, \ldots, W_m \in \mathbf{A}_n[u]$ such that:*

- $P = W_1 P_1 + \cdots + W_m P_m$,
- $\forall j = 1, \ldots, m \quad \exp_\prec(W_j P_j) \preceq \exp_\prec(P)$ *if* $W_j \neq 0$.

To construct such operators, we are going to introduce a new variable z and work in $\mathbf{A}_n[u]\langle z \rangle$ as in Castro and Narváez (1997).

Let $\mathbf{A}_n[u]\langle z \rangle$ be the $\mathbf{K}$-algebra generated by $x_i, \partial_{x_i}, u_i, z$ with the following relations:

- $\forall i = 1, \ldots, q,\ u_i$ commutes with the other variables,
- z commutes with the other variables,
- $\forall i, j \quad [x_i, x_j] = [\partial_{x_i}, \partial_{x_j}] = 0$,
- $\forall i, j \quad [\partial_{x_i}, x_j] = \delta_{ij} z^2$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

On $\mathbf{N}^{2n+q+1}$, we define the order $\prec^h$ by:

$$(\alpha,\beta,\gamma,i) \prec^h (\alpha',\beta',\gamma',i') \iff \begin{cases} & |\alpha|+|\beta|+|\gamma|+i<|\alpha'|+|\beta'|+|\gamma'|+i' \\ \text{or} & (|\alpha|+|\beta|+|\gamma|+i=|\alpha'|+|\beta'|+|\gamma'|+i' \\ & \text{and } \ (\alpha,\beta,\gamma)\prec(\alpha',\beta',\gamma')). \end{cases}$$

We note that $\prec^h$ is an order compatible with sums and a well-order.
For $P \in \mathbf{A}_n[u]$, let us write

$$P = \sum a_{\alpha,\beta,\gamma} x^\alpha \partial_x^\beta u^\gamma.$$

We denote $\text{ord}^T(P) = \max\{|\alpha|+|\beta|+|\gamma| / a_{\alpha,\beta,\gamma} \neq 0\}$. We define $h(P) \in \mathbf{A}_n[u]\langle z\rangle$ by:

$$h(P) = \sum a_{\alpha,\beta,\gamma} x^\alpha \partial_x^\beta u^\gamma z^{\text{ord}^T(P)-|\alpha|-|\beta|-|\gamma|}.$$

Thanks to the relations in $\mathbf{A}_n[u]\langle z\rangle$, we have:

$$\forall P,Q \in \mathbf{A}_n[u], \quad h(PQ) = h(P)h(Q).$$

We define the ideal $h(I) \subseteq \mathbf{A}_n[u]\langle z\rangle$ by the following:
Let $Q_1,\dots,Q_r$ be generators of I , we define

$$h(I) = \sum_{i=1}^{r} \mathbf{A}_n[u]\langle z\rangle \cdot h(Q_i).$$

Assertion 1. We can compute a $\prec^h$-standard basis $H_1(z),\dots,H_m(z)$ of $h(I)$ (in the sense of 2.1) consisting of T-homogeneous operators (see remark).

Remark. $H \in \mathbf{A}_n[u]\langle z\rangle$ is said to be T-homogeneous with a total order $\text{ord}^T(H) = d$ if it is written:

$$H = \sum_{\alpha,\beta,\gamma} a_{\alpha,\beta,\gamma} x^\alpha \partial_x^\beta u^\gamma z^{d-|\alpha|-|\beta|-|\gamma|}.$$

In order to prove Assertion 1, we just have to note that $\prec^h$ is a well-order and that divisions in $\mathbf{A}_n[u]\langle z\rangle$ preserve homogeneity. Then the H_j constructed from the $h(Q_i)$ (by division of semisyzygies, see Lejeune-Jalabert, 1985 for the commutative case; see also Castro and Granger, 1997) will be homogeneous.

Assertion 2. The system $H_1(1),\dots,H_m(1)$ gives the P_j of Proposition 2.1.
Let us prove Assertion 2.
Let $P \in I$, there exists $B_1,\dots,B_r \in \mathbf{A}_n[u]$ such that $P = \sum B_i Q_i$. There exists $k,k_1,\dots,k_r \in \mathbf{N}$ such that

$$z^k h(P) = z^{k_1} h(B_1 Q_1) + \dots + z^{k_r} h(B_r Q_r).$$

We have $h(B_i Q_i) = h(B_i)h(Q_i)$ then $z^k h(P) \in h(I)$. Let us divide $z^k h(P)$ by the $H_j(z)$, we have:

- $z^k h(P) = \sum U_j(z) H_j(z)$,
- $\forall j \quad \exp_{\prec^h}(z^k h(P)) \succeq^h \exp_{\prec^h}(U_j H_j)$ if $U_j \neq 0$,
- $\forall j \quad U_j$ is T-homogeneous.

Denote $(\alpha,\beta,\gamma,i) = \exp_{\prec^h}(z^k h(P))$ and $(\alpha_j,\beta_j,\gamma_j,i_j) = \exp_{\prec^h}(U_j H_j)$. Since all the terms in the sum are T-homogeneous (with order equal to $k + \text{ord}^T(P)$), we have

$(\alpha, \beta, \gamma) \succeq (\alpha_j, \beta_j, \gamma_j)$. And also because of homogeneity we have $(\alpha, \beta, \gamma) = \exp_{\prec^h}(P)$ and $(\alpha_j, \beta_j, \gamma_j) = \exp_{\prec^h}(U_j(1)H_j(1))$.
So $P = \sum U_j(1)H_j(1)$ with $\exp_{\prec}(P) \succeq \exp_{\prec}(U_j(1)H_j(1))$.
Assertion 2 is proved and so is Proposition 2.1.

3. V-multifiltration

Definition 3.1. Let $P \in \mathbf{A}_{n+p}$, $P \neq 0$, P can be written in only one way as follows:

$$P = \sum a_{\mu,\nu} t^{\mu} \partial_t^{\nu} \quad \text{with} \quad \mu, \nu \in \mathbf{N}^p,\ a_{\mu,\nu} \in \mathbf{A}_n.$$

We define:

- for $j = 1, \ldots, p$, the V_j-order of P
$$\mathrm{ord}^{V_j}(P) = \max\{\nu_j - \mu_j \ / \ a_{\mu,\nu} \neq 0\}.$$
- the V-order of P
$$\mathrm{ord}^{V}(P) = (\mathrm{ord}^{V_1}(P), \ldots, \mathrm{ord}^{V_p}(P)).$$
- for $j = 1, \ldots, p$, the V_j-principal symbol of P
$$\sigma^{V_j}(P) = \sum_{\nu_j - \mu_j = \mathrm{ord}^{V_j}(P)} a_{\mu,\nu} t^{\mu} \partial_t^{\nu}.$$
- for $m \in \mathbf{Z}^p$, the V-partial symbol of order m of P
$$\sigma_m^V(P) = \sum_{\nu - \mu = m} a_{\mu,\nu} t^{\mu} \partial_t^{\nu}.$$
- the V-principal symbol of P
$$\sigma^V(P) = \sigma^V_{\mathrm{ord}^V(P)}(P).$$

For $m \in \mathbf{Z}^p$, we denote

$$V_m(\mathbf{A}_{n+p}) = \{P \in \mathbf{A}_{n+p} \ / \ \forall j = 1, \ldots, p \quad \mathrm{ord}^{V_j}(P) \leq m_j\}.$$

Remark 3.2. We may have $\sigma^V(P) = 0$ without having $P = 0$. For instance, if $p = 2$ and $P = t_1 + t_2$, $\sigma^V(P) = 0$.

As usual, we have the following properties:

Lemma 3.3. *Let $P, Q \in \mathbf{A}_{n+p}$, $P, Q \neq 0$, we have:*

- $\mathrm{ord}^V(PQ) = \mathrm{ord}^V(P) + \mathrm{ord}^V(Q)$,
- $\sigma^V(PQ) = \sigma^V(P)\sigma^V(Q)$,
- $\sigma^{V_j}(PQ) = \sigma^{V_j}(P)\sigma^{V_j}(Q)$.

4. Bernstein–Sato Equations and Malgrange Point of View

Consider

$$\mathcal{L} = \mathbf{K}[x][f_1^{-1}, \ldots, f_p^{-1}, s] \cdot f^s$$

the free module generated by the symbol f^s where $f^s := f_1^{s_1} \cdots f_p^{s_p}$.

$\mathcal{L}$ has an $\mathbf{A}_n[s]$-module structure. Thus, for instance, we have:

$$\partial_{x_i}(g(x,s)f^{s-m}) = \left(\frac{\partial g}{\partial x_i} f^{-m} + \sum_{j=1}^{p} g(x,s)(s_j - m_j)\frac{\partial f_j}{\partial x_i} f_j^{-1} f^{-m}\right) f^s$$

where $g(x,s) \in \mathbf{K}[x][s]$ and $m \in \mathbf{N}^p$.
Now let us introduce an $\mathbf{A}_{n+p}$-module structure on $\mathcal{L}$:
if $j = 1, \ldots, p$ and $g(s) \in \mathbf{K}[x][f^{-1}, s]$, we define:

$$t_j \cdot (g(s)f^s) = g(s_1, \ldots, s_j + 1, \ldots, s_p) f_j f^s,$$

$$\partial_{t_j} \cdot (g(s)f^s) = -s_j g(s_1, \ldots, s_j - 1, \ldots, s_p) f_j^{-1} f^s.$$

We can easily check that $(\partial_{t_j} \cdot t_j) \cdot (g(s)f^s) = (t_j \cdot \partial_{t_j} + 1) \cdot (g(s)f^s)$.

We have the following relations:

(i) $-\partial_{t_j} t_j (g(s)f^s) = s_j g(s) f^s$ if $g(s) \in \mathbf{K}[x][f^{-1}, s]$,
(ii) $(t_j - f_j(x)) \cdot f^s = 0 \quad \forall j = 1, \ldots, p$,
(iii) $\left(\partial_{x_i} + \sum_j \frac{\partial f_j}{\partial x_i} \partial_{t_j}\right) \cdot f^s = 0 \quad \forall i = 1, \ldots, n.$

We have the inclusions:

$$\mathbf{A}_n[s]f^s \subseteq \mathbf{A}_{n+p} f^s \subseteq \mathcal{L}$$

Note that the first inclusion comes from (i).
As in Lemma 4.1 of Malgrange (1975), we have:

Lemma 4.1. *Let*

$$I = \sum_{j=1}^{p} \mathbf{A}_{n+p}\big(t_j - f_j(x)\big) + \sum_{i=1}^{n} \mathbf{A}_{n+p}\left(\partial_{x_i} + \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i} \partial_{t_j}\right).$$

Then I is maximal in $\mathbf{A}_{n+p}$.

Proof. Let $\phi : \mathbf{K}^{n+p} \longrightarrow \mathbf{K}^{n+p}$ defined by

$$\phi(x_1, \ldots, x_n, t_1, \ldots, t_p) = (x_1, \ldots, x_n, t_1 - f_1(x), \ldots, t_p - f_p(x)).$$

ϕ is bijective and induces an isomorphism ϕ_* on the ring $\mathbf{A}_{n+p}$. The image of I is

$$\phi_*(I) = \sum_{j=1}^{p} \mathbf{A}_{n+p} t_j + \sum_{i=1}^{n} \mathbf{A}_{n+p} \partial_{x_i}.$$

Moreover, it is easy to see that:

$$I \text{ maximal is equivalent to } \phi_*(I) \text{ maximal.}$$

We suppose, in the rest of the proof, that $I = \sum \mathbf{A}_{n+p} t_j + \sum \mathbf{A}_{n+p} \partial_{x_i}$.
Let $P \in \mathbf{A}_{n+p} \setminus I$, denote $I' = I + \mathbf{A}_{n+p} P$. We shall prove that $I' = \mathbf{A}_{n+p}$. Write

$$P = \sum a_{\alpha\beta\gamma\delta} x^\alpha \partial_t^\beta t^\gamma \partial_x^\delta \quad \text{with} \quad \alpha, \delta \in \mathbf{N}^n, \beta, \gamma \in \mathbf{N}^p,\ a_{\alpha\beta\gamma\delta} \in \mathbf{K},$$

then

$$P \in \sum a_{\alpha\beta 00} x^\alpha \partial_t^\beta + I.$$

Thus we can assume that P is of the following form:

$$P = \sum a_{\alpha\beta} x^{\alpha} \partial_t^{\beta} = \sum a_{\beta}(x) \partial_t^{\beta}.$$

Let $\mu(P) = \max\{|\beta|, a_\beta \neq 0\}$. We shall prove by induction on $\mu(P)$ that there exists $a(x) \in \mathbf{K}[x] \cap I'$, $a(x) \neq 0$.
If $\mu(P) = 0$ then the assertion is true. Assume that $\mu(P) > 0$.
Write

$$P = Q + a_{\beta'} \partial_t^{\beta'} \quad \text{with} \quad |\beta'| = \mu(P).$$

Let j be such that $\beta'_j \neq 0$ and denote by $C = [P, t_j]$ the commutator of P and t_j.
Then on one hand $C \in I'$ and $C \neq 0$, and on the other hand $\mu(C) < \mu(P)$. We then apply the induction hypothesis. At this point, we have proved the existence of $a(x) \in \mathbf{K}[x]$, $a(x) \neq 0$ in I'.
For such an a, consider $\nu(a) = \max\{\deg_{x_1}(a), \ldots, \deg_{x_n}(a)\}$, and let us show by induction on $\nu(a)$ that $1 \in I'$.
If $\nu(a) = 0$ then $1 \in I'$.
Assume $\nu(a) > 0$ and let i such that $\deg_{x_i}(a) = \nu(a)$, then

$$\frac{\partial a}{\partial x_i} = [a, \partial_{x_i}] \in I' \quad \text{and} \qquad \deg_{x_i}\left(\frac{\partial a}{\partial x_i}\right) < \deg_{x_i}(a).$$

After a finite number of steps we obtain the existence of $b(x) \in I'$, $b(x) \neq 0$ with $\nu(b) < \nu(a)$. We then apply the induction hypothesis which gives $1 \in I'$. □

Lemma 4.2. *I is the annihilator of f^s in $\mathbf{A}_{n+p}$.*

Proof. Denote by $I' = \{P \in \mathbf{A}_{n+p} / P \cdot f^s = 0\}$ the annihilator of f^s in $\mathbf{A}_{n+p}$.
Using relations (ii) and (iii), we have $I \subseteq I'$; moreover $I' \neq \mathbf{A}_{n+p}$ because $1 \notin I'$. Then by the previous lemma $I = I'$. □

As a consequence, we have:

Corollary 4.3. *Let $P(s) \in \mathbf{A}_n[s]$. We have:*

$$P(s) \cdot f^s = 0 \quad \textit{in} \quad \mathcal{L} \Longleftrightarrow P(-\partial_{t_1} t_1, \ldots, -\partial_{t_p} t_p) \in I.$$

Proof. By relation (i) we have: $P(s) f^s = P(-\partial_t t) f^s$, and the previous lemma ends the proof. □

5. Bernstein–Sato Ideals

Definition 5.1. Consider the ideals of $\mathbf{K}[s] = \mathbf{K}[s_1, \ldots, s_p]$:

- $\mathcal{B} = \{b(s) \in \mathbf{K}[s] \ / \ b(s) f^s \in \mathbf{A}_n[s] f^{s+1}\}$,
- $\mathcal{B}_j = \{b(s) \in \mathbf{K}[s] \ / \ b(s) f^s \in \mathbf{A}_n[s] f_j f^s\}$ for $j = 1, \ldots, p$,
- $\mathcal{B}_\Sigma = \{b(s) \in \mathbf{K}[s] \ / \ b(s) f^s \in \sum_{j=1}^{p} \mathbf{A}_n[s] f_j f^s\}$,

with $f^{s+1} := f_1^{s_1+1} \cdots f_p^{s_p+1}$.

We have the inclusions:

$$\mathcal{B} \subseteq \mathcal{B}_j \subseteq \mathcal{B}_\Sigma.$$

Our purpose is to give an algorithm for computing $\mathcal{B}_\Sigma$ and $\mathcal{B}_j$.

DEFINITION 5.2. Let $P \in \mathbf{A}_{n+p}$ such that $\mathrm{ord}^V = (m_1, \ldots, m_p)$. We define $\psi(P)(s) \in \mathbf{A}_n[s_1, \ldots, s_p]$ by the following:

$$\psi(P)(-\partial_{t_1} t_1, \ldots, -\partial_{t_p} t_p) = \sigma^V(S_1, \ldots, S_p P)$$

with

$$S_j = \begin{cases} t_j^{m_j} & \text{if} \quad m_j > 0 \\ \partial_{t_j}^{-m_j} & \text{if} \quad m_j \leq 0. \end{cases}$$

We define the following ideals of $\mathbf{A}_n[s]$ (we still denote by I the annihilator of f^s in $\mathbf{A}_{n+p}$) by:

- $\psi_{\mathcal{B}}(I) = \{\psi(P) \ / \ P \in I, \ \mathrm{ord}^V(P) = 0, \ \forall j \ \mathrm{ord}^{V_j}(P - \sigma^V(P)) < 0\} \cup (0)$,
- $\psi_{\mathcal{B}_\Sigma}(I) = \{\psi(P) \ / \ P \in I, \ \mathrm{ord}^V(P) = 0\} \cup (0)$,
- $\psi_{\mathcal{B}_j}(I) = \{\psi(P) \ / \ P \in I, \ \mathrm{ord}^V(P) = 0, \ \mathrm{ord}^{V_j}(P - \sigma^V(P) < 0\} \cup (0)$.

LEMMA 5.3. *Let $b(s) \in \mathbf{K}[s]$, we have:*

(a) $b(s) \in \mathcal{B} \iff b(-\partial_t t) \in I + V_0(\mathbf{A}_{n+p}) t_1 \cdots t_p$,
(b) $b(s) \in \mathcal{B}_j \iff b(-\partial_t t) \in I + V_0(\mathbf{A}_{n+p}) t_j$,
(c) $b(s) \in \mathcal{B}_\Sigma \iff b(-\partial_t t) \in I + \sum V_0(\mathbf{A}_{n+p}) t_j$.

PROOF. We shall now prove Assertion (b), the other assertions can be proved in a similar way.
$b(s) \in \mathcal{B}_j$ is equivalent to the existence of $P(s) \in \mathbf{A}_n[s]$ such that $b(s) f^s = P(s) f_j f^s$, or $(b(s) - P(s) f_j).f^s = 0$, then by Corollary 4.3, $b(-\partial_t t) - P(-\partial_t t) f_j \in I$, and using relation (ii) of Section 4, we obtain:

$$b(s) \in \mathcal{B}_j \iff b(-\partial_t t) - P(-\partial_t t) t_j \in I \tag{1}$$

thus the direct implication is proved.
Conversely, suppose that there exists $Q \in V_0(\mathbf{A}_{n+p})$ such that $b(-\partial_t t) - Q.t_j \in I$. Since $Q \in V_0(\mathbf{A}_{n+p})$, we can write it as:

$$Q = \sum_{m \in \mathbf{N}^p} Q_m(-\partial_t t) t^m$$

with $Q_m(-\partial_t t) \in \mathbf{A}_n[-\partial_t t]$. Using relation (ii) of Section 4, we obtain:

$$Q t_j \in I + \left(\underbrace{\sum Q_m(-\partial_t t) f^m}_{P(-\partial_t t)} \right) t_j.$$

Hence $b(-\partial_t t) - P(-\partial_t t) t_j \in I$. Using equation (1), we end the proof. □

As a consequence, we have:

Proposition 5.4. *We still denote by I the annihilator of f^s in $\mathbf{A}_{n+p}$.*

(a) $\psi_{\mathcal{B}}(I) \cap \mathbf{K}[s] = \mathcal{B}$,
(b) $\psi_{\mathcal{B}_j}(I) \cap \mathbf{K}[s] = \mathcal{B}_j$,
(c) $\psi_{\mathcal{B}_\Sigma}(I) \cap \mathbf{K}[s] = \mathcal{B}_\Sigma$.

Proof. We only prove Assertion (b) (the other assertions being similar).
Let $b(s) \in \psi_{\mathcal{B}_j}(I) \cap \mathbf{K}[s]$. Then there exists $Q \in I$ with $\mathrm{ord}^V(Q) = 0$ such that $\mathrm{ord}^{V_j}(Q - \sigma^V(Q)) < 0$ and such that $b(s) = \psi(Q)(s)$, or $b(-\partial_t t) = \sigma^V(Q)$. Write

$$Q - \sigma^V(Q) = Q' t_j \quad \text{with} \quad Q' \in V_0(\mathbf{A}_{n+p}).$$

We have $b(-\partial_t t) + Q' t_j \in I$. Then by the previous lemma, $b(s) \in \mathcal{B}_j$.
Let $b(s) \in \mathcal{B}_j$. Then by the previous lemma, there exists $Q \in I$ and $Q' \in V_0(\mathbf{A}_{n+p})$ such that

$$b(-\partial_t t) = Q - Q' t_j.$$

Thus we see that $\mathrm{ord}^V(Q) = (0, \dots, 0)$ and $b(s) = \psi(Q)(s)$, i.e. $b(s) \in \psi_{\mathcal{B}_j}(I) \cap \mathbf{K}[s]$. □

Remark 5.5. If we can compute generators of $\psi_{\mathcal{B}_j}(I)$ then we will easily compute generators of $\mathcal{B}_j$. Actually, if we compute a standard basis $G_1, \dots, G_q$ with respect to an order which eliminates the variables x_i and ∂_{x_i} then $\{G_1, \dots, G_q\} \cap \mathbf{K}[s]$ will generate $\mathcal{B}_j$ on $\mathbf{K}[s]$. The same remark can be applied to $\psi_{\mathcal{B}_\Sigma}(I)$ and $\mathcal{B}_\Sigma$. Thus, in the next section, we will compute $\psi_{\mathcal{B}_\Sigma}(I)$ and $\psi_{\mathcal{B}_j}(I)$.

6. Computation of $\mathcal{B}_\Sigma$ and $\mathcal{B}_j$

In this section, we present the main result of this paper. As we saw in Proposition 5.4 and in Remark 5.5, it is enough to compute $\psi_{\mathcal{B}_\Sigma}(I)$ and $\psi_{\mathcal{B}_j}(I)$ to obtain $\mathcal{B}_\Sigma$ and $\mathcal{B}_j$. The way we will compute $\psi_{\mathcal{B}_\Sigma}(I)$ and $\psi_{\mathcal{B}_j}(I)$ is the following:

- we introduce new variables $y_1, \dots, y_p$ which commute with $\mathbf{A}_{n+p}$,
- given an ideal J of $\mathbf{A}_{n+p}[y_{j_1}, \dots, y_{j_k}]$ and $j \in \{1, \dots, p\} \setminus \{j_1, \dots, j_k\}$, we define $h^{V_j}(J)$ ideal of $\mathbf{A}_{n+p}[y_{j_1}, \dots, y_{j_k}, y_j]$.
  Given generators of J, we show how to find generators of $h^{V_j}(J)$,
- we define an ideal $h(I)$ of $\mathbf{A}_{n+p}[y_2, \dots, y_p]$ as $h(I) = h^{V_2}(h^{V_3}(\cdots h^{V_p}(I) \cdots))$ (see Definition 6.7). It is an ideal obtained by $p-1$ homogenizations starting from $I \subset \mathbf{A}_{n+p}$. In fact we construct $h^{V_p}(I) \subset \mathbf{A}_{n+p}[y_p]$, then $h^{V_{p-1}}(h^{V_p}(I)) \subset \mathbf{A}_{n+p}[y_{p-1}]$, etc.,
- by definition, we can compute generators (finite in number) of $h(I)$,
- we introduce a multigraduation on $\mathbf{A}_{n+p}[y_2, \dots, y_p]$ given by a multiform $H = (H_2, \dots, H_p)$. For $G \in \mathbf{A}_{n+p}[y_2, \dots, y_p]$, we have a notion of being H-multihomogeneous. An ideal J of $\mathbf{A}_{n+p}[y_2, \dots, y_p]$ is said to be H-multihomogeneous if it can be generated by H-multihomogeneous elements,
- for an element of $\mathbf{A}_{n+p}$ (or $\mathbf{A}_{n+p}[y_2, \dots, y_p]$) we introduce a notion of V-regularity (resp. V_j-goodness),

- we introduce an order $\prec_{\mathcal{B}_\Sigma}$ (resp. $\prec_{\mathcal{B}_j}$) which is used to detect if an element is V-regular (resp. V_j-good) or not (see Propositions 6.16 and 6.22). If $G \in \mathbf{A}_{n+p}[y_2,\dots,y_p]$ is H-multihomogeneous, then thanks to Proposition 6.16 (resp. 6.22) and Lemma 6.11 we can express the fact that G is V-regular (resp. $\mathcal{B}_j$-good) or not by a property of non-divisibility by y_j (see Corollary 6.17),
- Lemma 6.12 shows that $h(I)$ is H-multihomogeneous. A consequence is that there exists a standard basis of $h(I)$ with respect to $\prec_{\mathcal{B}_\Sigma}$ (resp. $\prec_{\mathcal{B}_j}$) made of H-multihomogeneous elements,
- finally, Proposition 6.19 (resp. 6.23) shows that a standard basis of $h(I)$ with respect to $\prec_{\mathcal{B}_\Sigma}$ (resp. $\prec_{\mathcal{B}_j}$) gives a system of generators of $\psi_{\mathcal{B}_\Sigma}(I)$ (resp. $\psi_{\mathcal{B}_j}(I)$), by keeping only the V-regular (resp V_j-good) elements.

6.1. homogenizations

Let $y_1,\dots,y_p$ be p new variables. Let $k \in \{0,\dots,p-1\}$.
If $k \in \{1,\dots,p-1\}$, let $\{j_1,\dots,j_k\}$ be a part of $\{1,\dots,p\}$ of cardinal equal to k.
If $k=0$ then we set $\mathbf{A}_{n+p}[y_{j_1},\dots,y_{j_k}] = \mathbf{A}_{n+p}$ and $\{j_1,\dots,j_k\} = \varnothing$.
Let $j \in \{1,\dots,p\} \smallsetminus \{j_1,\dots,j_k\}$.

Definition 6.1. Let $P \in \mathbf{A}_{n+p}[y_{j_1},\dots,y_{j_k}]$, and denote $m_j = \mathrm{ord}^{V_j}(P)$. We define $h^{V_j}(P) \in \mathbf{A}_{n+p}[y_{j_1},\dots,y_{j_k},y_j]$ in the following way:
let us write

$$P = \sum a_{\mu\nu} t^\mu \partial_t^\nu \quad \mu,\nu \in \mathbf{N}^p,\ a_{\mu\nu} \in \mathbf{A}_n[y_1,\dots,y_{j_k}].$$

Set:

$$h^{V_j}(P) = \sum a_{\mu\nu} t^\mu \partial_t^\nu y_j^{m_j-(\nu_j-\mu_j)}.$$

Lemma 6.2. *For each* $P,Q \in \mathbf{A}_{n+p}[y_{j_1},\dots,y_{j_k}]$,

$$h^{V_j}(PQ) = h^{V_j}(P)h^{V_j}(Q).$$

Proof. The proof is based on the following:
If $V_j(t^\mu\partial_t^\nu) = m$ and $V_j(t^{\mu'}\partial_t^{\nu'}) = m'$ then:

- $V_j(t^\mu\partial_t^\nu t^{\mu'}\partial_t^{\nu'}) = m+m'$,
- $t^\mu\partial_t^\nu t^{\mu'}\partial_t^{\nu'} = \sum c_{\mu'',\nu''} t^{\mu''}\partial_t^{\nu''}$
  with $V_j(t^{\mu''}\partial_t^{\nu''}) = m+m'$ if $c_{\mu'',\nu''} \neq 0$.

This follows from the Leibniz rule:

$$\partial_{t_j}^k \cdot a = \sum_{i=0}^k \binom{k}{i} \frac{\partial^i a}{\partial t_j} \partial_{t_j}^{k-i}.$$

In fact we can say that V_j-homogeneity is preserved by products. □

Lemma 6.3. *Let* $P,P_1,\dots,P_d \in \mathbf{A}_{n+p}[y_{j_1},\dots,y_{j_k}]$ *be such that* $P = P_1+\cdots+P_d$ *with* $\forall i \quad \mathrm{ord}^{V_j}(P_i) \le \mathrm{ord}^{V_j}(P)$, *then there exists* $l_1,\dots,l_d \in \mathbf{N}$ *such that:*

$$h^{V_j}(P) = y_j^{l_1} h^{V_j}(P_1) + \cdots + y_j^{l_d} h^{V_j}(P_d).$$

In fact $l_i = \mathrm{ord}^{V_j}(P) - \mathrm{ord}^{V_j}(P_i)$.

Definition 6.4. Let J be an ideal of $\mathbf{A}_{n+p}[y_{j_1},\ldots,y_{j_k}]$. We define $h^{V_j}(J)$ as the ideal of $\mathbf{A}_{n+p}[y_{j_1},\ldots,y_{j_k},y_j]$ generated by $\{h^{V_j}(P)\ /\ P\in J\}$.

Proposition 6.5. *Let J be an ideal of $\mathbf{A}_{n+p}[y_{j_1},\ldots,y_{j_k}]$. There exists $P_1,\ldots,P_q\in J$ such that for each $P\in J$ there exists $C_1,\ldots,C_q\in\mathbf{A}_{n+p}[y_{j_1},\ldots,y_{j_k}]$ such that:*

$$P=\sum C_iP_i \quad \textit{with} \quad \mathrm{ord}^{V_j}(C_iP_i)\le \mathrm{ord}^{V_j}(P) \quad \textit{if} \quad C_i\neq 0.$$

Moreover, we can obtain these P_i in an algorithmic way if we know a finite system of generators of J.

Proof. We just apply Proposition 2.1 to the order $\prec_{V_j}$ defined on $\mathbf{N}^{2n+2p+k}$ by the following:
$(\alpha,\beta,\mu,\nu,\eta)\prec_{V_j}(\alpha',\beta',\mu',\nu',\eta')$

$$\iff \begin{cases} \quad \nu_j-\mu_j<\nu'_j-\mu'_j \\ \text{or} \quad (\nu_j-\mu_j=\nu'_j-\mu'_j \text{ and } (\alpha,\beta,\mu,\nu,\eta)\prec(\alpha',\beta',\mu',\nu',\eta')) \end{cases}$$

where $\prec$ is a total well-order compatible with sums, the parameters of Proposition 2.1 being $y_{j_1},\ldots,y_{j_k}$ here.□

Corollary 6.6. *Let J be an ideal of $\mathbf{A}_{n+p}[y_{j_1},\ldots,y_{j_k}]$ and let $P_1,\ldots,P_q\in J$ be as in Proposition 6.5, then $h^{V_j}(J)$ is generated by $h^{V_j}(P_1),\ldots,h^{V_j}(P_q)$.*

Proof. Let $P\in J$. By Proposition 6.5, there exists $C_1,\ldots,C_q\in\mathbf{A}_{n+p}[y_{j_1},\ldots,y_{j_k}]$ such that:

$$P=\sum C_iP_i \quad \text{with} \quad \mathrm{ord}^{V_j}(C_iP_i)\le \mathrm{ord}^{V_j}(P) \quad \text{if} \quad C_i\neq 0.$$

By Lemma 6.2 and Lemma 6.3, we have:

$$h^{V_j}(P)=y_j^{l_1}h^{V_j}(C_1)h^{V_j}(P_1)+\cdots+y_j^{l_q}h^{V_j}(C_q)h^{V_j}(P_q).\square$$

6.2. THE HOMOGENIZED IDEAL $h(I)$

From now on, y denotes $(y_2,\ldots,y_p)$.

Definition 6.7. We still denote by I the annihilator of f^s in $\mathbf{A}_{n+p}$. We define by a descending induction an ideal $h(I)$ by:

- $h_{p+1}(I)=I$,
- $h_k(I)=h^{V_k}(h_{k+1}(I))$,
- $h(I)=h_2(I)$.

We can write:

$$h(I)=h^{V_2}(h^{V_3}(\cdots h^{V_p}(I)\cdots)).$$

This is an ideal of $\mathbf{A}_{n+p}[y]=\mathbf{A}_{n+p}[y_2,\ldots,y_p]$.
Similarly, for $P\in\mathbf{A}_{n+p}$ we define $h(P)\in\mathbf{A}_{n+p}[y]$ by:

$$h(P)=h^{V_2}(h^{V_3}(\cdots h^{V_p}(P)\cdots)).$$

For example, for $p=3$, we have:

$$h^{V_3}(I) = \left\{ \sum_j E_j(y_3) h^{V_3}(Q_j) / Q_j \in I, E_j(y_3) \in \mathbf{A}_{n+p}[y_3] \right\}$$

$$h^{V_2}(h^{V_3}(I)) = \left\{ \sum_i F_i(y_2, y_3) h^{V_2} \left(\sum_j E_{i,j}(y_3) h^{V_3}(Q_{i,j}) \right) \right\}$$

with $Q_{i,j} \in I, E_{i,j} \in \mathbf{A}_{n+p}[y_3], F_i(y_2, y_3) \in \mathbf{A}_{n+p}[y_2, y_3]$.
It is easy to see that for $P \in I$ we have $h(P) \in h(I)$. Lemma 6.12 says that, conversely, $h(I)$ is generated by such elements, which is not completely direct.

Remark 6.8. Following Proposition 6.5 and Corollary 6.6, we can obtain in an algorithmic way generators (finite in number) of $h(I)$.

Definition 6.9.

- For $j = 2, \ldots, p$, we define the linear form H_j on $\mathbf{N}^{2n+2p+p-1}$ by:

$$H_j(\alpha, \beta, \mu, \nu, \eta) = \eta_j + \nu_j - \mu_j.$$

  We denote by H the multiform $(H_2, \ldots, H_p)$.
- Let $G = G(y) \in \mathbf{A}_{n+p}[y]$. Write:

$$G = \sum a_{\mu,\nu,\eta} t^\mu \partial_t^\nu y^\eta \quad \mu, \nu \in \mathbf{N}^p, \eta \in \mathbf{N}^{p-1}, a_{\mu,\nu,\eta} \in \mathbf{A}_n.$$

  — We define $\sigma_{d_j}^{H_j}(G)$ the partial H_j-symbol of G with order equal to $d_j \in \mathbf{Z}$ by :

$$\sigma_{d_j}^{H_j}(G) = \sum_{\eta_j + \nu_j - \mu_j = d_j} a_{\mu,\nu,\eta} t^\mu \partial_t^\nu y^\eta.$$

  — G is said to be H_j-homogeneous with $\mathrm{ord}^{H_j}(G) = d_j$ if $G = \sigma_{d_j}^{H_j}(G)$.
  — G is said to be H-homogeneous (or H-multihomogeneous) with $\mathrm{ord}^H(G) = d = (d_2, \ldots, d_p)$ if for each $j = 2, \ldots, p$, G is H_j-homogeneous of order equal to d_j. We define $\sigma_d^H(G)$ the H-symbol with order equal to $d = (d_2, \ldots, d_p)$ of G by:

$$\sigma_d^H(G) = \sum_{\forall j, \eta_j + \nu_j - \mu_j = d_j} a_{\mu,\nu,\eta} t^\mu \partial_t^\nu y^\eta.$$

  We have:

$$\sigma_d^H(G) = \sigma_{d_2}^{H_2}(\cdots \sigma_{d_p}^{H_p}(G) \cdots).$$

  Note that the operations $\sigma_{d_j}^{H_j}$ commute with each other.

This yields the four following results:

Lemma 6.10. *We take the notations of Definition* 6.4*:*
let J be an ideal of $\mathbf{A}_{n+p}[y_{j_1}, \ldots, y_{j_k}]$, let $G \in h^{V_j}(J)$. Then for each $m \in \mathbf{Z}$, we have $\sigma_m^{H_j}(G) \in h^{V_j}(J)$.

Proof. By definition of $h^{V_j}(J)$, there exists $P_1, \ldots, P_e \in J$ and $R_1, \ldots, R_e \in \mathbf{A}_{n+p}[y_{j_1}, \ldots, y_{j_k}, y_j]$ such that $G = \sum_i R_i h^{V_j}(P_i)$. Since $\sigma_m^{H_j}(G) = \sum_i \sigma_m^{H_j}(R_i h^{V_j}(P_i))$, we can assume that $G = Rh^{V_j}(P)$ with $P \in J$. Write $R = \sum_{l \in \mathbf{Z}} \sigma_l^{H_j}(R)$, we have:

$$Rh^{V_j}(P) = \sum_l \sigma_l^{H_j}(R) h^{V_j}(P).$$

By the same calculation as in 6.2, we can prove that $\sigma_l^{H_j}(R) h^{V_j}(P)$ is H_j-homogeneous with ord^{H_j} equal to $l + \mathrm{ord}^{H_j}(h^{V_j}(P)) = l + \mathrm{ord}^{V_j}(P)$ so we have:

$$\sigma_m^{H_j}(Rh^{V_j}(P)) = \sigma^{H_j}_{m - \mathrm{ord}^{V_j}(P)}(R) h^{V_j}(P)$$

which ends the proof. □

Lemma 6.11. *For each $G \in \mathbf{A}_{n+p}[y]$ H-homogeneous, there exists $l = (l_2, \ldots, l_p) \in \mathbf{N}^{p-1}$ such that $G = y^l h(G(1, \ldots, 1))$.*

Lemma 6.12. *$h(I)$ is the ideal of $\mathbf{A}_{n+p}[y_2, \ldots, y_p]$ generated by the set $\{h(P)/P \in I,\ P \neq 0\}$.*

Proof. First, it is clear that for each P in I, $h(P)$ is in $h(I)$.
Conversely, let $G \in h(I)$. Since $G = \sum_{k \in \mathbf{Z}^{p-1}} \sigma_k^H(G)$, we are reduced to prove that each $\sigma_k^H(G)$ is in the ideal generated by $\{h(P)/P \in I\}$. In fact we shall prove that for each $k = (k_2, \ldots, k_p) \in \mathbf{Z}^{p-1}$, there exists $l = (l_2, \ldots, l_p) \in \mathbf{N}^{p-1}$ and $Q \in I$ such that $\sigma_k^H(G) = y^l h(Q)$, which will prove the lemma. The proof will be constructed by induction on r the number of homogenizations (the claim in step r of this induction being the following: $h_{p+1-r}(I)$ is generated by all the $h^{V_{p+1-r}}(\ldots h^{V_p}(P) \ldots)$ with $P \in I$).

- If $r = 1$:
  in this case, we have $h(I) = h^{V_p}(I)$, $H = H_p$, $k \in \mathbf{Z}$. By 6.10 we have $\sigma_k^H(G) \in h(I)$, so $\sigma_k^H(G)_{|y_p=1} \in I$. As in Lemma 6.11, there exists $l \in \mathbf{N}$ such that $\sigma_k^H(G) = y_p^l h^{V_p}(\sigma_k^H(G)_{|y_p=1})$. This proves the result for $r = 1$.
- The induction step:
  in order to avoid too many notations, we will assume that the result is true for $r = p-2$ and we will prove it for $r = p-1$ (i.e. we only make the last step of the induction which is completely similar to the rth step). Let us assume the result true for $h^{V_3}(\cdots h^{V_p}(I))$.
  We have:
  $$\sigma_k^H(G) = \sigma_{k'}^{H'}(\sigma_{k_2}^{H_2}(G)) \quad \text{where} \quad H' = (H_3, \ldots, H_p) \text{ and } k' = (k_3, \ldots, k_p).$$
  Denote $G' = \sigma_{k_2}^{H_2}(G)_{|y_2=1}$ then using 6.10, we have $G' \in h^{V_3}(\cdots h^{V_p}(I) \cdots)$, moreover there exists $l_2' \in \mathbf{N}$ such that $\sigma_{k_2}^{H_2}(G) = y_2^{l_2'} h^{V_2}(G')$. Thus we have:
  $$\begin{aligned}\sigma_k^H(G) &= \sigma_{k'}^{H'}(y_2^{l_2'} h^{V_2}(G')) \\ &= y_2^{l_2'} \sigma_{k'}^{H'}(h^{V_2}(G')).\end{aligned}$$

Assertion. $\exists l_2'' \in \mathbf{Z}$ such that $\sigma_{k'}^{H'}(h^{V_2}(G')) = y_2^{l_2''} h^{V_2}(\sigma_{k'}^{H'}(G'))$.

Let G'' be such that $G' = \sigma^{H'}_{k'}(G') + G''$ then we have:

$$h^{V_2}(G') = y_2^{l''_2} h^{V_2}(\sigma^{H'}_{k'}(G')) + y_2^{l'''_2} h^{V_2}(G'')$$

with $l''_2 = 0$ or $l'''_2 = 0$, after which, $\sigma^{H'}_{k'}(h^{V_2}(G')) = y_2^{l''_2} h^{V_2}(\sigma^{H'}_{k'}(G'))$.
Let $l_2 = l'_2 + l''_2$, we have:

$$\sigma^{H}_{k}(G) = y_2^{l_2} h^{V_2}(\sigma^{H'}_{k'}(G')).$$

We apply the induction hypothesis:

$$\sigma^{H'}_{k'}(G') = y_3^{l_3} \cdots y_p^{l_p} h^{V_3}(\cdots h^{V_p}(Q)\cdots) \text{ with } Q \in I.$$

Finally we have

$$\sigma^{H}_{k}(G) = y^l h(Q) \text{ with } l = (l_2, \ldots, l_p) \quad \text{and} \quad Q \in I. \ \square$$

Corollary 6.13. *For each $G \in h(I)$, we have $G_{|y=(1,\ldots,1)} \in I$.*

Another consequence of Lemma 6.12 is that $h(I)$ is multihomogeneous: if we compute a standard basis of $h(I)$ with respect to some order, then we will be able to construct another standard basis whose elements will be H-homogeneous. Let us denote by $G_1, \ldots, G_q$ such a basis, then $\{\sigma^H_k(G_i)/k \in \mathbf{Z}^{p-1}, i = 1, \ldots, q\}$ will be another standard basis.
In fact we shall see that, in our case, only q elements of this system will be useful.

6.3. computation of $\mathcal{B}_\Sigma$

From now on, we denote by V^+ the linear form on $\mathbf{N}^{2p}$:

$$V^+(\mu, \nu) = \nu_2 - \mu_2 + \cdots + \nu_p - \mu_p.$$

We naturally extend V^+ on $\mathbf{N}^{2n+2p+p-1}$.

Definition 6.14. Let $G(y) \in \mathbf{A}_{n+p}[y]$. Then we say that $G(y)$ (or $G(1)$) is V-regular if $\sigma^V(G(1)) \neq 0$.

Definition 6.15. On $\mathbf{N}^{2n+2p+p-1}$ we define the following order:

$$(\alpha, \beta, \mu, \nu, \eta) \prec_{\mathcal{B}_\Sigma} (\alpha', \beta', \mu', \nu', \eta')$$

$$\Longleftrightarrow \begin{cases} \quad \nu_1 - \mu_1 < \nu'_1 - \mu'_1 \\ \text{or} \quad \big(= \text{ and } V^+(\mu,\nu) < V^+(\mu',\nu')\big) \\ \text{or} \quad \big(= \text{ and } = \text{ and } |\eta| < |\eta'|\big) \\ \text{or} \quad \big(= \text{ and } = \text{ and } = \text{ and } (\alpha,\beta,\mu,\nu,\eta) \prec (\alpha',\beta',\mu',\nu',\eta')\big) \end{cases}$$

where $\prec$ is a total well-order compatible with sums.

Proposition 6.16. *Let $P \in \mathbf{A}_{n+p}$,*

$$P \textit{ is } V\textit{-regular} \iff mp_{\prec_{\mathcal{B}_\Sigma}}(h(P)) \quad \textit{has no} \quad y_j \textit{ in its factorization,}$$

where $mp_{\prec_{\mathcal{B}_\Sigma}}(h(P))$ is the privileged monomial of $h(P)$ with respect to the order $\prec_{\mathcal{B}_\Sigma}$.

Proof. Denote $\gamma = \exp_{\prec_{\mathcal{B}_\Sigma}}(P)$. We have:

$$\mathrm{mp}_{\prec_{\mathcal{B}_\Sigma}}(h(P)) \text{ has no factors } y_j \iff \forall j = 2, \dots, p \quad V_j(\gamma) = \mathrm{ord}^{V_j}(P).$$

Assume that $\forall j = 2, \dots, p \quad V_j(\gamma) = \mathrm{ord}^{V_j}(P)$. By the definition of $\prec_{\mathcal{B}_\Sigma}$, we also have $V_1(\gamma) = \mathrm{ord}^{V_1}(P)$, hence $\gamma \in \mathrm{ND}(\sigma^V(P))$ and then $\sigma^V(P) \neq 0$, i.e. P is V-regular.
Conversely, assume that P is V-regular. Then let $\gamma_0 \in \mathrm{ND}(\sigma^V(P))$. By the definition of $\prec_{\mathcal{B}_\Sigma}$, we have $V_1(\gamma_0) = V_1(\gamma)$. Then, by the same definition, we have:

$$V_2(\gamma) + \cdots + V_p(\gamma) = V^+(\gamma) \geq V^+(\gamma_0) = \mathrm{ord}^{V_2}(P) + \cdots + \mathrm{ord}^{V_p}(P),$$

but, since $\forall j = 2, \dots, p \quad V_j(\gamma) \leq \mathrm{ord}^{V_j}(P)$ we have

$$\forall j = 2, \dots, p \quad V_j(\gamma) = \mathrm{ord}^{V_j}(P)$$

and then $\mathrm{mp}_{\prec_{\mathcal{B}_\Sigma}}(h(P))$ has no y_j in its factorization. □

Corollary 6.17. *Let $G = G(y) \in \mathbf{A}_{n+p}[y]$ such that G is H-homogeneous then:*

$$mp_{\prec_{\mathcal{B}_\Sigma}}(G) \text{ has no } y_j \text{ in its factorization} \Longrightarrow G \text{ is } V\text{-regular.}$$

Proof. This is a direct consequence of Lemma 6.11, Corollary 6.13 and Proposition 6.16. □

Computation of $\mathcal{B}_\Sigma$

We first give a lemma:

Lemma 6.18. *Let $q, P \in \mathbf{A}_{n+p}$ with q a monomial, such that* $\mathrm{ord}^V(qP) = (0, \dots, 0)$. *Then $\sigma^V(qP) \in \mathbf{A}_n[-\partial_t t]\psi(P)(-\partial_t t)$.*

Proof. We have:

(i) $t_j^k \partial_{t_j}^k \in \mathbf{K}[-\partial_{t_j} t_j]$,
(ii) $t^k c(-\partial_t t) \in \mathbf{K}[-\partial_t t] t^k$ with $c(x) \in \mathbf{K}[x]$.

We prove (i) by an induction on k (for $k = 0$, (i) is true):

$$\begin{aligned} t^k \partial_t^k &= t^k \partial_t \partial_t^{k-1} \\ &= (\partial_t t^k - k t^{k-1}) \partial_t^{k-1} \\ &= (\partial_t t - k) t^{k-1} \partial_t^{k-1}. \end{aligned}$$

Hence (i) is true by the induction hypothesis.
For (ii), we note that it is enough to prove the result for $c(x) = x$:

$$\begin{aligned} t^k \partial_t t &= (\partial_t t^k + k t^{k-1}) t \\ &= (\partial_t t + k) t^k. \end{aligned}$$

Using these relations, it is easy to see that $q \in \mathbf{A}_n[-\partial_t t] S_1 \cdots S_p$ (see the notations of Definition 5.2). Thus:

$$\sigma^V(qP) \in \mathbf{A}_n[-\partial_t t] \sigma^V(S_1 \cdots S_p P)(-\partial_t t). \ \square$$

Let $G_1(y), \dots, G_q(y), \dots, G_{q+r}(y)$ be a $\prec_{\mathcal{B}_\Sigma}$-standard basis of $h(I)$ (in the sense of Proposition 2.1) such that $G_1, \dots, G_q$ are V-regular and not the next ones. By Lemma 6.12, $h(I)$ is H-homogeneous so that for each G_i from this basis and each $k \in \mathbf{Z}^{p-1}$, we have $\sigma^H_k(G_i) \in h(I)$.
For each $i = 1, \dots, q+r$, let $k_i \in \mathbf{Z}^{p-1}$ such that

$$\exp_{\prec_{\mathcal{B}_\Sigma}}(G_i) = \exp_{\prec_{\mathcal{B}_\Sigma}}(\sigma^H_{k_i}(G_i))$$

and let $H_i(y) = \sigma^H_{k_i}(G_i)$, H_i is H-homogeneous. We shall prove that:

Proposition 6.19. *$\psi_{\mathcal{B}_\Sigma}(I)$ is generated by $\psi(H_1(1)), \dots, \psi(H_q(1))$.*

Note that $\forall i, \quad \psi(H_i(1)) \in \psi_{\mathcal{B}_\Sigma}(I)$ by Corollary 6.13

Proof. Let $P \in I$, $P \neq 0$, be V-regular and such that $\mathrm{ord}^V(P) = (0, \dots, 0)$. We have $h(P) \in h(I)$, and therefore:

$$\exists j_0 \in \{1, \dots, q+r\} \ / \ \exp_{\prec_{\mathcal{B}_\Sigma}}(h(P)) \in \exp(H_{j_0}) + \mathbf{N}^{2(n+p)+p-1}.$$

By Proposition 6.16 (applied to P) and Corollary 6.17 (applied to $H_1, \dots, H_{q+r}$), we have necessarily $j_0 \in \{1, \dots, q\}$. Let m_{j_0} be the monomial such that

$$\mathrm{mp}_{\prec_{\mathcal{B}_\Sigma}}(h(P)) = m_{j_0} \mathrm{mp}_{\prec_{\mathcal{B}_\Sigma}}(H_{j_0}).$$

Necessarily m_{j_0} and $\mathrm{mp}_{\prec_{\mathcal{B}_\Sigma}}(H_{j_0})$ have no y_j in their factorization.
We set $\tilde{P}_1 = h(P) - m_{j_0} H_{j_0}$, we have $\tilde{P}_1 \in h(I)$.
If $\sigma^V_{(0,\dots,0)}(\tilde{P}_1) = 0$ then we stop the process.
If $\sigma^V_{(0,\dots,0)}(\tilde{P}_1) \neq 0$ then:

- $\tilde{P}_1$ is H-homogenous,
- $\tilde{P}_1 = y^m h(\tilde{P}_{1|y=(1,\dots,1)}) \quad$ with $m \in \mathbf{N}^{p-1}$,
- $\forall j = 2, \dots, p$, $\mathrm{ord}^{V_j}(\tilde{P}_1) = 0$,
- it follows that $m = (0, \dots, 0) \quad$ and $\quad \tilde{P}_1 = h(\tilde{P}_{1|y=(1,\dots,1)})$ is V-regular,
- in this case, we can apply the process with P replaced by $\tilde{P}_{1|y=(1,\dots,1)}$.

We continue the process:

$$\tilde{P}_{k+1} = \tilde{P}_k - m_{j_k} H_{j_k}$$

with $\forall k$, $j_k \in \{1, \dots, q\}$,
and we stop the process as soon as $\sigma^V_{(0,\dots,0)}(\tilde{P}_{s+1}) = 0$, i.e. when all the monomials of $\sigma^V(P)$ are eliminated. Note that the restriction of $\prec_{\mathcal{B}_\Sigma}$ to $\{(\alpha, \beta, \mu, \nu, \eta) \in \mathbf{N}^{2n+2p+p-1} /$ $\forall j = 1, \dots, p \ \ \nu_j - \mu_j = 0\}$ is a well-order and that for all $k < s$

$$\exp_{\prec_{\mathcal{B}_\Sigma}}(\sigma^V_{(0,\dots,0)}(\tilde{P}_{k+1})) = \exp_{\prec_{\mathcal{B}_\Sigma}}(\tilde{P}_{k+1}) \prec_{\mathcal{B}_\Sigma} \exp_{\prec_{\mathcal{B}_\Sigma}}(\tilde{P}_k) = \exp_{\prec_{\mathcal{B}_\Sigma}}(\sigma^V_{(0,\dots,0)}(\tilde{P}_k)).$$

Thus we obtain:

$$h(P) = \sum_{k=0}^{s} m_{j_k} H_{j_k} + \tilde{P}_{s+1}$$

with

$$\begin{cases} \forall k,\ j_k \in \{1,\dots,q\} \\ \forall k,\ \mathrm{ord}^V(m_{j_k}H_{j_k}) = (0,\dots,0) \\ \sigma^V_{(0,\dots,0)}(\tilde{P}_{s+1}) = 0. \end{cases}$$

Finally

$$\sigma^V_{(0,\dots,0)}(P) = \sum_{k=0}^{s} \sigma^V_{(0,\dots,0)}(m_{j_k}(1)H_{j_k}(1))$$

and by Lemma 6.18, $\forall k$, $\sigma^V_{(0,\dots,0)}(m_{j_k}H_{j_k}(1)) \in \mathbf{A}_n[-\partial_t t]\psi(H_{j_k}(1))(-\partial_t t)$. Hence

$$\psi(P) \in \sum_{i=0}^{q} \mathbf{A}_n[s]\psi(H_i(1))(s). \ \square$$

We then find a finite system of generators of $\psi_{\mathcal{B}_\Sigma}$, and by Remark 5.5 we can compute $\mathcal{B}_\Sigma$.

6.4. COMPUTATION OF $\mathcal{B}_j$

Since we can permute f_1 and f_j, we can assume that $j = 1$. Thus we shall compute B_1.

DEFINITION 6.20. Let $G(y) \in \mathbf{A}_{n+p}[y]$. We say that $G(y)$ (or $G(1)$) is $\mathcal{B}_1$-good if we have the following condition:

$$\mathrm{ord}^{V_1}(G(1) - \sigma^V(G(1))) < \mathrm{ord}^{V_1}(G(1)).$$

DEFINITION 6.21. On $\mathbf{N}^{2n+2p+p-1}$ we define the following order:

$$(\alpha,\beta,\mu,\nu,\eta) \prec_{\mathcal{B}_1} (\alpha',\beta',\mu',\nu',\eta')$$

$$\iff \begin{cases} \quad \nu_1 - \mu_1 < \nu'_1 - \mu'_1 \\ \text{or} \quad \big(= \text{ and } V^+(\mu',\nu') < V^+(\mu,\nu)\big) \\ \text{or} \quad \big(= \text{ and } = \text{ and } |\eta| < |\eta'|\big) \\ \text{or} \quad \big(= \text{ and } = \text{ and } = \text{ and } (\alpha,\beta,\mu,\nu,\eta) \prec (\alpha',\beta',\mu',\nu',\eta')\big) \end{cases}$$

where $\prec$ is a total well-order compatible with sums.

PROPOSITION 6.22. *Let $P \in \mathbf{A}_{n+p}$,*

$$P \text{ is } \mathcal{B}_1\text{-good} \iff mp_{\prec_{\mathcal{B}_1}}(h(P)) \text{ has no } y_j \text{ in its factorization.}$$

PROOF. Denote $\gamma = \exp_{\prec_{\mathcal{B}_1}}(P)$. As in Proposition 6.16, we have:

$$\mathrm{mp}_{\prec_{\mathcal{B}_1}}(h(P)) \text{ has no factors } y_j \iff \forall j = 2,\dots,p \quad V_j(\gamma) = \mathrm{ord}^{V_j}(P).$$

Assume that P is not $\mathcal{B}_1$-good, i.e. there exists $\gamma' \in \mathrm{ND}(P) \smallsetminus \mathrm{ND}(\sigma^V(P))$ such that $V_1(\gamma') = \mathrm{ord}^{V_1}(P)$ then $V^+(\gamma) \le V^+(\gamma') < \mathrm{ord}^{V_2}(P) + \cdots + \mathrm{ord}^{V_p}(P)$ and then $\mathrm{mp}_{\prec_{\mathcal{B}_1}}$ $(h(P))$ has a y_j in its factorization.
Conversely, assume that P is $\mathcal{B}_1$-good, then $\sigma^V(P) = \sigma^{V_1}(P)$. But, by the definition of $\prec_{\mathcal{B}_1}$, $\gamma \in \mathrm{ND}(\sigma^{V_1}(P))$. Then $\gamma \in \mathrm{ND}(\sigma^V(P))$ and then $V_j(\gamma) = \mathrm{ord}^{V_j}(P) \quad \forall j = 2,\dots,p$, i.e. $\mathrm{mp}_{\prec_{\mathcal{B}_1}}(h(P))$ has no factors y_j. $\square$

Computation of $\mathcal{B}_1$

Let $G_1(y),\ldots,G_q(y),\ldots,G_{q+r}(y)$ a $\prec_{\mathcal{B}_1}$-standard basis of $h(I)$ (in the sense of Proposition 2.1) such that $G_1,\ldots,G_q$ are $\mathcal{B}_1$-good and not the next ones.
For each $i=1,\ldots,q+r$, let $k_i\in\mathbf{Z}^{p-1}$ such that

$$\exp_{\prec_{\mathcal{B}_1}}(G_i)=\exp_{\prec_{\mathcal{B}_1}}(\sigma^H_{k_i}(G_i))$$

and let $H_i(y)=\sigma^H_{k_i}(G_i)$: H_i is H-homogeneous, and we can prove in the same way as for $\psi_{\mathcal{B}_\Sigma}(I)$ that:

PROPOSITION 6.23. *$\psi_{\mathcal{B}_1}(I)$ is generated by $\psi(H_1(1)),\ldots,\psi(H_q(1))$.*

As before, we can obtain generators of $\mathcal{B}_1$ using Remark 5.5.

6.5. EXAMPLES

1. Let $f_1\in\mathbf{K}[x_1,\ldots,x_m]$, $f_2\in\mathbf{K}[x_{m+1},\ldots,x_n]$, with $m<n$. Let b_1 and b_2 be the Bernstein polynomials associated, respectively, with f_1 and f_2. Then we can show that:
$$\mathcal{B}_1=\mathbf{K}[s_1,s_2]b_1(s_1)\quad\text{and}\quad\mathcal{B}_\Sigma=\mathbf{K}[s_1,s_2](b_1(s_1),b_2(s_2)).$$
2. An example treated by hand.
Let $f_1(x_1,x_2)=x_1+x_2^2$ and $f_2(x_1,x_2)=x_1$.
The ideal I of $\mathbf{A}_4$ is generated by: $P_1=t_1-x_1-x_2^2$, $P_2=t_2-x_1$, $P_3=\partial_{x_1}+\partial_{t_1}+\partial_{t_2}$ and $P_4=\partial_{x_2}+2x_2\partial_{t_1}$.
Using semisyzygies, we can see that P_1,P_2,P_3,P_4 is already a V_2-standard basis of I. To make these calculations we decided that $\mathrm{mp}_{\prec_{V_2}}(P_1)=-x_2^2$ and $\mathrm{mp}_{\prec_{V_2}}(P_4)=\partial_{x_2}$. Thus $h(I)=h^{V_2}(I)$ is generated by $Q_1=h(P_1)$, $Q_2=h(P_2)$, $Q_3=h(P_3)$, $Q_4=h(P_4)$.
By a number of divisions of semisyzygies, we obtain the following elements of $h(I)$:
$Q_5=x_2\partial_{x_2}+2t_2\partial_{t_2}+2t_1\partial_{t_1}+2y_2t_2\partial_{x_1}+2$,
$Q_6=2x_2\partial_{t_2}+2x_2\partial_{x_1}y_2-\partial_{x_2}y_2$,
$Q_7=4t_1\partial_{t_1}^2+4t_2\partial_{t_2}\partial_{t_1}+6\partial_{t_1}-4\partial_{x_1}t_2\partial_{t_2}-4t_2y_2\partial_{x_1}^2-\partial_{x_2}^2$,
$Q_8=4t_1\partial_{t_1}\partial_{t_2}+4t_2\partial_{t_2}^2+6\partial_{t_2}+6\partial_{x_1}y_2+\partial_{x_2}^2y_2+8t_2\partial_{t_2}\partial_{x_1}y_2+4t_1\partial_{t_1}\partial_{x_1}y_2+4t_2\partial_{x_1}^2y_2^2$.
These divisions are made with respect to the order $\prec_{\mathcal{B}_\Sigma}$. Remember that, for two multi-indices ω and ω', we first compare $V_1(\omega)$ and $V_1(\omega')$ then $V_2(\omega)$ and $V_2(\omega')$ and then we compare ω and ω' using a well-order $\prec$. Here we decided that $\prec$ is a lexical order such that $\exp_\prec(x_2)\succ\exp_\prec(t_1)\succ\zeta$ where $\zeta=\max_\prec\{\exp_\prec(\xi)/\xi\in\{x_1,\partial_{x_1},\partial_{x_2},t_2,\partial_{t_1},\partial_{t_2},y_2\}\}$. In fact, the computation of a $\prec_{\mathcal{B}_\Sigma}$-standard basis of $h(I)$ is not complete, but at this point we can see that Q_7 and Q_8 are V-regular and we have:
$\psi(Q_7)=(s_1+1)(2s_1+2s_2+3)$ and $\psi(Q_8)=(s_2+1)(2s_1+2s_2+3)$.
Let E be the ideal of $\mathbf{K}[s_1,s_2]$ generated by $\psi(Q_7)$ and $\psi(Q_8)$. We have the inclusion $E\subset\mathcal{B}_\Sigma$. We also prove that $\mathcal{B}_\Sigma\subset\mathbf{K}[s_1,s_2](s_1+1,s_2+1)$ (by taking $s_1=-1$, $s_2=-1$ in a functional equation).

PROPOSITION 6.24. *We have $\mathcal{B}_\Sigma=E$.*

Proof. If the inclusion $E \subset \mathcal{B}_\Sigma$ is not an equality, then we can construct an element $b(s_1, s_2)$ in $\mathcal{B}_\Sigma$ such that for all $c \in \mathbf{Z} \setminus 4$, $(2s_1 + 2s_2 + c)$ does not divide b. Using an argument of Maynadier (see Maynadier, 1996), it would follow that

$$(s_1 + 1)(s_2 + 1)\Pi_{d_1,d_2\in\mathbf{N},d_1+d_2\leq l-1} b(s_1 + d_1, s_2 + d_2) \text{ where } l \in \mathbf{N}$$

is in $\mathcal{B}$. But, according to Maynadier, B is principal and generated by $(s_1+1)(s_2+1)(2s_1 + 2s_2 + 3)(2s_1 + 2s_2 + 5)$. Contradiction. □

3. The next examples have been made using the software KAN (see Takayama, 1991). The results for $\mathcal{B}$ come from Maynadier (1996).
   $(f_1, f_2)(x_1, x_2) = (x_1, x_1 + x_2^2)$
   - $\mathcal{B}_\Sigma = \langle (s_1 + 1)(2s_1 + 2s_2 + 3), (s_2 + 1)(2s_1 + 2s_2 + 3)\rangle$
   - $\mathcal{B}_1 = \langle (s_1 + 1)(2s_1 + 2s_2 + 3)\rangle$
   - $\mathcal{B}_2 = \langle (s_2 + 1)(2s_1 + 2s_2 + 3)\rangle$
   - $\mathcal{B} = \langle (s_1 + 1)(s_2 + 1)(2s_1 + 2s_2 + 3)(2s_1 + 2s_2 + 5)\rangle$
4. $(f_1, f_2)(x_1, x_2) = (x_1, x_1 + x_2^3)$
   - $\mathcal{B}_\Sigma = \langle (s_1+1)(3s_1+3s_2+4)(3s_1+3s_2+5), (s_2+1)(3s_1+3s_2+4)(3s_1+3s_2+5)\rangle$
   - $\mathcal{B}_1 = \langle (s_1 + 1)(3s_1 + 3s_2 + 4)(3s_1 + 3s_2 + 5)\rangle$
   - $\mathcal{B}_2 = \langle (s_2 + 1)(3s_1 + 3s_2 + 4)(3s_1 + 3s_2 + 5)\rangle$
   - $\mathcal{B} = \langle (s_1+1)(s_2+1)(3s_1+3s_2+4)(3s_1+3s_2+5)(3s_1+3s_2+7)(3s_1+3s_2+8)\rangle$
5. $(f_1, f_2)(x_1, x_2) = (x_1, x_1^2 + x_2^3)$
   - $\mathcal{B}_\Sigma = \langle (s_1+1)(3s_1+6s_2+5)(3s_1+6s_2+7), (s_2+1)(3s_1+6s_2+5)(3s_1+6s_2+7)\rangle$
   - $\mathcal{B}_1 = \langle (s_1 + 1)(3s_1 + 6s_2 + 5)(3s_1 + 6s_2 + 7)\rangle$
   - $\mathcal{B}_2 = \langle (s_2 + 1)(3s_1 + 6s_2 + 5)(3s_1 + 6s_2 + 7)(3s_1 + 6s_2 + 8)(3s_1 + 6s_2 + 10)\rangle$
   - $\mathcal{B} = \langle (s_1 + 1)(s_2 + 1)(3s_1 + 6s_2 + 5)(3s_1 + 6s_2 + 7)(3s_1 + 6s_2 + 8)(3s_1 + 6s_2 + 10)(3s_1 + 6s_2 + 11)(3s_1 + 6s_2 + 13)\rangle$
6. $(f_1, f_2)(x_1, x_2) = (x_1, x_1^3 + x_2^2)$
   - $\mathcal{B}_\Sigma = \langle (s_1 + 1)(2s_1 + 6s_2 + 5), (s_2 + 1)(2s_1 + 6s_2 + 5)\rangle$
   - $\mathcal{B}_1 = \langle (s_1 + 1)(2s_1 + 6s_2 + 5)\rangle$
   - $\mathcal{B}_2 = \langle (s_2 + 1)(2s_1 + 6s_2 + 5)(2s_1 + 6s_2 + 7)(2s_1 + 6s_2 + 9)\rangle$
   - $\mathcal{B} = \langle (s_1+1)(s_2+1)(2s_1+6s_2+5)(2s_1+6s_2+7)(2s_1+6s_2+9)(2s_1+6s_2+11)\rangle$
7. $(f_1, f_2)(x_1, x_2) = (x_1^2 + x_2^3, x_1^3 + x_2^2)$
   - $\mathcal{B}_\Sigma = \langle (s_1 + 1)(4s_1 + 6s_2 + 5)(4s_1 + 6s_2 + 7)(6s_1 + 4s_2 + 5)(6s_1 + 4s_2 + 7), (s_2 + 1)(4s_1 + 6s_2 + 5)(4s_1 + 6s_2 + 7)(6s_1 + 4s_2 + 5)(6s_1 + 4s_2 + 7)\rangle$
   - $\mathcal{B}_1 = \langle (s_1 + 1)(4s_1 + 6s_2 + 5)(4s_1 + 6s_2 + 7)(6s_1 + 4s_2 + 5)(6s_1 + 4s_2 + 7)(6s_1 + 4s_2 + 9)\rangle$
   - $\mathcal{B}_2 = \langle (s_2 + 1)(4s_1 + 6s_2 + 5)(4s_1 + 6s_2 + 7)(6s_1 + 4s_2 + 5)(6s_1 + 4s_2 + 7)(4s_1 + 6s_2 + 9)\rangle$

Some remarks:
— the computation of $\mathcal{B}$ for example 7 seems to be hard. We tried using the algorithm of Oaku (in Oaku and Takayama, 1999) but the computer did not succeed;
— we can see that in examples 4, 5, 6 and 7, we do not have $\mathcal{B}_\Sigma = \mathcal{B}_1 + \mathcal{B}_2$;

— in these examples, $\mathcal{B}$ can be obtained as follows:
take $\mathcal{B}_2(s_1+1, s_2) := \{b(s_1+1, s_2)/b(s_1, s_2) \in \mathcal{B}_2\}$, then $\mathcal{B} = \mathcal{B}_1.\mathcal{B}_2(s_1+1, s_2)$. Note that we also have $\mathcal{B} = \mathcal{B}_1(s_1, s_2+1).\mathcal{B}_2$ where $\mathcal{B}_1(s_1, s_2+1) = \{b(s_1, s_2+1)/b(s_1, s_2) \in \mathcal{B}_1\}$.

Acknowledgement

I wish to thank Prof. Michel Granger for his advice.

References

Bernstein, I. N. (1972). The analytic continuation of generalized functions with respect to a parameter. *Funct. Anal.*, **6**, 273–285.

Björk, J. E. (1979). *Rings of Differential Operators*, volume 21 of North-Holland Math. Libr., Amsterdam, North-Holland.

Buchberger, B. (1970). Ein algorithmisches kriterium fur die lösbarkeit eines algebraischen gleichungssystems (Ph.D. Thesis). *Aequationes Math.*, **4**, 374–383.

Castro, F., Granger, M. (1997). Explicit calculations in rings of differential operators, *Prépublication de l'université d'Angers*, no**40**, juin.

Castro, F., Narváez, L. (1997). Homogenising differential operators, *Prepublicaciones de la Facultad de Matemáticas*, Universidad de Sevilla, **36**.

Lejeune-Jalabert, M. (1985). *Effectivité Des Calculs Polynomiaux, Cours de DEA*, Université de Grenoble I.

Malgrange, B. (1975). *Le Polynôme de Bernstein d'une Singularité Isolée*, volume 459 of Lecture Notes in Math., pp. 98–119. Springer Verlag.

Maynadier, H. (1996). Equations fontionnelles pour une intersection complète quasi-homogène à singularité isolée et un germe semi-quasi-homogène. Thèse, Nice-sophia Antipolis.

Maynadier, H. (1997). Polynômes de Bernstein–Sato associés à une intersection complète quasi-homogène à singularité isolée. *Bull. Soc. Math. France*, **125**, 547–571.

Oaku, T. (1997). An algorithm of computing b-functions. *Duke Math. J.*, **87**, 115–132.

Oaku, T., Takayama, N. (1999). An algorithm for de Rham cohomology groups of the complement of an affine variety via D-module computation. *J. Pure Appl. Algebra*, **139**, 201–233.

Sabbah, C. (1987). Proximité évanescente II. Équations fonctionnelles pour plusieurs fonctions analytiques. *Compositio Math.*, **64**, 213–241.

Takayama, N. (1991). Kan: a system for computation in algebraic analysis, 1991-. See `www.math.kobe-u.ac.jp/KAN/`.

Received 15 October 2000
Accepted 31 March 2001