Propagating phase boundaries and capillary fluids

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Abstract

The aim is to give an overview of recent advancements in the theory of Euler–Korteweg model for liquid-vapour mixtures. This model takes into account the surface tension of interfaces by means of a capillarity coefficient. The interfaces are not sharp fronts. Their width, even though extremely small for values of the capillarity compatible with the measured, physical surface tension, is nonzero. We are especially interested in non-dissipative isothermal models, in which the viscosity of the fluid is neglected and therefore the (extended) free energy, depending on the density and its gradient, is a conserved quantity. From the mathematical point of view, the resulting conservation law for the momentum of the fluid involves a third order, dispersive term but no parabolic smoothing effect. We present recent results about well-posedness and propagation of solitary waves.

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1 Introduction to Korteweg's theory of capillarity

1.1 Historical background

The mathematical theory of phase boundaries dates back to the 19th century. One of the first achievements is the famous *Young–Laplace* relation, stating that the pressure difference across a fluid interface (between e.g. water and air, or water and vapour) equals the *surface tension* (a volumic force actually localized on the surface, as its name indicates) times the sum of principal curvatures of the surface (which implies in particular that there is no pressure difference for a flat interface). It was independently established by Young[1] [76] and Laplace[2] [52] in the early 1800s, and was later revisited by Gauß[3] [37]. Poisson[4] [64], Maxwell[5] [58], Gibbs[6] [40], Thomson[7] [72], and Rayleigh[8] [65] then contributed to develop the theory of nonzero thickness interfaces, in which capillarity comes into play, before it was formalized by van der Waals[9] [74] and his student (the only one known) Korteweg[10] [49]: we refer

[1] Thomas Young [1773–1829]
[2] Pierre-Simon Laplace [1749–1827]
[3] Carl Friedrich Gauß [1777–1855]
[4] Siméon-Denis Poisson [1781–1840]
[5] James Clerk Maxwell [1831–1879]
[6] Josiah Willard Gibbs [1839–1903]
[7] James Thomson [1822–1892] (elder brother of Lord Kelvin)
[8] John William Strutt, best known as Lord Rayleigh [1842–1919]
[9] Johannes Diederik van der Waals [1837–1923]
[10] Diederik Johannes Korteweg [1848–1941]

the reader to the book by Rowlinson and Widom [71] for more historical and physical details on *capillarity*, as well as for further information on reprinted and/or translated/commented editions of those ancient papers (which were originally written in as various languages as dutch, english, french, german, and latin).

1.2 What is capillarity?

In everyday life, capillarity effects may be observed in thin tubes. For instance, if a straw is filled with your favourite drink, this liquid will usually exhibit a concave meniscus at end-points (even though some liquids, like mercury, yield a convex meniscus, if you remember how looked like those old medical thermometers). Capillarity is also involved when you use a paper towel to wipe off spilled coffee on your table, and (before you clean up), it is surface tension that maintains the non-flat shape of coffee drops spread on the table: as we shall see, capillarity and surface tension are intimately linked (the words are even sometimes used as synonyms, at least by mathematicians). Less likely to be seen in your kitchen are the super-fluids (such as liquid helium at very low temperature), which would spontaneously creep up the wall of your cup and eventually spill over the table. Again, capillarity is suspected to play a role in this weird phenomenon.

Of course these observations do not make a definition. As far as we are concerned, capillarity will occur as a 'coefficient', possibly depending on density, in the energy of the fluid. We will consider only isothermal fluids (which seems to be physically justified in the case of superfluids, and for liquid-vapour mixtures in standard conditions). For this reason, by energy we will actually mean *free energy*. A 'regular' isothermal fluid (at rest) of density ρ and temperature T has an energy density $F_0(\rho, T)$, and its total energy in a volume Ω is

$$\mathscr{F}_0[\rho, T] = \int_\Omega F_0(\rho, T)\,\mathrm{d}\mathbf{x}.$$

In a capillary fluid, regions with large density gradients (typically phase boundaries of small but nonzero thickness), are assumed to be responsible for an additional energy, which is usually taken of the form

$$\tfrac{1}{2}\int_\Omega K(\rho, T)\,|\nabla\rho|^2\,\mathrm{d}\mathbf{x},$$

in such a way that the *total energy density* is

$$F = F(\rho, T, \nabla\rho) = F_0(\rho, T) + \tfrac{1}{2}K(\rho, T)\,|\nabla\rho|^2.$$

1.3 Where are the phase boundaries?

If F_0 is a convex function of ρ with a unique (global) minimum $\overline{\rho}$, the fluid will not exhibit phase boundaries. For, the constant state $\rho \equiv \overline{\rho}$ is an obvious global minimum of the total energy $\mathscr{F} = \int_\Omega F\,\mathrm{d}\mathbf{x}$ under the mass constraint $\int_\Omega(\rho - \overline{\rho})\mathrm{d}\mathbf{x} = 0$. Now things are very different if F_0 is a *double-well potential* , which happens to be so for instance in *van der Waals fluids* below critical temperature. By double-well potential we mean that F_0 has a bitangent at some points ρ_v and ρ_ℓ, called *Maxwell points*, with $\rho_v < \rho_\ell$ to fix the ideas (the subscript v standing for vapour and ℓ for liquid).Then the total energy has many types of minimisers where the two values, each one corresponding to a 'phase', ρ_v and ρ_ℓ co-exist. The investigation of minimisers in general is out of scope, but we shall come back to this topic in §3 from the special point of view of planar traveling waves. This will be a way to consider not only stationary phase boundaries in a fluid at rest but also propagating phase boundaries in a moving fluid.

1.4 Equations of motion for capillary fluids

The motion of a 'regular', compressible and inviscid isothermal fluid is known (see for instance [31]) to be governed by the Euler equations, consisting of

conservation of mass

$$\partial_t \rho + \mathrm{div}(\rho \mathbf{u}) = 0\,, \tag{1}$$

conservation of momentum

$$\partial_t(\rho \mathbf{u}) + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) \,=\, \mathrm{div}\boldsymbol{\Sigma}\,, \tag{2}$$

where ρ denotes as before the density, $\mathbf{u}$ is the velocity, and $\boldsymbol{\Sigma}$ is the *stress tensor* of the fluid, given by

$$\boldsymbol{\Sigma} \,=\, -p\,\mathbf{I}\,, \quad p := \rho\,\frac{\partial F}{\partial \rho} - F\,.$$

Here above, the energy density F is assumed to depend only on (ρ, T), and p is the actual pressure in the fluid. For capillary fluids, or more generally if we allow F to depend not only on (ρ, T) but also on $\nabla\rho$, we can still define a (generalised) pressure by

$$p := \rho\,\frac{\partial F}{\partial \rho} - F\,.$$

However, it turns out that in this situation the stress tensor is not merely given by $-p\mathbf{I}$. Additional terms are to be defined in terms of the vector field $\mathbf{w}$, of components

$$w_i := \frac{\partial F}{\partial \rho_{,i}}\,,$$

where for $i \in \{1,\dots,d\}$, $\rho_{,i}$ stands for $\partial_i \rho$, the i-th component of $\nabla\rho$. By variational arguments detailed in the appendix (also see [67]), we can justify that for capillary fluids the stress tensor $\boldsymbol{\Sigma}$ has to be modified into

$$\boldsymbol{\Sigma} = (-p \,+\, \rho\,\mathrm{div}\mathbf{w})\,\mathbf{I} - \mathbf{w} \otimes \nabla\rho\,. \tag{3}$$

In particular, when

$$F = F_0(\rho) \,+\, \tfrac{1}{2}\,K(\rho)\,|\nabla\rho|^2\,, \tag{4}$$

(we can forget about the dependency of F on T since we consider only isothermal motions) we have

$$\mathbf{w} = K\nabla\rho\,,\quad p = p_0 + \tfrac{1}{2}\,(\rho\,K'_\rho - K)\,|\nabla\rho|^2\,,$$

where

$$p_0 := \rho\,\frac{\partial F_0}{\partial\rho} - F_0$$

is the standard pressure. After substitution of $\mathbf{w}$ and p for their expressions in terms of K, ρ, and $\nabla\rho$ in $\mathbf{\Sigma}$, the momentum equation (2) may seem overcomplicated at first glance. To get a simpler point of view, it is in fact better to return to the abstract form of $\mathbf{\Sigma}$, and observe that by the generalised *Gibbs relation*

$$\mathrm{d}F = -S\,\mathrm{d}T + g\,\mathrm{d}\rho + \sum_{i=1}^{d} w_i\,\mathrm{d}\rho_{,i}\,,$$

we have (by definition) $p = \rho g - F$, hence

$$\mathrm{d}p = \rho\,\mathrm{d}g + S\,\mathrm{d}T - \sum_{i=1}^{d} w_i\,\mathrm{d}\rho_{,i}\,.$$

(Here above, S is the entropy density, and g is the - generalised - chemical potential of the fluid.) In particular, along isothermal, smooth enough motions we have

$$\partial_j p = \rho\,\partial_j g - \sum_{i=1}^{d} w_i\,\partial_j\rho_{,i} = \rho\,\partial_j g - \sum_{i=1}^{d} w_i\,\partial_i\rho_{,j}$$

by the Schwarz lemma. This (almost readily) yields the identity

$$\mathrm{div}\mathbf{\Sigma} = \rho\,\nabla(-g + \mathrm{div}\mathbf{w})\,.$$

Therefore, the Euler equations (1)-(2) with the modified stress tensor (3) may alternatively be written in conservative form

$$\left\{\begin{array}{l} \partial_t\rho + \mathrm{div}(\rho\mathbf{u}) = 0\,,\\ \partial_t(\rho\mathbf{u}) + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) = \nabla(-p + \rho\mathrm{div}\mathbf{w}) - \mathrm{div}(\mathbf{w}\otimes\nabla\rho)\,,\end{array}\right. \tag{5}$$

or (using in a standard manner the conservation of mass to cancel out terms in the left hand side of the momentum equation), in convection form

$$\left\{\begin{array}{l} \partial_t\rho + \mathbf{u}\cdot\nabla\rho + \rho\,\mathrm{div}\mathbf{u} = 0\,,\\ \partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla(-g + \mathrm{div}\mathbf{w})\,.\end{array}\right. \tag{6}$$

In particular, when $\mathbf{w} = K\nabla\rho$ (that is, if (4) holds true),

$$g = g_0 + \tfrac{1}{2}\,K'_\rho\,|\nabla\rho|^2\,,$$

where g_0 is independent of $\nabla\rho$, and more precisely g_0 is such that $p_0 = \rho g_0 - F_0$, and the second equation in (6) reads

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla(-g_0 + \tfrac{1}{2}K'_\rho\,|\nabla\rho|^2 + K\,\Delta\rho)\,. \tag{7}$$

An apparently simple, special case is when K is a constant. Then Equation (7) reduces to

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla g_0 = K\,\nabla\Delta\rho\,, \tag{8}$$

which has a linear principal part. Nevertheless, we shall see in the analysis of the Cauchy problem that it is not the easiest case. A somehow physical explanation for this fact is the following. Since the total energy of the fluid, including kinetic energy, is

$$F + \tfrac{1}{2}\rho\,|\mathbf{u}|^2 = F_0 + \tfrac{1}{2}\rho\,(|\mathbf{u}|^2 + |\mathbf{v}|^2)\,, \quad \mathbf{v} := \sqrt{\frac{K}{\rho}}\,\nabla\rho\,,$$

it seems reasonable to try and reformulate (7) in terms of the vector field $\mathbf{v}$ instead of $\mathbf{w} = K\nabla\rho$. This works indeed, thanks to the identity (shown in the appendix)

$$\tfrac{1}{2}K'_\rho\,|\nabla\rho|^2 + K\,\Delta\rho = a\,\mathrm{div}\mathbf{v} + \tfrac{1}{2}|\mathbf{v}|^2\,, \quad a := \sqrt{\rho K}\,,$$

so that (7) equivalently reads

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla g_0 = \nabla(a\,\mathrm{div}\mathbf{v} + \tfrac{1}{2}|\mathbf{v}|^2)\,. \tag{9}$$

Even though this equation looks more complicated than (8), it points out another special case, namely when a is constant, that is when K is proportional to $1/\rho$ (which we shall merely write $K \propto 1/\rho$). Indeed, for such an a, the principal part in (9) is $a\nabla\mathrm{div}\mathbf{v}$, obviously linear in $\mathbf{v}$ with a constant coefficient. (We could also have noticed that $\rho K \equiv$ constant simplifies the principal part, $\nabla(\rho K\Delta\rho)$, of the momentum equation in (5) when $\mathbf{w} = K\nabla\rho$.) By the way, we shall see in §1.5 below that the special case $K \propto 1/\rho$ is a physical one, in the framework of Quantum HydroDynamics (QHD) for semiconductors or for Bose–Einstein condensates. In any case, Eq. (9) and its counterpart for $\mathbf{v}$ (see §2.2.1) will be play a crucial role later on in the analysis.

1.5 Euler–Korteweg equations and related models

For F as in (4), $\mathbf{w} = K(\rho)\nabla\rho$, and p, p_0, g, g_0 defined as in the previous subsection by

$$\left\{\begin{array}{ll} p = p_0 + \tfrac{1}{2}(\rho K'_\rho - K)\,|\nabla\rho|^2\,, & p_0 = \rho g_0 - F_0\,, \\[2ex] g = g_0 + \tfrac{1}{2}K'_\rho\,|\nabla\rho|^2\,, & g_0 = \dfrac{\partial F_0}{\partial\rho}\,, \end{array}\right. \tag{10}$$

we shall refer indifferently to (5) or to (6) as the *Euler–Korteweg equations*, and call *Korteweg stress tensor*

$$\mathbf{K} := p_0\,\mathbf{I} + \mathbf{\Sigma} = \rho\,K\,\Delta\rho\,\mathbf{I} + \tfrac{1}{2}(K + \rho K'_\rho)\,|\nabla\rho|^2\,\mathbf{I} - K\,\nabla\rho\otimes\nabla\rho\,. \tag{11}$$

It is to be noted that originally Korteweg [49] (see also [73, pp. 513–515] for a modern account and discussion) considered more general stress tensors, of the form

$$\mathbf{K} = \alpha(\rho)\,|\nabla\rho|^2\,\mathbf{I} + \beta(\rho)\,\nabla\rho\otimes\nabla\rho + \gamma(\rho)\,\Delta\rho\,\mathbf{I} + \delta(\rho)\,\nabla^2\rho\,. \tag{12}$$

The tensor obtained above corresponds to $\alpha = \frac{1}{2}\,(K+\rho K'_\rho)$, $\beta = -K$, $\gamma = \rho K$, $\delta = 0$. Another class of examples pertaining to the general form (12) is

$$\mathbf{K} = \mu(\rho)\,\nabla^2\xi(\rho)\,,$$

corresponding to $\alpha = 0$, $\beta = 0$, $\gamma = \mu\xi''_\rho$, $\delta = \mu\xi'_\rho$. We shall meet of one them in the QHD paragraph.

Incompressible inhomogeneous Euler–Korteweg equations An *incompressible* fluid is characterized by a *solenoidal* (that is, divergence-free) velocity. It can be homogeneous, and then its density ρ is merely a constant, or inhomogeneous, in which case ρ is just transported by the flow, obeying the law

$$\partial_t\rho + \mathbf{u}\cdot\nabla\rho = 0\,.$$

(This equation is a special case of the mass conservation law (1) when $\mathrm{div}\mathbf{u} = 0$.) Recently [56], local-in-time well-posedness in Sobolev spaces was shown for the model of incompressible inhomogeneous fluids with a Korteweg tensor of the form (12) with $\alpha > 0$, $\beta = -3\alpha$, $\gamma = 0$, $\delta = 0$. It is to be noted that for these values of parameters, the momentum equation does not contain third order derivatives. The nasty quadratic terms in $\nabla\rho$ are dealt with by using elliptic regularity estimates for the modified pressure $\Pi = p + 2\alpha|\nabla\rho|^2$.

Navier–Stokes–Korteweg equations The (compressible) Navier–Stokes–Korteweg equations are made of (1)-(2) with a stress tensor

$$\mathbf{\Sigma} = -p_0\mathbf{I} + \mathbf{K} + \mathbf{D}\,,$$

including a *viscous stress tensor*, generally of the form

$$\mathbf{D} = \lambda(\rho)\,(\mathrm{div}\mathbf{u})\mathbf{I} + \mu(\rho)\,(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)\,.$$

In the mathematical analysis of the Navier–Stokes–Korteweg equations, the positivity of the viscosity coefficient μ plays a crucial role (basically lying in a parabolic smoothing). More precisely, it was shown in [30] that for constant λ and μ such that $\mu > 0$ and $\lambda + 2\mu > 0$, and for $K > 0$ also constant, the Cauchy problem is locally well-posed in critical Besov spaces, and globally well-posed for 'small' data (in fact, for densities close to a *stable state*, *i.e.* where F_0 is strictly convex). On the other hand, it was shown in [24] that for $\lambda = 0$, $\mu = \nu\rho$, $\nu > 0$ constant, and $K > 0$ constant, the Cauchy problem admits global weak solutions. The special form of μ enabled indeed the authors to derive an a priori estimate for the *effective velocity* $\mathbf{u} + \nu\nabla\ln\rho$. Another important ingredient in the analysis of the Navier–Stokes–Korteweg (and also of the Euler–Korteweg) equations is a further *a priori* estimate on $\nabla\rho$, due to the term $\frac{1}{2}K|\nabla\rho|^2$ in the energy, which allows more general pressure laws – in particular nonmonotone ones, like for van der Waals fluids – than for compressible Navier–Stokes equations (dealt with by Lions [55] and Feireisl [35]).

Quantum hydrodynamic equations The mathematical theory of *semi-conductors* involves models analogous to Euler–Korteweg and Navier–Stokes–Korteweg equations, referred to as *quantum hydrodynamic equations* (QHD) [44, 45], in which the 'Korteweg' tensor is of the form

$$\mathbf{K} = \varepsilon\rho\,\nabla^2 \ln(\rho)\,.$$

When expanded as in (12), this tensor involves in particular a term $\beta\,\nabla\rho\otimes\nabla\rho$ with $\beta = -\varepsilon/\rho$, similarly as the Korteweg tensor we introduced for capillary fluids (11) when $K \propto 1/\rho$. Let us mention that in QHD, the higher order term div$\mathbf{K}$ in the momentum equation is usually written by using the remarkable identity (shown in the appendix)

$$\mathrm{div}(\rho\,\nabla^2 \ln(\rho)) = 2\,\rho\,\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right).$$

Recently [46], a global existence result of weak solutions was obtained for the quantum Navier–Stokes equations, namely (1)-(2) with $\mathbf{\Sigma} = -p_0\mathbf{I} + \mathbf{K} + \mathbf{D}$, $\mathbf{K}$ as above and $\mathbf{D} = \nu\rho\,(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$, using a reformulation of the equations in terms of the same effective velocity $\mathbf{u} + \nu\nabla\ln\rho$ as in [24]. As regards inviscid quantum equations, local-in-time existence results for smooth solutions were obtained in the early 2000s [43, 53] on the quantum Euler-Poisson equations, in which the momentum equation is coupled with a *Poisson equation* through the force associated to the electrostatic potential. (The literature on QHD is of course much wider than the few references given here above.)

Gross–Pitaevskiĭ equation A model for *Bose–Einstein condensates* [63] (which also have to do with quantum mechanics, and among which we find for instance superfluid Helium-4) is the *Gross–Pitaevskiĭ equation*

$$i\partial_t\psi + \tfrac{1}{2}\Delta\psi = (|\psi|^2 - 1)\,\psi\,,$$

a particular *nonlinear Schrödinger equation* (usually referred to as NLS). More generally, we may consider the NLS

$$i\partial_t\psi + \tfrac{1}{2}\Delta\psi = g_0(|\psi|^2)\,\psi\,. \tag{13}$$

A bit of algebra shows that for (smooth enough) complex-valued solutions ψ of this equation, the vector-valued function ($\rho = |\psi|^2, \mathbf{u} = \nabla\varphi$), where φ denotes the argument of ψ, satisfies (at least formally) the Euler–Korteweg equations (6) with

$$K = \frac{1}{4\rho}\,.$$

Conversely, a solution $(\rho,\mathbf{u})$ of (6) with $\rho > 0$, $K = 1/(4\rho)$, and $\mathbf{u}$ irrotational, yields a solution ψ of (13) through the *Madelung transform*

$$(\rho,\mathbf{u}) \mapsto \psi = \sqrt{\rho}\,\mathrm{e}^{i\varphi}\,; \quad \nabla\varphi = \mathbf{u}\,.$$

The theory of the Gross–Pitaevskiĭ equation is a very active field of research in itself, see *e.g.* [2, 14, 15, 16, 17, 18, 39].

Water waves equations The original *Boussinesq*[11] equation [23]

$$\partial_t^2 h - g\,H\partial_x^2 h - \tfrac{3}{2}\,g\,\partial_x^2(h^2) - g\,H^3\,\partial_x^4 h = 0$$

was derived as a modification of the wave equation in order to describe long water waves of height h propagating on the surface of a shallow river, of depth H, under the gravity g. Since then, generalised Boussinesq equations

$$\partial_t^2 h + \partial_x^2 \pi(h) + \kappa\,\partial_x^4 h = 0 \tag{14}$$

have been considered [20]. We speak of the 'good Boussinesq' equation when $\kappa > 0$, because the linear operator $\partial_t^2 + \kappa\,\partial_x^4$ yields well-posed Cauchy problems in that case. As in [20], one may consider the associated system

$$\left\{\begin{array}{l} \partial_t h = \partial_x u\,, \\ \partial_t u + \partial_x \pi(h) = -\kappa\,\partial_x^3 h\,, \end{array}\right. \tag{15}$$

(If (h,u) is a smooth enough solution of (15) then h is a solution of (14).) Now, it is remarkable that the Euler–Korteweg equations in one space dimension

$$\left\{\begin{array}{l} \partial_t \rho + \partial_x(\rho u) = 0\,, \\ \partial_t(\rho\,u) + \partial_x(\rho\,u^2 + p_0(\rho)) = \partial_x(\rho\,K\,\partial_x^2\rho + \tfrac{1}{2}\,(\rho\,K_\rho' - K)\,(\partial_x\rho)^2)\,, \end{array}\right. \tag{16}$$

give rise to a generalised version of (15) when reformulated in Lagrangian coordinates. More precisely, if (ρ, u) is a smooth enough solution[12] of (16) with $\rho > 0$, if we denote by $v = 1/\rho$ the specific volume of the fluid, and y the mass Lagrangian coordinate, characterised by $\mathrm{d}y = \rho\,\mathrm{d}x - \rho\,u\,\mathrm{d}t$, we find that (see details in the appendix) (v, u) satisfies

$$\left\{\begin{array}{l} \partial_t v = \partial_y u\,, \\ \partial_t u + \partial_y p_0 = -\partial_y(\kappa\,\partial_y^2 v + \tfrac{1}{2}\,\kappa_v'\,(\partial_y v)^2)\,, \end{array}\right. \tag{17}$$

with $\kappa := \rho^5\,K$. In particular when κ is constant, that is $K \propto \rho^{-5}$ (even though it is not clear what this should mean physically), (17) reduces to (15) with $h = v$ and $\pi(h) = p_0(\rho)$.

Another asymptotic model for water waves is known as the *Saint-Venant*[13] *system* (at least by the French; otherwise it is called the *shallow water equations*), which coincides with the Euler equations for a quadratic 'pressure' (the height of the waves playing the role of density). When surface tension is taken into account, higher order terms are involved that are similar to those in the Korteweg tensor, see [24, 25] for more details.

Finally, the complete (incompressible Euler) equations for water waves with surface tension are also close to the Euler–Korteweg equations, and some of the techniques used in the recent work by Alazard, Burq and Zuily[1] are to some extent similar to those we shall describe later on.

[11] Joseph Valentin Boussinesq [1842–1929]

[12] For Euler equations, it is known that the two formulations are equivalent also for weak solutions [75], but for Euler–Korteweg equations we refrain from talking about weak solutions because their meaning is not clear.

[13] Adhémar Jean-Claude Barré de Saint-Venant [1797–1886] (advisor of Boussinesq)

1.6 Hamiltonian structures

From now on, we concentrate on the (compressible) Euler–Korteweg equations. Let us more specifically look at these equations written (for smooth motions) as in (1)(7), which we recall here for convenience

$$\left\{\begin{array}{l}\partial_t\rho + \mathrm{div}(\rho\,\mathbf{u}) = 0\,,\\ \partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u} = \nabla(-g_0 + \tfrac{1}{2}K'_\rho\,|\nabla\rho|^2 + K\,\Delta\rho)\,.\end{array}\right. \tag{18}$$

A remarkable fact about this system is that it admits a *Hamiltonian structure*, associated with the *total energy* (including kinetic energy)

$$\mathscr{H} := \int H\,\mathrm{d}\mathbf{x},\quad H(\rho,\nabla\rho,\mathbf{u}) := F(\rho,\nabla\rho) + \tfrac{1}{2}\,\rho\,|\mathbf{u}|^2 = F_0(\rho) + \tfrac{1}{2}\,K(\rho)\,|\nabla\rho|^2 + \tfrac{1}{2}\,\rho\,|\mathbf{u}|^2\,.$$

The expression of $\mathscr{H}$ is formal if the integral is taken on the whole space, because H does not need being integrable (it may even not tend to zero at infinity). However, we shall see later on that we can modify H in a suitable manner, at least in one space dimension, to ensure the integral is well-defined. The *variational gradient* of the functional $\mathscr{H}$ is the vector-valued function $\delta\mathscr{H}$ (at least formally) defined by

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\mathscr{H}[\rho+\theta\dot\rho,\mathbf{u}+\theta\dot{\mathbf{u}}]_{|\theta=0} = \int \delta\mathscr{H}[\rho,\mathbf{u}]\cdot\begin{pmatrix}\dot\rho\\ \dot{\mathbf{u}}\end{pmatrix}\mathrm{d}\mathbf{x}$$

for perturbations $(\dot\rho,\dot{\mathbf{u}})$ vanishing sufficiently rapidly on the boundary of the domain or at infinity in the whole space. Equivalently (see for instance [60, p. 244–247]), the components of $\delta\mathscr{H}$ are $\mathscr{E}_\rho H$, $\mathscr{E}_{u_1}H,\dots,\mathscr{E}_{u_d}H$ where the *Euler operators* are defined by

$$\mathscr{E}_\rho H := \frac{\partial H}{\partial\rho} - \sum_{i=1}^{d}\mathrm{D}_i\left(\frac{\partial H}{\partial\rho_{,i}}\right),\quad \mathscr{E}_{u_j}H := \frac{\partial H}{\partial u_j}$$

(the latter being simpler than the former because H does not depend on the derivatives of $\mathbf{u}$). Here above (as in the appendix, first paragraph), D_i stands for the *total derivative* with respect to $x_i\in\mathbb{R}^d$. Therefore, recalling that $g_0 = (F_0)'_\rho$, we find that

$$\delta\mathscr{H}[\rho,\mathbf{u}] = \begin{pmatrix}\tfrac{1}{2}\,|\mathbf{u}|^2 + g_0(\rho) - \tfrac{1}{2}\,K'_\rho\,|\nabla\rho|^2 - K\,\Delta\rho\\ \rho\mathbf{u}\end{pmatrix}.$$

Let us now introduce the differential operator

$$\mathscr{J} := \begin{pmatrix}0 & -\mathrm{div}\\ -\nabla & 0\end{pmatrix}.$$

It is skew-adjoint on $L^2(\mathbb{R}^d)$ (with domain $H^1(\mathbb{R}^d)$). Its skew-symmetry can be seen from the identity, merely coming from integrations by parts, $\int(\rho_1\mathrm{div}\mathbf{u}_2 + \mathbf{u}_1\cdot\nabla\rho_2)\,\mathrm{d}\mathbf{x} = -\int(\rho_2\mathrm{div}\mathbf{u}_1 + \mathbf{u}_2\cdot\nabla\rho_1)\,\mathrm{d}\mathbf{x}$). Another way is to remark that its *symbol*

$$\mathbf{J}(\boldsymbol{\xi}) := \begin{pmatrix}0 & -i\,\boldsymbol{\xi}^T\\ -i\,\boldsymbol{\xi} & 0\end{pmatrix}$$

is skew-symmetric. Then we readily see that (18) is equivalent to

$$\partial_t \begin{pmatrix} \rho \\ \mathbf{u} \end{pmatrix} = \mathscr{J}\, \delta\mathscr{H}[\rho, \mathbf{u}]\,, \tag{19}$$

provided that $(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla(\frac{1}{2}|\mathbf{u}|^2)$, which is the case in particular (see (61) in the appendix) if the velocity field $\mathbf{u}$ is *irrotational* (that is, $\partial_i u_j = \partial_j u_i$ for all indices $i, j \in \{1, \ldots, d\}$).

Remark 1. *By Poincaré's lemma, a vector field is* irrotational *(or curl-free) in a simply connected domain if and only it is* potential*, which means that it is the gradient of a potential function. Of course this is always the case in one space dimension. Observe also that for the Gross–Pitaevskiĭ equation* (13)*, the 'velocity' field is potential by construction.*

System (19) is the prototype of an *infinite dimensional hamiltonian system* (the infinite dimensions being due to the fact that $\mathscr{J}$ and $\mathscr{H}$ act on functions of $\mathbf{x}$). It is associated with the *Poisson bracket*

$$\{\mathscr{Q}, \mathscr{R}\} := \int \delta\mathscr{Q} \cdot \mathscr{J}\, \delta\mathscr{R}\, \mathrm{d}\mathbf{x}$$

for functionals $\mathscr{Q}$ and $\mathscr{R}$. The skew-symmetry of $\{,\}$ readily follows from the skew-symmetry of $\mathscr{J}$. Furthermore, $\{,\}$ automatically satisfies the *Jacobi identity*

$$\{\mathscr{Q}, \{\mathscr{R}, \mathscr{S}\}\} + \{\mathscr{R}, \{\mathscr{S}, \mathscr{Q}\}\} + \{\mathscr{S}, \{\mathscr{Q}, \mathscr{R}\}\} = 0$$

because $\mathscr{J}$ has constant coefficients (see [60, p. 438]).

Remark 2. *The Hamiltonian formulation in* (19) *is in fact not specific to the energy as in* (4)*. Going back to the definition of the Euler operator E_ρ we see that in general*

$$\delta\mathscr{H}[\rho, \mathbf{u}] = \begin{pmatrix} \frac{1}{2}|\mathbf{u}|^2 + g(\rho, \nabla\rho) - \mathrm{div}\mathbf{w} \\ \rho\mathbf{u} \end{pmatrix},$$

and the system (6) *is indeed* (19) *with this expression of $\delta\mathscr{H}$.*

In three space dimensions ($d = 3$), it turns out that (18) admits another Hamiltonian formulation associated with the Hamiltonian $\mathscr{H}$, with a *constant* skew-symmetric operator $\mathbf{J}$ instead of $\mathscr{J}$, for general flows (*i.e.* without the assumption $\nabla \times \mathbf{u} = 0$). This is even true for the more general system (6). We follow here the approach described by Benjamin [7, §7.2] (also see references to much earlier work therein, in particular the original paper by Clebsch[14] [27] and the book by Lamb[15] [51, §167]). The idea is to make use of the *Clebsch transformation* and write the velocity field as

$$\mathbf{u} = \nabla\varphi + \lambda\nabla\mu\,,$$

[14] Rudolf Friedrich Alfred Clebsch [1833-1872]

[15] Sir Horace Lamb [1849–1934]

where $\nabla\varphi$ is clearly potential (and thus irrotational), and λ, μ have some degrees of freedom we shall not discuss here. Note however that $\lambda\nabla\mu$ is not solenoidal [16] in general, its divergence being equal to $\lambda\Delta\mu+\nabla\lambda\cdot\nabla\mu$, so this way of decomposing the velocity field is not directly linked to the *Leray–Helmholtz projector* onto divergence-free vector fields. Note also that all the information on the *vorticity*, defined as the *curl* of the velocity

$$\boldsymbol{\omega} := \nabla\times\mathbf{u}\,,$$

is then contained in λ and μ:

$$\boldsymbol{\omega} = \nabla\lambda\times\nabla\mu\,.$$

Let us compute the variational gradient of $\mathscr{H}$ when H is viewed as a function of $(\rho,\nabla\rho,\Lambda,\nabla\varphi,\nabla\mu)$ with $\Lambda := \rho\,\lambda$, that is

$$H = F(\rho,\nabla\rho) + \tfrac{1}{2}\,\rho\,|\nabla\varphi + (\Lambda/\rho)\,\nabla\mu|^2\,.$$

The components of $\delta\mathscr{H}$ are

$$\mathscr{E}_\rho H := \frac{\partial H}{\partial\rho} - \sum_{i=1}^{3}\mathrm{D}_i\left(\frac{\partial H}{\partial\rho_{,i}}\right) = -\,\lambda\,\mathbf{u}\cdot\nabla\mu + \tfrac{1}{2}\,|\mathbf{u}|^2 + g(\rho,\nabla\rho) - \mathrm{div}\mathbf{w}\,,$$

$$\mathscr{E}_\Lambda H := \frac{\partial H}{\partial\Lambda} = \mathbf{u}\cdot\nabla\mu\,,$$

$$\mathscr{E}_\varphi H := -\sum_{i=1}^{3}\mathrm{D}_i\left(\frac{\partial H}{\partial\varphi_{,i}}\right) = -\,\mathrm{div}(\rho\,\mathbf{u})\,,$$

$$\mathscr{E}_\mu H := -\sum_{i=1}^{3}\mathrm{D}_i\left(\frac{\partial H}{\partial\mu_{,i}}\right) = -\,\mathrm{div}(\Lambda\,\mathbf{u})\,.$$

Let us consider the Hamiltonian system

$$\partial_t\begin{pmatrix}\rho\\ \Lambda\\ \varphi\\ \mu\end{pmatrix} = \mathbf{J}\,\delta\mathscr{H}[\rho,\Lambda,\varphi,\mu]\,,\quad \mathbf{J} := \begin{pmatrix}0&0&1&0\\ 0&0&0&1\\ -1&0&0&0\\ 0&-1&0&0\end{pmatrix}. \tag{20}$$

The first row in (20) is nothing but the conservation of mass (1), and the third row reads

$$\partial_t\varphi - \lambda\,\mathbf{u}\cdot\nabla\mu + \tfrac{1}{2}\,|\mathbf{u}|^2 + g(\rho,\nabla\rho) - \mathrm{div}\mathbf{w} = 0\,,$$

which can be combined with the fourth row

$$\partial_t\mu + \mathbf{u}\cdot\nabla\mu = 0$$

to give

$$\partial_t\varphi + \lambda\,\partial_t\mu + \tfrac{1}{2}\,|\mathbf{u}|^2 + g(\rho,\nabla\rho) - \mathrm{div}\mathbf{w} = 0\,. \tag{21}$$

[16] By Poincaré's lemma, a field is *solenoidal*, which means divergence-free, in a simply connected domain if and only if it can be written as the curl of a vector potential.

Finally, the fourth row

$$\partial_t \Lambda + \operatorname{div}(\Lambda \mathbf{u}) = 0$$

combined with the conservation of mass (1) shows that

$$\partial_t \lambda + \mathbf{u} \cdot \nabla \lambda = 0,$$

which means that λ, like μ, is transported by the flow. This implies, together with the identity (see (61) in the appendix)

$$\nabla \left(\tfrac{1}{2}|\mathbf{u}|^2\right) = \mathbf{u} \times (\nabla \times \mathbf{u}) + (\mathbf{u} \cdot \nabla)\mathbf{u},$$

that (21) is an integrated version of the velocity equation in (6). More precisely, we can derive the velocity equation by applying the gradient operator ∇ to (21) and by using that (see (62) the appendix)

$$\mathbf{u} \times (\nabla \lambda \times \nabla \mu) - (\mathbf{u} \cdot \nabla \mu)\, \nabla \lambda + (\mathbf{u} \cdot \nabla \lambda)\, \nabla \mu = 0.$$

2 Well-posedness issues for the Euler–Korteweg equations

2.1 Nature of the equations

To discuss well-posedness for classical solutions we can either consider (18) or the conservative form of the Euler–Korteweg equations, which in view of the expression of p in (10) reads

$$\left\{ \begin{array}{l} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) = \nabla \left(-p_0 + \rho K \Delta \rho + \tfrac{1}{2}(K + \rho K_\rho') \, |\nabla \rho|^2 \right) - \operatorname{div}(K \, \nabla \rho \otimes \nabla \rho), \end{array} \right. \tag{22}$$

with $p_0 = p_0(\rho)$ and $K = K(\rho)$. Clearly, (18) looks nicer than (22). However, both are third order systems with no parabolic smoothing effect, even though we expect some smoothing on the density thanks to the conservation of the total energy

$$\mathscr{H} = \int (F_0(\rho) + \tfrac{1}{2}\, K(\rho)\, |\nabla \rho|^2 + \tfrac{1}{2}\, \rho\, |\mathbf{u}|^2)\, \mathrm{d}\mathbf{x}.$$

This conservation property readily follows from the Hamiltonian structures evidenced in §1.6. It can also be deduced from the local conservation law for the energy

$$\begin{aligned} &\partial_t \left(F_0 + \tfrac{1}{2}\, K \, |\nabla \rho|^2 + \tfrac{1}{2}\, \rho\, |\mathbf{u}|^2 \right) + \\ &\operatorname{div}\left(\left(F_0 + p_0 - \rho K \Delta \rho - \tfrac{1}{2}\, (K + \rho K_\rho') \, |\nabla \rho|^2 + \tfrac{1}{2}\, \rho\, |\mathbf{u}|^2 \right) \mathbf{u} + K \operatorname{div}(\rho \mathbf{u})\, \nabla \rho \right) = 0, \end{aligned}$$

which is a special case of

$$\partial_t \left(F + \tfrac{1}{2}\, \rho\, |\mathbf{u}|^2 \right) + \operatorname{div}\left((F + \tfrac{1}{2}\, \rho\, |\mathbf{u}|^2)\, \mathbf{u} \right) = \operatorname{div}(\boldsymbol{\Sigma} \mathbf{u} - (\rho \operatorname{div} \mathbf{u})\, \mathbf{w}),$$

where $\boldsymbol{\Sigma} = (-p + \rho \operatorname{div} \mathbf{w})\, \mathbf{I} - \mathbf{w} \otimes \nabla \rho$ is the stress tensor (as in (3)) and the additional term $-(\rho \operatorname{div} \mathbf{u})\, \mathbf{w}$ has been called *interstitial working* by Dunn and Serrin [33].

If one wants to understand the mathematical nature of (18) or (22), the first thing to do is to examine the linearised equations about constant states (which are obvious solutions). Linearising (18) about $(\rho,\mathbf{u}) = (\underline{\rho},\underline{\mathbf{u}})$, we get

$$\tag{23}\left\{\begin{array}{l}\partial_t\,\rho + \underline{\mathbf{u}}\cdot\nabla\rho + \underline{\rho}\;\mathrm{div}\mathbf{u} = 0\,,\\[2ex] \partial_t\mathbf{u} + (\,\underline{\mathbf{u}}\cdot\nabla\,)\,\mathbf{u} + \underline{\gamma}\,\nabla\rho = \underline{K}\,\nabla\Delta\rho\,,\end{array}\right.$$

where we have used the notations $\underline{\gamma} = g_0'(\underline{\rho}\,)$ and $\underline{K} = K(\underline{\rho}\,)$ for simplicity. We can write (23) under the abstract form

$$\tag{24}\partial_t\left(\begin{array}{c}\rho\\ \mathbf{u}\end{array}\right) = \mathbf{A}(\nabla)\left(\begin{array}{c}\rho\\ \mathbf{u}\end{array}\right),$$

where the *symbol* of the differential operator $\mathbf{A}(\nabla)$ is

$$\tag{25}\mathbf{A}(i\boldsymbol{\xi}) = -\left(\begin{array}{cc} i\,\underline{\mathbf{u}}\cdot\boldsymbol{\xi} & i\,\underline{\rho}\,\boldsymbol{\xi}^T\\ i\,(\underline{\gamma}+\underline{K}\,|\boldsymbol{\xi}|^2)\,\boldsymbol{\xi} & i\,(\underline{\mathbf{u}}\cdot\boldsymbol{\xi})\,\mathbf{I}_d\end{array}\right),$$

of which the characteristic polynomial is

$$\tag{26}\chi(\tau;\boldsymbol{\xi}) = (\tau + i\,\underline{\mathbf{u}}\cdot\boldsymbol{\xi})^{d-1}\left((\tau + i\,\underline{\mathbf{u}}\cdot\boldsymbol{\xi})^2 + \underline{\rho}\,|\boldsymbol{\xi}|^2\,(\underline{\gamma} + \underline{K}\,|\boldsymbol{\xi}|^2)\right).$$

A necessary and sufficient condition for $\chi(\cdot;\boldsymbol{\xi})$ to have only purely imaginary roots τ is

$$\tag{27}\underline{\rho}\,|\boldsymbol{\xi}|^2\,(\underline{\gamma} + \underline{K}\,|\boldsymbol{\xi}|^2\,) \geq 0\,.$$

For $\underline{K} = 0$ we would be left the usual *hyperbolicity* condition for the (pure) Euler equations, namely $\underline{\rho}\,\underline{\gamma} = p_0'(\underline{\rho}\,) \geq 0$. This condition means that F_0 is convex at $\underline{\rho}$, and if it is the case, the *sound speed* is well defined at $\underline{\rho}$ by $c(\underline{\rho}\,) := \sqrt{p_0'(\underline{\rho}\,)}$. If we are to consider fluids for which F_0 is not convex, in particular those for which F_0 is a double-well potential, there are states $\underline{\rho}$ in between the Maxwell points violating the hyperbolicity condition for the Euler equations (this is a well-known drawback of the Euler equations when applied to van der Waals fluids below critical temperature for instance). For $\underline{K} > 0$, those states also violate (27) but only for 'small' frequencies, that is, for

$$|\boldsymbol{\xi}|^2 < \frac{-\underline{\gamma}}{\underline{K}} = \frac{-p_0'(\underline{\rho}\,)}{\underline{\rho}\,\underline{K}}\,.$$

This leaves hope that the Cauchy problem be well-posed for the Euler–Korteweg equations even with initial data containing *unstable states*, where p_0' is negative. As we shall see, this is far from being trivial though.

Sticking for the moment to the constant-coefficient linearised problem (23) about a strictly *hyperbolic state*, that is where $p_0' > 0$, we can convince ourselves that the Cauchy problem is well-posed in $H^1 \times L^2$ by using *semi-group theory*. Indeed, the operator $\mathbf{A}(\nabla)$ (where $\mathbf{A}$ is the matrix-valued symbol defined in (25)) turns out to be skew-adjoint for the inner product associated with the rescaled norm defined by

$$\|(\rho,\mathbf{u})\|^2 = \int (\underline{\gamma}\,\rho^2 + \underline{K}\,|\nabla\rho|^2 + \underline{\rho}\,|\mathbf{u}|^2)\,\mathrm{d}\mathbf{x}$$

on $H^1(\mathbb{R}^d)\times L^2(\mathbb{R}^d;\mathbb{R}^d)$. Therefore, by Stone's theorem [61, p. 41], $\mathbf{A}(\nabla)$ is the infinitesimal generator of a semi-group of unitary operators. In the degenerate case when $\underline{\gamma}=0$ (as though the fluid were 'pressureless'), we can in fact solve (24) by hand on the whole space $\mathbb{R}^d$ thanks to Fourier transform. As a consequence, we find *a priori* estimates in the form of *dispersion inequalities* for ρ, $\nabla\rho$, and $\mathscr{Q}\mathbf{u}$, where $\mathscr{Q}$ denotes the *Leray–Helmholtz projector* onto curl-free fields orthogonally to divergence-free fields, defined in Fourier variables by

$$\widehat{\mathscr{Q}\mathbf{u}}(\boldsymbol{\xi}) = \frac{\boldsymbol{\xi}\cdot\widehat{u}(\boldsymbol{\xi})}{|\boldsymbol{\xi}|^2}\,\boldsymbol{\xi}\,.$$

Those dispersion inequalities read

$$\begin{aligned}
\|\rho(t)\|_{L^\infty} &\leq C\,t^{-d/2}\left(\|\rho(0)\|_{L^1} + \|\nabla^{-1}\mathscr{Q}\mathbf{u}(0)\|_{L^1}\right),\\
\|\nabla\rho(t)\|_{L^\infty} &\leq C\,t^{-d/2}\left(\|\nabla\rho(0\|_{L^1} + \|\mathscr{Q}\mathbf{u}(0)\|_{L^1}\right),\\
\|\mathscr{Q}\mathbf{u}(t)\|_{L^\infty} &\leq C\,t^{-d/2}\left(\|\nabla\rho(0\|_{L^1} + \|\mathscr{Q}\mathbf{u}(0)\|_{L^1}\right),
\end{aligned}$$

where $C\propto \underline{a}^{-d}$, $\underline{a}:=\sqrt{\underline{\rho}\,\underline{K}}$ (this notation is chosen on purpose for later use). They mean that for sufficiently 'localised' initial data (namely, $(\rho_0, \mathscr{Q}\mathbf{u}_0=\nabla\varphi_0)$ with $(\rho_0,\varphi_0)\in W^{1,1}(\mathbb{R}^d;\mathbb{R}^2)$), the solution $(\rho,\nabla\rho,\mathscr{Q}\mathbf{u})$ decays algebraically to zero in L^∞ norm as t goes to infinity: this is the property usually required from what is called a dispersion inequality in the theory of dispersive PDEs. As to the divergence-free part of $\mathbf{u}$, if not zero initially it remains bounded away from zero, since it is only transported by $\underline{\mathbf{u}}$:

$$\|(\mathbf{u}-\mathscr{Q}\mathbf{u})(t)\|_{L^\infty} = \|(\mathbf{u}-\mathscr{Q}\mathbf{u})(0)\|_{L^\infty}\,.$$

See [12, § 3.1] for the technical details. These partial dispersive features are linked to the so-called *Kato smoothing effect* that will be discussed in § 2.2.4 below.

Dispersion is also (more directly) visible on the characteristic polynomial χ (defined in (26)), which admits two roots $\tau = iW(\boldsymbol{\xi})$ such that the associated *group velocities* $\nabla_{\boldsymbol{\xi}}W$ are not *phase velocities* if $\underline{\rho}\,\underline{K}$ is nonzero. Indeed, for

$$\begin{aligned}
W(\xi) &= -\underline{\mathbf{u}}\cdot\boldsymbol{\xi} \pm |\boldsymbol{\xi}|\sqrt{\underline{\rho}\,(\underline{\gamma}+\underline{K}|\boldsymbol{\xi}|^2)}\,,\\
\nabla_{\boldsymbol{\xi}}W &= -\underline{\mathbf{u}} \pm \frac{\underline{\rho}\,(\underline{\gamma}+2\underline{K}|\boldsymbol{\xi}|^2)}{\sqrt{\underline{\rho}\,(\underline{\gamma}+\underline{K}|\boldsymbol{\xi}|^2)}}\,\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\,,\\
\boldsymbol{\xi}\cdot\nabla_{\boldsymbol{\xi}}W - W(\boldsymbol{\xi}) &= \pm\frac{\underline{\rho}\,\underline{K}}{\sqrt{\underline{\rho}\,(\underline{\gamma}+\underline{K}|\boldsymbol{\xi}|^2)}}\,|\boldsymbol{\xi}|^3\,,
\end{aligned}$$

which is nonzero unless $\underline{\rho}\,\underline{K}=0$ (for Euler equations it is well-known that group velocities are phase velocities) or $\boldsymbol{\xi}=0$.

To be retained from this paragraph is that the Euler–Korteweg equations display both *hyperbolic* and *dispersive* features.

2.2 Cauchy problem

2.2.1 Extended systems

At present day it is not known how to deal with well-posedness directly on the system (22) (nor on its nonconservative form (18)). We have to consider an extended one.

As far as smooth solutions are concerned, if $(\rho, \mathbf{u})$ is one of them, by taking the gradient of the mass conservation law (1) in (22) we can always write the following equation for $\nabla\rho$:

$$\partial_t(\nabla\rho) + \nabla(\mathbf{u}\cdot\nabla\rho) + \nabla(\rho\,\mathrm{div}\mathbf{u}) = 0. \tag{28}$$

When F_0 is strictly convex (that is, $F_0'' > 0$), if $K > 0$ satisfies the further condition $KK'' \geq 2K'^2$, it turns out (see appendix) that the total energy density

$$H = F_0 + \tfrac{1}{2}K|\nabla\rho|^2 + \tfrac{1}{2}\rho|\mathbf{u}|^2$$

is a strictly convex function of $(\rho, \rho\mathbf{u}, \nabla\rho)$, and that the Euler–Korteweg equations (22) supplemented with the equation (28) for $\nabla\rho$ are *symmetrizable* by means of the Hessian of H. More precisely, writing $(\rho, \rho\mathbf{u}, \nabla\rho)$ as a column vector $\mathbf{W}$, the system (22)(28) can be written in abstract form as a *non-dissipative* second order system (using Einstein's convention on repeated indices)

$$\partial_t\mathbf{W} + \mathbf{A}_k(\mathbf{W})\,\partial_k\mathbf{W} + \partial_k\Big(\mathbf{B}_{k,\ell}(\mathbf{W})\,\partial_\ell\mathbf{W}\Big) = 0, \tag{29}$$

where the matrices $\mathbf{A}_k(\mathbf{W})$ and $\mathbf{B}_{k,\ell}(\mathbf{W})$ are such that

$$\mathbf{S}(\mathbf{W})\,\mathbf{A}_k(\mathbf{W}) \text{ is symmetric for all } k \in \{1,\dots,d\},$$

and for all vectors $\mathbf{X}^1, \dots, \mathbf{X}^d$ in $\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d$,

$$\langle \mathbf{X}^k, \mathbf{S}(\mathbf{W})\,\mathbf{B}_{k,\ell}(\mathbf{W})\,\mathbf{X}^\ell\rangle = 0, \tag{30}$$

where $\mathbf{S}(\mathbf{W}) := \mathrm{D}^2_{\mathbf{W}}H$. This was pointed out by Gavrilyuk and Gouin [38], see [12, § 2.1] for more details. It is (30) that characterizes the lack of dissipativity. Up to our knowledge there is no general theory of the Cauchy problem for such systems.

In fact, we can deal with the Cauchy problem for (18) with no convexity assumption on F_0. The analysis is based on another *extended system*, which is actually a reformulation of (22)(28) in slightly differently variables. We write the velocity equation as in (9), where

$$a = \sqrt{\rho K}, \quad \mathbf{v} = \sqrt{\frac{K}{\rho}}\nabla\rho,$$

and we introduce ζ so that $\mathbf{v} = \nabla\zeta$. Obviously this amounts to defining (up to a constant) $\zeta = R(\rho)$ where R is a primitive of $\rho \mapsto \sqrt{K(\rho)/\rho}$, which is well defined away from vacuum (that is, for $\rho > 0$) provided that K is continuous (in pratice, we assume K is $\mathscr{C}^\infty$) and $K(\rho) > 0$. The equation satisfied by ζ is merely obtained by multiplying the mass conservation law

(1) by $\sqrt{K(\rho)/\rho}$, and by differentiation in $\mathbf{x}$ we obtain an equation for $\mathbf{v}$ (equivalently, we can multiply (28) by $\sqrt{K(\rho)/\rho}$). The resulting extended system reads

$$(31)\qquad \begin{cases} \partial_t\zeta + \mathbf{u}\cdot\nabla\zeta + a\,\mathrm{div}\mathbf{u} = 0\,, \\ \partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla(\frac{1}{2}\,|\mathbf{v}|^2) - \nabla(a\,\mathrm{div}\mathbf{v}) = -\nabla g_0\,, \\ \partial_t\mathbf{v} + \nabla(\mathbf{u}\cdot\mathbf{v}) + \nabla(a\,\mathrm{div}\mathbf{u}) = 0\,. \end{cases}$$

The gradient term ∇g_0 has been put into the right-hand side because it can be written - and will be considered - as a term of order zero. Indeed, viewing g_0 as a function of ζ instead of ρ we have

$$\nabla g_0 = (g_0)'_\zeta\,\mathbf{v}.$$

Clearly, if $(\zeta,\mathbf{u},\mathbf{v})$ is a smooth solution of (31) for which $\mathbf{v}$ is curl-free initially (which will be the case if we take $\nabla\zeta_0$ as initial condition for $\mathbf{v}$), it remains so for all times (by applying the curl operator to the third equation we see that $\partial_t\nabla\times\mathbf{v} = 0$). So we may manipulate the equation for $\mathbf{v}$ under the compatibility assumption $\partial_j v_k = \partial_k v_j$. This allows us to write

$$\nabla(\mathbf{u}\cdot\mathbf{v}) = (\mathbf{u}\cdot\nabla)\mathbf{v} + (\nabla\mathbf{u})\,\mathbf{v}.$$

So the final form of the extended system is

$$(32)\qquad \begin{cases} \partial_t\zeta + \mathbf{u}\cdot\nabla\zeta + a\,\mathrm{div}\mathbf{u} = 0\,, \\ \partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla(\frac{1}{2}\,|\mathbf{v}|^2) - \nabla(a\,\mathrm{div}\mathbf{v}) = q\,\mathbf{v}, \\ \partial_t\mathbf{v} + (\mathbf{u}\cdot\nabla)\mathbf{v} + (\nabla\mathbf{u})\,\mathbf{v} + \nabla(a\,\mathrm{div}\mathbf{u}) = 0\,, \end{cases}$$

with $a = a(\zeta)$ and $q = q(\zeta) := -(g_0)'(\zeta)$. Note that in this form it is no longer obvious that curl-free fields are preserved by the equation on $\mathbf{v}$. Nevertheless, this is true thanks to the transport equation satisfied by the curl of $\mathbf{v}$ (*i.e.* $\nabla\times\mathbf{v}$ in three space dimensions) along the fluid flow, which is more safely found by using coordinates. For, the j-th component of the third equation in (32) is

$$\partial_t v_j + u_k\partial_k v_j + v_k\partial_j u_k + \partial_j(a\,\partial_k u_k) = 0\,,$$

which implies by differentiation that for any $i, j \in \{1,\ldots,d\}$,

$$\partial_t(\partial_i v_j - \partial_j v_i) + u_k\partial_k(\partial_i v_j - \partial_j v_i) + (\partial_i u_k)\,(\partial_k v_j - \partial_j v_k) + (\partial_i v_k - \partial_k v_i)\,(\partial_j u_k) = 0\,.$$

So the matrix-valued function $\mathbf{\Omega} = \mathrm{curl}\mathbf{v} := (\partial_i v_j - \partial_j v_i)_{i,j}$ solves the transport equation

$$\partial_t\mathbf{\Omega} + (\mathbf{u}\cdot\nabla)\mathbf{\Omega} + (\nabla\mathbf{u})\mathbf{\Omega} + \mathbf{\Omega}\,(\nabla\mathbf{u})^T = 0\,.$$

Therefore, integrating by parts we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}|\mathbf{\Omega}|^2\,\mathrm{d}\mathbf{x} - \frac{1}{2}\int_{\mathbb{R}^d}(\mathrm{div}\mathbf{u})\,|\mathbf{\Omega}|^2\,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^d}(\nabla\mathbf{u})\mathbf{\Omega} + \mathbf{\Omega}\,(\nabla\mathbf{u})^T) : \mathbf{\Omega}\,\mathrm{d}\mathbf{x} = 0\,,$$

hence

$$\|\mathbf{\Omega}(t)\|_{L^2}^2 \leq \|\mathbf{\Omega}(0)\|_{L^2}^2 \, \mathrm{e}^{\int_0^t (\|\mathrm{div}\mathbf{u}(s)\|_{L^\infty} + 4\|\nabla\mathbf{u}(s)\|_{L^\infty})\,\mathrm{d}s}$$

as long as $\mathbf{u}$ belongs to $L^1_t W^{1,\infty}_{\mathbf{x}}$. This shows that $\mathbf{\Omega}$ remains null if it is so initially.

The advantage of (32) over (31) lies in the observation that for a solution of (31) the complex-valued field $\mathbf{z} := \mathbf{u} + i\mathbf{v}$ satisfies a (degenerate) Schrödinger-like equation, namely

$$\partial_t \mathbf{z} + (\mathbf{u}\cdot\nabla)\mathbf{z} + i\,(\nabla\mathbf{z})\,\mathbf{v} + i\,\nabla(a\,\mathrm{div}\mathbf{z}) = q\,\mathbf{v}. \tag{33}$$

To be more explicit (33) reads, using coordinates,

$$\partial_t z_j + u_k \partial_k z_j + i\,v_k \partial_j z_k + i\,\partial_j(a\partial_k z_k) = q\,v_j\,, \quad j \in \{1,\ldots,d\}. \tag{34}$$

Would $\mathbf{u}$, $\mathbf{v}$, a, and q be fixed functions of $\mathbf{x}$, Eq. (34) could be seen, at least in the case $d = 1$, as a (generalised) *Schrödinger equation*. It is to be noted though that the first order terms in (34) are more 'complex' than usual (if one compares with the equations considered in [66] for instance, the complex-valued coefficient $i\,v$ in the first order part is nonstandard). In fact, it is well-known that we can get into trouble with such terms, even with constant coefficients. As pointed out for instance in [50, ch. 2], the generalised Schrödinger equation

$$\partial_t z + i\,v\,\partial_x z + i\,\partial_x^2 z = 0 \tag{35}$$

does not satisfy the *Petrowsky condition* if $v > 0$ (by Fourier transform we find that $\widehat{z}(t,\xi) = \mathrm{e}^{t(v\xi + i\xi^2)}\widehat{z}(0,\xi)$ is exponentially growing in time t), and thus the Cauchy problem is ill-posed in all Sobolev spaces.

Furthermore, in several space dimensions ($d \geq 2$), the higher order coupling in the system (34) is somewhat degenerate. A less degenerate situation would be if we had $\mathrm{div}(a\nabla\mathbf{z})$ in (33) instead of $\nabla(a\mathrm{div}\mathbf{z})$.

2.2.2 Main well-posedness result

We are concerned here with the Cauchy problem for (18) in the whole space $\mathbb{R}^d$, from the point of view of 'smooth', classical solutions, and more precisely of solutions whose difference with a reference one (*e.g.* a constant but not only) are in Sobolev spaces $H^s(\mathbb{R}^d)$ (to be more correct, velocities are in H^s and densities are in H^{s+1}) . The main result, stated below in detail, contains *local-in-time well-posedness* for $s > d/2+1$, together with a *blow-up criterion*

$$\lim_{t\nearrow T_*} \|(\nabla^2\rho, \nabla\mathbf{u})\|_{L^1(0,t;L^\infty(\mathbb{R}^d))} = +\infty$$

for maximal solutions defined only on $[0, T_*)$, as though $(\nabla\rho, \mathbf{u})$ were solution of a *symmetrizable hyperbolic* (first order) system (see for instance [13, 57]).

We assume that $(\underline{\rho}, \underline{\mathbf{u}})$ is a special, smooth solution of (18) on a fixed time interval $[0, \underline{T}]$ having the following properties

- the density $\underline{\rho}$ is bounded away from zero, and we denote by I an open interval such that

$$\underline{\rho}([0,\underline{T}] \times \mathbb{R}^d) \Subset I\,,$$

- both $\nabla^2\underline{\rho}$ and $\nabla\underline{\mathbf{u}}$ belong to $\mathscr{C}([0,\underline{T}];H^{s+3}(\mathbb{R}^d;\mathbb{R}^{d\times d}))$ with $s>d/2+1$.

Constants are obviously admissible, and we shall see in § 3 that travelling waves provide less trivial examples (with $\underline{T}$ arbitrarily large).

For simplicity, we introduce the notation

$$H_s := H^{s+1}(\mathbb{R}^d)\times H^s(\mathbb{R}^d;\mathbb{R}^d)\,.$$

Theorem 1 ([10])**. Existence and uniqueness** *For all initial data* $(\rho_0,\mathbf{u}_0)\in(\underline{\rho},\underline{\mathbf{u}})(0)+H_s$ *such that*

$$\rho_0([0,\underline{T}]\times\mathbb{R}^d)\Subset I\,,$$

there exists $T\in(0,\underline{T}]$ *and a unique solution* $(\rho,\mathbf{u})$ *of* (18) *on the time interval* $[0,T]$ *such that*

$$(\rho,\mathbf{u})\in(\underline{\rho},\underline{\mathbf{u}})+\mathscr{C}([0,T];H_s)\cap\mathscr{C}^1([0,T];H_{s-2})\,.$$

Well-posedness *There exists a neighbourhood of* $(\rho_0,\mathbf{u}_0)$ *in* $(\underline{\rho},\underline{\mathbf{u}})(0)+H_s$ *such that the existence time is uniform for initial data in this neighbourhood, and the solution map*

$$\begin{array}{ccc}(\underline{\rho},\underline{\mathbf{u}})(0)+H_s & \longrightarrow & (\underline{\rho},\underline{\mathbf{u}})+\mathscr{C}([0,T];H_s)\cap\mathscr{C}^1([0,T];H_{s-2})\\ (\rho_0,\mathbf{u}_0) & \longmapsto & (\rho,\mathbf{u})\end{array}$$

is continuous.

Blow-up criterion *If the maximal time of existence* T_* *is finite, then one of the following conditions fails (in the case of a potential flow, one of the first two must fail):*

$$\int_0^{T_*}(\|\Delta\rho(t)\|_{L^\infty}+\|\mathrm{curl}\mathbf{u}(t)\|_{L^\infty}+\|\mathrm{div}\mathbf{u}(t)\|_{L^\infty}\,\mathrm{d}t<+\infty\,,\tag{36}$$

$$\rho([0,T_*)\times\mathbb{R}^d)\Subset I\,,\tag{37}$$

$$\exists\alpha\in(0,1)\,,\ \sup_{t\in[0,T_*)}\|\rho(t)\|_{C^\alpha}<+\infty\,,\tag{38}$$

where C^α *denotes the Hölder space of index* α.

The proof is rather long and technical. It is based on the extended system in (32), and the main ingredients are the following.

- *A priori* estimates without loss of derivatives. These are the crucial part and require a lot of care. At the energy level (*i.e.* in H_0), they are rather easy to obtain thanks to an L^2 estimate for $\sqrt{\rho}\mathbf{z}$ in (33). Note however that a brutal L^2 estimate of $\mathbf{z}$ (without $\sqrt{\rho}$ in factor) would not work, because of the first order terms in (33). So we already see that we need use some weighted norms, the 'weight' (or *gauge function*, using the same term as in [54]) being $\sqrt{\rho}$ at the first level. This is also true for the higher order estimates (in H_s, $s>0$), all the more so that, at least for non-constant a, the second order term in (33) yields bad commutators. The fix lies in the use of two weights, one ($\sqrt{\rho\,a^s}$) for the curl-free part of $\mathbf{z}$, and the other one (more complicated in general but degenerating to a constant if a is constant) for the divergence-free part of $\mathbf{z}$.

- Construction of approximate solutions by means of a regularised system. Using the semi-group associated with the biharmonic operator $-\varepsilon\,\Delta^2$ and a great deal of *a priori* estimates (obtained using the weights introduced for the $\varepsilon = 0$ case), it is possible to construct (by means of the *contraction method*) approximate solutions $(\rho_\varepsilon, \mathbf{u}_\varepsilon)$ on a time interval independent of $\varepsilon > 0$, which depend Lipschitz-continuously on the initial data and satisfy the blow-up criterion announced for the $\varepsilon = 0$ case.

- Uniqueness proof and blow-up criterion are obtained by means of *a priori* estimates and Gronwall's inequality. Tricky inequalities involving the homogeneous Besov space $\dot{B}^0_{\infty,\infty}$ are used to obtain the sharp criterion in (36) - which may be viewed as the analogue of the celebrated *Beale–Kato–Majda* criterion [6] for Euler equations. A cruder and easier to obtain one is $\int_0^{T_*} \|\nabla \mathbf{z}(t)\|_{L^\infty}\,\mathrm{d}t < +\infty$.

- Existence proof is by solving the regularised system for mollified initial data, with a *Friedrichs mollifier* parametrised by a suitably chosen fractional power of ε - which is actually part of the so-called *Bona–Smith method* [21]. The resulting family of approximate solutions is then shown to satisfy the Cauchy criterion, and the limit is the sought, exact solution.

- Continuity of the solution map is shown by using the Bona–Smith method mentioned above.

In addition, provided that $(\underline{\rho}, \underline{\mathbf{u}})(t)$ is defined for all $t \geq 0$ and has constant Sobolev and Hölder norms (*e.g.* $(\underline{\rho}, \underline{\mathbf{u}})$ is constant or is a travelling wave), there exists a lower bound for the maximal time of existence depending continuously on $\eta := \|(\rho_0, \mathbf{u}_0) - (\underline{\rho}, \underline{\mathbf{u}})(0)\|_{H_s}$, that bound being at least of the order of $-\log\eta$ for small η. This is a by-product of the lower bound for the existence time of approximate solutions.

2.2.3 A priori estimates and gauge functions

Let us focus on the first step in the proof of Theorem 1, namely the derivation of a priori estimates without loss of derivatives by means of suitably chosen weights (gauge functions). In fact, the key ingredient lies in a priori estimates for a linear version of (33) with arbitrary right-hand side, namely

$$\partial_t \mathbf{z} + (\mathbf{u}\cdot\nabla)\mathbf{z} + i\,(\nabla \mathbf{z})\,\mathbf{v} + i\,\nabla(a\,\mathrm{div}\mathbf{z}) = \mathbf{f}, \tag{39}$$

where $\mathbf{z}$ is independent of the coefficients $\mathbf{u}$, $\mathbf{v}$, and a. As remarked above about the failure of the Petrowsky condition for (35), the first order terms, and especially $i\,(\nabla\mathbf{z})\,\mathbf{v}$, forbid a priori estimates in general. Nevertheless, we can derive a priori estimates under appropriate assumptions, which happen to fit the actual construction of solutions to (32). We start with the estimates of order 0. Recall that, as a primitive of $\rho \mapsto \sqrt{K(\rho)/\rho}$, R defines a diffeomorphism from I to another interval of $\mathbb{R}$, and thus $a = \sqrt{\rho K}$ can be seen as a function of $\zeta = R(\rho)$.

Lemma 1. *If* $\mathbf{z} \in \mathscr{C}^1([0,T];L^2(\mathbb{R}^d;\mathbb{C}^d)) \cap \mathscr{C}([0,T];H^2(\mathbb{R}^d;\mathbb{C}^d))$ *satisfies* (39) *with* $(\mathbf{u},\mathbf{v},\mathbf{f})$ *such that*

$$\mathbf{v} = \nabla\zeta\,, \quad \zeta = R\circ\rho\,, \quad \|(\partial_t\rho + \mathrm{div}(\rho\mathbf{u}))/\rho\|_{L^\infty(\mathbb{R}^d\times[0,T])} \leq C\,, \quad \rho(\mathbb{R}^d\times[0,T]) \Subset I\,,$$

and $\sqrt{\rho}\mathbf{f} \in L^\infty([0,T];L^2(\mathbb{R}^d;\mathbb{C}^d))$ *then*

$$\|\sqrt{\rho}\,\mathbf{z}(t)\|_{L^2} \leq \mathrm{e}^{Ct}\|\sqrt{\rho}\,\mathbf{z}(0)\|_{L^2} + \int_0^t \mathrm{e}^{C(t-\tau)}\|\sqrt{\rho}\,\mathbf{f}(\tau)\|_{L^2}\,\mathrm{d}\tau$$

for all $t \in [0,T]$.

Proof. Multiplying (39) by $\rho\,\bar{z}_j$ and summing on j we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho\,z_j\,\bar{z}_j = \int(\partial_t\rho)\,z_j\,\bar{z}_j - 2\,\mathrm{Re}\int\rho\,\bar{z}_j\,(u_k\partial_k z_j + i\,v_k\partial_j z_k + i\,\partial_j(a(\zeta)\partial_k z_k)$$

$$+2\,\mathrm{Re}\int\rho\,\bar{z}_j\,f_j\,.$$

Integrating by parts we find that

$$2\,\mathrm{Re}\int\rho\,\bar{z}_j\,u_k\partial_k z_j = -\int\partial_k(\rho u_k)\,z_j\,\bar{z}_j\,,$$

$$2\,\mathrm{Re}\int i\,\rho\,\bar{z}_j\,v_k\partial_j z_k = \int i\,\rho\,v_k\,(\bar{z}_k\partial_j z_j - z_k\partial_j\bar{z}_j)\,,$$

$$2\,\mathrm{Re}\int i\,\rho\,\bar{z}_j\,\partial_j(a(\zeta)\partial_k z_k) = -\int i\,a(\zeta)\,(\partial_k\rho)\,(\bar{z}_k\partial_j z_j - z_k\partial_j\bar{z}_j)$$

In the last but one equality here above we have used that $\rho\mathbf{v}$ is a gradient (as a function of ρ times $\nabla\rho$, by assumption), hence $\partial_j v_k = \partial_k v_j$ for all indices j, k. In addition, we have

$$\rho\,v_k = a(\zeta)\,\partial_k\rho \quad \forall k \in \{1,\ldots,d\}\,.$$

Therefore there is a cancellation between two integrals, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\rho\,z_j\,\bar{z}_j = \int(\partial_t\rho + \partial_k(\rho u_k))\,z_j\,\bar{z}_j + 2\,\mathrm{Re}\int\rho\,\bar{z}_j\,f_j\,.$$

The conclusion follows in a standard way from Cauchy-Schwarz inequality and Gronwall's lemma. □

For higher order estimates, we are going to make intense use of the Fourier multipliers Λ^s in $\mathbb{R}^d$, of symbol $\lambda^s(\boldsymbol{\xi}) := (1+|\boldsymbol{\xi}|^2)^{s/2}$. In particular, the H^s norms are easily defined by

$$\|u\|_{H^s(\mathbb{R}^d)} = \|\Lambda^s u\|_{L^2(\mathbb{R}^d)}\,.$$

Applying Λ^s to (39) we obtain

$$\partial_t \Lambda^s z_j + u_k \partial_k \Lambda^s z_j + i\, v_k \partial_j \Lambda^s z_k + i\, \partial_j (a \partial_k \Lambda^s z_k) = \tag{40}$$

$$\Lambda^s f_j + [u_k \partial_k, \Lambda^s]\, z_j + i\, [v_k \partial_j, \Lambda^s]\, z_k + i\, [\partial_j a \partial_k, \Lambda^s]\, z_k .$$

Here above, to simplify the writing, we have denoted a for $a\circ\zeta$: this abuse of notation will be used repeatedly in what follows. The left-hand side in (40) consists of the same operator as in (39), applied to $\Lambda^s\mathbf{z}$ instead of $\mathbf{z}$. In the right-hand side, there are several *commutators*[17]. Those involving u_k and v_k are basically of order s in $\mathbf{z}$ (as we learn from pseudodifferential calculus, the commutator of operators of order s and r is of order $s+r-1$, see for instance [3]), which is good because s is less than the order of operators on the left-hand side. But the last commutator is too demanding (being of order $s+1$) in general. It is its principal part that will dictate the appropriate choice of a weight for $s>0$. To compute it, we can use symbolic calculus (see again [3]), and we see that the principal part of $[a, \Lambda^s]$ is $s\,(\partial_m a)\Lambda^{s-2}\partial_m$ (with again Einstein's convention of summation over repeated indices). Therefore, since the operators Λ^s, ∂_j and ∂_k commute with each other, the main contribution of the commutator term $[\partial_j a \partial_k, \Lambda^s]\, z_k$ should be contained in

$$s\,\partial_j\big((\partial_m a) \Lambda^{s-2} \partial_m \partial_k z_k \big)\,.$$

This expression is not quite nice in general. Remarkably enough, it simplifies when $\mathbf{z}$ is potential, because then Re$\mathbf{z}$ and Im$\mathbf{z}$ are both *curl-free*, which implies

$$\Lambda^{s-2} \nabla \partial_k z_k = \Lambda^{s-2} \Delta \mathbf{z}$$

and thus, by writing $\Delta = -\Lambda^2 + 1$,

$$\Lambda^{s-2} \nabla \partial_k z_k = -\,\Lambda^s \mathbf{z} + \Lambda^{s-2} \mathbf{z}.$$

Therefore, in the case of a potential field $\mathbf{z}$, the expected main contribution of $[\partial_j a \partial_k, \Lambda^s]\, z_k$ is

$$-\,s\,(\partial_m a)\,\partial_j(\Lambda^s z_m)\,.$$

Thanks to this observation, the appropriate weight for the H^s estimate of a potential $\mathbf{z}$ turns out to be $\sqrt{\rho a^s}$, as stated in the following.

Lemma 2. *If* $\mathbf{z} \in \mathscr{C}^1([0,T]; H^s(\mathbb{R}^d;\mathbb{C}^d)) \cap \mathscr{C}([0,T]; H^{s+2}(\mathbb{R}^d;\mathbb{C}^d))$ *is such that*

$$\partial_j z_k = \partial_k z_j$$

for all indices j, $k \in \{1,\dots,d\}$ *and satisfies* (39) *with* $(\mathbf{u},\mathbf{v},\mathbf{f})$ *such that*

$$\mathbf{v} = \nabla\zeta\,,\quad \zeta = R\circ\rho\,,\quad \|(\partial_t\rho + \mathrm{div}(\rho\mathbf{u}))/\rho\|_{L^\infty(\mathbb{R}^d\times[0,T])} \leq C\,,\quad \rho(\mathbb{R}^d\times[0,T]) \Subset I\,,$$

[17] The notation $[A,B]$ for two operators that can be composed with each other means $A\circ B - B\circ A$.

and $\sqrt{\rho a^s}\Lambda^s \mathbf{f} \in L^\infty([0,T];L^2(\mathbb{R}^d;\mathbb{C}^d))$ *then*

$$\|\mathbf{z}(t)\|_s \le \mathrm{e}^{C_s t}\|\mathbf{z}(0)\|_s + \int_0^t \mathrm{e}^{C_s(t-\tau)}\|\mathbf{f}(\tau)\|_s \,\mathrm{d}\tau$$

for all $t \in [0,T]$*, where* $\|\cdot\|_s$ *stands for the norm defined by*

$$\|\mathbf{z}\|_s = \|\sqrt{\rho a^s}\,\Lambda^s \mathbf{z}\|_{L^2},$$

and

$$C_s = C + \|(\nabla^2 a, \nabla \mathbf{u}, \nabla \mathbf{v})\|_{L^\infty([0,T];E_s)}, \quad E_s = \begin{cases} H^{d/2}\cap L^\infty & \text{if } s \le d/2+1, \\ H^{s-1} & \text{if } s > d/2+1. \end{cases}$$

Sketch of proof. Similarly as in the case $s=0$, the aim is to estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho\, a^s\,(\Lambda^s z_j)\,(\Lambda^s \bar{z}_j^s).$$

For this we rewrite (40), as suggested by the observation made above, in the equivalent form

$$\partial_t \Lambda^s z_j + u_k \partial_k \Lambda^s z_j + i\,v_k \partial_j \Lambda^s z_k + i\,\partial_j(a\partial_k \Lambda^s z_k) + i\,s(\partial_k a)\partial_j(\Lambda^s z_k) = \Lambda^s f_j + r_j^s, \tag{41}$$

the remainder term being given by

$$r_j^s := [u_k\partial_k, \Lambda^s]\,z_j + i\,[v_k\partial_j, \Lambda^s]\,z_k + i\,[\partial_j a\partial_k, \Lambda^s]\,z_k + i\,s\,(\partial_k a)\,\partial_j(\Lambda^s z_k).$$

A preliminary task would be to estimate r_j^s. We skip this (very) technical part here, and refer the interested reader to [10].

In what follows, we use the simplifying notation

$$z_j^s = \Lambda^s z_j.$$

Multiplying (41) by $\rho\, a^s\, \bar{z}_j^s$ and summing on j we get, after integration by parts (similar as those performed in the proof of Lemma 1),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho\, a^s\, z_j^s\, \bar{z}_j^s &= \int (\partial_t(\rho\, a^s) + \partial_k(\rho\, a^s\, u_k))\, z_j^s\, \bar{z}_j^s \\ &- \int i\,\rho\, a^s\, v_k\,(\bar{z}_k^s \partial_j z_j^s - z_k^s \partial_j \bar{z}_j^s) + \int i\,a\partial_k(\rho\, a^s)\,(\bar{z}_k^s \partial_j z_j^s - z_k^s \partial_j \bar{z}_j^s) \\ &- \int i\,s\,\rho\, a^s\,(\partial_k a)\,(\bar{z}_j^s \partial_j z_k^s - z_j^s \partial_j \bar{z}_k^s) + 2\,\mathrm{Re}\int \rho\, \bar{z}_j\,(f_j^s + r_j^s).\end{aligned}$$

Using once more that $\rho\, v_k = a\partial_k \rho$, and integrating by parts in the penultimate integral, we see that three middle integrals cancel out, hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho\, a^s\, z_j^s\, \bar{z}_j^s = \int (\partial_t(\rho\, a^s) + \partial_k(\rho\, a^s\, u_k))\, z_j^s\, \bar{z}_j^s + 2\,\mathrm{Re}\int \rho\, \bar{z}_j\,(f_j^s + r_j^s).$$

The conclusion follows from the estimate of the remainder r_j^s (which has been omitted here), Cauchy-Schwarz' inequality and Gronwall's lemma. □

For nonpotential fields $\mathbf{z}$, the derivation of *a priori* estimates without loss of derivatives is even trickier, and in general necessitates different weights for the potential part $\mathscr{Q}\mathbf{z}$ and for the divergence-free part $\mathscr{P}\mathbf{z} := \mathbf{z} - \mathscr{Q}\mathbf{z}$ of $\mathbf{z}$. More precisely, it turns out that it is possible to estimate $\|\mathscr{Q}(\sqrt{\rho a^s}\Lambda^s\mathbf{z})\|_{L^2}$ (similarly as above), as well as $\|\mathscr{P}(\sqrt{A^s}\Lambda^s\mathbf{z})\|_{L^2}$, where A_s is a primitive of

$$\rho \mapsto a(\rho)^s - \rho\,\frac{\mathrm{d}}{\mathrm{d}\rho}a(\rho)^s .$$

Note that in the special case $a \equiv$constant (which means $K \propto 1/\rho$), both weights can be taken equal to $\sqrt{\rho}$, the weight of estimates of order 0. To get final H^s estimates, we then invoke the following equivalence inequalities

$$\|\mathscr{Q}(\sqrt{\rho a^s}\Lambda^s\mathbf{z})\|_{L^2}^2 + \|\mathscr{P}(\sqrt{A^s}\Lambda^s\mathbf{z})\|_{L^2}^2 \lesssim \|\mathbf{z}\|_{H^s}^2 ,$$

$$\|\mathbf{z}\|_{H^s}^2 \lesssim \|\mathscr{Q}(\sqrt{\rho a^s}\Lambda^s\mathbf{z})\|_{L^2}^2 + \|\mathscr{P}(\sqrt{A^s}\Lambda^s\mathbf{z})\|_{L^2}^2 + \|\nabla\rho\|_{C^{-\alpha}}^2\,\|\mathbf{z}\|_{H^{s-1+\alpha}}^2 ,$$

for $\alpha \in [0,1)$. To complete this (very) sketchy description, the reader is referred to [10].

2.2.4 Kato smoothing effect

After the work of Kato [47] on the Korteweg–de Vries equation, we usually call *Kato smoothing effect* for an evolution PDE

$$\partial_t u = P(\nabla)\,u ,$$

the property that for any cut-off function

$$\chi(t,\mathbf{x}) = \chi_0(t)\,\chi_1(x_1)\ldots\chi_d(x_d) , \quad \chi_j \in \mathscr{C}_0^\infty(\mathbb{R}) ,$$

there exists $C > 0$ so that the solutions $u(t) = \mathrm{e}^{tP(\nabla)}\,u_0$ satisfy the space-time estimates

$$\|\chi\,\Lambda^{s+\varepsilon}\,u\|_{L^2(\mathbb{R}^{d+1})} \leq C\,\|u_0\|_{H^s(\mathbb{R}^d)}$$

for some $\varepsilon > 0$ independent of $s \in \mathbb{R}$ (in general ε depends only on the order of P), where $\Lambda^{s+\varepsilon}$ operates in the space $\mathbb{R}^d$ only. Recall that for all $s > 0$, Λ^s plays the role of a (fractional) 'differentiation' operator of order s, and that

$$\|v\|_{H^s(\mathbb{R}^d)} = \|\Lambda^s v\|_{L^2(\mathbb{R}^d)}$$

(even for nonpositive s). Thus the Kato smoothing effect as described above means a (local) gain of ε derivatives. It was proved by Constantin and Saut [28] to hold true for a rather large class of dispersive PDEs, namely for operators P of symbol $P(i\boldsymbol{\xi}) \in i\mathbb{R}$ such that $iP(i\boldsymbol{\xi}) \sim |\boldsymbol{\xi}|^m$ for $|\boldsymbol{\xi}| \gg 1$, with $\varepsilon = (m-1)/2$ (if $m > 1$). Since then, there have been important efforts to derive micro-local smoothing effects for variable-coefficient operators, see in particular the seminal work by [29]. As far as the Euler–Korteweg equations are concerned, smoothing effects are not well understood yet (even though recent results on the water wave equations with surface tension [1] give promising insight). However, it is possible to show a partial smoothing effect for the linearised system about constant states. More precisely, let us consider the extended version of (23),

$$
\left\{
\begin{aligned}
&\partial_t \zeta + \underline{\mathbf{u}} \cdot \nabla \zeta + \underline{a}\, \mathrm{div} \mathbf{u} = 0\,,\\
&\partial_t \mathbf{u} + (\underline{\mathbf{u}} \cdot \nabla) \mathbf{u} - \nabla(\underline{a}\, \mathrm{div} \mathbf{v}) = \underline{q}\, \mathbf{v}\,,\\
&\partial_t \mathbf{v} + (\underline{\mathbf{u}} \cdot \nabla) \mathbf{v} + \nabla(\underline{a}\, \mathrm{div} \mathbf{u}) = 0\,,
\end{aligned}
\right.
\tag{42}
$$

which must be satisfied by $\zeta := R'(\underline{\rho})\,\rho$, $\mathbf{u}$, and $\mathbf{v} := \nabla \zeta$ for $(\rho, \mathbf{u})$ to be solution to (23). (Alternatively, (42) may be viewed as the linearisation of (32) about $\zeta := R(\underline{\rho})$, $\mathbf{v} = 0$.) Up to the change of Galilean frame $\mathbf{x} \mapsto \mathbf{x} - \underline{\mathbf{u}}\, t$, we may assume without loss of generality that $\underline{\mathbf{u}} = 0$. Then we are left with the degenerate Schrödinger system

$$
\left\{
\begin{aligned}
&\partial_t \mathbf{u} - \nabla(\underline{a}\, \mathrm{div} \mathbf{v}) = \underline{q}\, \mathbf{v}\,,\\
&\partial_t \mathbf{v} + \nabla(\underline{a}\, \mathrm{div} \mathbf{u}) = 0\,,
\end{aligned}
\right.
\tag{43}
$$

which turns out to have a smoothing effect on potential parts of $(\mathbf{u}, \mathbf{v})$.

Theorem 2 (Audiard[5]). *Assuming that* $\underline{a} > 0$, *if* $\mathbf{u}_0$, $\mathbf{v}_0 \in H^s(\mathbb{R}^d)$ *are curl-free, then for any cut-off function*

$$
\chi(t, \mathbf{x}) = \chi_0(t)\, \chi_1(x_1) \ldots \chi_d(x_d)\,, \quad \chi_j \in \mathscr{C}_0^\infty(\mathbb{R})\,,
$$

there exists $C > 0$ *so that the solution* $(\mathbf{u}, \mathbf{v})$ *of* (43) *satisfies the space-time estimates*

$$
\|\chi\, \Lambda^{s+1/2}\, (\mathbf{u}, \mathbf{v})\|_{L^2(\mathbb{R}^{d+1})} \leq C\, \|(\mathbf{u}_0, \mathbf{v}_0)\|_{H^s(\mathbb{R}^d)}\,.
$$

2.3 Initial-boundary value problem

Physically, flows hardly ever occur in the whole space. Moreover, if applied scientists want to make numerical simulations, they also need boundaries even for the Cauchy problem, just because computers can only handle a finite number of data. We then speak of *artificial boundaries* . This makes at least two reasons for investigating not only the Cauchy problem in $\mathbb{R}^d$ but also *mixed problems*, involving both initial data (at $t = 0$) and boundary data, on the boundary of the spatial domain in which the fluid flows. This boundary can be 'physical' (a solid one like a wall, or an immaterial one like the entrance/exit of a tube), in which case we are interested in physically achievable boundary data (thermodynamical ones like pressure or temperature for instance, or fluid flow, etc.). If it is artificial (for numerical purposes) then the important topic is to find appropriate boundary conditions that do not alter too much the expected solution on the whole space. We speak of *transparent boundary conditions* if they do not affect at all the solution, and of *absorbing boundary conditions* if they affect it moderately (*i.e.* with few, or small, reflected waves inside the domain). In fact, the two situations can be mixed together, as is well-known in the theory of hyperbolic PDEs: for example in the case of Euler equations, depending on the nature of the flow, we cannot prescribe all the unknowns at the entrance or exit of a tube, otherwise the initial-boundary value problem would be ill-posed from a mathematical point of view; but for numerical experiments we do need a full set of data on the boundary, and the extra data they had better be as transparent as possible.

For hyperbolic PDEs, initial-boundary value problems are tough but already rather well understood (see [13] and references therein). For dispersive PDEs, the mathematical analysis of initial-boundary value problems is in its infancy, see for instance the articles [19, 34], and the books [36, 41], also see [4] for the more focused topic of artificial boundary conditions for the Schrödinger equation.

Regarding the Euler–Korteweg equations, we can first investigate the mixed problem for the linearised system (23), which we recall here for convenience

$$\left\{\begin{array}{l} \partial_t \rho + \underline{\mathbf{u}}\cdot\nabla\rho + \underline{\rho}\ \mathrm{div}\mathbf{u} = 0\,, \\[2ex] \partial_t\mathbf{u} + (\underline{\mathbf{u}}\cdot\nabla)\,\mathbf{u} + \underline{\gamma}\,\nabla\rho = \underline{K}\,\nabla\Delta\rho\,, \end{array}\right.$$

on a half-space $\{\mathbf{x}\in\mathbb{R}^d;\ x_d > 0\}$. Following the usual method for initial-boundary value problems we may take the *Fourier–Laplace transform* of this system, and more precisely its Laplace transform in t and its Fourier transform in directions $x_1,\ldots,x_{d-1}$. This yields the ODE system for the Fourier–Laplace transforms of ρ, $\check{\mathbf{u}} := (u_1,\ldots,u_{d-1})^T$, $u := u_d$, to which we do not put hats for readability,

$$\left\{\begin{array}{l} \tau\,\rho + i\,(\underline{\mathbf{u}}\cdot\boldsymbol{\eta})\rho + \underline{u}\partial_d\rho + i\,\underline{\rho}\ \boldsymbol{\eta}\cdot\mathbf{u} + \underline{\rho}\ \partial_d u = \rho_0\,, \\[2ex] \tau\,\check{\mathbf{u}} + i\,(\underline{\mathbf{u}}\cdot\boldsymbol{\eta})\,\check{\mathbf{u}} + \underline{u}\partial_d\check{\mathbf{u}} + \ i\,\underline{\gamma}\,\rho\,\check{\boldsymbol{\eta}} + i\,\underline{K}\,(|\boldsymbol{\eta}|^2\rho - \partial_d^2\rho)\,\check{\boldsymbol{\eta}} = \check{\mathbf{u}}_0\,, \\[2ex] \tau\,u + i\,(\underline{\mathbf{u}}\cdot\boldsymbol{\eta})\,u + \underline{u}\partial_d u + \underline{\gamma}\,\partial_d\rho + \ \underline{K}\,(|\boldsymbol{\eta}|^2\,\partial_d\rho - \partial_d^3\rho) = u_0\,, \end{array}\right.$$

where $\tau\in\mathbb{C}$ (of real part bounded by below) denotes the dual variable to t, $\check{\boldsymbol{\eta}} = (\eta_1,\ldots,\eta_{d-1})^T \in \mathbb{R}^{d-1}$ is the dual variable to $\mathbf{y} := (x_1,\ldots,x_{d-1})$, and $\boldsymbol{\eta} = (\eta_1,\ldots,\eta_{d-1},0)^T \in \mathbb{R}^d$ (so that we actually have $|\boldsymbol{\eta}|^2 = |\check{\boldsymbol{\eta}}|^2$, $\mathbf{u}\cdot\boldsymbol{\eta} = \check{\mathbf{u}}\cdot\check{\boldsymbol{\eta}}$, and similarly with the underlined, reference velocities). In the system here above, $\rho_0, \check{\mathbf{u}}_0, u_0$ stand for the initial values of $\rho, \check{\mathbf{u}}, u$ in the space variables (*i.e.* before Fourier transformation), whereas in the left-hand side, $\rho, \check{\mathbf{u}}, u$, are in the Fourier–Laplace variables (recall that for readability we have refrained from putting hats). By change of Galilean frame ($\mathbf{y}\mapsto\mathbf{y}-\underline{\check{\mathbf{u}}}\,t$) or, equivalently, by changing τ into $\tau + i\,(\underline{\mathbf{u}}\cdot\boldsymbol{\eta})$ - which obviously does not change its real part -, we can assume that $\underline{\check{\mathbf{u}}} = 0$. This simplifies a little bit the above system, which can be rewritten as the following first order system

$$\mathbf{B}\,\partial_d \mathbf{U} = \mathbf{A}(\tau,\boldsymbol{\eta})\,\mathbf{U} + \mathbf{f}, \tag{44}$$

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \underline{K} & 0 & -\underline{u} \\ 0 & 0 & 0 & \underline{u}\mathbf{I}_{d_1} & 0 \\ 0 & 0 & 0 & 0 & \underline{\rho} \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} \rho \\ \partial_d \rho \\ \partial_d^2 \rho \\ i\check{\mathbf{u}} \\ u \end{pmatrix}, \tag{45}$$

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & (\underline{\gamma}+\underline{K}|\boldsymbol{\eta}|^2) & 0 & \tau \\ (\underline{\gamma}+\underline{K}|\boldsymbol{\eta}|^2)\check{\boldsymbol{\eta}} & 0 & -\underline{K}\check{\boldsymbol{\eta}} & -\tau\mathbf{I}_{d_1} & 0 \\ -\tau & -\underline{u} & 0 & -\underline{\rho}\,\check{\boldsymbol{\eta}}^T & 0 \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} 0 \\ 0 \\ -\rho_0 \\ i\check{\mathbf{u}}_0 \\ u_0 \end{pmatrix}. \tag{46}$$

The boundary $\{x_d = 0\}$ is *non-characteristic* for (44) provided that $\mathbf{B}$ be invertible, which is equivalent to $\underline{\rho}\,\underline{K}\,\underline{u} \neq 0$. From now on we assume this holds true, and more precisely that $\underline{\rho} > 0$ (the reference state is away from vacuum), $\underline{K} > 0$ (there is capillarity), $\underline{u} \neq 0$ (in the reference state, the fluid is moving transversally to the boundary). Then the characteristic polynomial of $\mathbf{B}^{-1}\mathbf{A}(\tau,\boldsymbol{\eta})$ is, unsurprisingly,

$$\pi(\omega;\tau,\boldsymbol{\eta}) = \chi(\tau;\check{\boldsymbol{\eta}},-i\omega) = (\tau+\underline{u}\omega)^{d-1}\Big((\tau+\underline{u}\omega)^2 + \underline{\rho}\,(|\boldsymbol{\eta}|^2-\omega^2)\,(\underline{\gamma}+\underline{K}\,(|\boldsymbol{\eta}|^2-\omega^2))\Big), \tag{47}$$

where χ is the polynomial defined in (26) (in which $|\boldsymbol{\xi}|^2$ is really polynomial and equals $\sum_j \xi_j^2$, with no moduli when extended to complex components, hence the expression here above of $\chi(\tau;\boldsymbol{\xi})$ for $\boldsymbol{\xi} = (\eta_1,\ldots,\eta_{d-1},-i\omega)^T$).

Proposition 1. *We assume here that $\underline{\gamma}$ is nonnegative (which means that the reference density $\underline{\rho}$ is a thermodynamically weakly* stable state*). Then for all $(\tau,\boldsymbol{\eta}) \in \mathbb{C}\times\mathbb{R}^{d-1}$ with* $\mathrm{Re}\tau > 0$, *the matrix $\mathbf{B}^{-1}\mathbf{A}(\tau,\boldsymbol{\eta})$ is* hyperbolic, i.e. *has no purely imaginary eigenvalue, and its stable subspace $E_s(\tau,\boldsymbol{\eta})$ is such that*

Inflow case $\dim E_s(\tau,\boldsymbol{\eta}) = d+1$ *if* $\underline{u} > 0$,

Outflow case $\dim E_s(\tau,\boldsymbol{\eta}) = 2$ *if* $\underline{u} < 0$.

Proof. Obviously, the roots of $\pi(\cdot;\tau,\boldsymbol{\eta})$ are

- if $d \geq 2$, $\omega_0 = -\tau/\underline{u}$, which contributes to the stable subspace if and only if $\underline{u} > 0$ (recall that we have assumed $\underline{u} \neq 0$),
- and the roots of the fourth order polynomial

$$\widetilde{\pi}(\omega;\tau,\boldsymbol{\eta}) = (\tau+\underline{u}\omega)^2 + \underline{\rho}\,(|\boldsymbol{\eta}|^2-\omega^2)\,(\underline{\gamma}+\underline{K}\,(|\boldsymbol{\eta}|^2-\omega^2))\,,$$

  which has no purely imaginary root unless τ itself is purely imaginary (recall that we have assumed $\underline{\rho}\,\underline{K} > 0$, and $\underline{\gamma} \geq 0$). By a continuity/connectedness argument we can then count the number of roots of negative real parts by considering the special case $\boldsymbol{\eta} = 0$, $|\tau| \gg 1$, for which we have asymptotically $\omega^4 \sim -\tau^2/(\underline{\rho}\,\underline{K})$. Thus we find two of them of positive real parts and two of negative real parts. This count is of course with multiplicity.

Furthermore, the corresponding eigenvectors are of the form

$$\mathbf{U}_0(\tau,\boldsymbol{\eta};\mathbf{v}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \tau\mathbf{v} \\ \underline{u}\,\check{\boldsymbol{\eta}}\cdot\mathbf{v} \end{pmatrix}$$

with $\mathbf{v} \in \mathbb{R}^{d-1}$, which clearly span an eigenspace of dimension $d-1$ for ω_0, and

$$\mathbf{U}(\omega;\tau,\boldsymbol{\eta}) = \begin{pmatrix} \underline{\rho} \\ \underline{\rho}\,\omega \\ \underline{\rho}\,\omega^2 \\ \dfrac{\tau+\underline{u}\omega}{\omega^2-|\boldsymbol{\eta}|^2}\,\check{\boldsymbol{\eta}} \\ -\dfrac{\tau+\underline{u}\omega}{\omega^2-|\boldsymbol{\eta}|^2}\,\omega \end{pmatrix}$$

for the roots of $\widetilde{\pi}(\cdot;\tau,\boldsymbol{\eta})$, at least when $\omega^2 \neq |\boldsymbol{\eta}|^2$. When $\omega^2 - |\boldsymbol{\eta}|^2$ goes to zero with $\widetilde{\pi}(\omega;\tau,\boldsymbol{\eta}) = 0$, necessarily $\tau + \underline{u}\omega$ goes to zero too - which means that ω goes to ω_0 -, and we see that

$$\frac{(\tau+\underline{u}\omega)^2}{\omega^2-|\boldsymbol{\eta}|^2} \to \underline{\rho}\,\underline{\gamma},$$

therefore

$$\widetilde{\mathbf{U}}(\omega;\tau,\boldsymbol{\eta}) := (\tau+\underline{u}\omega)\,\mathbf{U}(\omega;\tau,\boldsymbol{\eta}) \to \begin{pmatrix} 0 \\ 0 \\ 0 \\ \underline{\rho}\,\underline{\gamma}\,\check{\boldsymbol{\eta}} \\ \underline{\rho}\,\underline{\gamma}\,\frac{\tau}{\underline{u}} \end{pmatrix},$$

which coincides with $\mathbf{U}_0(\tau,\boldsymbol{\eta};\mathbf{v})$ for $\mathbf{v} = \underline{\rho}\,\underline{\gamma}\check{\boldsymbol{\eta}}/\tau$. Away from those points where ω and ω_0 collide, the vectors $\mathbf{U}(\omega;\tau,\boldsymbol{\eta})$ are clearly independent of $\mathbf{U}_0(\tau,\boldsymbol{\eta};\mathbf{v})$. In addition, if ω_1 and ω_2 are two distinct roots of $\widetilde{\pi}(\cdot;\tau,\boldsymbol{\eta})$, then by definition, $\mathbf{U}(\omega_1;\tau,\boldsymbol{\eta})$ and $\mathbf{U}(\omega_2;\tau,\boldsymbol{\eta})$ are also independent. Therefore, the dimension of the stable subspace of $\mathbf{B}^{-1}\mathbf{A}(\tau,\boldsymbol{\eta})$ only depends on the sign of $\mathrm{Re}\omega_0$. It equals 2 if $\mathrm{Re}\omega_0 > 0$, and $d-1+2 = d+1$ if $\mathrm{Re}\omega_0 < 0$. □

As a consequence of Proposition 1, we have the following.

Corollary 1. *For a boundary value associated with* (44) *to be well-posed, the number of prescribed boundary conditions must equal either* $d+1$ *if* $\underline{u} > 0$, *or* 2 *if* $\underline{u} < 0$, *and there cannot be nontrivial elements of* $\dim E_s(\tau, \boldsymbol{\eta})$ *meeting those conditions, or more precisely, being in the kernel of the (linear or linearised) boundary operator.*

If we compare with the usual Euler equations (which correspond to $\underline{K} = 0$), this makes one more condition for supersonic outflows, two more for subsonic outflows, one more for subsonic inflows, and the same number for supersonic inflows (see for instance [13, pp. 411-412]).

We can easily find examples of boundary operators satisfying the necessary conditions given in Corollary 1. In particular, we can take either the *Dirichlet* or the *Neumann* operator on $(\rho, \mathbf{u})$ for inflows, and on (ρ, u) for outflows.

To obtain sufficient conditions is much more delicate. We must pay attention to *a priori* estimates. This is where the notion of (generalised) *Kreiss symmetrizer* can come into play. However, many difficulties occur, in particular because of the non-homogeneity of the equations, contrary to those in hyperbolic BVPs like for the Euler equations, conjugated with very limited smoothing properties, contrary to what happens for hyperbolic-parabolic BVPs dealt with by Métivier and Zumbrun [59]. We refer to the doctoral thesis of C. Audiard for more details [5].

3 Traveling wave solutions to the Euler–Korteweg system

3.1 Profile equations

A traveling wave is characterized by a permanent profile propagating at a constant speed in some direction. We shall in fact concentrate on *planar traveling waves* , whose profiles depend on a single variable, and are governed by systems of ODEs in that variable.Among them, we find in particular *homoclinic profiles*, *heteroclinic profiles*, and *periodic profiles*. Homoclinic traveling waves are usually called *solitary waves*, or even *solitons* to account for their remarkable stability, which makes them viewed as almost material things like elementary particles (hence the suffix 'ton'). They occur in a variety of physical frameworks, the most famous being the water wave described by Russel[18] in 1844. His description is reproduced in many papers and textbooks on solitary waves, and was the motivation for the work by Boussinesq [23], and later on by Korteweg and de Vries[19] [48], who derived model equations nowadays bearing their names for the propagation of solitary water waves. We have already met the Boussinesq equation in these notes, which can be viewed as a special case of the Euler-Korteweg equations expressed in Lagrangian coordinates. The Korteweg–de Vries is another celebrated dispersive PDE known to admit solitons together with a Hamiltonian structure (in fact, *an infinity* of Hamiltonian structures, but we shall not enter into these algebraic features here, we refer for instance to [60]).

[18] John Scott Russel [1808–1882]
[19] Gustav de Vries [1866–1934]

Let us derive the profile equations for the Euler–Korteweg equations in (18). Consider a fixed direction $\mathbf{n} \in \mathbb{S}^{d-1}$ (the unit sphere of $\mathbb{R}^d$), a speed $\sigma \in \mathbb{R}$, and assume that $(\rho,\mathbf{u}) = (\rho,\mathbf{u})(\mathbf{x}\cdot\mathbf{n} - \sigma t)$ is a (smooth) solution of

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho\,\mathbf{u}) = 0\,, \\ \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = \nabla(-g_0 + \frac{1}{2} K'_\rho\, |\nabla\rho|^2 + K\,\Delta\rho)\,. \end{cases}$$

The first equation implies that $j := \rho\,(\mathbf{u}\cdot\mathbf{n} - \sigma)$ is a constant, *i.e.* is independent of $\xi = \mathbf{x}\cdot\mathbf{n} - \sigma t$. The quantity j is well-known in compressible fluid dynamics, and corresponds to the mass flux across the wave. It will be a parameter in the profile equations. From the remaining equations in (18), we see that

$$(\mathbf{u}\cdot\mathbf{n} - \sigma)\,\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\xi} = \frac{\mathrm{d}}{\mathrm{d}\xi}\left(-g_0 + \tfrac{1}{2} K'_\rho \left(\frac{\mathrm{d}\rho}{\mathrm{d}\xi}\right)^2 + K\,\frac{\mathrm{d}^2\rho}{\mathrm{d}\xi^2}\right)\mathbf{n}\,,$$

or equivalently

$$\frac{j}{\rho}\,\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\xi} = \frac{\mathrm{d}}{\mathrm{d}\xi}\left(-g_0 + \tfrac{1}{2} K'_\rho \left(\frac{\mathrm{d}\rho}{\mathrm{d}\xi}\right)^2 + K\,\frac{\mathrm{d}^2\rho}{\mathrm{d}\xi^2}\right)\mathbf{n}\,.$$

If $j = 0$ (no mass transfer across the wave), this readily implies an ODE for ρ,

$$K(\rho)\,\frac{\mathrm{d}^2\rho}{\mathrm{d}\xi^2} + \frac{1}{2}\,K'(\rho)\left(\frac{\mathrm{d}\rho}{\mathrm{d}\xi}\right)^2 = g_0(\rho) - \mu\,,$$

where μ is a constant. If $j \neq 0$, the equation above implies that $\mathbf{u}$ is constant in all directions orthogonal to $\mathbf{n}$. Then by change of Galilean frame we can always assume that $\mathbf{u}$ is zero in all directions orthogonal to $\mathbf{n}$. Now, denoting by $u = \mathbf{u}\cdot\mathbf{n}$ the normal velocity of the fluid, we have

$$u = \frac{j}{\rho} + \sigma\,,\quad \text{and} \quad \frac{j}{\rho}\,\frac{\mathrm{d}u}{\mathrm{d}\xi} = \frac{\mathrm{d}}{\mathrm{d}\xi}\left(-g_0 + \tfrac{1}{2} K'_\rho \left(\frac{\mathrm{d}\rho}{\mathrm{d}\xi}\right)^2 + K\,\frac{\mathrm{d}^2\rho}{\mathrm{d}\xi^2}\right),$$

which imply that

$$K(\rho)\,\frac{\mathrm{d}^2\rho}{\mathrm{d}\xi^2} + \frac{1}{2}\,K'(\rho)\left(\frac{\mathrm{d}\rho}{\mathrm{d}\xi}\right)^2 = g_0(\rho) + \frac{j^2}{2\rho^2} - \mu\,, \tag{48}$$

where again μ is a constant. Observe that the ODE obtained above for $j = 0$ is just a special case of this one. It turns out that the *phase portrait* of equation (48) does not depend on K. Indeed, we can equivalently write (48) as the system

$$\begin{cases} \dfrac{\mathrm{d}\rho}{\mathrm{d}\xi} = \frac{1}{\sqrt{K(\rho)}}\,\psi\,, \\ \dfrac{\mathrm{d}\psi}{\mathrm{d}\xi} = \frac{1}{\sqrt{K(\rho)}}\,(g_0(\rho) + \frac{j^2}{2\rho^2} - \mu)\,, \end{cases} \tag{49}$$

of which the phase portrait is the one of the vector field

$$\psi\,\frac{\partial}{\partial\rho} + \left(g_0(\rho) + \frac{j^2}{2\rho^2} - \mu\right)\frac{\partial}{\partial\psi}\,.$$

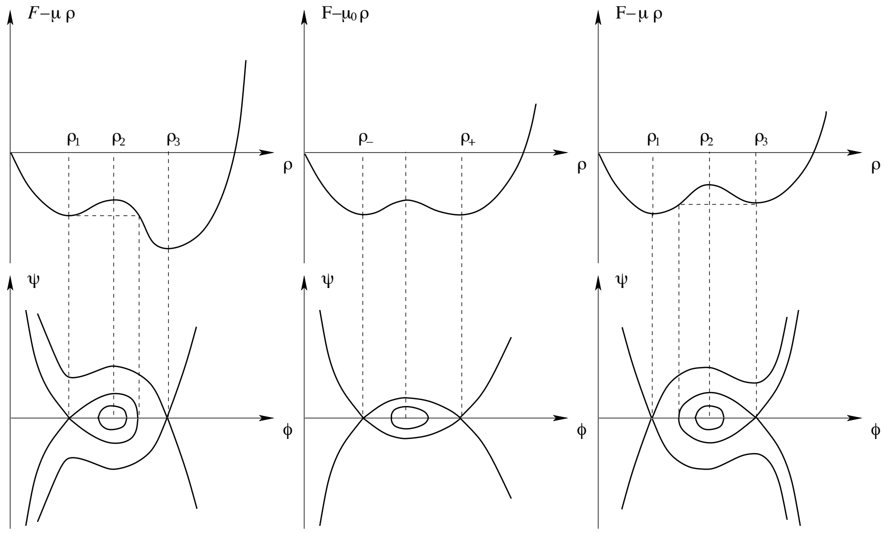

Figure 1: Double-well potentials and associated phase portraits: supersaturated ($\mu > \mu_0$, left), saturated ($\mu = \mu_0$, center), subsaturated ($\mu < \mu_0$, right).

Furthermore, recalling that g_0 is the derivative of F_0, we see that a first integral of (49) is

$$\frac{1}{2}\psi^2 - F_0(\rho) + \mu\rho + \frac{j^2}{2\rho},$$

so that the phase portrait of (48) can easily be obtained from the plotting of the function $\rho \mapsto F_0(\rho) - \mu\rho - \frac{j^2}{2\rho}$, see examples on Fig. 3.1 in the case $j = 0$. We can see orbits corresponding to each kind of wave mentioned at the beginning, namely homoclinic, heteroclinic, and periodic ones. In particular, the heteroclinic waves are prototypes of *propagating phase boundaries*, the endpoints being in different phases (on Fig. 3.1, ρ_- and ρ_+ are exactly the Maxwell points, and μ_0 is the slope of the bitangent of F_0 at those points). The physical interpretation of solitary and periodic waves is less clear, as far as capillary fluids are concerned. (Nevertheless, for related models such as the Gross–Pitaevskiĭ equation or the Boussinesq equation, solitary waves are of special interest.) A theoretical classification of solitary waves for van der Waals-like fluids has been given in [11] using Lagrangian coordinates, in which it is slightly easier to take into account the fourth fixed point occurring when $j \neq 0$ (on Fig. 3.1 there are only three fixed points because $j = 0$).

Eulerian *vs* Lagrangian coordinates. Even though this may not seem obvious at first glance, there is indeed a one-to-one correspondance between travelling waves in Eulerian coordinates and travelling waves in Lagrangian coordinates (for which ρ and $v = 1/\rho$ are both bounded). More precisely, to each wave with mass transfer flux j in Eulerian coordinates corresponds a wave with “speed” $-j$ in Lagrangian coordinates, see [11, p.386-387] for more details.

Anyhow, the existence of travelling waves follows from the phase portrait analysis of (49), which is easy since we have a first integral at hand. In fact, it is a general property that the governing ODEs for travelling profiles associated with Hamiltonian PDEs are themselves Hamiltonian: this was pointed out by Benjamin [7], see Proposition 4 in appendix for more details.

Abstract form of profile equations. From the computations made above, if a planar travelling wave $(\rho,\mathbf{u})$ of speed σ in the direction $\mathbf{n}$ is solution to the multi-dimensional Euler–Korteweg equations then $(\rho, u=\mathbf{u}\cdot\mathbf{n})$ is a travelling wave solution to the one-dimensional Euler–Korteweg equations

$$\left\{\begin{array}{l} \partial_t\rho + \partial_x(\rho\, u) = 0\,, \\ \partial_t u + u\partial_x u = \partial_x(-g_0 + \frac{1}{2}K'_\rho\,(\partial_x\rho)^2 + K\partial_x^2\rho)\,, \end{array}\right. \tag{50}$$

where x stands for $\mathbf{x}\cdot\mathbf{n}$. System (50) admits as a *Hamiltonian structure*, which is nothing but the one-dimensional version of the one in (19), namely

$$\partial_t\begin{pmatrix}\rho\\ u\end{pmatrix} = \mathscr{J}\,\delta\mathscr{H}[\rho,u]\,, \tag{51}$$

with

$$\mathscr{J} := -\mathrm{D}_x\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \mathscr{H} := \int_{\mathbb{R}} H\,\mathrm{d}x\,,\quad H(\rho,\partial_x\rho,u) := F_0(\rho) + \tfrac{1}{2}K(\rho)\,(\partial_x\rho)^2 + \tfrac{1}{2}\rho\,u^2\,.$$

So another way to write the profile equations is

$$\partial_\xi\left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\delta\mathscr{H}[\rho,u] - \sigma\begin{pmatrix}\rho\\ u\end{pmatrix}\right) = \begin{pmatrix}0\\ 0\end{pmatrix},$$

with x replaced by $\xi = x-\sigma t$ in $\delta\mathscr{H}[\rho,u]$, or equivalently,

$$\partial_\xi\left(\delta\mathscr{H}[\rho,u] - \sigma\begin{pmatrix}u\\ \rho\end{pmatrix}\right) = \begin{pmatrix}0\\ 0\end{pmatrix}.$$

Noting that $(u,\rho)^T = \delta\mathscr{I}[\rho,u]$ with $\mathscr{I} := \int_{\mathbb{R}}(\rho\,u)\,\mathrm{d}x$ (formally[20]), the profile equations amount to

$$\delta\mathscr{H}[\rho,u] - \sigma\,\delta\mathscr{I}[\rho,u] = \text{constant}. \tag{52}$$

The functional $\mathscr{I}$ is what has been called an *impulse* by Benjamin [7]. The fact that it is a conserved quantity of the underlying PDEs (which can readily be seen here from the local conservation law in (22)) is linked to their *translation invariance*. This can be seen in the following, both elementary and abstract way. For simplicity we shall use the notation $\mathbf{U} = (\rho,u)$. By definition of variational gradients we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{I}[\mathbf{U}] = \int_{\mathbb{R}}\delta\mathscr{I}[\mathbf{U}]\cdot\partial_t\mathbf{U}\,\mathrm{d}x$$

[20]We shall make $\mathscr{H}$ and $\mathscr{I}$ well defined, with convergent integrals, later on.

Now, since $\delta\mathscr{I}[\mathbf{U}] = \mathbf{J}\mathbf{U}$ where the matrix

$$\mathbf{J} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is symmetric and such that $\mathbf{J}^2 = \mathbf{I}$, if $\partial_t \mathbf{U} = \mathscr{J}\,\delta\mathscr{H}[\mathbf{U}] = -\mathrm{D}_x \mathbf{J}\,\delta\mathscr{H}[\mathbf{U}]$ then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{I}[\mathbf{U}] = -\int_{\mathbb{R}} \mathbf{U}\cdot \mathrm{D}_x \delta\mathscr{H}[\mathbf{U}]\,\mathrm{d}x\,,$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{I}[\mathbf{U}] = \int_{\mathbb{R}} \delta\mathscr{H}[\mathbf{U}]\cdot \partial_x \mathbf{U}\,\mathrm{d}x$$

after an integration by parts. Here comes into play the translation invariance. Indeed, (provided that we make rigourous the definition of $\mathscr{H}$) for all $s \in \mathbb{R}$, $\mathscr{H}[\mathbf{U}] = \mathscr{H}[\mathbf{U}_s]$, where $\mathbf{U}_s : (x,t) \mapsto \mathbf{U}(x+s,t)$, so that by differentiation with respect to s at $s=0$ we find

$$\int_{\mathbb{R}} \delta\mathscr{H}[\mathbf{U}]\cdot \partial_x \mathbf{U}\,\mathrm{d}x = 0\,.$$

The abstract form (52) of the profile equations will be useful in their subsequent stability analysis (by means of adapted versions of classical methods for Hamiltonian PDEs). Another useful, more explicit remark is that the endpoints of either heteroclinic or homoclinic waves correspond to *saddle fixed points*. In particular, saddles points of (49) are characterized as follows. A pair $(\rho_\infty, 0)$ is a saddle point for (49) if and only if the Jacobian matrix of the vector field

$$\psi\,\frac{\partial}{\partial\rho} + \left(g_0(\rho) + \frac{j^2}{2\rho^2} - \mu\right)\frac{\partial}{\partial\psi}$$

has real eigenvalues of opposite signs, which is clearly equivalent to

$$\det\begin{pmatrix} 0 & 1 \\ g_0'(\rho_\infty) - \frac{j^2}{\rho_\infty} & \end{pmatrix} < 0\,,$$

or, recalling that $j = \rho_\infty(u_\infty - \sigma)$,

$$p_0'(\rho_\infty) = \rho_\infty\, g_0'(\rho_\infty) > (u_\infty - \sigma)^2\,.$$

In other words, saddles correspond to *thermodynamically stable* ($p_0'(\rho_\infty) > 0$), *subsonic states* (the relative speed of the fluid with respect to the wave is lower than the *sound speed* $c(\rho_\infty) := \sqrt{p_0'(\rho_\infty)}$). This observation will be used in particular to localise *essential spectra* (a notion that will be recalled in due time). As a consequence of the *hyperbolicity*[21] of the endpoints, we also have that the limits $\lim_{x\to\pm\infty}(\underline{\rho}, u)(x)$ are attained exponentially fast.

[21] Recall that a fixed point of a system of ODEs, or equivalently of a vector field, is *hyperbolic* if the Jacobian matrix of this field has no purely imaginary spectrum. Saddle points are particular types of hyperbolic fixed points.

3.2 Stability issues

The stability of travelling waves in large time is crucial to know whether they have any chance to be observed experimentally. By stability we actually mean *orbital stability*, in say Sobolev spaces on $\mathbb{R}^d$. Indeed, both the equations and the Sobolev norms being invariant by spatial translations, we cannot expect *asymptotic stability* (which would mean that solutions starting close enough to a travelling profile tend to this very profile when time goes to infinity) but only that the solutions asymptotically approach the set of translates of travelling profiles. This is what happens for instance to stable viscous travelling waves (of reaction-diffusion equations or of second order conservation laws). In the absence of dissipation, we can only prove a weaker stability, namely that solutions starting close enough to a travelling profile remain arbitrarily close to the set of translated profiles. In our framework, this is what we shall call orbital stability.

As we shall see, there are several ways to tackle stability. The most efficient, when applicable, is the Lyapunov method: the energy functional $\mathscr{H}$ conserved by our equations being a good candidate for a *Lyapunov function*[22], the main difficulty is then to identify profiles that are local minimizers, up to translations[23], of that functional. This is hopeless, however, in several space dimensions because planar travelling waves do not fit the nice, variational interpretation (as critical points of $\mathscr{H}$ under constraints) they have in one dimension, and the functionals $\mathscr{H}$, $\mathscr{I}$ are not even well defined along those travelling waves (the integrals diverge in transverse directions).

In one space dimension, the variational approach does work.

- For heteroclinic profiles, it was successfully applied in [11], see §3.3 below for more details; the stability result obtained there is not complete though, because the energy does not control enough derivatives to ensure global existence in general (by contrast, global existence - and thus genuine orbital stability - was obtained for the Boussinesq solitary waves in [20], using the fact that the principart of the Boussinesq equation has constant coefficients).

- For homoclinic profiles, by (slightly) adapting the method of Grillakis, Shatah and Strauss [42] to nonzero endstates, we can characterize orbital stability by means of the so-called *moment of instability of Boussinesq* [11, 78, 9], see §3.4 below.

An alternative, and often complementary approach is to investigate the *spectral stability* of waves, which means that the linearised equations about the waves (in Galilean frames attached to them) do not admit exponentially growing modes. Sometimes spectral stability can be achieved by means of *a priori* estimates. This is the case, together with Sturm–Liouville arguments, for the heteroclinic, monotone profiles of the Euler–Korteweg equations [8] (see § 3.3). Another useful tool, introduced for solitary waves by Pego and Weinstein [62], is the so-called *Evans function*. This is basically a *Wronskian* $D = D(\tau;\boldsymbol{\eta})$ for the

[22] By definition, a Lyapunov function for a given dynamical system is monotone along its orbits. It can be used to prove stability of states that are local minimisers of that function. When in addition it is strictly monotone, it yields asymptotic stability.

[23] We shall make this statement more precise below.

eigenvalue ODEs, obtained by Fourier–Laplace transformation of the linearised equations (Fourier transform in the transverse variables $\mathbf{y} \perp \mathbf{n}$, and Laplace transform in time, the dual variable τ then being a candidate for an unstable eigenvalue). For one dimensional Hamiltonian PDEs, it has been pointed out in several frameworks that the low-frequency ($\tau \to 0$) behavior of Evans functions $D = D(\tau)$ associated with solitary waves is linked to the moment of instability of Boussinesq [9, 26, 62, 78].

Evans functions are especially useful to prove spectral *instability*. In one space dimension, this is usually done by using the mean value theorem (if the real valued Evans function $D(\tau)$ is negative near zero and positive at infinity then it must vanish somewhere). Furthermore, as was shown by Zumbrun and Serre [77] for viscous travelling waves, a Rouché-type argument can also serve to prove instability in several space dimensions (when one has one-dimensional stability). This is how the spectral instability to transverse perturbations of solitary wave solutions to the Euler–Korteweg equations has been shown in [9] (see § 3.4).

A different way of deriving spectral instability was pointed by Rousset and Tzvetkov [70] for rather general Hamiltonian PDEs, which happen to include the Euler–Korteweg equations. Besides, these authors have also done a series of work showing that spectral instability to transverse perturbations implies *nonlinear instability* [68, 69] for dispersive/Hamiltonian PDEs.

Before stating more detailed results, let us summarise what is known concerning planar travelling waves for the Euler–Korteweg equations.

Propagating phase boundaries (heteroclinic, monotone profiles). In several space dimensions, propagating phase boundaries are spectrally (neutrally) stable [8]. In one space dimension, they are orbitally stable (in finite time).

Solitary waves (homoclinic profiles, one bump). In one space dimension, there are stable ones and unstable ones (depending on their speed σ, on the mass transfer j, and on the constant μ). Stable ones are destabilised in several space dimensions by transverse perturbations (of intermediate to large wavelength).

Periodic waves They are unstable in one space dimension.

For later use, let us write the linearised equations. In a Galilean frame attached to a wave of speed σ in the direction $\mathbf{n}$, the Euler–Korteweg equations (22) read

$$\left\{ \begin{array}{l} \partial_t\rho + \mathrm{div}(\rho\,(\mathbf{u}-\sigma\mathbf{n})) = 0\,, \\ \partial_t\mathbf{u} + (\mathbf{u}-\sigma\mathbf{n})\cdot\nabla\mathbf{u} = \nabla(-g_0 + \tfrac{1}{2}K'_\rho\,|\nabla\rho|^2 + K\,\Delta\rho)\,, \end{array}\right. \tag{53}$$

of which the wave is a stationary solution depending only on $x = \mathbf{x}\cdot\mathbf{n} = x_d$ if we assume, without loss of generality, that $\mathbf{n}$ is the d-th basis vector in $\mathbb{R}^d$. From now on, we denote by $(\underline{\rho},\underline{\mathbf{u}}) = (\underline{\rho},\underline{\mathbf{u}})(x)$ a profile of this wave. Since $\check{\underline{\mathbf{u}}}$ is constant (at least for dynamical waves, as we have seen above), we may also assume (by a further Galilean change of frame in transverse directions) that $\check{\underline{\mathbf{u}}}$ equals zero[24]. Then, linearising about $(\underline{\rho},\underline{\mathbf{u}})$ we get an evolution

[24]The symbol $\check{}$ has the same meaning as in §2.3: its stands for the projection onto $\mathbb{R}^{d-1} \simeq \mathbb{R}^{d-1}\times\{0\}$ in $\mathbb{R}^d$.

system $\partial_t \mathbf{U} = \mathbf{L}^\sigma \mathbf{U}$ with

$$\mathbf{U} = \begin{pmatrix} \rho \\ \check{\mathbf{u}} \\ u \end{pmatrix}, \quad \mathbf{L}^\sigma \mathbf{U} := - \begin{pmatrix} \check{\nabla} \cdot (\underline{\rho}\, \check{\mathbf{u}}) + \partial_x(\underline{\rho}\, u + \rho\,(\underline{u} - \sigma)) \\ (\underline{u} - \sigma)\, \partial_x \check{\mathbf{u}} + \check{\nabla}(\,\underline{\alpha}\, \rho - \underline{K}'\, \partial_x \rho - \underline{K}\, \Delta \rho\,) \\ \partial_x(\,(\underline{u} - \sigma)\, u + \underline{\alpha}\, \rho - \underline{K}'\, \partial_x \rho - \underline{K}\, \Delta \rho\,) \end{pmatrix}, \tag{54}$$

where

$$\underline{K} = K(\underline{\rho}\,), \quad \underline{K}' := K'(\underline{\rho}\,), \quad \underline{\alpha} := g_0'(\underline{\rho}\,) - K'(\underline{\rho}\,)\, \partial_x^2 \underline{\rho} - \frac{1}{2} K''(\underline{\rho}\,)\, (\partial_x \underline{\rho}\,)^2 .$$

In one space dimension, the definitions above reduce to

$$\mathbf{U} = \begin{pmatrix} \rho \\ u \end{pmatrix}, \quad \mathbf{L}^\sigma \mathbf{U} := - \begin{pmatrix} \partial_x(\underline{\rho}\, u + \rho\,(\underline{u} - \sigma)) \\ \partial_x(\,(\underline{u} - \sigma)\, u + \underline{\alpha}\, \rho - \underline{K}'\, \partial_x \rho - \underline{K}\, \partial_x^2 \rho\,) \end{pmatrix}. \tag{55}$$

Remarkably enough, we have in this case

$$\mathbf{L}^\sigma \;=\; \mathscr{J}\, (\mathrm{Hess}\mathscr{H} - \sigma \mathrm{Hess}\mathscr{I}\,)\,[\underline{\rho}\,, \underline{u}]\,, \tag{56}$$

where $\mathscr{J}$, $\mathscr{H}$, and $\mathscr{I}$ are respectively the differential operator and the functionals considered on p. 32, and the Hessians of these functionals are the differential operators defined (at least formally) by the general formula

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} \mathscr{H}[\underline{\rho} + \theta\rho, \underline{u} + \theta u]_{|\theta=0} = \int \begin{pmatrix} \rho \\ u \end{pmatrix} \cdot \mathrm{Hess}[\underline{\rho}\,, \underline{\mathbf{u}}] \begin{pmatrix} \rho \\ u \end{pmatrix} \mathrm{d}x\,,$$

which gives that

$$\mathrm{Hess}\mathscr{H}[\underline{\rho}\,, \underline{u}] = \begin{pmatrix} \underline{\mathscr{M}} & u \\ \underline{u} & \underline{\rho} \end{pmatrix}, \quad \underline{\mathscr{M}} := -\partial_x \underline{K} \partial_x + \underline{\alpha}\,, \quad \mathrm{Hess}\mathscr{I}[\underline{\rho}\,, \underline{u}] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The expression of the linearised operator $\mathbf{L}^\sigma$ in (56) is of course not a coincidence, since the abstract, Hamiltonian form of the equations in (51) (on p. 32 again) becomes

$$\partial_t \begin{pmatrix} \rho \\ u \end{pmatrix} = \mathscr{J}\, (\delta\mathscr{H} - \sigma\delta\mathscr{I}\,)[\rho, u]\,,$$

after change of Galilean frame, and thus, by linearisation about $(\underline{\rho}\,, \underline{u})$,

$$\partial_t \begin{pmatrix} \rho \\ u \end{pmatrix} = \mathscr{J}\, (\mathrm{Hess}\mathscr{H} - \sigma \mathrm{Hess}\mathscr{I}\,)[\rho, u] \begin{pmatrix} \rho \\ u \end{pmatrix},$$

since $\mathscr{J}$ has constant coefficients. Now, we can state crucial facts about the differential operator

$$\mathscr{L}^\sigma \;:=\; (\mathrm{Hess}\mathscr{H} - \sigma \mathrm{Hess}\mathscr{I}\,)\,[\underline{\rho}\,, \underline{u}] = \begin{pmatrix} \underline{\mathscr{M}} & \underline{u} - \sigma \\ \underline{u} - \sigma & \underline{\rho} \end{pmatrix}.$$

Lemma 3. *Let $(\underline{\rho}\,, \underline{u})$ be a saddle-saddle connecting profile (thus with subsonic endstates).*

- *If $\underline{\rho}$ is monotone (that is, for a heteroclinic connection), then $\mathscr{L}^\sigma$ is a* monotone operator*, which means that*
$$\langle \mathbf{U}, \mathscr{L}^\sigma \mathbf{U}\rangle \geq 0$$
for all $\mathbf{U} = (\rho, u)^T \in H^1 \times L^2$ and more precisely there exists $c > 0$ so that
$$\langle \mathbf{U}, \mathscr{L}^\sigma \mathbf{U}\rangle \geq c\,(\|\rho\|_{H^1}^2 + \|u\|_{L^2}^2)$$
for all $\mathbf{U} = (\rho, u)^T \in H^1 \times L^2$ such that $\int_{\mathbb{R}} \rho\, \partial_x \underline{\rho}\, \mathrm{d}x = 0$.

- *If $\underline{\rho}$ is nonmonotone, and more precisely if $\partial_x \underline{\rho}$ vanishes exactly once, then there exists $\mathbf{U} = (\rho, u)^T \in H^1 \times L^2$ such that $\langle \mathbf{U}, \mathscr{L}^\sigma \mathbf{U}\rangle < 0$ with $\rho > 0$ and $j\,u < 0$ everywhere (recall that $j = \underline{\rho}\,(\underline{u} - \sigma)$).*

Sketch of proof. By direct computation we have
$$\langle \mathbf{U}, \mathscr{L}^\sigma \mathbf{U}\rangle = \langle \rho, \mathscr{M}^\sigma \rho\rangle_{L^2} + (1/\underline{\rho})\,\|\underline{\rho}\, u + \rho(\underline{u} - \sigma)\|_{L^2}^2,$$
with
$$\mathscr{M}^\sigma := \underline{\mathscr{M}} - (\underline{u} - \sigma)^2/\underline{\rho}\,.$$
The operator $\mathscr{M}^\sigma$ is (like $\underline{\mathscr{M}}$) a *Sturm–Liouville operator* (that is, a formally self-adjoint, second order, differential operator). The subsonicity of the endstates $\lim_{x\to\pm\infty} \underline{\rho}\,(x)$ implies that the constant-coefficient operators obtained in the limits, and thus also $\mathscr{M}^\sigma$, have positive *essential spectrum.*[25] Furthermore, $\partial_x \underline{\rho}$ is in the kernel of $\mathscr{M}^\sigma$. This can be seen on the abstract form of the profile equation in (52). Differentiating with respect to x we find indeed that
$$\mathscr{L}^\sigma \begin{pmatrix} \partial_x \underline{\rho} \\ \partial_x \underline{u} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
of which the first row implies $\mathscr{M}^\sigma \partial_x \underline{\rho} = 0$ by elimination of $\partial_x \underline{u}$ thanks to the second row. By the classical Sturm–Liouville theory (see [32, pp. 1479–1481]), if $\partial_x \underline{\rho}$ does not vanish then 0 must be the lowest eigenvalue of $\mathscr{M}^\sigma$, while if $\partial_x \underline{\rho}$ vanishes once then 0 must be the second lowest eigenvalue of $\mathscr{M}^\sigma$.

In the first case (that is, $\underline{\rho}$ strictly monotone), we can obtain the claimed lower bound for $\langle \mathbf{U}, \mathscr{L}^\sigma \mathbf{U}\rangle_{L^2\times L^2}$ by noting that (since $\underline{\rho}$ and $\underline{u}$ are bounded, and $\underline{\rho}$ is bounded away from zero)
$$\|\rho\|_{L^2}^2 + (1/\underline{\rho})\,\|\underline{\rho}\, u + \rho(\underline{u} - \sigma)\|_{L^2}^2 \gtrsim \|\rho\|_{L^2}^2 + \|u\|_{L^2}^2,$$

[25] Several definitions of the essential spectrum are available in the literature. We define here the essential spectrum of a differential operator $\mathscr{M}$, viewed as an unbounded operator on $L^2(\mathbb{R})$ with domain $H^m(\mathbb{R})$ if it is of order m, as the set $\sigma_{\mathrm{ess}}(\mathscr{M})$ of complex numbers z such that $\mathscr{M} - z\mathbf{I}$ is not a *Fredholm operator* of *index* zero, which means that one of the following properties must fail: the kernel $\ker(\mathscr{M} - z\mathbf{I}) = \{f \in H^m\,;\ \mathscr{M} f = z f\}$ is finite-dimensional; the range $\mathrm{R}(\mathscr{M} - z\mathbf{I}) = \{\mathscr{M} f - z f\,;\ f \in H^m\}$ is closed and has finite co-dimension; $\dim\ker(\mathscr{M} - z\mathbf{I}) = \operatorname{codim}\mathrm{R}(\mathscr{M} - z\mathbf{I})$. Note that if $z \notin \sigma_{\mathrm{ess}}(\mathscr{M})$ is not an eigenvalue, then $(\mathscr{M} - z\mathbf{I})$ is an isomorphim from H^m to L^2, which means that z belongs to the *resolvent set* of $\mathscr{M}$. As far as we are concerned, the essential spectrum of our variable-coefficient differential operators is given by the union of the essential spectra of the constant-coefficient operators obtained in the limits $x \to \pm\infty$, the latter being easily localised by means of Fourier transform.

and by showing that

$$\langle \rho, \mathscr{M}^\sigma \rho \rangle \gtrsim \|\rho\|_{H^1}^2$$

for all $\rho \in (\partial_x \underline{\rho})^\perp$ (subspace of the domain H^1, orthogonal to the line generated by $\partial_x \underline{\rho}$ with respect to the L^2-inner product). The latter estimate can be obtained by using that

- the derivative of the profile, $\partial_x \underline{\rho}$ spans the (generalised) kernel of $\mathscr{M}^\sigma$,
- the 'restriction'[26] of $\mathscr{M}^\sigma$ to $(\partial_x \underline{\rho})^\perp$ has it spectrum[27] in some interval $[m, +\infty)$ with $m > 0$,
- because $\underline{K}$, $\underline{\alpha}$, $\underline{\rho}$ and $\underline{u}$ are bounded, for λ large enough, $\langle \zeta, (\mathscr{M}^\sigma + \lambda \mathbf{I})\rho \rangle$ defines an inner product on H^1 in such a way that the associated norm is equivalent to the usual one, and that $(\partial_x \underline{\rho})^\perp$ coincides (precisely because $\mathscr{M}^\sigma \partial_x \underline{\rho} = 0$) with the orthogonal to the line generated by $\partial_x \underline{\rho}$ in H^1 with respect to that modified inner product.

In the second case (when $\partial_x \underline{\rho}$ vanishes exactly once), there exists a unique negative eigenvalue ω of $\mathscr{M}^\sigma$, associated with a positive-valued eigenfunction ρ. Then taking $u := -\rho(\underline{u} - \sigma)/\underline{\rho}$, we get

$$\langle \mathbf{U}, \mathscr{L}^\sigma \mathbf{U} \rangle = \omega \|\rho\|_{L^2}^2 < 0.$$

□

Another result of general interest is the following, which makes clearer how to 'factor out translation invariance'. Its proof is merely based on the implicit function theorem, see [11, p. 407], or its original versions in[22, 42].

Lemma 4. *For any non-constant function $\underline{\rho}$ tending exponentially fast to $\rho_\pm$ at $\pm\infty$, for all $k \geq 0$, there exists $\varepsilon > 0$ and a smooth function*

$$\mathscr{S}_k : \mathrm{C}_\varepsilon = \left\{ \rho \in \underline{\rho} + H^k(\mathbb{R})\,;\, \inf_{s \in \mathbb{R}} \| \rho_s - \underline{\rho} \|_{H^k(\mathbb{R})} < \varepsilon \right\} \to \mathbb{R}$$

such that

- *for all $\rho \in \mathrm{C}_\varepsilon$, $\displaystyle\int_{\mathbb{R}} (\rho_{\mathscr{S}_k(\rho)} - \underline{\rho}) \partial_x \underline{\rho} \, \mathrm{d}x = 0$,*
- *and $\mathscr{S}_k(\rho_r) = \mathscr{S}_k(\rho) - r$ for all $r \in \mathbb{R}$.*

This result quantifies how much we have to translate a function ρ to pull it back to the orthogonal subset $(\partial_x \underline{\rho})^\perp$ (in which we expect good estimates, by Lemma 3).

[26] In the sense of the decomposition of a symmetric operator according to orthogonal subspaces, as defined for instance by Kato [47, p. 277].

[27] See [47, p. 178] on the separation of the spectrum.

3.3 Propagating phase boundaries

We have two main results regarding the stability of propagating phase boundaries.

Theorem 3 (Spectral stability [8])**.** *Propagating phase boundaries are spectrally* neutrally stable*, in that the spectrum of operator* $\mathbf{L}^\sigma$ *defined in* (54) *for a monotone*[28] *profile* $(\underline{\rho}, \underline{u})$ *is purely imaginary.*

Theorem 4 (Orbital stability in one space dimension [11])**.** *Propagating phase boundaries are orbitally stable in one space dimension in the following sense. Given a monotone profile* $(\underline{\rho}, \underline{u})$ *, for all (small enough)* $\varepsilon > 0$*, there exists* $\delta > 0$ *so that if* (ρ, u) *is a smooth solution of* (50) *such that*

$$\|(\rho(0)-\underline{\rho}, u(0)-\underline{u})\|_{H^1\times L^2} < \delta \quad \textit{and} \quad \|(\rho(0)-\underline{\rho}, u(0)-\underline{u})\|_{(L^\infty\cap L^1)\times L^1} < \delta\,,$$

then

$$\inf_{s\in\mathbb{R}} \|(\rho_s(t)-\underline{\rho}, u_s(t)-\underline{u})\|_{H^1\times L^2} < \varepsilon$$

for all t *as long as the solution exists.*

The proof of Theorem 3 is mostly based on algebraic arguments. For, by Fourier transform in transverse directions we are left with a family of differential operators $\mathbf{L}^\sigma(\boldsymbol{\eta})$ parametrised by the wave vector $\boldsymbol{\eta} \in \mathbb{R}^{d-1}$. The essential spectrum of $\mathbf{L}^\sigma(\boldsymbol{\eta})$ is found to be $i\mathbb{R}$ by Fourier transform in $x = x_d$, using the observation made at the end of § 3.1 (on p. 34). That $\mathbf{L}^\sigma(\boldsymbol{\eta})$ has no point spectrum (that is, eigenvalues) outside $i\mathbb{R}$ relies on the following identity (for nonzero $\boldsymbol{\eta}$, otherwise it is even simpler, even though we cannot brutally set $\boldsymbol{\eta} = 0$ in what follows)

$$\tau^2 \langle N(\boldsymbol{\eta})^{-1}\rho, \rho\rangle + i\,\tau\,\mathrm{Im}\langle \rho, N(\boldsymbol{\eta})^{-1}\partial_x(\rho(\underline{u}-\sigma))\rangle + \langle (M(\boldsymbol{\eta}) + (\underline{u}-\sigma)\,P(\boldsymbol{\eta})\,(\underline{u}-\sigma))\rho, \rho\rangle = 0$$

for a possible eigenvalue τ, and for ρ, u, respectively the first and last components of an associated eigenvector in $\mathbb{C}^{d+1}$, where

$$M(\boldsymbol{\eta}) := -\partial_x \underline{K} \partial_x + \underline{\alpha} + \underline{K}\,\|\boldsymbol{\eta}\|^2, \quad N(\boldsymbol{\eta}) := -\partial_x \underline{\rho}\,\partial_x + \underline{\rho}\,\|\boldsymbol{\eta}\|^2, \quad P(\boldsymbol{\eta}) := \partial_x N(\boldsymbol{\eta})^{-1}\partial_x\,.$$

It turns out that the the monotonicity of $\mathscr{M}^\sigma$ (see the proof of Lemma 3) implies the monotonicity of $M(\boldsymbol{\eta}) + (\underline{u}-\sigma)\,P(\boldsymbol{\eta})\,(\underline{u}-\sigma)$ (see [8, p. 247]). Therefore, τ solves a second-order polynomial equation $a\tau^2 + b\tau + c = 0$ with $a \in \mathbb{R}^{+*}$ (because the *Sturm–Liouville operator* $N(\boldsymbol{\eta})$ itself is monotone and ρ is nonzero), $b \in i\mathbb{R}$, and $c \in \mathbb{R}^+$, which implies $\tau \in i\mathbb{R}$.

The proof of Theorem 4 uses the following ingredients. First of all, in the chosen functional setting, the Hamiltonian structure in (51) is fully justified if $\mathscr{H}$ is defined by

$$\mathscr{H} := \int_{\mathbb{R}} (H(\rho, \partial_x\rho, u) - H(\underline{\rho}, \partial_x\underline{\rho}, \underline{u}))\,\mathrm{d}x\,.$$

[28] The monotonicity referring to $\underline{\rho}$, and in fact also to $\underline{u}$ since $\underline{\rho}\,(\underline{u}-\sigma)$ =constant.

Then the abstract form of the profile equations in (52) is equivalent to the fact that $(\underline{\rho}, \underline{u})$ is a critical point of $\mathscr{H}$ under constraints on the functionals

$$\mathscr{I} := \int_{\mathbb{R}} (\rho u - \underline{\rho}\,\underline{u})\,\mathrm{d}x\,, \quad \mathscr{R} := \int_{\mathbb{R}} (\rho - \underline{\rho})\,\mathrm{d}x\,, \quad \mathscr{U} := \int_{\mathbb{R}} (u - \underline{u})\,\mathrm{d}x\,,$$

the Lagrange multipliers being respectively σ, the first component say λ and the second component say μ of $(\delta\mathscr{H} - \sigma\delta\mathscr{I})[\underline{\rho}, \underline{u}]$. Once these notations are fixed, we can enumerate four crucial properties thanks to which the proof of Theorem 4 is a generalisation (using in particular Lemma 4) of the usual Lyapunov method.

1. the endpoints $(\rho_\pm, u_\pm)$ of $(\underline{\rho}, \underline{u})$ are hyperbolic fixed points of the profile equations, which read

$$\delta(\mathscr{H} - \sigma\mathscr{I} - \lambda\mathscr{R} - \mu\mathscr{U})[\underline{\rho}, \underline{u}] = (0,0)^T\,, \tag{57}$$

2. all solutions of (57) tending to $(\rho_\pm, u_\pm)$ at $\pm\infty$ are translates of a reference profile $(\underline{\rho}, \underline{u})$,

3. the functional $\mathscr{F} := \mathscr{H} - \sigma\mathscr{I} - \lambda\mathscr{R} - \mu\mathscr{U}$ vanishes on all translates of $(\underline{\rho}, \underline{u})$, and it is conserved along solutions of (50),

4. the Hessian $\mathrm{Hess}\mathscr{F}[\underline{\rho}, \underline{u}] = \mathscr{L}^\sigma$ is monotone and is bounded below on $(\partial_x\underline{\rho})^\perp$ (see Lemma 3).

3.4 Solitary waves

We start by discussing *one-dimensional stability* of solitary waves. By the second statement of Lemma 3 we see that when $\underline{\rho}$ is not monotone, the profile $(\underline{\rho}, \underline{u})$ has no chance to be a local minimiser of $\mathscr{H}$ without further constraints. However, it can be so under the (natural) constraint dictated by $\mathscr{I}$, provided that a further stability condition encoded by the *moment of instability of Boussinesq* is satisfied. To define properly this 'moment of instability', let us first make the Hamiltonian structure in (51) rigorous in functional settings suitable to perturbations of a solitary wave with endstate (ρ_∞, u_∞). The following choice of functionals

$$\mathscr{H} := \int_{\mathbb{R}} (H(\rho, \partial_x\rho, u) - H(\rho_\infty, 0, u_\infty) - \partial_\rho H(\rho_\infty, 0, u_\infty)(\rho - \rho_\infty) - \partial_u H(\rho_\infty, 0, u_\infty)(u - u_\infty))\,\mathrm{d}x\,,$$

$$\mathscr{I} := \int_{\mathbb{R}} (\rho - \rho_\infty)(u - u_\infty)\,\mathrm{d}x\,,$$

has the advantage of implying that the endstate itself is a critical point of both $\mathscr{H}$ and $\mathscr{I}$, which simplifies the abstract form of the profile equations. They become indeed (57) with $\lambda = \mu = 0$. Then we define the moment of instability of Boussinesq at a travelling of speed σ and profile $(\underline{\rho}, \underline{u})$ as

$$m(\sigma) := \mathscr{H}[\underline{\rho}, \underline{u}] - \sigma\mathscr{I}[\underline{\rho}, \underline{u}]\,.$$

This is indeed a function of σ (and of the endstate (ρ_∞, u_∞), which is fixed in all what follows), which does not depend on the chosen, possibly translated profile. Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}s}(\mathscr{H}[\underline{\rho}_s, \underline{u}_s] - \sigma\mathscr{I}[\underline{\rho}_s, \underline{u}_s]) = \int_{\mathbb{R}} \delta(\mathscr{H}[\underline{\rho}, \underline{u}] - \sigma\mathscr{I}[\underline{\rho}, \underline{u}]) \cdot \partial_x(\underline{\rho}, \underline{u})^T \,\mathrm{d}x = 0 .$$

(This was for the very same reason that $\mathscr{F}$ in §3.3 was taking the same value on all translated profiles.) As was in particular pointed out by Grillakis, Shatah and Strauss [42], the convexity of m (or not) is linked the variational properties of $\mathscr{H}$ under the constraint $\mathscr{I}$. Using Lemma 3 and the implicit function theorem, we can indeed show the following (see [11, p. 399]).

Proposition 2. *If $m''(\sigma) \le 0$, then the profile $(\underline{\rho}, \underline{u})$ is not a local minimiser of $\mathscr{H}$ under the constraint $\mathscr{I}$, in that there exist (a curve of) functions (ρ, u) close to $(\underline{\rho}, \underline{u})$ such that $\mathscr{I}[\rho, u] = \mathscr{I}[\underline{\rho}, \underline{u}]$ and $\mathscr{H}[\rho, u] < \mathscr{I}[\underline{\rho}, \underline{u}]$.*

This gives a hint, even though not a proof, that profiles for which $m''(\sigma) \le 0$ are unstable. What we can actually prove is the following.

Theorem 5.

(Sufficient condition) *A solitary wave for which $m''(\sigma)$ is positive is orbitally stable.*

(Necessary condition) *A solitary wave for which $m''(\sigma)$ is negative is linearly unstable.*

The proof of the sufficient part follows from an adaptation to nonzero endstates of the main result of Grillakis, Shatah and Strauss in [42]. This result shows indeed that the 'unstable' directions of $\mathscr{L}^\sigma = \mathrm{Hess}(\mathscr{H} - \sigma\mathscr{I})$ (that this, the $\mathbf{U}$ satisfying $\langle \mathbf{U}, \mathscr{L}^\sigma \mathbf{U}\rangle < 0$ as in Lemma 3), can be ruled out since they are transverse to the manifold defined by the constraint associated with $\mathscr{I}$, unlike what happens when $m''(\sigma) \le 0$ as in Proposition 2 above. This is linked to the identity

$$\langle \partial_\sigma \underline{\mathbf{U}}, \mathscr{L}^\sigma \partial_\sigma \underline{\mathbf{U}} \rangle = \int_{\mathbb{R}} \delta\mathscr{I}[\underline{\mathbf{U}}] \cdot \partial_\sigma \underline{\mathbf{U}} \,\mathrm{d}x = -m''(\sigma) ,$$

which shows that if $m''(\sigma) > 0$ then $\partial_\sigma \underline{\mathbf{U}}$ is an unstable direction and does not belong to $(\delta\mathscr{I}[\underline{\mathbf{U}}])^\perp$.

The proof of the necessary part does not directly follow from the method in [42], because the operator $\mathscr{J}$ is not onto (the same problem occurs for the Hamiltonian structure of the Korteweg–de Vries equation, and it was fixed almost at the same time by Bona, Souganidis and Strauss [22] for that specific equation). A general way to overcome the problem is to use Evans functions techniques, as what pointed out by Zumbrun [78]. The starting point is that the linearised operator $\mathbf{L}$ has a *Jordan block* at $\tau = 0$. Indeed, as already noted in the proof of Lemma 3, the derivative $\partial_x \underline{\mathbf{U}}$ of the profile is an eigenvector of $\mathscr{L}^\sigma$ for the eigenvalue 0, which also implies that it is an eigenvector of $\mathbf{L}^\sigma = \mathscr{J}\mathscr{L}^\sigma$ for the eigenvalue 0. Furthermore, as used to get the identity above, we have

$$\mathscr{L}^\sigma \partial_\sigma \underline{\mathbf{U}} = \delta\mathscr{I}[\underline{\mathbf{U}}] ,$$

which implies that

$$\mathbf{L}^{\sigma}\partial_{\sigma}\underline{\mathbf{U}} = \mathscr{J}\,\delta\mathscr{I}[\underline{\mathbf{U}}] = -\partial_x\underline{\mathbf{U}}$$

(the last equality is exactly the reason why $\mathscr{I}$ has been called an impulse associated with translation invariance). This means that 0 is an eigenvalue of $\mathbf{L}^{\sigma}$ of at least 2 as algebraic multiplicity. Without trying to construct here the Evans function (see [9] and references therein for more details), this implies that $D(0) = D'(0) = 0$. More precisely, we can show the following (see [9, Lemma 1]).

Lemma 5. *In one space dimension, the essential spectrum of the linearised operator $\mathbf{L}^{\sigma}$ about a solitary wave of speed σ is purely imaginary. The nonnegative real point spectrum of $\mathbf{L}^{\sigma}$ consists of the zeros of an analytic function $D : \tau \in [0,+\infty) \mapsto D(\tau) \in \mathbb{R}$ such that*

$$D(0) = D'(0) = 0, \quad D(\lambda) > 0 \text{ for } \lambda \gg 1, \quad \operatorname{sgn} D''(0) = \operatorname{sgn} m''(\sigma).$$

With this lemma, the end of the proof of Theorem 5 merely follows from the mean value theorem: if $m''(0) < 0$ then D is negative near 0 and positive at infinity, and thus must vanish somewhere.

Remark 3. *We can give a more explicit expression of m, namely*

$$m(\sigma) := \int_{\mathbb{R}} \underline{K}\,(\partial_x\underline{\rho})^2\,\mathrm{d}x.$$

Would $\underline{\rho}$ be a phase transition profile, any specialist in phase transitions would recognise $m(\sigma)$ as being the surface tension *across the 'interface'. It is not clear though if this is meaningful for solitary waves. Anyhow, it can be computed up to a quadrature, without actually solving the profile ODEs. In this way it is easy to test numerically the convexity of m. This was done in [11], where both cases of convexity and concavity were shown.*

Let us now turn to the *multi-dimensional stability* question of solitary waves, at least those known to be stable in one space dimension, that is, for which $m''(\sigma)$ is positive. By Fourier transform in transverse directions we are led to consider (as for phase boundaries) a family of operators $\mathbf{L}^{\sigma}(\boldsymbol{\eta})$ parametrised by the wave vector $\boldsymbol{\eta} \in \mathbb{R}^{d-1}$. Their essential spectra are still purely imaginary. Furthermore, we can show the following (see again [9, Lemma 1]).

Lemma 6. *The point spectrum of $\mathbf{L}^{\sigma}(\boldsymbol{\eta})$ in the half-plane $\{\tau\,;\,\operatorname{Re}\tau > 0\}$ consists of the zeros of a function $D = D(\tau,\boldsymbol{\eta})$, which is analytic along rays $\{(\lambda\tau,\lambda\boldsymbol{\eta})\,;\,\lambda > 0\}$ for $\operatorname{Re}\tau > 0$, and has the asymptotic behavior for low frequencies ($\lambda \to 0$)*

$$D(\lambda\tau,\lambda\boldsymbol{\eta}) \sim \lambda^2 P(\tau,\boldsymbol{\eta}), \quad P(\tau,\boldsymbol{\eta}) = m''(\sigma)\,\tau^2 - r^2\|\boldsymbol{\eta}\|^2, \quad r > 0.$$

As a consequence, if $m''(\sigma) > 0$ we find by a Rouché argument that for small $\|\boldsymbol{\eta}\|$, there exists $\tau \simeq r\|\boldsymbol{\eta}\|/\sqrt{m''(\sigma)} > 0$ for which $D(\tau,\boldsymbol{\eta}) = 0$.

This proves that solitary wave solutions to the Euler–Korteweg equations are linearly unstable in several space dimensions.This result was obtained in a much less technical/more elegant way by Rousset and Tzvetkov in [70]. Applying a general method of theirs, they found indeed (in two space dimensions, on the Bernouilli form of the equations for potential flows) a specific nonzero η (not necessarily small, determined by a mean value argument) and a small $\tau > 0$ (by means of a *Lyapunov–Schmidt reduction*) that is an eigenvalue of $\mathbf{L}^{\sigma}(\eta)$.

3.5 Periodic solutions

The stability analysis of periodic waves require different tools, which are out the scope of these notes. As a by-product of a work of Serre (unpublished), periodic waves of the Euler–Korteweg equations are known to be unstable.

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Appendix

Derivation of the Korteweg tensor

In what follows, we use variational arguments to derive the stress tensor of an isothermal capillary fluid at rest, of stationary density ρ. We assume that its (free) energy density F depends on ρ and $\nabla\rho$. Then its total energy $\int F(\rho, \nabla\rho)\, \mathrm{d}x$ is invariant under space translations. More specifically, its invariance under the vector field $\mathbf{v}^k := \partial/\partial x_k$ implies, by Noether's theorem (see for instance [60, p. 272]) that

$$(58) \qquad \mathrm{div} P^k - \rho_{,k}\, \mathscr{E} F = 0\,, \quad \text{where } P_i^k := \delta_i^k\, F - \rho_{,k}\, \frac{\partial F}{\partial \rho_{,i}} \quad \text{and } \mathscr{E} F := \frac{\partial F}{\partial \rho} - \sum_{i=1}^{d} \mathrm{D}_i \left(\frac{\partial F}{\partial \rho_{,i}} \right).$$

Here above, the notation D_i stands for the *total derivative* with respect to $x_i \in \mathbb{R}^d$. If ρ were an *extremal* of $\mathscr{F}$, we would have by definition of the *Euler operator* $\mathscr{E}$ that $(\mathscr{E} F)(\rho) = 0$, and (58) would imply the conservation law $\mathrm{div} P^k = 0$. However, physically ρ can only be an extremal under mass constraint $\int \rho$. Hence $(\mathscr{E} F)(\rho) = \lambda$, a Lagrange multiplier associated with this constraint. Therefore, using that λ is indeed a constant, (58) implies the conservation law $\mathrm{div}\Sigma^k = 0$ with

$$\Sigma_i^k := \delta_i^k\, (F - \rho\, (\mathscr{E} F)(\rho)\,) - \rho_{,k}\, \frac{\partial F}{\partial \rho_{,i}}\,,$$

or, more explicitely,

$$\Sigma_i^k := \delta_i^k\, (F - \rho\, \frac{\partial F}{\partial \rho} + \rho \mathrm{div}\mathbf{w}) - \rho_{,k}\, \frac{\partial F}{\partial \rho_{,i}}\,, \quad \text{where } w_i := \frac{\partial F}{\partial \rho_{,i}}\,.$$

In another words, we have

$$\mathrm{div}\boldsymbol{\Sigma} = 0\,, \quad \text{with } \boldsymbol{\Sigma} := \left(F - \rho\, \frac{\partial F}{\partial \rho} \right) \mathbf{I} + \rho \mathrm{div}\mathbf{w}\, \mathbf{I} - \mathbf{w} \otimes \nabla\rho\,.$$

When F does not depend on $\nabla\rho$, the stress tensor $\boldsymbol{\Sigma}$ reduces to $-p\,\mathbf{I}$, where

$$(59) \qquad p := \rho\, \frac{\partial F}{\partial \rho} - F$$

is the usual pressure in the fluid. When F depends on $\nabla\rho$, the modified tensor is

$$\boldsymbol{\Sigma} = -p\,\mathbf{I} + \rho \mathrm{div}\mathbf{w}\, \mathbf{I} - \mathbf{w} \otimes \nabla\rho\,,$$

where p defined by (59) is a generalised pressure (depending also on $\nabla\rho$).

Calculus identities

With the notations $a := \sqrt{\rho K}$ and $\mathbf{v} := \sqrt{K/\rho}\, \nabla\rho$ we have

$$\tfrac{1}{2}\, K'_\rho\, |\nabla\rho|^2 + K\, \Delta\rho \;=\; a\, \mathrm{div}\mathbf{v} + \tfrac{1}{2}\, |\mathbf{v}|^2\,.$$

Proof. Observing that $\mathbf{v} = \nabla R(\rho)$ with R a primitive of $\rho \mapsto \sqrt{K(\rho)/\rho}$, we have

$$\mathrm{div}\mathbf{v} = \Delta R(\rho) \,=\, R'(\rho)\,\Delta\rho \,+\, R''(\rho)\,|\nabla\rho|^2\,.$$

The conclusion follows from the equalities

$$R' = \sqrt{K/\rho}\,, \quad R'' = \tfrac{1}{2}K'_\rho/\sqrt{\rho K} - \tfrac{1}{2}\sqrt{K/\rho^3}\,.$$

□

The identity

$$\mathrm{div}(\rho\,\nabla^2\ln(\rho)) \,=\, 2\,\rho\,\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)$$

is a special case of

$$\mathrm{div}(\mu(\rho)\,\nabla^2\xi(\rho)) \,=\, \lambda(\rho)\,\nabla\,\big(\sigma'(\rho)\,\Delta\sigma(\rho)\big)\,,$$

which is valid under the conditions

$$\mu\,\xi' \,=\, \lambda\,\sigma'^2 \equiv \text{constant}\,, \quad \lambda'\,\sigma'^2 = \mu'\,\xi'\,, \quad \lambda^2\,\sigma'\,\sigma'' \equiv \text{constant}\,, \tag{60}$$

these being satisfied if and only if there exist $a, b, c, d \in \mathbb{R}$ with $acd \neq 0$ such that

$$\sigma = \sqrt{a\rho + b} + \text{constant}\,, \quad \lambda = \frac{4c}{a^2}\,(a\rho+b)\,, \quad \mu = d\,(a\rho+b)\,, \quad \xi \,=\, \frac{c}{ad}\,\ln(a\rho+b) + \text{constant}\,.$$

(The special case above corresponds to $a = c = d = 1$ and $b = 0$.)

Proof. To simplify notations we use Einstein's convention on repeated indices. By definition, we have

$$\mathrm{div}(\mu\,\nabla^2\xi)_j \,=\, \partial_i(\mu\,\partial^2_{i,j}\xi)\,,$$

and by the chain rule we find that

$$\partial_i(\mu\partial^2_{i,j}\xi) \,=\, \mu\,\xi'\,\partial_j\Delta n + (2\mu\xi'' + \mu'\xi')\,(\partial_i\rho)\partial^2_{i,j}\rho + \mu\,\xi''\,\partial_j\Delta\rho + (\mu\,\xi'')'\,|\nabla\rho|^2\,\partial_j\rho\,,$$

or, without indices,

$$\mathrm{div}(\mu\nabla^2\xi) \,=\, \mu\,\xi'\,\nabla\Delta\rho + (2\mu\,\xi'' + \mu'\,\xi')\,\nabla^2\rho\cdot\nabla\rho + \mu\,\xi''\,\Delta\rho\,\nabla\rho + (\mu\,\xi'')'\,|\nabla\rho|^2\,\nabla\rho\,.$$

On the other hand, by the chain rule again we have that

$$\Delta\sigma \,=\, \sigma'\,\Delta\rho + \sigma''\,|\nabla\rho|^2\,,$$

$$\nabla(\sigma'\,\Delta\sigma) \,=\, \sigma'^2\,\nabla\Delta\rho + 2\,\sigma'\,\sigma''\,\nabla^2\rho\cdot\nabla\rho + 2\,\sigma'\,\sigma''\,\Delta\rho\,\nabla\rho + (\sigma'\sigma'')'\,|\nabla\rho|^2\,\nabla\rho\,.$$

Equating coefficients of the four terms, we find the following sufficient conditions for the equality $\mathrm{div}(\mu\,\nabla^2\xi) \,=\, \lambda\,\nabla\,\big(\sigma'\,\Delta\sigma\big)$,

$$\mu\,\xi' \,=\, \lambda\,\sigma'^2\,, \quad \mu\,\xi'' \,=\, 2\,\lambda\,\sigma'\,\sigma'' \,=\, 2\,\mu\,\xi'' \,+\, \mu'\,\xi'\,, \quad (\mu\,\xi'')' \,=\, \lambda\,(\sigma'\,\sigma'')'\,,$$

which are equivalent to

$$\mu \xi' = \lambda \sigma'^2, \quad (\mu \xi')' = 0, \quad \lambda' \sigma'^2 = \mu' \xi', \quad 2\lambda' \sigma' \sigma'' + \lambda (\sigma' \sigma'')' = 0,$$

themselves obviously equivalent to (60). Now a nontrivial solution $(\mu, \xi, \lambda, \sigma)$ of (60) is necessarily such that $\sigma''/\sigma'^3 \equiv$ constant, hence $1/\sigma'^2$ is an affine function of ρ, and then λ, μ, and $1/\xi'$ are affine as well. The explicit coefficients are easily found by inspection of the equations. □

Let us verify the identity

$$\nabla\left(\tfrac{1}{2}|\mathbf{u}|^2\right) = \mathbf{u} \times (\nabla \times \mathbf{u}) + (\mathbf{u} \cdot \nabla)\mathbf{u}. \tag{61}$$

Proof. By the chain rule,

$$\partial_j\left(\tfrac{1}{2}|\mathbf{u}|^2\right) = \sum_{i=1}^{d} u_i \partial_j u_i = (\mathbf{u} \cdot \nabla) u_j + \sum_{i=1}^{d} u_i (\partial_j u_i - \partial_i u_j).$$

When $d = 3$, we easily check that the sum (which comprises actually only two terms, since the $i = j$ term is null) is the j-th component of $\mathbf{u} \times (\nabla \times \mathbf{u})$. □

The algebraic identity

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\, \mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\, \mathbf{w} \tag{62}$$

is in the same spirit.

Proof. The j-th component of both sides is easily identified with

$$\sum_{i=1}^{3} u_i (v_j w_i - v_i w_j).$$

□

Eulerian *vs* Lagrangian coordinates

Let $\kappa := \rho^5 K$ be viewed as function of $v = 1/\rho$. The system

$$\left\{ \begin{array}{l} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p_0(\rho)) = \partial_x(\rho K \partial_x^2 \rho + \tfrac{1}{2}(\rho K'_\rho - K)(\partial_x \rho)^2), \end{array} \right. \tag{63}$$

is (formally) equivalent to

$$\left\{ \begin{array}{l} \partial_t v = \partial_y u, \\ \partial_t u + \partial_y p_0(v) = -\partial_y(\kappa \partial_y^2 v + \tfrac{1}{2}\kappa'_v (\partial_y v)^2), \end{array} \right. \tag{64}$$

in Lagrangian coordinates.

Proof. Recall that the mass Lagrangian coordinate y is characterised by $\mathrm{d}y = \rho\,\mathrm{d}x - \rho\,u\,\mathrm{d}t$. Now, a conservation law in Eulerian coordinates

$$\partial_t\varphi + \partial_x\psi = 0$$

is equivalent to the fact that the differential form $\varphi\,\mathrm{d}x - \psi\,\mathrm{d}t$ is closed, and since $\mathrm{d}x = v\,\mathrm{d}y + u\,\mathrm{d}t$, it is also equivalent the closedness of $\varphi\,\mathrm{d}y + (u\varphi - \psi)\,\mathrm{d}t$, which is itself equivalent to the conservation law in Lagrangian coordinates

$$\partial_t(v\,\varphi) + \partial_y(\psi - u\,\varphi) = 0\,.$$

For $\varphi \equiv 1$ and $\psi \equiv 0$, this yields the first equation in (64) (note that the first equation in (63) has already served to define y). The second conservation law in (63) corresponds to

$$\varphi = \rho\,u\,,\quad \psi = p_0 - \rho\,K\,\partial_x^2\rho + \tfrac{1}{2}\,(-\rho\,K'_\rho + K)\,(\partial_x\rho)^2 = p_0 - \rho^3\,K\,\partial_y^2\rho - \tfrac{1}{2}\,\rho^2\,(\rho\,K'_\rho + K)\,(\partial_y\rho)^2\,.$$

To eventually obtain the second equation in (64), we compute that

$$\partial_y\rho = -\rho^2\,\partial_y v\,,\quad \partial_y^2\rho = 2\,\rho^3\,(\partial_y v)^2 - \rho^2\,\partial_y^2 v\,,$$

and

$$\kappa'_v = -5\,\rho^6\,K - \rho^7\,K'_\rho\,.$$

□

Convexity of the total energy density

Proposition 3. *Assume that $F'' > 0$, $K > 0$, and $KK'' \geq 2K'^2$ everywhere, then the total energy density*

$$H = F(\rho) + \tfrac{1}{2}\,K(\rho)\,|\mathbf{v}|^2 + \tfrac{1}{2}\,\rho\,|\mathbf{u}|^2$$

is a strictly convex function of $(\rho, \mathbf{m} := \rho\mathbf{u}, \mathbf{v})$.

Proof. A standard trick in the theory of Euler equations consists in noticing that the (strict) convexity of the (volumic) energy density in the conservative variables $(\rho, \mathbf{m})$ is equivalent to the (strict) convexity of the specific energy in the variables $(v = 1/\rho, \mathbf{u})$. This works for the Euler–Korteweg equations too. Clearly (by characterising convex functions as upper envelopes of affine functions), $H = H(\rho, \mathbf{m}, \mathbf{v})$ is convex if and only if

$$h := H/\rho = f + \tfrac{1}{2}\,(\rho\,K)\,|\mathbf{w}|^2 + \tfrac{1}{2}\,|\mathbf{u}|^2$$

is a convex function of $(v, \mathbf{u}, \mathbf{w})$ with $f = F/\rho$, $\mathbf{w} = \mathbf{v}/\rho$. Since we have $f''(v) = \rho^3 F''(\rho)$, the (strict) convexity of $F = F(\rho)$ is equivalent to the (strict) convexity of $f = f(v)$. Since $\mathbf{u} \mapsto |\mathbf{u}|^2$ is obviously strictly convex, it remains to determine a convexity condition for the function $(v, \mathbf{w}) \mapsto \kappa(v)\,|\mathbf{w}|^2$, with $\kappa := \rho\,K$. By computation of its Hessian we find the sufficient condition $\kappa''(v)\,\kappa(v) \geq 2\,\kappa'(v)^2$. To show that this is equivalent to $K''(\rho)\,K(\rho) \geq 2\,K'(\rho)^2$, we can write

$$v\kappa(v) = K(\rho)\,,\quad v^2\,\kappa'(v) = -\,(K(\rho) + \rho\,K'(\rho))\,,\quad v^3\,\kappa''(v) = 2\,K(\rho) + 4\rho\,K'(\rho) + \rho^2\,K''(\rho)\,,$$

and thus see that

$$v^4\,(\kappa\,\kappa''-2\,(\kappa')^2)\;=\;\rho^2\,(K''\,K-2\,(K')^2)\,.$$

$\square$

Remark 4. *The somewhat mysterious condition $KK''\geq 2K'^2$ is clearly satisfied (as an equality) by constant functions K, and also by $K\propto 1/\rho$. However, its is violated by $K\propto 1/\rho^5$, which corresponds to a constant coefficient κ in Lagrangian coordinates.*

Hamiltonian structure of travelling profiles ODEs for Hamiltonian PDEs

Proposition 4 (Benjamin)**.** *Let us consider a Hamiltonian system of PDEs in one space dimension*

$$\partial_t\mathbf{u}=\mathscr{J}\,\delta\mathscr{H}[\mathbf{u}]\,,\tag{65}$$

with $\mathscr{J}$ of the form $\mathscr{J}=\mathrm{D}_x\mathbf{J}$, where $\mathbf{J}$ is a symmetric, invertible real matrix. We assume that there exists a functional $\mathscr{I}:\mathbf{u}\mapsto\mathscr{I}[\mathbf{u}]=\int I(\mathbf{u})$ such that

$$\mathbf{u}=-\mathbf{J}\,\delta\mathscr{I}[\mathbf{u}]$$

for all $\mathbf{u}\in\mathbb{R}^n$, and that $\mathscr{H}$ depends only on $\mathbf{u}$ and $\mathbf{u}_x$ (the first order derivative of $\mathbf{u}$ with respect to x). For simplicity we also assume that $\delta(\mathscr{H}-\sigma\mathscr{I})[0]=0$. Then

- *the (smooth) solutions of* (65) *satisfy a local conservation law of the form*

  $$I_t+S_x=0\,,$$

  where S depends only on $\mathbf{u}$, $\mathbf{u}_x$ et $\mathbf{u}_{xx}$;

- *the profile equations for solitary wave solutions to* (65) *of speed σ are the Euler–Lagrange equations associated with the Lagrangian $H-\sigma I$, and the corresponding Hamiltonian is $S-\sigma I$, which is therefore a first integral of the profile equations.*

Remark 5. *The functional $\mathscr{I}$ is linked to the invariance by translations in x, in that the (generalised)* Hamiltonian vector field *[60, p. 435]*

$$\widehat{\mathbf{v}}_{\mathscr{I}}:=\sum_{j=1}^n(\mathscr{J}\,\delta\mathscr{I}[\mathbf{u}])_j\,\frac{\partial}{\partial u_j}\;=-\sum_{j=1}^n(\mathrm{D}_x u_j)\,\frac{\partial}{\partial u_j}$$

is the so-called evolutionary representative *[60, p. 291] of $\frac{\partial}{\partial x}$, the infinitesimal generator of the group of translations $x\mapsto x+a$, $a\in\mathbb{R}$.*

Proof. We first look for S, which must be such that

$$S_x=-I_t=-\sum_{j=1}^n u_{j,t}\frac{\partial I}{\partial u_j}\;=-\sum_{j=1}^n u_{j,t}\frac{\partial I}{\partial u_j}$$

along solutions of (65), that is of

$$u_{j,t} = \sum_{i=1}^{n} J_{ji} (\mathscr{E}_i H)_x\,,$$

where $\mathscr{E}_i$ denotes the *Euler operator* associated with the component u_i of $\mathbf{u}$. Since $\mathscr{H}$ depends only on $\mathbf{u}$ and $\mathbf{u}_x$, we have

$$\mathscr{E}_i H = -\mathrm{D}_x\Big(\frac{\partial H}{\partial u_{i,x}}\Big) + \frac{\partial H}{\partial u_i}\,.$$

By using the symmetry of $\mathbf{J}$ and the definition of I (through the one of $\mathscr{I}$) we infer that

$$S_x = \sum_{i=1}^{n} u_i (\mathscr{E}_i H)_x\,,$$

hence

$$S = \sum_{i=1}^{n} u_i (\mathscr{E}_i H) - H + \sum_{i=1}^{n} u_{i,x} \frac{\partial H}{\partial u_{i,x}}$$

up to a constant, which can be taken to zero. Now the profile equations are

$$-\sigma \mathbf{u}' = \mathbf{J}\delta\mathscr{H}[\mathbf{u}]'\,,$$

or, using again the definition of $\mathscr{I}$,

$$\mathbf{J}\delta(\mathscr{H} - \sigma\mathscr{I})[\mathbf{u}]' = 0\,.$$

Since $\mathbf{J}$ is invertible, we obtain for profiles that are homoclinic to zero that

$$\delta(\mathscr{H} - \sigma\mathscr{I})[\mathbf{u}] = 0\,,$$

which equivalently reads

$$\mathscr{E}_i (H - \sigma I) = 0\,, \quad i = 1, \dots, n\,.$$

These are the Euler–Lagrange equation for the Lagrangian $H - \sigma I$. The corresponding Hamiltonian is the *Legendre transform* of $H - \sigma I$, namely (since I does not depend on $\mathbf{u}_x$),

$$P := \sum_{i=1}^{n} u_{i,x} \frac{\partial H}{\partial u_{i,x}} - (H - \sigma I)$$

in the variables u_i et $p_i = \frac{\partial H}{\partial u_{i,x}}$. To conclude that $P = S - \sigma I$, we observe that from the profile equations

$$\mathscr{E}_i H = \sigma \mathscr{E}_i I = \sigma \frac{\partial I}{\partial u_i}\,,$$

hence

$$\sum_{i=1}^{n} u_i (\mathscr{E}_i H) = \sigma \sum_{i=1}^{n} u_i (\mathscr{E}_i I) = 2\sigma I$$

since I is quadratic (by definition). □

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