Optimal transport: introduction, applications and derivation
Lecture 3: two applications of optimal transport

Contents

In this lecture, our goal is to provide two applications of optimal transport for nonlinear analysis. The first one concerns hyperbolic PDEs, the second one concerns calculus of variations.

1 Dobrushin's contraction estimate

Here, we present a result due to Dobrushin in [1]. Our goal is to give existence and uniqueness for the following PDE:

$$\begin{cases} \partial_t \rho + \operatorname{div}\Big(\rho(A + B * \rho)\Big) = 0, \\ \rho|_{t=0} = \rho_0. \end{cases} \tag{1}$$

Before describing the exact context in which we will actually show things let us give two preliminary remarks. First, if $B(0) = 0$ and $(X_i)_{i\in\mathbb{N}}$ is a family of curves on the torus satisfying the ODE

$$\dot{X}_i(t) = A(X_i(t)) + \frac{1}{N}\sum_{j\neq i} B(X_i(t) - X_j(t)),$$

then $\rho(t) := \frac{1}{N}\{\delta_{X_1(t)} + \dots \delta_{X_n(t)}\}$ solves our PDE in the sense of distributions.

Second, if the base space on which we look at this PDE is the set of positions and velocity $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$, if $A(x, v) = (v, F(x))$ and $B(x, v) = (0, G(x))$, using as usually done f the unknown measure, we rewrite the PDE

$$\partial_t f + v \cdot \nabla_x f + \left(F + G * \int f \,\mathrm{d}v\right) \cdot \nabla_v f = 0.$$

This is the Vlasov equation modelling the evolution of a population of particles under the action of an external force field F, and of a pairwise interaction force G.

We will work in the following context: A and B are two Lipschitz-continuous functions from the d-dimensional torus $\mathbb{T}^d$ to $\mathbb{R}^d$, with $d \in \mathbb{N}^*$. We assume that $B(0) = 0$. This is reasonable if we assume it to be odd, which is often the case. The unknown is the time dependent measure $\rho = \rho(t) \in \mathcal{P}(\mathbb{T}^d)$, whose initial condition $\rho_0 \in \mathcal{P}(\mathbb{T}^d)$ is prescribed. Of course, the solution needs to be understood in the sense of distributions. We define solutions as follows.

Definition 1. A solution of the PDE 1 is a continuous curve $t \mapsto \rho(t) \in \mathcal{P}(\mathbb{T}^d)$ such that for all $\varphi \in C^1_c(\mathbb{R}^*_+ \times \mathbb{T}^d)$,

$$\int_0^{+\infty}\!\!\int \nabla\varphi(t) \cdot \Big\{A + B * \rho(t)\Big\} \,\mathrm{d}\rho(t)\,\mathrm{d}t = 0.$$

Note that because of the regularity of B, $B * \rho$ is a bounded function and hence the space-time integral is well defined.

We prove the following theorem.

Theorem 2. *For all $\rho_0 \in \mathcal{P}(\mathbb{T}^d)$, there exists a unique solution of the PDE* (1) *in the sense of Definition 1.*

Proof. The idea is very similar to the proof of the Cauchy-Lipschitz theorem, but in the space of probability measures, and with the Wasserstein 1 distance. More precisely, we set for all $t \in \mathbb{R}_+$, $\rho^0(t) := \rho_0$, and for all $n \in \mathbb{N}$, ρ^{n+1} is the unique solution of the continuity equation

$$\partial_t \rho^{n+1} + \operatorname{div}\Big(\rho^{n+1}(A + B * \rho^n)\Big) = 0$$

as built in the previous chapter: calling $\phi^n : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{T}^d \to \mathbb{T}^d$ the flow associated to the vector field $v^n := A + B * \rho^n$ (which clearly satisfies the conditions of the Cauchy-Lipschitz theorem because A and B

are Lipschitz), for all $t \in \mathbb{R}_+$, we set $\rho^{n+1}(t) := \phi^n(t;0,\cdot)_\# \rho_0$. For all $n \in \mathbb{N}$, $t \mapsto \rho^n(t)$ is continuous for the topology of narrow convergence, satisfies $\rho^n(0) = \rho_0$, and whenever $n \geq 1$ for all $\varphi \in C_c^1(\mathbb{R}_+^* \times \mathbb{T}^d)$,

$$\int_0^{+\infty}\int \nabla\varphi(t) \cdot \Big\{ A + B * \rho^{n-1}(t) \Big\} \, \mathrm{d}\rho^n(t) \, \mathrm{d}t = 0. \tag{2}$$

Now, we show a contraction estimate in short times for the family (ρ^n), that is, we will show that for a sufficiently small $\varepsilon > 0$ to be defined later, and which is independent of ρ_0, for all $n \in \mathbb{N}^*$,

$$\sup_{t\leq\varepsilon} W_1\Big(\rho^{n+1}(t), \rho^n(t)\Big) \leq \frac{1}{2} \sup_{t\leq\varepsilon} W_1\Big(\rho^n(t), \rho^{n-1}(t)\Big). \tag{3}$$

So let $n \in \mathbb{N}^*$ and $t \in \mathbb{R}_+$. We want to estimate

$$W_1\Big(\rho^{n+1}(t), \rho^n(t)\Big) = W_1\Big(\phi^n(t;0,\cdot)_\#\rho_0, \phi^{n-1}(t;0,\cdot)_\#\rho_0\Big).$$

There is an obvious coupling between $\rho^{n+1}(t)$ and $\rho^n(t)$: $(\phi^n(t;0,\cdot), \phi^{n-1}(t;0,\cdot))_\#\rho_0$. Therefore,

$$W_1\Big(\rho^{n+1}(t), \rho^n(t)\Big) \leq \int d(\phi^n(t;0,x), \phi^{n-1}(t;0,x)) \, \mathrm{d}\rho_0(x). \tag{4}$$

But for all $x \in \mathbb{T}^d$,

$$\begin{aligned}
&\phi^n(t;0,x) - \phi^{n-1}(t;0,x) \\
&\quad = \int_0^t \Big\{ (A + B * \rho^n(s))(\phi^n(s;0,x)) - (A + B * \rho^{n-1}(s))(\phi^{n-1}(s;0,x)) \Big\} \, \mathrm{d}s \\
&\quad = \int_0^t \Big\{ (A + B * \rho^n(s))(\phi^n(s;0,x)) - (A + B * \rho^{n-1}(s))(\phi^n(s;0,x)) \Big\} \, \mathrm{d}s \\
&\qquad\qquad + \int_0^t \Big\{ (A + B * \rho^{n-1}(s))(\phi^n(s;0,x)) - (A + B * \rho^{n-1}(s))(\phi^{n-1}(s;0,x)) \Big\} \, \mathrm{d}s \\
&\quad = \int_0^t \Big\{ (A + B * \rho^n(s))(\phi^n(s;0,x)) - (A + B * \rho^n(s))(\phi^{n-1}(s;0,x)) \Big\} \, \mathrm{d}s \\
&\qquad\qquad + \int_0^t \Big\{ B * \rho^n(s)(\phi^{n-1}(s;0,x)) - B * \rho^{n-1}(s)(\phi^{n-1}(s;0,x)) \Big\} \, \mathrm{d}s.
\end{aligned}$$

So by the triangle inequality

$$\begin{aligned}
d(\phi^n(t;0,x), \phi^{n-1}(t;0,x)) \leq \Big(\mathrm{Lip}(A) + \mathrm{Lip}(B)\Big) &\int_0^t d(\phi^n(s;0,x), \phi^{n-1}(s;0,x)) \, \mathrm{d}s \\
&+ \int_0^t \|B * \rho^n(s) - B * \rho^{n-1}(s)\|_\infty \, \mathrm{d}s.
\end{aligned}$$

Let us call $L := \mathrm{Lip}(A) + \mathrm{Lip}(B)$. By the Gronwall lemma, we deduce that

$$d(\phi^n(t;0,x), \phi^{n-1}(t;0,x)) \leq \int_0^t \exp\big(L(t-s)\big) \|B * \rho^n(s) - B * \rho^{n-1}(s)\|_\infty \, \mathrm{d}s. \tag{5}$$

Now, we need to estimate for a given $s \in [0,t]$ the quantity $\|B * \rho^n(s) - B * \rho^{n-1}(s)\|_\infty$. For a given $y \in \mathbb{T}^d$

and a given $\gamma \in \Pi(\rho^n(s), \rho^{n-1}(s))$, we have

$$\begin{aligned}
\big|B * \rho^n(s)(y) - B * \rho^{n-1}(s)(y)\big| &= \left|\int B(y-z)\,\mathrm{d}\rho^n(s,z) - \int B(y-z)\,\mathrm{d}\rho^{n-1}(s,z)\right| \\
&= \left|\int \Big\{B(y-z) - B(y-z')\Big\}\,\mathrm{d}\gamma(z,z')\right| \\
&\leq \int \Big|B(y-z) - B(y-z')\Big|\,\mathrm{d}\gamma(z,z') \\
&\leq \mathrm{Lip}(B)\int d(z,z')\,\mathrm{d}\gamma(z,z').
\end{aligned}$$

Taking the sup in y on the l.h.s. and inf in $\gamma \in \Pi(\rho^n(s), \rho^{n-1}(s))$ on the r.h.s. we get

$$\|B * \rho^n(s) - B * \rho^{n-1}(s)\|_\infty \leq \mathrm{Lip}(B) W_1(\rho^n(s), \rho^{n-1}(s)). \tag{6}$$

Plugging this estimate into (5), we get

$$d(\phi^n(t;0,x), \phi^{n-1}(t;0,x)) \leq \mathrm{Lip}(B)\int_0^t \exp\big(L(t-s)\big) W_1(\rho^n(s), \rho^{n-1}(s))\,\mathrm{d}s.$$

It remains to integrate w.r.t. ρ_0, to deduce with the help of (4) our main estimate

$$W_1(\rho^{n+1}(t), \rho^n(t)) \leq \mathrm{Lip}(B)\int_0^t \exp\big(L(t-s)\big) W_1(\rho^n(s), \rho^{n-1}(s))\,\mathrm{d}s.$$

Therefore, for a given $\varepsilon > 0$, we have

$$\sup_{t\leq\varepsilon} W_1(\rho^{n+1}(t), \rho^n(t)) \leq \frac{\mathrm{Lip}(B)}{L}\Big(\exp(L\varepsilon) - 1\Big)\sup_{t\leq\varepsilon} W_1(\rho^n(t), \rho^{n-1}(t)).$$

So if ε is so small that $\frac{\mathrm{Lip(B)}}{L}(\exp(L\varepsilon)-1) \leq \frac{1}{2}$, the contraction estimate (3) follows.

Hence, the family of continuous maps $t \in [0,\varepsilon] \mapsto \rho^n(t) \in (\mathcal{P}(\mathbb{T}^d), W_1)$, form a Cauchy sequence in the complete space $C^0([0,\varepsilon]; (\mathcal{P}(\mathbb{T}^d), W_1))$, and hence admit a limit ρ in this space. Of course, $\rho(0) = \rho_0$, and we easily pass to the limit in (2) (note that because of (6), for all $t \in [0,\varepsilon]$, $(B * \rho^n(t))$ converges uniformly towards $B * \rho(t)$, which is enough to pass to the limit). We have therefore built a solution on the set of times $[0,\varepsilon]$. To build a solution on the whole $\mathbb{R}_+$, we just have to iterate this construction.

To show uniqueness, we just need to remark that with the same computations, we can show that if ρ, ρ' are two solutions, for all $t \in [0,\varepsilon]$,

$$\sup_{t\leq\varepsilon} W_1(\rho(t), \rho'(t)) \leq \frac{1}{2}\sup_{t\leq\varepsilon} W_1(\rho(t), \rho'(t)),$$

so that ρ and ρ' have to coincide on the set of times $[0,\varepsilon]$. We can therefore iterate the argument. □

2 McCann's geodesic convexity

Our second application of optimal transport concerns the uniqueness of the minimizers of the energy functional

$$\mathcal{E} : \rho \in \mathcal{P}(\mathbb{R}^d) \mapsto \begin{cases} \displaystyle\int \Big\{V + W * \rho + \log\frac{\mathrm{d}\rho}{\mathrm{d}x}\Big\}\,\mathrm{d}\rho, & \text{if } \rho \ll \mathrm{d}x, \\ +\infty, & \text{otherwise,} \end{cases} \tag{7}$$

under various convexity assumptions on the external potential $V : \mathbb{R}^d \to \mathbb{R}_+ \cup \{+\infty\}$ and pairwise interaction potential $W : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$. This result is originally due to McCann [2]. A priori, it is not clear how to define properly the last term in the definition of $\mathcal{E}$. Namely, how to deal with entropy on the whole $\mathbb{R}^d$ is a bit technical, and therefore we will not prove everything. We admit the following result.

Proposition 3. *The following formula*

$$H : \rho \in \mathcal{P}_2(\mathbb{R}^d) \mapsto \begin{cases} \displaystyle\int \log \frac{\mathrm{d}\rho}{\mathrm{d}x}\,\mathrm{d}\rho = \int \frac{\mathrm{d}\rho}{\mathrm{d}x} \log \frac{\mathrm{d}\rho}{\mathrm{d}x}\,\mathrm{d}x, & \text{if } \rho \ll \mathrm{d}x, \\ +\infty, & \text{otherwise.} \end{cases} \tag{8}$$

provides a good definition in $(-\infty, +\infty]$*. In addition, for all* $M > 0$*,* H *is lower semicontinuous on the sets* $\{\rho \in \mathcal{P}(\mathbb{R}^d) \text{ with } \int |x|^2 \,\mathrm{d}\rho(x) \leq M\}$*.*

Moreover, calling (τ_s) *the heat kernel on* $\mathbb{R}^d$*, for all* $\rho \in \mathcal{P}_2(\mathbb{R}^d)$ *the map* $s \in \mathbb{R}_+ \mapsto H(\rho * \tau_s) \in (-\infty, +\infty]$ *is continuous and nonincreasing, and has finite values whenever* $s > 0$*.*

Proof. Several points of the proof will rely on the following convex inequality holding for all $z \geq 0$ and $u \in \mathbb{R}$:

$$uz \leq z \log z + \exp(u - 1), \tag{9}$$

with equality if and only if $u = 1 + \log z$.

Let us consider $M > 0$ and let us call $\mathcal{P}_M := \{\rho \in \mathcal{P}(\mathbb{R}^d) \text{ with } \int |x|^2 \,\mathrm{d}\rho(x) \leq M\}$. Let us show that H is well defined in $(-\infty, +\infty]$ and l.s.c. on $\mathcal{P}_M$. For a given $\rho \in \mathcal{P}_M$, we decompose $\rho = \rho_1 + \rho_2$ with $\rho_1 \ll \mathrm{d}x$ and $\rho_2 \perp \mathrm{d}x$, and we call $A := {}^c\,\mathrm{Supp}(\rho_2)$. Finally, we call

$$f := -\frac{\mathrm{d}\rho_1}{\mathrm{d}x} \log \frac{\mathrm{d}\rho_1}{\mathrm{d}x} \times \mathbb{1}_A \times \mathbb{1}_{\frac{\mathrm{d}\rho_1}{\mathrm{d}x} \leq 1}.$$

Our first task is to show that f, which is nonnegative, is integrable, and that its integral outside of a ball decreases to zero as the radius of the ball goes to zero, uniformly in $\rho \in \mathcal{P}_M$. For almost all $x \in A$ such that $\frac{\mathrm{d}\rho_1}{\mathrm{d}x} \leq 1$, using (9) with $u = -|x|$ and $z = \frac{\mathrm{d}\rho_1}{\mathrm{d}x}(x)$, we find

$$f \leq |x| \frac{\mathrm{d}\rho_1}{\mathrm{d}x}(x) + \exp(-|x| - 1).$$

It is easy to see that this inequality holds almost everywhere on $\mathbb{R}^d$. Integrating it on ${}^cB(0, r)$, for some $r \geq 0$, we deduce

$$\int f(x)\,\mathrm{d}x \leq \int_{{}^cB(0,r)} |x| \,\mathrm{d}\rho(x) + \int_{{}^cB(0,r)} \exp(-|x| - 1)\,\mathrm{d}x,$$

which is finite for all $r \geq 0$. For $r = 0$, this is enough to conclude that when $\rho \ll \mathrm{d}x$, the negative part of $\frac{\mathrm{d}\rho}{\mathrm{d}x} \log \frac{\mathrm{d}\rho}{\mathrm{d}x}$ is integrable, and therefore that H is well defined with values in $(-\infty, +\infty]$. Then, if $r > 0$, we can go further and find

$$\int f(x)\,\mathrm{d}x \leq \frac{1}{r} \int_{{}^cB(0,r)} |x|^2 \,\mathrm{d}\rho(x) + \int_{{}^cB(0,r)} \exp(-|x| - 1)\,\mathrm{d}x \leq \frac{M}{r} + \int_{{}^cB(0,r)} \exp(-|x| - 1)\,\mathrm{d}x$$

which goes to zero as $r \to +\infty$, uniformly in $\rho \in \mathcal{P}_M$.

From now on, we fix $\varepsilon > 0$, and we consider r_0 large enough so that for all $\rho \in \mathcal{P}_M$, $\int_{{}^cB(0,r)} f(x)\,\mathrm{d}x \leq \varepsilon$.

Next, we show that for all $\rho \in \mathcal{P}_M$, H is l.s.c. at ρ, and so we consider $(\rho_n) \in \mathcal{P}_M^{\mathbb{N}}$ converging narrowly towards ρ. First, if $H(\rho) < +\infty$, identifying ρ and its density w.r.t. Lebesgue, then we can find $r \geq r_0$ such that

$$H(\rho) \leq \int_{B(0,r)} \rho(x) \log \rho(x)\,\mathrm{d}x + \varepsilon.$$

But using a density argument relying on (9), we find that there exists $\varphi \in C_c(B(0, r))$ such that

$$\int_{B(0,r)} \rho(x) \log \rho(x)\,\mathrm{d}x \leq \int \varphi(x)\,\mathrm{d}\rho(x) - \int_{B(0,r)} \exp(\varphi(x) - 1)\,\mathrm{d}x + \varepsilon.$$

Hence,

$$H(\rho) \leq \int \varphi(x)\,\mathrm{d}\rho(x) - \int_{B(0,r)} \exp(\varphi(x) - 1)\,\mathrm{d}x + 2\varepsilon.$$

On the other hand, for all $n \geq 0$,

$$H(\rho_n) \geq \int \varphi(x)\,\mathrm{d}\rho_n(x) - \int_{B(0,r)} \exp(\varphi(x)-1)\,\mathrm{d}x - \varepsilon.$$

Thus, as $\int \varphi\,\mathrm{d}\rho_n$ pass to the limit, we find

$$\liminf_{n\to+\infty} H(\rho_n) \geq H(\rho) - 3\varepsilon,$$

and we get the result sending $\varepsilon \to 0$.

Finally, if $H(\rho) = +\infty$, the proof follows the same steps, but replacing

$$H(\rho) \leq \int \varphi(x)\,\mathrm{d}\rho(x) - \int_{B(0,r)} \exp(\varphi(x)-1)\,\mathrm{d}x + 2\varepsilon$$

by

$$\int \varphi(x)\,\mathrm{d}\rho(x) - \int_{B(0,r)} \exp(\varphi(x)-1)\,\mathrm{d}x \geq \frac{1}{\varepsilon},$$

and I leave the details for the reader.

Let $s > 0$ and $\rho \in \mathcal{P}_M$. Then $\rho * \tau_s \in \mathcal{P}_{2M+2s}$. Indeed,

$$\int |x|^2\,\mathrm{d}\tau_s * \rho(x) = \iint |y-x|^2 \tau_s(y)\,\mathrm{d}y\,\mathrm{d}\rho(x) \leq 2\left(\int |y|^2\tau_s(y)\,\mathrm{d}y + \int |x|^2\,\mathrm{d}\rho(x)\right) < +\infty.$$

Therefore, $H(\rho*\tau_s)$ is well defined in $(-\infty,+\infty]$. Moreover,

$$H(\rho*\tau_s) = \int \rho*\tau_s(x)\log\rho*\tau_s(x)\,\mathrm{d}x \leq \int (\rho\log\rho)*\tau_s(x)\,\mathrm{d}x = H(\rho),$$

where we used Jensen's inequality. By the flow property of the heat flow (for all $t,s \geq 0$, $\rho*\tau_{s+t} = (\rho*\tau_t)*\tau_s$), we deduce that $H(\rho*\tau_s)$ is nondecreasing in s.

Let us show that whenever $s > 0$, $H(\rho*\tau_s) < +\infty$. Once again by Jensen, we have

$$H(\rho*\tau_s) = \int \rho*\tau_s(x)\log\rho*\tau_s(x)\,\mathrm{d}x \leq \int (\tau_s\log\tau_s)*\rho(x)\,\mathrm{d}x = \int \tau_s(x)\log\tau_s(x)\,\mathrm{d}x.$$

But as for all $x \in \mathbb{R}^d$,

$$\tau_s(x) = \frac{1}{\sqrt{2\pi s}^d}\exp\left(-\frac{|x|^2}{2s}\right),$$

we have

$$\tau_s(x)\log\tau_s(x) = \left(-\frac{d}{2}\log(2\pi s) - \frac{|x|^2}{2s}\right)\times\frac{1}{\sqrt{2\pi s}^d}\exp\left(-\frac{|x|^2}{2s}\right),$$

which is clearly integrable. □

We will also need the so-called Li-Yau inequality, that we state here in a weak form.

Proposition 4. *For all $s > 0$, the quantity defined for $\mu \in \mathcal{P}(\mathbb{R}^d)$ by*

$$\frac{1}{2}\int |\nabla\log(\mu*\tau_s)|^2\,\mathrm{d}\mu*\tau_s \leq \frac{1}{2s}.$$

Proof. In a similar way as in the last lecture, we can prove that for a given $\mu \in \mathcal{P}(\mathbb{R}^d)$ and $s > 0$, noticing that $\mu * \tau_s \nabla \log \mu * \tau_s = \nabla(\mu * \tau_s)$, we have

$$\begin{aligned}
\frac{1}{2}\int |\nabla \log(\mu * \tau_s)|^2 \,\mathrm{d}\mu * \tau_s &= \sup_{\xi \in C_b(\mathbb{R}^d;\mathbb{R}^d)} \int \xi \cdot \nabla(\mu * \tau_s) - \frac{1}{2}\int |\xi|^2 \mu * \tau_s \\
&= \sup_{\xi \in C_b(\mathbb{R}^d;\mathbb{R}^d)} \int (\xi * \mu) \cdot \nabla \tau_s - \frac{1}{2}\int (|\xi|^2 * \mu)\tau_s \\
&\leq \sup_{\xi \in C_b(\mathbb{R}^d;\mathbb{R}^d)} \int (\xi * \mu) \cdot \nabla \tau_s - \frac{1}{2}\int |\xi * \mu|^2 \tau_s \\
&\leq \sup_{\xi \in C_b(\mathbb{R}^d;\mathbb{R}^d)} \int \xi \cdot (\nabla \log \tau_s)\tau_s - \frac{1}{2}\int |\xi|^2 \tau_s = \frac{1}{2}\int |\nabla \log \tau_s|^2 \tau_s.
\end{aligned}$$

The result is therefore a direct computation. □

We do not treat the question of existence, what we are interested in is the question of uniqueness. The result is the following.

Theorem 5. *Let us assume that V and W are convex and that there exists $c > 0$ such that*

$$V(x) \geq c|x|^2 \tag{10}$$

for $|x|$ sufficiently large.

1. *If V is strictly convex, then $\mathcal{E}$ admits at most one minimizer.*
2. *If W is strictly convex, then $\mathcal{E}$ admits at most one minimizer up to translation.*

The proof of the theorem is a consequence of the following lemmas.

Lemma 6. *Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ and $t \mapsto \rho(t)$ a geodesic between μ and ν in the sense of Benamou-Brenier. If $V : \mathbb{R}^d \to \mathbb{R}_+ \cup \{+\infty\}$ is convex, then*

$$t \in [0,1] \mapsto \int V \,\mathrm{d}\rho(t)$$

is convex. Moreover, if V is strictly convex and $\mu \neq \nu$, then the map above is strictly convex.

Lemma 7. *Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ and $t \mapsto \rho(t)$ a geodesic between μ and ν in the sense of Benamou-Brenier. If $W : \mathbb{R}^d \to \mathbb{R}_+ \cup \{+\infty\}$ is convex, then*

$$t \in [0,1] \mapsto \int W * \rho(t) \,\mathrm{d}\rho(t)$$

is convex. Moreover, if W is strictly convex and ν is not a translation of μ, then the map above is strictly convex.

Lemma 8. *Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ and $t \mapsto \rho(t)$ a geodesic between μ and ν in the sense of Benamou-Brenier. The map*

$$t \in [0,1] \mapsto H(\rho(t))$$

is convex.

With these lemma, Theorem 5 is very easy to prove.

Proof of Theorem 5. Let us assume that $\mathcal{E}$ admits two minimizers $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$. Because of (10), $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$. Let $t \mapsto \rho(t)$ be a geodesic between μ and ν in the sense of Benamou-Brenier. Note that $\mathcal{E}(\rho(0)) = \mathcal{E}(\rho(1))$, and for all $t \in [0,1]$, $\mathcal{E}(\rho(t)) \geq \mathcal{E}(\rho(0))$ by optimality. So the map $t \mapsto \mathcal{E}(\rho(t))$ cannot be strictly convex.
1. If V is strictly convex, then we have $\mu = \nu$, otherwise $t \mapsto \mathcal{E}(\rho(t))$ is strictly convex and we get a contradiction.
2. If W is strictly convex, then ν is a translation of μ, otherwise $t \mapsto \mathcal{E}(\rho(t))$ is strictly convex and we get a contradiction.

So the result is proved. □

So it remains to prove our three lemmas.

Proof of Lemma 6. By the results of the previous lecture, we know that to the geodesic ρ is associated a generalized flow $P \in \mathcal{P}(C^0([0,1];\mathbb{R}^d))$ that is optimal for the dynamical optimal transport problem. The optimality of P enforces it to only charge straight line trajectories. Therefore, for P-almost ω, $t \mapsto V(\omega(t))$ is convex. In particular, as a sum of convex functions is convex,

$$t \mapsto \int V \,\mathrm{d}\rho(t) = \int V(\omega(t)) \,\mathrm{d}P(\omega)$$

is convex.

In the case when V is strictly convex and $\mu \neq \nu$, then necessarily the set of non constant ω is not P-negligible: there is a subset A of $C^0([0,1];\mathbb{R}^d)$ with $P(A) > 0$, and for P-almost all $\omega \in A$, ω is not a constant. But for such ω, $t \mapsto V(\omega(t))$ is strictly convex. So for all $s,t \in [0,1]$ and $\lambda \in (0,1)$, for all $\omega \in \mathrm{Supp}(P)$

$$V(\omega((1-\lambda)s + \lambda t)) \leq (1-\lambda)V(\omega(s)) + \lambda V(\omega(t)),$$

with a strict inequality whenever $\omega \in A$. Integrating w.r.t. P and using $P(A) > 0$ gives the result. □

Proof of Lemma 7. The proof if very similar to the previous one, but this time, for all $t \in [0,1]$

$$\int W * \rho(t) \,\mathrm{d}\rho(t) = \int W(\omega(t) - \omega'(t)) \,\mathrm{d}P(\omega) \,\mathrm{d}P(\omega').$$

When W is convex as for $P \otimes P$-almost all (ω, ω'), $t \mapsto \omega(t) - \omega'(t)$ is affine, $t \mapsto W(\omega(t) - \omega'(t))$ is convex, and the map under interest is convex.

If W is strictly convex, this map is strictly convex unless for $P \otimes P$-almost all (ω, ω'), $t \mapsto \omega(t) - \omega'(t)$ is constant. Assume that this is true. Recalling that P only charge straight line trajectories, then for P-almost all ω', for P-almost all ω, the (constant) speed of ω coincides with the (constant) speed of ω', that is,

$$\dot{\omega} = \dot{\omega}'. \tag{11}$$

Calling v the speed of an ω' for which (11) holds for P-almost all ω, we find that P-almost all ω has speed v. In particular, for P-almost all ω, $\omega(1) = \omega(0) + v$, and therefore $\nu = (\mathrm{Id} + v)_{\#}\mu$. Hence, the result. □

Proof of Lemma 8. This is the most difficult part of the proof of the theorem, and this is the only term of $\mathcal{E}$ that is convex because of the structure of optimal transport solutions, and not only because at the optimizer, particles travel along straight lines. What we will actually prove is that whenever μ and ν have finite entropy and $\alpha \in (0,1)$, then

$$H(\rho(\alpha)) \leq (1-\alpha)H(\mu) + \alpha H(\nu).$$

The actual result is then just a matter of reparametrization. So let us assume $H(\mu)$ and $H(\nu)$ to be finite, and $\alpha \in (0,1)$. Notice that because of Lemma 6, $\rho(\alpha) \in \mathcal{P}_2(\mathbb{R}^d)$ and $H(\rho(\alpha))$ is well defined.

Let v be the vector field associated with ρ, that is, such that (ρ, v) is a solution of the continuity equation, and $W_2^2(\mu,\nu) = \iint |v|^2 \,\mathrm{d}\rho(t)\,\mathrm{d}t$. The idea is to use the optimality of (ρ, v) in order to compare its action with a well chosen alternative competitor. We build a two parameter family of approximated solutions of the continuity equation as follows: for given $\varepsilon > 0$ and $\delta \geq 0$ we build $\rho^{\varepsilon,\delta}$ as:

$$\rho^{\varepsilon,\delta}(t) = \begin{cases} \rho(t) * \tau_{\delta+\varepsilon\frac{t}{\alpha}}, & \text{if } t \in [0,\alpha], \\ \rho(t) * \tau_{\delta+\varepsilon\frac{1-t}{1-\alpha}}, & \text{if } t \in [\alpha,1], \end{cases}$$

where (τ_s) is the heat kernel. It is easy to see that $\rho^{\varepsilon,\delta}$ is narrowly continuous (remember that ρ is narrowly continuous). Moreover, a quick computation shows that in the sense of distributions,

$$\partial_t \rho^{\varepsilon,\delta} + \operatorname{div}(\rho^{\varepsilon,\delta} c^{\varepsilon,\delta}) = 0,$$

where

$$c^{\varepsilon,\delta}(t) := \begin{cases} \dfrac{(\rho(t)v(t)) * \tau_{\delta+\varepsilon\frac{t}{\alpha}}}{\rho^{\varepsilon,\delta}(t)} - \dfrac{\varepsilon}{\alpha}\nabla \log \rho^{\varepsilon,\delta}(t), & \text{if } t \in [0,\alpha], \\ \dfrac{(\rho(t)v(t)) * \tau_{\delta+\varepsilon\frac{1-t}{1-\alpha}}}{\rho^{\varepsilon,\delta}(t)} + \dfrac{\varepsilon}{1-\alpha}\nabla \log \rho^{\varepsilon,\delta}(t), & \text{if } t \in [\alpha,1]. \end{cases}$$

Hence, we have built a two parameter family of solutions of the continuity equation $(\rho^{\varepsilon,\delta}, c^{\varepsilon,\delta})$. Moreover, when $\delta = 0$, $\rho^{\varepsilon,0}$ has endpoints μ and ν, so by optimality of (ρ, v), we know that for all $\varepsilon > 0$

$$\frac{1}{2}\int_0^1\!\!\int |v|^2 \,\mathrm{d}\rho(t)\,\mathrm{d}t \leq \frac{1}{2}\int_0^1\!\!\int |c^{\varepsilon,0}|^2 \,\mathrm{d}\rho^{\varepsilon,0}(t)\,\mathrm{d}t. \tag{12}$$

To lighten the computations, let us call $v^{\varepsilon,\delta}$ the term involving the convolution of ρv in the definition of $c^{\varepsilon,\delta}$, and $w^{\varepsilon,\delta}$ the other term so that $c^{\varepsilon,\delta} = v^{\varepsilon,\delta} + w^{\varepsilon,\delta}$. Our goal is now to estimate for given ε, δ the quantity $\iint |c^{\varepsilon,\delta}|^2 \,\mathrm{d}\rho^{\varepsilon,\delta}\,\mathrm{d}t$. Notice that whenever $\delta > 0$, everything is smooth w.r.t. space with uniform bounds in time, and all the computations are easily justified. So let us make the computations for both $\varepsilon > 0$ and $\delta > 0$. We have

$$\begin{aligned} \frac{1}{2}\int_0^1\!\!\int |c^{\varepsilon,\delta}|^2 \,\mathrm{d}\rho^{\varepsilon,\delta}(t)\,\mathrm{d}t &= \frac{1}{2}\int_0^1\!\!\int |v^{\varepsilon,\delta}|^2 \,\mathrm{d}\rho^{\varepsilon,\delta}(t)\,\mathrm{d}t + \int_0^1\!\!\int c^{\varepsilon,\delta}\cdot w^{\varepsilon,\delta} \,\mathrm{d}\rho^{\varepsilon,\delta}(t)\,\mathrm{d}t - \frac{1}{2}\int_0^1\!\!\int |w^{\varepsilon,\delta}|^2 \,\mathrm{d}\rho^{\varepsilon,\delta}(t)\,\mathrm{d}t \\ &\leq \frac{1}{2}\int_0^1\!\!\int |v^{\varepsilon,\delta}|^2 \,\mathrm{d}\rho^{\varepsilon,\delta}(t)\,\mathrm{d}t + \int_0^1\!\!\int c^{\varepsilon,\delta}\cdot w^{\varepsilon,\delta} \,\mathrm{d}\rho^{\varepsilon,\delta}(t)\,\mathrm{d}t. \end{aligned}$$

On the other hand, the map $t \mapsto H(\rho^{\varepsilon,\delta})$ is of regularity $W^{1,1}$ and distributionaly

$$\frac{\mathrm{d}}{\mathrm{d}t}H(\rho^{\varepsilon,\delta}(t)) = \int c^{\varepsilon,\delta}(t)\cdot\nabla\log\rho^{\varepsilon,\delta}(t)\,\mathrm{d}\rho^{\varepsilon,\delta}(t).$$

Therefore,

$$\begin{aligned} \int_0^1\!\!\int c^{\varepsilon,\delta}\cdot w^{\varepsilon,\delta} \,\mathrm{d}\rho^{\varepsilon,\delta}(t)\,\mathrm{d}t &= -\frac{\varepsilon}{\alpha}\int_0^\alpha \frac{\mathrm{d}}{\mathrm{d}t}H(\rho^{\varepsilon,\delta}(t))\,\mathrm{d}t + \frac{\varepsilon}{1-\alpha}\int_\alpha^1 \frac{\mathrm{d}}{\mathrm{d}t}H(\rho^{\varepsilon,\delta}(t))\,\mathrm{d}t \\ &= \frac{\varepsilon}{\alpha(1-\alpha)}\Big((1-\alpha)H(\rho^{\varepsilon,\delta}(0)) + \alpha H(\rho^{\varepsilon,\delta}(1)) - H(\rho^{\varepsilon,\delta}(\alpha))\Big). \end{aligned}$$

Finally, the same argument as in the previous lecture provides

$$\frac{1}{2}\int_0^1\!\!\int |v^{\varepsilon,\delta}|^2 \,\mathrm{d}\rho^{\varepsilon,\delta}(t)\,\mathrm{d}t \leq \frac{1}{2}\int_0^1\!\!\int |v|^2 \,\mathrm{d}\rho(t)\,\mathrm{d}t$$

All in all, we have the estimate

$$\frac{1}{2}\int_0^1\!\!\int |c^{\varepsilon,\delta}|^2 \,\mathrm{d}\rho^{\varepsilon,\delta}(t)\,\mathrm{d}t \leq \frac{1}{2}\int_0^1\!\!\int |v|^2 \,\mathrm{d}\rho(t)\,\mathrm{d}t + \frac{\varepsilon}{\alpha(1-\alpha)}\Big((1-\alpha)H(\rho^{\varepsilon,\delta}(0)) + \alpha H(\rho^{\varepsilon,\delta}(1)) - H(\rho^{\varepsilon,\delta}(\alpha))\Big).$$

Notice that because of Proposition 3, the last term is continuous in ε, δ, even at $\varepsilon = 0$ and/or $\delta = 0$.

Now, we just need to justify that

$$\liminf_{\delta \to 0} \frac{1}{2} \int_0^1 \int |c^{\varepsilon,\delta}|^2 \, \mathrm{d}\rho^{\varepsilon,\delta}(t) \, \mathrm{d}t \geq \frac{1}{2} \int_0^1 \int |c^{\varepsilon,0}|^2 \, \mathrm{d}\rho^{\varepsilon,0}(t) \, \mathrm{d}t. \tag{13}$$

Indeed, if it is true, because of (12) and our last estimate, we find

$$(1-\alpha) H(\rho^{\varepsilon,0}(0)) + \alpha H(\rho^{\varepsilon,0}(1)) - H(\rho^{\varepsilon,0}(\alpha)) \geq 0.$$

Taking the limit $\varepsilon \to 0$, we therefore get the expected result. So let us justify (13). By the same proof as in the previous lecture, for all $(\bar{\rho}, \bar{c})$,

$$\frac{1}{2} \int_0^1 \int |\bar{c}|^2 \, \mathrm{d}\bar{\rho}(t) \, \mathrm{d}t = \sup_{\xi \in C_b([0,1] \times \mathbb{R}^d; \mathbb{R}^d)} \int_0^1 \int \xi \cdot \mathrm{d}\bar{\rho}\bar{c} - \frac{1}{2} \int_0^1 \int |\xi|^2 \, \mathrm{d}\bar{\rho}(t) \, \mathrm{d}t.$$

This functional is therefore l.s.c. with respect to narrow convergence of $(\bar{\rho}, \overline{\rho c})$. As clearly, $(\rho^{\varepsilon,\delta}, \rho^{\varepsilon,\delta} c^{\varepsilon,\delta})$ converges narrowly towards $(\rho^{\varepsilon,0}, \rho^{\varepsilon,0} c^{\varepsilon,0})$ as $\delta \to 0$, we get exactly what we needed, and the result follows. □

References

[1] R.L. Dobrushin. Vlasov equations. *Functional Analysis and Its Applications*, 13(2):115–123, 1979.

[2] R.J. McCann. A convexity principle for interacting gases. *Advances in mathematics*, 128(1):153–179, 1997.