DISCOURAGING RESULTS FOR ULTRAIMAGINARY INDEPENDENCE THEORY

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Abstract. Dividing independence for ultraimaginaries is neither symmetric nor transitive.
Moreover, any notion of independence satisfying certain axioms (weaker than those for independence in a simple theory) and defined for all ultraimaginary sorts, is necessarily trivial.

Introduction

Assume that we work in a first order simple theory (see [Wag00] for a general exposition). Then dividing, or rather non-dividing, defines a ternary independence relation $\mathop{\smile\!\!\!\!\!\!|}$ on possibly infinite tuples, satisfying:

Invariance: $a \mathop{\smile\!\!\!\!\!\!|}_c b$ depends solely on $\operatorname{tp}(a,b,c)$.
Symmetry: $a \mathop{\smile\!\!\!\!\!\!|}_c b \Longleftrightarrow b \mathop{\smile\!\!\!\!\!\!|}_c a$.
Transitivity: $a \mathop{\smile\!\!\!\!\!\!|}_c bd \Longleftrightarrow a \mathop{\smile\!\!\!\!\!\!|}_c b \wedge a \mathop{\smile\!\!\!\!\!\!|}_{bc} d$.
Monotonicity: If $a \mathop{\smile\!\!\!\!\!\!|}_c b$ and $b' \in \operatorname{dcl}(b)$ then $a \mathop{\smile\!\!\!\!\!\!|}_c b'$.
Finite character: $a \mathop{\smile\!\!\!\!\!\!|}_c b$ if and only if $a' \mathop{\smile\!\!\!\!\!\!|}_c b'$ for every finite sub-tuples $a' \subseteq a$, $b' \subseteq b$.
Extension: For every a, b, c there is $a' \equiv_c a$ such that $a' \mathop{\smile\!\!\!\!\!\!|}_c b$.

In fact, $\mathop{\smile\!\!\!\!\!\!|}$ satisfies two additional properties, namely the local character and the independence theorem, with which we do not deal here.

In Kim's original paper [Kim98], these properties were proved for tuples of real (or imaginary) elements. In [HKP00], a new kind of elements was introduced: a *hyperimaginary element* a_E is an equivalence class of a possibly infinite tuple a modulo a type-definable equivalence relation E.

One naïve approach to the extension to independence theory to such equivalence classes would be to define that $a_E \mathop{\smile\!\!\!\!\!\!|}_c b_F$ holds if and only if there are representatives a' and b', respectively, such that $a' \mathop{\smile\!\!\!\!\!\!|}_c b'$: here a_E and b_E are hyperimaginaries, but c has to be a real tuple. In our view, one of the conceptually fundamental breakthroughs of [HKP00] was to extend simplicity theory, that is independence theory, defining dividing independence *over* hyperimaginaries, and proving that it satisfies the same axioms as for reals. In order to do this one seems to have to develop some logic (not entirely first order) for hyperimaginaries, and then re-develop some of simplicity theory in this context.

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Following common terminology, if E is just an invariant equivalence relation, then an equivalence class a_E is an *ultraimaginary.*

The naïve approach mentioned above allows us to consider independence of ultraimaginaries over a real (or hyperimaginary) tuple, and would seem to give rather satisfactory results (see [BTW]). Our goal in this paper is to show that in a sense, this is the best that can be done: it is impossible to define a notion of independence over ultraimaginaries satisfying the axioms mentioned above, and when considering dividing independence the situation is even worse.

More precisely, we prove:

- Dividing independence for ultraimaginaries is neither symmetric nor transitive.
- Any relation of independence defined on ultraimaginaries, satisfying a subset of the axioms required for a simple independence relation, is necessarily trivial.

1. Dividing independence for ultraimaginaries

We recall that two ultraimaginaries a_E and a'_E have the same type if they correspond by an automorphism of the universal domain, or equivalently if they have representatives $b\ E\ a$ and $b'\ E\ a'$ which have the same type as real tuples.

One problem in defining dividing for ultraimaginaries is the definition of indiscernible sequences, since the type of an infinite tuple of ultraimaginaries is not necessarily determined by the types of its finite sub-tuples. There are two *a priori* non-equivalent definitions that come to mind:

Definition 1.1. Let $(a_{iE} : i < \alpha)$ be an infinite sequence of ultraimaginaries, and b_F another ultraimaginary. Then:

(i) (a_{iE}) is *strongly* b_F-indiscernible if there are $a'_i\ E\ a_i$ and $b'\ F\ b$ such that (a'_i) is b'-indiscernible.

(ii) (a_{iE}) is *weakly* b_F-indiscernible if for every κ there is a sequence $(a'_{jE} : j < \kappa)$ similar over b_F to (a_{iE}), meaning that:
For every $n < \omega$ and $i_0 < \cdots < i_{n-1} < \alpha$, $j_0 < \cdots < j_{n-1} < \kappa$, $a_{i_0E}, \ldots, a_{i_{n-1}E}, b_F \equiv a'_{j_0E}, \ldots, a'_{j_{n-1}E}, b_F$.

Fact 1.2. (i) *Strong indiscernibility implies weak indiscernibility.*

(ii) *A sequence is weakly indiscernible if and only if it is similar in the sense of Definition 1.1 to a strongly indiscernible sequence.*

Proof. (i) Since an indiscernible sequence of real (or hyperimaginary) elements can be extended to a similar sequence of arbitrary length.

(ii) Right to left is by the previous item. For the converse, assume that $(a_{iE} : i < \alpha)$ is weakly indiscernible over b_F, so for κ arbitrarily big it is similar over b_F to some sequence $(a'_{jE} : j < \kappa)$. Taking κ big enough, there is a third sequence $(a''_i : i < \omega)$ which is b-indiscernible and such that for every $n < \omega$ there are $i_0 < \cdots < i_{n-1} < \kappa$ such that $a''_0, \ldots, a''_{n-1} \equiv_b a'_{i_0}, \ldots, a'_{i_{n-1}}$. Then $(a''_{iE} : i < \omega)$ is strongly indiscernible and similar over b_F to $(a_{iE} : i < \alpha)$.

QED

Definition 1.3. Let $p(x, a_E) = \mathrm{tp}(c_G/a_E)$. Then $p(x, a_E)$ *strongly (weakly) divides* over b_F if there is a strongly (weakly) b_F-indiscernible sequence (a_{iE}) in $\mathrm{tp}(a_E/b_F)$ such that $\bigwedge p(x, a_{iE})$ is inconsistent.

(The definition of strong and weak dividing is given solely for the purposes of this paper, and is *a priori* unrelated to any other definition of strong or weak dividing that may appear in the literature.)

Notation 1.4. We write $a_E \mathop{\smile\!\!\!\!\!\!\mid}^{s}_{c_G} b_F$ ($a_E \mathop{\smile\!\!\!\!\!\!\mid}^{w}_{c_G} b_F$) to say that $\mathrm{tp}(a_E/b_F c_G)$ does not strongly (weakly) divide over c_G (so $\mathop{\smile\!\!\!\!\!\!\mid}^{w}$ is actually stronger than $\mathop{\smile\!\!\!\!\!\!\mid}^{s}$).

We aim to prove that symmetry and transitivity fail for dividing, when considered on ultraimaginary sorts. Since we know from [Kim01] that if T is non-simple then they already fail for real sorts, we might as well assume that T is simple.

Definition 1.5. Let $\bar{a} = (a_i : i \in \mathbb{Z})$ and $\bar{b} = (b_i : i \in \mathbb{Z})$. Then:

(i) $\bar{a}\ R_0\ \bar{b}$ if there is $n \in \mathbb{Z}$ such that $a_i = b_{i+n}$ for all $i \in \mathbb{Z}$.

(ii) $\bar{a}\ R_1\ \bar{b}$ if there are $\bar{a}'$ and $\bar{b}'$ obtained from $\bar{a}$ and $\bar{b}$, respectively, through omission of finitely many elements, such that $\bar{a}'\ R_0\ \bar{b}'$.

Let $\bar{a} = (a_i : i \in \mathbb{Z})$ be a Morley sequence over $\varnothing$ in some non-bounded (that is, non-algebraic) type, and set $a = a_0$. Let $a'_i = a_{i-1}$ for $i \le 0$ and $a'_i = a_i$ for $i > 0$, so $\bar{a}' = (a'_i : i \in \mathbb{Z}) = \bar{a} \setminus \{a\}$. Note that $\bar{a}'\ R_1\ \bar{a}$ and $a \mathop{\smile\!\!\!\!\!\!\mid} \bar{a}'$.

Lemma 1.6. (i) $a \mathop{\smile\!\!\!\!\!\!\mid}^{w}_{\bar{a}_{R_1}} \bar{a}_{R_0}$

(ii) $a \mathop{\smile\!\!\!\!\!\!\mid}^{s} \bar{a}_{R_1}$

(iii) $a \not\mathop{\smile\!\!\!\!\!\!\mid}^{s} \bar{a}_{R_0}$

(iv) $\bar{a}_{R_0} \not\mathop{\smile\!\!\!\!\!\!\mid}^{s}_{\bar{a}_{R_1}} a$

(v) $\mathrm{tp}(a/\bar{b})$ *divides over* $\bar{a}'$ *for every* $\bar{b}\ R_0\ \bar{a}$.

Proof. (i) Write $p(x, \bar{a}_{R_0}\bar{a}_{R_1}) = \mathrm{tp}(a/\bar{a}_{R_0}\bar{a}_{R_1})$: then all it says is that $x \in \bar{a}_{R_0}$, namely that x appears somewhere on the (indiscernible) sequence $\bar{a}_{R_0}$, and we might as well write it as $p(x, \bar{a}_{R_0})$. We need to prove that $\mathrm{tp}(a/\bar{a}_{R_0})$ does not weakly divide over $\bar{a}_{R_1}$.

Assume first that $(\bar{c}_{jR_0} : j \in J)$ is strongly $\bar{a}_{R_1}$-indiscernible in $\mathrm{tp}(\bar{a}_{R_0}/\bar{a}_{R_1})$. Then we may assume that $(\bar{c}_j)$ is $\bar{b}$-indiscernible in $\mathrm{tp}(\bar{c}/\bar{b})$, where $\bar{b}\ R_1\ \bar{a}$ and $\bar{c}\ R_0\ \bar{a}$. By indiscernibility of $(\bar{c}_j)$ there is a set $A \subseteq \mathbb{Z}$ such that $i \in A \iff \forall j \neq j' \in J\, c_{j,i} \neq c_{j',i}$; furthermore, since $\bar{c}_j\ R_1\ \bar{b}$ for every $j \in J$, A must be finite.

Set $n = |A| + 1$. Then $n < \omega$ and:

(*) If $d \in \bar{c}_j$ for n distinct values of j, then $d \in \bar{c}_j$ for all j, and moreover there exists d satisfying this.

Due to its finitary character, the property (*) holds for every sequence which is similar to $(\bar{c}_{jR_0})$ (over $\bar{a}_{R_1}$, or even over $\varnothing$). In particular, by Fact 1.2, (*) holds if $(\bar{c}_{jR_0})$ is just weakly indiscernible over $\bar{a}_{R_1}$, so there is some d satisfying $d \in \bar{c}_j$ for all j, and $d \models \bigwedge p(x, \bar{c}_{jR_0})$.

(ii) We have $a \mathop{\smile\!\!\!\!\!\!\mid} \bar{a}'$, and $\bar{a}\ R_1\ \bar{a}'$: it follows that $\mathrm{tp}(a/\bar{a}_{R_1})$ cannot strongly divide over $\varnothing$.

(iii) Evident.
(iv) It would suffice to prove that $q(y, a) = \mathrm{tp}(\bar{a}_{R_0}/a)$ strongly divides over $\bar{a}_{R_1}$. Let $(b_j : j < \omega_1)$ be any non-constant $\bar{a}'$-indiscernible sequence in $\mathrm{tp}(a/\bar{a}')$: such a sequence exists since $\bar{a}$ was assumed to be a Morley sequence in a non-algebraic type. Clearly, $\bigwedge q(y, b_j)$ is inconsistent, as any realisation would have to be uncountable.
(v) The formula $x = a$ divides over $\bar{a}'$ since it divides over $\varnothing$ and $a \mathop{\smile\!\!\!\!\!\mid}\, \bar{a}'$. If $\bar{b}\ R_0\ \bar{a}$ then $a \in \bar{b}$, so $\mathrm{tp}(a/\bar{b}) \vdash x = a$ divides over $\bar{a}'$.

QED

Whereby:

Theorem 1.7. (i) *Strong (weak) dividing is not symmetric for ultraimaginaries.*
(ii) *Strong dividing is not transitive for ultraimaginaries. For weak dividing, either transitivity or monotonicity fails.*

Proof. (i) We showed that $a \mathop{\smile\!\!\!\!\!\mid}^{w}_{\bar{a}_{R_1}} \bar{a}_{R_0}$ and $\bar{a}_{R_0} \not\mathop{\smile\!\!\!\!\!\mid}^{s}_{\bar{a}_{R_1}} a$.
(ii) We showed that $a \mathop{\smile\!\!\!\!\!\mid}^{w}_{\bar{a}_{R_1}} \bar{a}_{R_0}$, $a \mathop{\smile\!\!\!\!\!\mid}^{s} \bar{a}_{R_1}$ but $a \not\mathop{\smile\!\!\!\!\!\mid}^{s} \bar{a}_{R_0}$, which gives the first statement. If moreover $a \mathop{\smile\!\!\!\!\!\mid}^{w} \bar{a}_{R_1}$, then transitivity fails for weak dividing as well. Otherwise, monotonicity fails since $a \mathop{\smile\!\!\!\!\!\mid}\, \bar{a}'$.

QED

In addition, we observe that although $a \mathop{\smile\!\!\!\!\!\mid}^{w}_{\bar{a}_{R_1}} \bar{a}_{R_0}$, we have $\bar{a}'\ R_1\ \bar{a}$ and $\mathrm{tp}(a/\bar{b})$ divides over $\bar{a}'$ for every $\bar{b}\ R_0\ \bar{a}$.

2. Ultraimaginary independence relations are trivial

We proved that dividing (or at least, those notions of dividing that we could imagine) fails to give a good notion of independence for ultraimaginaries. Still, one may ask whether there may be a notion of independence, defined in some other manner, which would satisfy at least some nice properties. As the question is not well defined we cannot give a precise answer. Still, we propose to give a negative answer to a rather natural instance of this question, namely show that there can be no interesting independence relation for ultraimaginaries satisfying a rather minimal set of axioms:

Theorem 2.1. *Let $\mathop{\smile\!\!\!\!\!\mid}$ be a notion of independence defined for ultraimaginaries, satisfying invariance, symmetry, transitivity, monotonicity, extension, and the finite character restricted to real tuples. Then $\mathop{\smile\!\!\!\!\!\mid}$ is trivial, namely $a_E \mathop{\smile\!\!\!\!\!\mid}_{c_G} b_F$ whatever be a_E, b_F and c_G.*

Proof. Let a be some possibly infinite real tuple. By extension, finite character and standard arguments for extraction of indiscernible sequences, there is a $\mathop{\smile\!\!\!\!\!\mid}$-Morley sequence $\bar{a} = (a_i : i \in \mathbb{Z})$ over $\varnothing$, with $a = a_0$. Let $\bar{a}'$ be as above, so $a \mathop{\smile\!\!\!\!\!\mid}\, \bar{a}'$, and by monotonicity $a \mathop{\smile\!\!\!\!\!\mid}\, \bar{a}_{R_1}$.
By extension, there is $\bar{b}\ R_1\ \bar{a}$ such that $\bar{b} \mathop{\smile\!\!\!\!\!\mid}_{\bar{a}_{R_1}} \bar{a}$. Then there is $d \in \bar{b} \cap \bar{a}$, and by monotonicity: $d \mathop{\smile\!\!\!\!\!\mid}_{\bar{a}_{R_1}} d$. By invariance, we have $a \mathop{\smile\!\!\!\!\!\mid}_{\bar{a}_{R_1}} a$, so by transitivity and monotonicity: $a \mathop{\smile\!\!\!\!\!\mid}\, a$.

Finally, given any a_E, b_F, c_G, we have $abc \mathop{\smile\!\!\!\!\!\!\,|}\, abc$, from which we obtain $a_E \mathop{\smile\!\!\!\!\!\!\,|}_{c_G} b_F$ by transitivity and monotonicity. QED

Corollary 2.2. *Let T be a simple theory, and $\mathop{\smile\!\!\!\!\!\!\,|}$ ordinary dividing independence on reals (or hyperimaginaries) in T. Then there is no ultraimaginary independence relation extending $\mathop{\smile\!\!\!\!\!\!\,|}$ satisfying invariance, symmetry, transitivity, monotonicity and extension.*

Corollary 2.3. *Recall the definitions of strong and weak dividing from the previous section. For ultraimaginaries a_E, b_F and c_G, say that* $\operatorname{tp}(a_E/b_F c_G)$ *does not* strongly (weakly) fork *over b_F if it has extensions to every set (or ultraimaginary) that do not strongly (weakly) divide over b_F, and write $a \mathop{\smile\!\!\!\!\!\!\,|}_b^{sf} c$ ($a \mathop{\smile\!\!\!\!\!\!\,|}_b^{wf} c$). Then neither $\mathop{\smile\!\!\!\!\!\!\,|}^{sf}$ nor $\mathop{\smile\!\!\!\!\!\!\,|}^{wf}$ can satisfy symmetry, transitivity and monotonicity.*

Proof. As usual, we may assume that T is simple, otherwise both symmetry and transitivity fail for real tuples. Let $\mathop{\smile\!\!\!\!\!\!\,|}$ be either $\mathop{\smile\!\!\!\!\!\!\,|}^{sf}$ or $\mathop{\smile\!\!\!\!\!\!\,|}^{wf}$, and let forking and dividing mean strong or weak, accordingly. Then $\mathop{\smile\!\!\!\!\!\!\,|}$ clearly satisfies invariance, and extends dividing independence for real tuples.
Assume it satisfies symmetry, transitivity and monotonicity as well. By either definition of dividing, $\operatorname{tp}(c_G/a_E c_G)$ cannot divide over c_G. Thus, for every a_E and c_G, $\operatorname{tp}(c_G/a_E c_G)$ has a (unique) extension to any set which does not divide over c_G, so $c_G \mathop{\smile\!\!\!\!\!\!\,|}_{c_G} a_E$. By symmetry, $a_E \mathop{\smile\!\!\!\!\!\!\,|}_{c_G} c_G$.
Let b_F be any ultraimaginary, and assume that every extension q of $\operatorname{tp}(a_E/c_G)$ to $b_F c_G$ forks over c_G. Then for every such q there exists A_q such that every extension of q to $b_F c_G A_q$ divides over c_G. Then every extension of $\operatorname{tp}(a_E/c_G)$ to $b_F c_G \cup \bigcup_q A_q$ divides over c_G, contradicting $a_E \mathop{\smile\!\!\!\!\!\!\,|}_{c_G} c_G$. Therefore there is $a'_E \models \operatorname{tp}(a_E/c_G)$ such that $a'_E \mathop{\smile\!\!\!\!\!\!\,|}_{c_G} b_F$.
We obtained an ultraimaginary independence relation extending dividing independence and satisfying invariance, symmetry, transitivity, monotonicity and extension, which is impossible. QED

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