On the well-posedness for the Euler-Korteweg model in several space dimensions

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Introduction

We call Korteweg model a system of conservation laws governing the motion of liquid-vapor mixtures, which takes into account the surface tension of interfaces by means of a capillarity coefficient; see [16] and [11] for the early developments of the theory of capillarity, and for instance [18, 9] for the derivation of the equations of motion. In this kind of model, the interfaces are not sharp fronts. Their width, even though extremely small for values of the capillarity compatible with the measured, physical surface tension, is nonzero. We call them diffuse interfaces. We are especially interested in *non-dissipative isothermal* models, in which the viscosity of the fluid is neglected and therefore the (extended) free energy, depending on the density and its gradient, is a conserved quantity.

From the mathematical point of view, the resulting conservation law for the momentum of the fluid involves a third order, dispersive term but no parabolic smoothing effect. The system made up with the conservation of mass and of momentum is thus the compressible Euler system modified by the adjunction of the so-called Korteweg stress, and we call it the Euler-Korteweg model. The well-posedness of the Cauchy problem for the Euler-Korteweg model is a challenging issue, which has been addressed in [4] in the one-dimensional case by reformulating the equations in Lagrangian coordinates. Here we consider the multi-dimensional case (in Eulerian formulation). As in [4], we allow the capillarity coefficient K to depend (in a smooth way) on the density ρ, which makes the system quasilinear and therefore more difficult to apprehend than in the special case $K \equiv$ constant. However, the case $K \equiv$ constant appears not to be the easiest one for the analysis. In fact, using a reformulation of the system involving a variable coefficients (degenerate) Schrödinger equation, we point out that K proportional to $1/\rho$ is the most peculiar case, for which we have a standard, "flat" Schrödinger operator and the derivation of *a priori* estimates greatly simplifies. It is remarkable that in a different physical framework, namely in Quantum Hydrodynamics (QHD), the very same system of PDEs arises (usually coupled with a Poisson equation), with precisely K proportional to $1/\rho$. Our approach (in fact its simplified version due to $\rho K \equiv$ constant) is thus applicable in that framework. However, we do not address here problems due to vacuum, which is as far as we know a crucial issue in QHD (related to singularities of the field associated with $(\rho, \mathbf{u})$ through the Madelung's transformation), nor the coupling with other equations; recent references on this topic are [14, 8]. As this might be confusing, we draw the attention of the reader on the fact that the Schrödinger equation which arises in our reformulation has nothing to do with the nonlinear Schrödinger equation (known as the Gross-Pitaevskii equation) the QHD case comes from : ours involves a non-linear Burgers-type first order term and the degenerate second order operator $i \nabla a \,\mathrm{div}$ (with $a = \sqrt{\rho K}$) whereas the Gross-Pitaevskii equation involves only $i \Delta +$ zeroth order term.

Our main purpose here is to prove the (local) well-posedness of the Euler-Korteweg model in all Sobolev spaces of supercritical index. In fact, density and velocity, or more precisely, the perturbations of density and velocity with respect to a reference state or a special solution, will not have the same index of regularity. Rather, the velocity $\mathbf{u}$ and the *gradient* of the density will have the same index. This is natural in view of the fact that $(\mathbf{u}, \nabla\rho)$ already satisfies a L^2 estimate at the linearized level: by considering the pressure-linked term in the total energy as a source term, the other term

$$\mathcal{K}[\rho, \mathbf{u}] := \int \left(\tfrac{1}{2}\rho|\mathbf{u}|^2 + \tfrac{1}{2}K(\rho)|\nabla\rho|^2\right) dx$$

can be bounded *a priori*; equivalently, away from vacuum and for positive capillarity $K > 0$, this gives a L^2 estimate for $(\mathbf{u}, \nabla\rho)$. Roughly speaking, we shall prove the local-in-time well-posedness of the Euler-Korteweg model and a blow-up criterion, as though $(\mathbf{u}, \nabla\rho)$ were solution of a *symmetrizable hyperbolic* system. For a precise statement, see Section 1. In passing, let us emphasize that we do not need any assumption on the monotonicity of the pressure, which is basically dealt with as a source term. This means that our result applies in the pure phases (liquid or vapor, where the pressure law is monotone) and in the presence of (diffuse) interfaces between liquid and vapor. Our method of proof is based on an extended formulation where $\nabla\rho$ is considered as an additional dependent variable. The extended system of conservation laws is *second order* and *non-dissipative*. In particular, we have to handle bad commutators due to second order terms. This is done by taking

$$\mathbf{w} := \sqrt{\frac{K}{\rho}}\,\nabla\rho$$

instead of $\nabla\rho$ as additional dependent variable and by estimating $(\mathbf{u}, \mathbf{w})$ in weighted Sobolev spaces (with weights depending on the solution). Unsurprisingly, the zeroth order weight (also named "gauge" function after Lim and Ponce [15]) is just $\sqrt{\rho}$: note indeed that

$$\|\sqrt{\rho}\,(\mathbf{u}, \mathbf{w})\|^2_{\mathrm{L}^2} = \mathcal{K}[\rho, \mathbf{u}]$$

The higher order weights appear to depend on the product ρK, which explains why the QHD case (where ρK is constant) is to some extent simpler. Once we have suitable *a priori* estimates, without loss of derivatives, we basically have uniqueness. For proving existence, we use a fourth order regularization of the *non-linear* system on $(\rho, \mathbf{u}, \mathbf{w})$. The regularized system involves the operator $\varepsilon\Delta^2$, where ε is a small parameter, and we show the time of existence is independent of ε. Then, using an idea of Bona and Smith [6], we show that for suitably mollified initial data, depending on ε, the solution of the regularized problem converges to a solution of the original problem. The continuous dependence on initial data uses the same kind of arguments.

In Section 1, we specify our notations, assumptions and state our main result. Section 2 introduces the extended formulation and the underlying Schrödinger equation. In Section 3 we derive a priori estimates for the linearized version of that equation, using suitable gauge functions. Section 4 is devoted the regularized system: we prove there its local well-posedness and derive a lower bound for the time of existence. The proof of our main result is given in Sections 5 and 6. Some technical results needed (inequalities, commutator estimates and mollifier properties) are stated and proved in the appendix for completeness.

1 Main result

1.1 Notations

For convenience, we introduce here the notations used repeatedly in the paper.

1.1.a Calculus

- For $f:\mathbb{R}^N\to\mathbb{C}$, we denote $Df:=(\partial_1 f,\cdots,\partial_N f)$ and $\nabla f:=(Df)^t$ where ∂_j stands for the partial derivative with respect to the space variable x_j. For $k\in\mathbb{N}^*$, the notation $D^k f$ stands for the family of all partial derivatives of f of order k. And $\nabla^2 f$ denotes the Hessian matrix of f.

- For $\mathbf{f}:\mathbb{R}^N\to\mathbb{C}^N$, we denote by $D\mathbf{f}$ the Jacobian matrix of $\mathbf{f}$, with coefficient $(D\mathbf{f})_{ij}=\partial_j f^i$ on the i-th row and the j-column if $f^1,\ldots,\ f^N$ are the components of $\mathbf{f}$, and by $\nabla\mathbf{f}:=(D\mathbf{f})^t$ the transposed matrix. The divergence $\operatorname{div}\mathbf{f}$ of $\mathbf{f}$ is the trace of $D\mathbf{f}$. The traceless gradient is denoted by ∇_0 , that is,

$$\nabla_0\,\mathbf{f}\;=\;\nabla\mathbf{f}\;-\;(\operatorname{div}\mathbf{f})\,\mathbf{I}_{\mathbb{C}^N}\,.$$

  The curl (or rotational) of $\mathbf{f}$ is $\operatorname{curl}\mathbf{f}:=D\mathbf{f}-\nabla\mathbf{f}$.

- For $\mathbf{z}$ a vector-field with complex valued components, we denote $\mathbf{z}^*:=(\overline{z}^1,\cdots,\overline{z}^N)$.

- For $\mathbf{z}$ and $\mathbf{u}$ two vector-fields with real or complex valued components, we denote by $(\mathbf{u}^*\cdot\nabla)\mathbf{z}$ the vector-field with components $\sum_{j=1}^N\overline{u}^j\partial_j z^i$, which is also denoted $\overline{u}^j\partial_j z^i$ using Einstein's convention of summation on repeated indices.

- For $\mathbf{K}:\mathbb{R}^N\to\mathbb{C}^{N\times N}$, $\operatorname{div}\mathbf{K}$ is the row matrix made up with the divergence of the column vectors of $\mathbf{K}$.

1.1.b Pseudodifferential calculus

For all $s\in\mathbb{R}$, Λ^s denotes the fractional derivative operator of symbol

$$\lambda^s(\xi)\;=\;(1+|\xi|^2)^{s/2}\,,\quad \xi\in\mathbb{R}^N\,,$$

that is, $\Lambda^s=\mathcal{F}^{-1}\lambda^s\mathcal{F}$, where $\mathcal{F}$ denotes the Fourier transform. The "standard" norm in the Sobolev space $\mathrm{H}^s(\mathbb{R}^N)$ thus reads

$$\|u\|_{\mathrm{H}^s}\;=\;\|\Lambda^s u\|_{\mathrm{L}^2}\,.$$

We shall also use the zeroth order operators $\mathcal{Q}$ and $\mathcal{P}=\mathbf{I}_{\mathrm{L}^2}-\mathcal{Q}$, where $\mathcal{Q}$ is of symbol $\xi\xi^*/|\xi|^2$. In other words, $\mathcal{Q}=-(-\Delta)^{-1}\nabla\mathrm{div}$ is the L^2 orthogonal projector on potential (or curl-free, or irrotational) vector-fields, and $\mathcal{P}$ is the L^2 orthogonal projector on solenoidal (or divergence-free, or "incompressible") vector-fields.

1.2 The Euler-Korteweg model

The model we consider takes the following form:

$$\left\{\begin{array}{l}\partial_t\,\rho\,+\,\operatorname{div}(\rho\,\mathbf{u})\;=\;0\,,\\[2ex] \partial_t(\rho\,\mathbf{u}^*)\,+\,\operatorname{div}(\,\rho\,\mathbf{u}\,\mathbf{u}^*\,+\,p\,\mathbf{I}_{\mathbb{R}^N}\,)\;=\;\operatorname{div}\mathbf{K}\,,\end{array}\right. \tag{1.1}$$

where $\rho>0$ is the density of the fluid, $\mathbf{u}\in\mathbb{R}^N$ is its velocity field, p is an "extended" pressure depending on both ρ and $\nabla\rho$ and $\mathbf{K}$ is the so-called Korteweg stress tensor, also depending on ρ and $\nabla\rho$:

$$p(\rho,\nabla\rho)\;=\;p_0(\rho)\;-\;\frac{1}{2}\left(K(\rho)\;-\;\rho\,\frac{\mathrm{d}K}{\mathrm{d}\rho}\right)|\nabla\rho|^2\,,$$

$$\mathbf{K}(\rho,\nabla\rho)\;=\;\rho\operatorname{div}(K(\rho)\,\nabla\rho)\,\mathbf{I}_{\mathbb{R}^N}\;-\;K(\rho)\,\nabla\rho\cdot D\rho\,.$$

(See for instance [4] for more details.) Both p_0 and K are assumed to be given smooth functions of ρ, with K positive and bounded away from zero on some open range for density $\mathbb{J}_\rho := (J_\rho^-, J_\rho^+) \subset \mathbb{R}^+$. Combining the two equations in (1.1), we may equivalently rewrite this system as

$$
\left\{ \begin{aligned} &\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ &\partial_t \mathbf{u} + (\mathbf{u}^* \cdot \nabla) \mathbf{u} = \nabla(K \Delta \rho + \tfrac{1}{2} K'_\rho |\nabla \rho|^2 - g_0), \end{aligned} \right. \tag{1.2}
$$

where g_0 is the bulk chemical potential of the fluid, by definition such that

$$
\frac{\mathrm{d}p_0}{\mathrm{d}\rho} = \rho \frac{\mathrm{d}g_0}{\mathrm{d}\rho}.
$$

This system is known to admit special smooth (that is, $\mathcal{C}^\infty$) solutions: constant states of course, but also planar traveling waves representing either diffuse interfaces or solitons. Indeed, the system of differential equations governing planar traveling waves reduces to a planar Hamiltonian system, for which a simple phase portrait analysis exhibits heteroclinic/homoclinic orbits; see [5] for more details. This is why in what follows we consider a smooth reference solution $(\underline{\rho}, \underline{\mathbf{u}})$ whose derivatives have a sufficient decay at infinity [1] (see Theorem 6.1 for more details). Our main result is the following, where C^α stands for the Hölder space of index α.

Theorem 1.1 *Take $N \geq 1$. For initial data $(\rho_0, \mathbf{u}_0) \in (\underline{\rho}, \underline{\mathbf{u}}) + \mathrm{H}^{s+1}(\mathbb{R}^N) \times \mathrm{H}^s(\mathbb{R}^N)$ with $s > 1 + \frac{N}{2}$ and ρ_0 taking its values in a compact subset of $\mathbb{J}_\rho$, there exists $T > 0$ and a unique solution $(\rho, \mathbf{u})$ of (1.1) such that $(\rho, \mathbf{u}) - (\underline{\rho}, \underline{\mathbf{u}})$ belongs to*

$$
\mathrm{E}_T^s := \mathcal{C}([0,T]; \mathrm{H}^{s+1}(\mathbb{R}^N) \times \mathrm{H}^s(\mathbb{R}^N)) \cap \mathcal{C}^1([0,T]; \mathrm{H}^{s-1}(\mathbb{R}^N) \times \mathrm{H}^{s-2}(\mathbb{R}^N)).
$$

Besides, $(\rho_0, \mathbf{u}_0) \mapsto (\rho, \mathbf{u})$ maps (continuously) $(\underline{\rho}_0, \underline{\mathbf{u}}_0) + \mathrm{H}^{s+1} \times \mathrm{H}^s$ into $(\underline{\rho}, \underline{\mathbf{u}}) + \mathrm{E}_T^s$.
Finally, any solution $(\rho, \mathbf{u})$ on $[0, T^*) \times \mathbb{R}^N$ which belongs to E_T^s for all $T < T^*$ and satisfies $\rho([0,T^*) \times \mathbb{R}^N) \subset\subset \mathbb{J}_\rho$, $\sup_{t \in [0,T^*)} \|\rho(t)\|_{\mathrm{C}^\alpha} < \infty$ for some $\alpha > 0$ and

$$
\int_0^{T^*} \left(\|\Delta \rho(t)\|_{\mathrm{L}^\infty} + \|\operatorname{curl} \mathbf{u}(t)\|_{\mathrm{L}^\infty} + \|\operatorname{div} \mathbf{u}(t)\|_{\mathrm{L}^\infty} \right) dt < \infty
$$

may be continued beyond T^*.

Remark 1.1 *The system (1.2) is obsviously time-reversible. Therefore a similar result may be stated for negative times.*

Remark 1.2 *It may be shown that for data which are perturbations of size η of a traveling profile, the lifespan is of order (at least) $-\log \eta$ (see Remark 6.1).*

2 An extended formulation for the Euler-Korteweg model

We expect the Euler-Korteweg system (1.2) to have smoother solutions that the Euler system (corresponding to $K = 0$). However, this is far from being easy to prove, as the third order terms do not imply a clear smoothing effect. Additionally, we also have to cope with the (high) nonlinearity (for nonconstant K) of those terms. Our strategy is to consider an extended system

[1] Note that this reference solution might of course be merely a constant.

involving $\nabla\rho$ as a new dependent variable. It appears that the "good" new dependent variable is not the gradient of ρ itself but

$$\mathbf{w} = \sqrt{\frac{K}{\rho}}\,\nabla\rho\,,$$

whose dimension is a velocity, like $\mathbf{u}$. The corresponding extended system contains a (degenerate) Schrödinger equation for the complex valued vector-field $\mathbf{z} := \mathbf{u} + i\,\mathbf{w}$, with variable coefficients depending on

$$a := \sqrt{\rho\,K}.$$

This follows from easy manipulations on (1.2), as we show now.

Observing that $\mathbf{w} = \nabla L$ with $L := \mathcal{L}(\rho)$ and $\mathcal{L}$ being a primitive of the function $\rho \mapsto a(\rho)/\rho$, we first write an equation for L. Multiplying the first equation in (1.2) by a/ρ, we easily get

$$\partial_t L + \mathbf{u}^* \cdot \nabla L + a \,\mathrm{div}\,\mathbf{u} = 0\,.$$

By differentiation in space this readily gives

$$\partial_t \mathbf{w} + \nabla(\,\mathbf{u}^* \cdot \mathbf{w}\,) + \nabla(\,a\,\mathrm{div}\,\mathbf{u}\,) = 0\,.$$

And since $\mathbf{w} = \mathcal{L}'(\rho)\,\nabla\rho$ and $K = \rho\,\mathcal{L}'(\rho)^2$, we have

$$K\,\Delta\rho + \frac{1}{2}\,K'_\rho\,|\nabla\rho|^2 = a\,\mathrm{div}\,\mathbf{w} + \frac{1}{2}\,|\mathbf{w}|^2\,.$$

Substituting this equality in the second equation in (1.2), we end up with the following system for $(\mathbf{u}, \mathbf{w})$:

$$\left\{\begin{array}{llllll} \partial_t\mathbf{u} & + & (\,\mathbf{u}^*\cdot\nabla\,)\,\mathbf{u} - \nabla\left(\frac{1}{2}\,|\mathbf{w}|^2\right) & -\nabla(\,a\,\mathrm{div}\,\mathbf{w}\,) & = & -\nabla g_0\,,\\ \partial_t\mathbf{w} & + & \nabla(\,\mathbf{u}^*\cdot\mathbf{w}\,) & +\nabla(\,a\,\mathrm{div}\,\mathbf{u}\,) & = & \mathbf{0}\,. \end{array}\right.$$

Using that $\mathbf{w}$ is potential, we may rewrite

$$\begin{aligned} \nabla(\,\mathbf{u}^*\cdot\mathbf{w}\,) &= (\,\mathbf{u}^*\cdot\nabla\,)\,\mathbf{w} + (\nabla\mathbf{u})\cdot\mathbf{w}\,,\\ \nabla\left(|\mathbf{w}|^2\right) &= (\,\mathbf{w}^*\cdot\nabla\,)\,\mathbf{w} + (\nabla\mathbf{w})\cdot\mathbf{w} = 2\,(\nabla\mathbf{w})\cdot\mathbf{w}, \end{aligned}$$

hence the above system reduces to the following equation for $\mathbf{z} = \mathbf{u} + i\,\mathbf{w}$:

$$\partial_t\mathbf{z} + (\,\mathbf{u}^*\cdot\nabla\,)\,\mathbf{z} + i\,(\nabla\mathbf{z})\cdot\mathbf{w} + i\,\nabla(\,a\,\mathrm{div}\,\mathbf{z}\,) = -\nabla g_0\,.$$

Finally, since we have assumed K positive, the function $\mathcal{L}$ is invertible. So we can change the dependent variable ρ into L, and introduce

$$a_\sharp := a\circ\mathcal{L}^{-1}, \quad q(\rho) = -\rho g_0'(\rho)/a(\rho) \quad \text{and} \quad q_\sharp = q\circ\mathcal{L}^{-1}.$$

Eventually, the Euler-Korteweg system (1.2) is equivalent to the extended system

$$\text{(ES)} \qquad \left\{\begin{array}{l} \partial_t L + \mathbf{u}^*\cdot\nabla L + a_\sharp(L)\mathrm{div}\,\mathbf{u} = 0,\\ \partial_t\mathbf{z} + (\mathbf{u}^*\cdot\nabla)\mathbf{z} + i\,\nabla\mathbf{z}\cdot\mathbf{w} + i\,\nabla(\,a\,\mathrm{div}\,\mathbf{z}\,) = q_\sharp(L)\mathbf{w}, \end{array}\right.$$

together with the compatibility conditions $\mathrm{Im}\,\mathbf{z} = \nabla L = \mathbf{w}$ and $\mathrm{Re}\,\mathbf{z} = \mathbf{u}$.

In what follows we shall always assume that the functions $a_\sharp$ and $q_\sharp$ are smooth functions defined on an open interval $\mathbb{J} := (J^-, J^+) \subset \mathbb{R}$ with $J^\pm := \mathcal{L}(J_\rho^\pm)$.

3 A priori estimates for the linearized equations

Here we focus on the second line in (ES), and more specifically on a "linearized" version of that equation. Namely, we consider real-valued vector-fields $\mathbf{v}$ and $\mathbf{w}$ such that $\mathbf{w} = a(\rho)\nabla \log \rho$ for some function ρ which satisfies

$$\text{(T)} \qquad \partial_t \rho + \operatorname{div}(\rho \mathbf{v}) = \rho g,$$

and we look for a priori estimates on a complex valued vector-field $\mathbf{z}$ satisfying the linear (degenerate) Schrödinger equation

$$\text{(LS)} \qquad \partial_t \mathbf{z} + (\mathbf{v}^* \cdot \nabla)\mathbf{z} + i\,\nabla \mathbf{z} \cdot \mathbf{w} + i\,\nabla(\,a \operatorname{div} \mathbf{z}\,) = \mathbf{f}.$$

(Note that (LS) is genuinely linear, as we do not assume here the compatibility relation $\operatorname{Im} \mathbf{z} = \mathbf{w}$.) Of course we are interested in estimates without loss of derivatives on the source terms g and $\mathbf{f}$.

3.1 The energy equality

The "natural" method which would consist in trying to estimate $\|\mathbf{z}\|_{\mathrm{L}^2}$ by multiplying (LS) on the left by $\mathbf{z}^*$ fails because of the term $\nabla \mathbf{z} \cdot \mathbf{w}$. In fact, it is most natural to estimate $\|\sqrt{\rho}\mathbf{z}\|_{\mathrm{L}^2}$, recalling that

$$\mathcal{K}[\rho, \mathbf{u}] := \int \left(\tfrac{1}{2}\rho|\mathbf{u}|^2 + \tfrac{K(\rho)}{2}|\nabla\rho|^2\right) dx = \tfrac{1}{2} \int \rho |\mathbf{z}|^2 \, dx .$$

Denoting $\phi_0 := \sqrt{\rho}$ and using the fact that $\phi_0 \mathbf{w} = 2a\nabla\phi_0$, one gets the following equation for $\mathbf{Z} := \phi_0 \mathbf{z}$

$$D_t \mathbf{Z} + i\nabla(a\operatorname{div}\mathbf{Z}) = \mathbf{F} + (D_t\phi_0)\mathbf{z} + i\big(\nabla(a\, D\phi_0 \cdot \mathbf{z}) + (a\operatorname{div}\mathbf{z})\nabla\phi_0 - 2a\nabla\mathbf{z}\cdot\nabla\phi_0\big)$$

where $\mathbf{F} := \phi_0 \mathbf{f}$ and $D_t := \partial_t + (\mathbf{v}^* \cdot \nabla)$.

On one hand, equation (T) insures that

$$(D_t\phi_0)\mathbf{z} = \frac{1}{2}(g - \operatorname{div}\mathbf{v})\mathbf{Z},$$

on the other hand, easy computations yield

$$\nabla(aD\phi_0 \cdot \mathbf{z}) + (a\operatorname{div}\mathbf{z})\nabla\phi_0 - 2a\nabla\mathbf{z}\cdot\nabla\phi_0 = \phi_0^{-1}\nabla(aD\phi_0)\cdot\mathbf{Z} + (a\operatorname{div}\mathbf{Z})\,\nabla\log\phi_0 - a\nabla\mathbf{Z}\cdot\nabla\log\phi_0.$$

We eventually get the following equation for $\mathbf{Z}$:

$$\text{(3.1)} \qquad D_t\mathbf{Z} + i\nabla(a\operatorname{div}\mathbf{Z}) = \mathbf{F} + (g - \operatorname{div}\mathbf{v})\frac{\mathbf{Z}}{2} + i\left(\frac{\nabla(aD\phi_0)}{\phi_0}\right)\cdot\mathbf{Z} - ia\nabla_0\mathbf{Z}\cdot\nabla\log\phi_0.$$

To get an L^2 estimate for $\mathbf{Z}$, it now suffices to multiply (3.1) on the left by $\mathbf{Z}^*$, to take twice the real part and integrate over $\mathbb{R}^N$. After integrating by parts, the left-hand side reduces to

$$\frac{d}{dt}\|\mathbf{Z}\|_{\mathrm{L}^2}^2 - \int |\mathbf{Z}|^2 \operatorname{div}\mathbf{v}\, dx.$$

Since $\phi_0^{-1}\nabla(aD\phi_0)$ is real symmetric (as a combination of $\nabla^2\rho$ and $\nabla\rho\cdot D\rho$) one easily gathers that

$$\operatorname{Im}\int \mathbf{Z}^*\cdot\phi_0^{-1}\nabla(aD\phi_0)\cdot\mathbf{Z} = 0.$$

Finally, one may apply the following lemma to the last term in (3.1).

Lemma 3.1 *For all* $\mathbf{W}\in\mathcal{C}^1(\mathbb{R}^N;\mathbb{R}^N)$ *and* $\mathbf{Z}\in\mathcal{C}^1(\mathbb{R}^N;\mathbb{C}^N)$ *tending to* 0 *at infinity,*

$$2i\,\mathrm{Im}\int \mathbf{Z}^*\cdot\nabla_0\mathbf{Z}\cdot\mathbf{W}\,dx=\int \mathbf{Z}^*\cdot\mathrm{curl}\,\mathbf{W}\cdot\mathbf{Z}\,dx.$$

Proof. Integrating by parts, we get

$$\begin{aligned}2i\mathrm{Im}\int \mathbf{Z}^*\cdot\nabla\mathbf{Z}\cdot\mathbf{W}\,dx &= \int\big(\overline{Z^j}\,(\partial_j Z^k)\,W^k - Z^j\,(\partial_j\overline{Z^k})\,W^k\big)\,dx\\ &=\int\big((\mathrm{div}\,\mathbf{Z})\,\mathbf{Z}^*\!\cdot\mathbf{W}-(\mathrm{div}\,\overline{\mathbf{Z}})\,\mathbf{W}^*\!\cdot\mathbf{Z}\big)\,dx+\int Z^j\overline{Z^k}(\partial_j W^k-\partial_k W^j)\,dx,\end{aligned}$$

where we have used Einstein's convention on summation over repeated indices. □

Now, Lemma 3.1 applies to $\mathbf{W}=a\,\nabla\log\phi_0=\frac{1}{2}\mathbf{w}$, which is curl-free by assumption. Therefore, Eqs (3.1) and (T) eventually imply the equality

$$\frac{d}{dt}\,\|\mathbf{Z}\|^2_{\mathrm{L}^2}=\int g|\mathbf{Z}|^2\,dx+2\,\mathrm{Re}\int\mathbf{Z}^*\cdot\mathbf{F}\,dx.$$

This is reformulated in the following.

Proposition 3.1 *Let* $\mathbf{z}$ *be a solution to* (LS) *with* $a=a(\rho)$, $\mathbf{w}=a\nabla\log\rho$ *for some function* ρ *which satisfies* $\partial_t\rho+\mathrm{div}\,(\rho\mathbf{v})=\rho g$. *Then we have*

$$\frac{d}{dt}\,\|\sqrt{\rho}\mathbf{z}\|^2_{\mathrm{L}^2}=\int\rho g|\mathbf{z}|^2\,dx+2\,\mathrm{Re}\int\rho\mathbf{z}^*\cdot\mathbf{f}\,dx.$$

3.2 Higher order estimates

In order to get H^s estimates for $\mathbf{z}$, we apply to (LS) the fractional derivative operator Λ^s. We get

$$D_t\Lambda^s\mathbf{z}+i\,\nabla\Lambda^s\mathbf{z}\cdot\mathbf{w}\;+\;i\,\nabla(\,a\,\mathrm{div}\,\Lambda^s\mathbf{z}\,)\;=\;\Lambda^s\mathbf{f}+[v^j,\Lambda^s]\partial_j\mathbf{z}+i[w^j,\Lambda^s]\nabla z^j+i\nabla[\,a,\Lambda^s]\mathrm{div}\,\mathbf{z},$$

where we have used again Einstein's convention on summation over repeated indices, and delimiters $[\,,\,]$ stand for commutators. Up to the three commutator terms in the right-hand side, this equation for $\Lambda^s\mathbf{z}$ resembles the one we have for $\mathbf{z}$.

On the one hand, as far as $\mathbf{v}$ and $\mathbf{w}$ are smooth enough, we do not have to worry about the first two commutators. Indeed, Lemma A.2 in the appendix insures that they are of order 0 with respect to $\Lambda^s\mathbf{z}$. On the other hand, the last commutator induces a loss of one derivative.

According to symbolic calculus (using the Poisson bracket of a and λ^s, see for instance [1], p. 38; also see Lemma A.3), we easily find that for a smooth enough, the principal part of the commutator $[\,a,\Lambda^s]$ is $s\,Da\cdot\Lambda^{s-2}\nabla$. Now, writing $\nabla\mathrm{div}=\Delta\mathcal{Q}$ where $\mathcal{Q}$ is the L^2 orthogonal projector on potential vector-fields, we observe that

$$\Lambda^{s-2}\nabla\mathrm{div}\,=\Lambda^{s-2}\Delta\mathcal{Q}\;=\;-\Lambda^s\,\mathcal{Q}+\Lambda^{s-2}\mathcal{Q}.$$

Using that Λ^{s-2} commutes with ∇ and div, we thus see that, up to a remainder term of order 0 with respect to $\Lambda^s\mathbf{z}$, we have

$$\nabla[\,a,\Lambda^s]\mathrm{div}\,\mathbf{z}=-s\nabla^2a\cdot\mathcal{Q}\Lambda^s\mathbf{z}-s\nabla(\mathcal{Q}\Lambda^s\mathbf{z})\cdot\nabla a.$$

Note that the first term $s\nabla^2a\cdot\mathcal{Q}\Lambda^s\mathbf{z}$ is also of order 0 (precise estimates will be given later).

Following the line of the previous section, we introduce the functions $\mathbf{Z}_s := \phi_0 \Lambda^s \mathbf{z}$ and $\mathbf{F}_s := \phi_0 \Lambda^s \mathbf{f}$. After a few calculations, we eventually get

$$
\begin{aligned}
(3.2) \qquad D_t \mathbf{Z}_s + i\nabla(a\mathrm{div}\,\mathbf{Z}_s) &= \mathbf{F}_s + (g - \mathrm{div}\,\mathbf{v})\tfrac{\mathbf{Z}_s}{2} + i\phi_0^{-1}\nabla(aD\phi_0)\cdot \mathbf{Z}_s \\
&\quad -ia\nabla_0 \mathbf{Z}_s \cdot \nabla \log \phi_0 - is\phi_0 \nabla(\mathcal{Q}\Lambda^s \mathbf{z})\cdot \nabla a + \mathbf{R}_s,
\end{aligned}
$$

with $\mathbf{R}_s := \phi_0\left(\mathbf{R}_1 + i\mathbf{R}_2 + i\mathbf{R}_0 - is\nabla^2 a \cdot \mathcal{Q}\Lambda^s \mathbf{z} + is\nabla^2 a \cdot \mathcal{Q}\Lambda^{s-2}\mathbf{z} + is\nabla \mathcal{Q}\Lambda^{s-2}\mathbf{z}\cdot Da\right)$ and

$$
\begin{aligned}
\mathbf{R}_0 &= \nabla(a\,\mathrm{div}\,\Lambda^s \mathbf{z}) - \Lambda^s \nabla(a\mathrm{div}\,\mathbf{z}) - s\nabla(Da\cdot \nabla \mathrm{div}\,\Lambda^{s-2}\mathbf{z}),\\
\mathbf{R}_1 &= [v^j, \Lambda^s]\partial_j \mathbf{z},\\
\mathbf{R}_2 &= [w^j, \Lambda^s]\nabla z^j.
\end{aligned}
$$

Unfortunately, the last but one term in (3.2), $\mathbf{G} := \nabla(\mathcal{Q}\Lambda^s \mathbf{z})\cdot \nabla a$ induces a loss of one derivative. However, one can try to cancel out this bad term by estimating $\psi_s \mathbf{Z}_s$ with ψ_s a convenient positive function of ρ (that we shall call a *gauge* after the one-dimensional case treated in [4]).

Rewriting $\mathbf{G} = \nabla \Lambda^s \mathbf{z}\cdot \nabla a - \nabla \mathcal{P}\Lambda^s \mathbf{z}\cdot \nabla a$ where $\mathcal{P}$ is the L^2 projector on divergence-free vector-fields, and computing as in the case $s = 0$, we get the following equation:

$$
\begin{aligned}
D_t(\psi_s \mathbf{Z}_s) + i\nabla(a\mathrm{div}\,(\psi_s \mathbf{Z}_s)) &= \psi_s\big(\mathbf{F}_s + \mathbf{R}_s + is\phi_0\psi_s \nabla \mathcal{P}\Lambda^s \mathbf{z}\cdot \nabla a\big) + (D_t \log \psi_s^2 + g - \mathrm{div}\,\mathbf{v})\frac{\psi_s \mathbf{Z}_s}{2}\\
+i\left[\left(\frac{\nabla(aD\phi_0)}{\phi_0} + \frac{\nabla(aD\psi_s)}{\psi_s} + a\nabla \log \psi_s D\log\left(\frac{a^s}{\psi_s^2}\right)\right)\cdot(\psi_s \mathbf{Z}_s)\right] &+ ia\nabla_0(\psi_s \mathbf{Z}_s)\cdot \nabla \log\left(\frac{\psi_s}{a^s \phi_0}\right)\\
&+ia\,\mathrm{div}\,(\psi_s \mathbf{Z}_s)\nabla \log\left(\frac{\psi_s^2}{a^s}\right).
\end{aligned}
$$

This looks rather complicated. For clarity, we shall first deal with the case when $\mathbf{z}$ is curl-free.

3.2.a Higher order estimates in the potential case

We assume here $\mathbf{z}$ is *potential*, that is $\mathrm{curl}\,\mathbf{z} = \mathbf{0}$. This will make possible a "direct" estimate of $\psi_s \mathbf{Z}_s$, provided that ψ_s is well chosen.

First of all, in the previous equation for $D_t(\psi_s \mathbf{Z}_s)$, the term $\nabla(\mathcal{P}\Lambda^s \mathbf{z})\cdot \nabla a$ vanishes. Furthermore, multiplying that equation on the left by $\psi_s \mathbf{Z}_s^*$, taking twice the real part, integrating over $\mathbb{R}^N$ and using Lemma 3.1 and that a, ϕ_0 and ψ_s are functions of ρ, we discover that the second line above has no contribution, that the terms corresponding to the first line may be computed exactly like in the case $s = 0$ and that the last line is non zero (hence entails the loss of one derivative) unless ψ_s^2 is proportional to a^s.

We thus set $\psi_s = a^{\frac{s}{2}}$. As

$$
D_t \psi_s^2 = \frac{s\rho a'(\rho)}{a(\rho)}(g - \mathrm{div}\,\mathbf{v}),
$$

we eventually obtain the following equality:

$$
\begin{aligned}
(3.3) \qquad \frac{d}{dt}\|\psi_s \mathbf{Z}_s\|_{\mathrm{L}^2}^2 &= 2\,\mathrm{Re}\int \psi_s^2 \mathbf{Z}_s^* \cdot \mathbf{F}_s + \int g\left(1 + \frac{s\rho a'}{a}\right)|\psi_s \mathbf{Z}_s|^2\,dx\\
&\quad - s\int \frac{\rho a'}{a(\rho)}(\mathrm{div}\,\mathbf{v})|\psi_s \mathbf{Z}_s|^2\,dx + 2\,\mathrm{Re}\int \psi_s^2 \mathbf{Z}_s^* \cdot \mathbf{R}_s\,dx.
\end{aligned}
$$

In order to conclude, we need that ρ and a be bounded and bounded away from zero. In what follows, we shall assume the following.

$$
(\mathcal{H}) \qquad 0 < \underline{\rho} \le \rho(t,x) \le \widetilde{\rho} < \infty \quad \text{and} \quad 0 < \underline{a} \le a(t,x) \le \widetilde{a} < \infty \quad \text{for} \quad (t,x) \in [0,T]\times \mathbb{R}^N.
$$

This obviously implies that

$$\widetilde{\|\mathbf{z}\|}_{\mathrm{H}^s} := \left\|\sqrt{\rho a^s}\,\Lambda^s\mathbf{z}\right\|_{\mathrm{L}^2}$$

defines a norm on H^s, equivalent to the standard one. We claim that the equality (3.3) combined with the commutator estimates of the appendix leads to the following.

Proposition 3.2 *Let* $\mathbf{z}$ *satisfy equation* (LS) *with* $a = a(\rho)$ *and* $\mathbf{w} = a\nabla\log\rho$ *for some function* ρ *such that* $\partial_t\rho + \mathrm{div}\,(\rho\mathbf{v}) = \rho g$. *Assume that* $(\mathcal{H})$ *is satisfied and that* $\mathrm{curl}\,\mathbf{z} = \mathbf{0}$.

- *If* $-N/2 < s < N/2+1$ *then the following estimate holds true for all* $t \in [0,T]$:

$$\widetilde{\|\mathbf{z}(t)\|}_{\mathrm{H}^s} \le \mathrm{e}^{C\int_0^t A(\tau)\,d\tau}\left(\widetilde{\|\mathbf{z}_0\|}_{\mathrm{H}^s} + \int_0^t \mathrm{e}^{-C\int_0^\tau A(\tau')\,d\tau'}\widetilde{\|\mathbf{f}(\tau)\|}_{\mathrm{H}^s}\,d\tau\right) \tag{3.4}$$

  with $\widetilde{\|\mathbf{z}\|}_{\mathrm{H}^s} := \|\sqrt{\rho a^s}\,\Lambda^s\mathbf{z}\|_{\mathrm{L}^2}$, C *depending only on* N, s, $\underline{\rho}$ *and* $\widetilde{\rho}$, *and*

$$A(t) = \|g(t)\|_{\mathrm{L}^\infty} + \|Da(t)\|_{\mathrm{L}^\infty} + \|D\mathbf{v}(t)\|_{\mathrm{H}^{\frac N2}\cap\mathrm{L}^\infty} + \|D\mathbf{w}(t)\|_{\mathrm{H}^{\frac N2}\cap\mathrm{L}^\infty} + \|\nabla^2 a(t)\|_{\mathrm{H}^{\frac N2}\cap\mathrm{L}^\infty}.$$

- *If* $s > N/2+1$ *then* (3.4) *holds true with*

$$A(t) = \|g(t)\|_{\mathrm{L}^\infty} + \|Da(t)\|_{\mathrm{L}^\infty} + \|D\mathbf{v}(t)\|_{\mathrm{H}^{s-1}} + \|D\mathbf{w}(t)\|_{\mathrm{H}^{s-1}} + \|\nabla^2 a(t)\|_{\mathrm{H}^{s-1}}.$$

- *If, additionally,* $\mathbf{v} = \mathrm{Re}\,\mathbf{z}$ *and* $\mathbf{w} = \mathrm{Im}\,\mathbf{z}$, *then we have for all* $s > 0$

$$\widetilde{\|\mathbf{z}(t)\|}_{\mathrm{H}^s} \le \mathrm{e}^{C\int_0^t A(\tau)\,d\tau}\left(\widetilde{\|\mathbf{z}_0\|}_{\mathrm{H}^s} + \int_0^t \mathrm{e}^{-C\int_0^\tau A(\tau')\,d\tau'}\widetilde{\|\mathbf{f}\|}_{\mathrm{H}^s}\,d\tau\right) \tag{3.5}$$

  with $A(t) = 1 + \|g(t)\|_{\mathrm{L}^\infty} + \|D\mathbf{z}(t)\|_{\mathrm{L}^\infty}$, *provided the function* $a_\sharp := a\circ\mathcal{L}^{-1}$ *is in* $\mathrm{W}^{\sigma+2,\infty}$ *with* σ *the smallest integer such that* $\sigma \ge s$.

  In the particular case where $g = 0$ *and* $\mathbf{f} = q_\sharp(L)\mathbf{w}$ *for some* $q_\sharp$ *in* $\mathrm{W}^{\sigma+1,\infty}$, *we have*

$$\widetilde{\|\mathbf{z}(t)\|}_{\mathrm{H}^s} \le \mathrm{e}^{C\int_0^t\left(1+\|D\mathbf{z}(\tau)\|_{\mathrm{L}^\infty}\right)d\tau}\widetilde{\|\mathbf{z}_0\|}_{\mathrm{H}^s}. \tag{3.6}$$

Proof. This is only a matter of bounding the remainder term $\mathbf{R}_s$.

Clearly, we have for all $s \in \mathbb{R}$,

$$\left\|\nabla^2 a\cdot\mathcal{Q}\Lambda^s\mathbf{z}\right\|_{\mathrm{L}^2} + \left\|\nabla^2 a\cdot\mathcal{Q}\Lambda^{s-2}\mathbf{z}\right\|_{\mathrm{L}^2} + \left\|\nabla\mathcal{Q}\Lambda^{s-2}\mathbf{z}\cdot Da\right\|_{\mathrm{L}^2} \lesssim (\left\|D^2 a\right\|_{\mathrm{L}^\infty} + \|Da\|_{\mathrm{L}^\infty})\|\mathbf{z}\|_{\mathrm{H}^s} \tag{3.7}$$

where it is understood that (in what follows) the notation $A \lesssim B$ means that $A \le CB$ for some harmless constant C.

Bounding the remainders $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{R}_0$ relies on the results of the appendix. We have to proceed differently according to the value of s.

Let us first assume that $-N/2 < s < N/2+1$. Applying the inequality (A.6) we get

$$\|\mathbf{R}_1\|_{\mathrm{L}^2} \lesssim \|D\mathbf{v}\|_{\mathrm{H}^{\frac N2}\cap\mathrm{L}^\infty}\|D\mathbf{z}\|_{\mathrm{H}^{s-1}}, \tag{3.8}$$

$$\|\mathbf{R}_2\|_{\mathrm{L}^2} \lesssim \|D\mathbf{w}\|_{\mathrm{H}^{\frac N2}\cap\mathrm{L}^\infty}\|D\mathbf{z}\|_{\mathrm{H}^{s-1}}. \tag{3.9}$$

For bounding the commutator $\mathbf{R}_0$, we apply Lemma A.3 with $u = \mathrm{div}\,\mathbf{z}$ and $m = 1$. We get

$$\|\mathbf{R}_0\|_{\mathrm{L}^2} \lesssim \|\nabla^2 a\|_{\mathrm{H}^{\frac N2}\cap\mathrm{L}^\infty}\|\mathrm{div}\,\mathbf{z}\|_{\mathrm{H}^{s-1}}. \tag{3.10}$$

Plugging (3.7), (3.8), (3.9) and (3.10) into (3.3), we end up with

$$(3.11)\quad \frac{1}{2}\frac{d}{dt}\widetilde{\|\mathbf{z}\|}_{\mathrm{H}^s}^2 \le \widetilde{\|\mathbf{z}\|}_{\mathrm{H}^s}\widetilde{\|\mathbf{f}\|}_{\mathrm{H}^s} + C\widetilde{\|\mathbf{z}\|}_{\mathrm{H}^s}^2\left(\|g\|_{\mathrm{L}^\infty}+\|Da\|_{\mathrm{L}^\infty}+\|D\mathbf{v}\|_{\mathrm{H}^{\frac{N}{2}}\cap\mathrm{L}^\infty}+\|D\mathbf{w}\|_{\mathrm{H}^{\frac{N}{2}}\cap\mathrm{L}^\infty}+\|\nabla^2 a\|_{\mathrm{H}^{\frac{N}{2}}\cap\mathrm{L}^\infty}\right)$$

for some constant C depending only on s, N, $\underline{\rho}$, $\widetilde{\rho}$, and on the function a.

If $s>0$, we apply the inequality (A.7) to v^j and $\partial_j z^i$, or w^j and $\partial_i z^j$, and obtain

$$(3.12)\quad \|\mathbf{R}_1\|_{\mathrm{L}^2} \lesssim \|D\mathbf{z}\|_{\mathrm{L}^\infty}\|D\mathbf{v}\|_{\mathrm{H}^{s-1}} + \|D\mathbf{v}\|_{\mathrm{L}^\infty}\|D\mathbf{z}\|_{\mathrm{H}^{s-1}},$$

$$(3.13)\quad \|\mathbf{R}_2\|_{\mathrm{L}^2} \lesssim \|D\mathbf{z}\|_{\mathrm{L}^\infty}\|D\mathbf{w}\|_{\mathrm{H}^{s-1}} + \|D\mathbf{w}\|_{\mathrm{L}^\infty}\|D\mathbf{z}\|_{\mathrm{H}^{s-1}}.$$

For $\mathbf{R}_0$, the second part of Lemma A.3 (with $u=\operatorname{div}\mathbf{z}$ and $m=1$) yields

$$(3.14)\quad \|\mathbf{R}_0\|_{\mathrm{L}^2} \lesssim \|\operatorname{div}\mathbf{z}\|_{\mathrm{L}^\infty}\|\nabla^2 a\|_{\mathrm{H}^{s-1}} + \left\|\nabla^2 a\right\|_{\mathrm{L}^\infty}\|\operatorname{div}\mathbf{z}\|_{\mathrm{H}^{s-1}}.$$

Plugging (3.7), (3.12), (3.13) and (3.14) into (3.3), we end up with

$$(3.15)\quad \tfrac{1}{2}\tfrac{d}{dt}\widetilde{\|\mathbf{z}\|}_{\mathrm{H}^s}^2 \le \widetilde{\|\mathbf{z}\|}_{\mathrm{H}^s}\widetilde{\|\mathbf{f}\|}_{\mathrm{H}^s} + C\widetilde{\|\mathbf{z}\|}_{\mathrm{H}^s}\|D\mathbf{z}\|_{\mathrm{L}^\infty}\left(\|D\mathbf{v}\|_{\mathrm{H}^{s-1}}+\|D\mathbf{w}\|_{\mathrm{H}^{s-1}}+\|\nabla^2 a\|_{\mathrm{H}^{s-1}}\right) + C\widetilde{\|\mathbf{z}\|}_{\mathrm{H}^s}^2\left(\|g\|_{\mathrm{L}^\infty}+\|Da\|_{\mathrm{L}^\infty}+\|D\mathbf{v}\|_{\mathrm{L}^\infty}+\|D\mathbf{w}\|_{\mathrm{L}^\infty}+\left\|\nabla^2 a\right\|_{\mathrm{L}^\infty}\right)$$

for some constant C depending only on s, N, $\underline{\rho}$, $\widetilde{\rho}$, $\underline{a}$ and on the function a.

Combining Gronwall's inequality with either (3.11) or (3.15) (and the fact that $\mathrm{H}^{s-1}\hookrightarrow\mathrm{L}^\infty$ if $s>1+N/2$) completes the proof of (3.4).

Let us now assume that $\mathbf{z}=\mathbf{u}+i\mathbf{w}$ and $s>0$. Remind that $\mathbf{w}=\nabla L$ with $L=\mathcal{L}(\rho)$ and that $a=a_\sharp(L)$. On the one hand, by Proposition B.1, we have

$$\|\nabla^2 a\|_{\mathrm{H}^{s-1}} \le C\|D\mathbf{w}\|_{\mathrm{H}^{s-1}}.$$

On the other hand, we have $\nabla^2 a = a_\sharp''(L)\nabla L\cdot DL + a_\sharp'(L)D\mathbf{w}$ so that by the interpolation inequality

$$(3.16)\quad \|DL\|_{\mathrm{L}^\infty}^2 \le C\,\|L\|_{\mathrm{L}^\infty}\left\|\nabla^2 L\right\|_{\mathrm{L}^\infty},$$

we get

$$\left\|D^2 a\right\|_{\mathrm{L}^\infty} \le C\,\|D\mathbf{w}\|_{\mathrm{L}^\infty} \quad\text{and}\quad \|Da\|_{\mathrm{L}^\infty} \le C(1+\|D\mathbf{w}\|_{\mathrm{L}^\infty})$$

for some constant C depending only on the function $a_\sharp$ and on the bounds for ρ.

Inserting the above inequalities in (3.15), we end up with

$$\frac{1}{2}\frac{d}{dt}\widetilde{\|\mathbf{z}\|}_{\mathrm{H}^s}^2 \le \widetilde{\|\mathbf{z}\|}_{\mathrm{H}^s}\widetilde{\|\mathbf{f}\|}_{\mathrm{H}^s} + C\widetilde{\|\mathbf{z}\|}_{\mathrm{H}^s}^2\left(\|g\|_{\mathrm{L}^\infty}+\|D\mathbf{z}\|_{\mathrm{L}^\infty}\right),$$

which gives (3.5) by Gronwall's lemma.

The proof of (3.6) stems from Corollary B.2 and Gronwall's lemma. The details are left to the reader. □

3.2.b Higher order estimates in the general case

This section is devoted to the proof of estimates in Sobolev spaces for equation (LS) in the general case. The counterpart of Proposition 3.2 is the following [2].

[2] Below, it is understood that $\|\cdot\|_{\mathrm{C}^{-\alpha}}=\|\cdot\|_{\mathrm{L}^\infty}$ if $\alpha=0$, and that $\|\cdot\|_{\mathrm{C}^{-\alpha}}$ is the norm in the *Besov space* $\mathrm{B}^{-\alpha}_{\infty,\infty}$ if $\alpha>0$ (see the definition in [17]).

Proposition 3.3 *Let* $\mathbf{z}$ *satisfy equation* (LS) *on* $[0,T]\times\mathbb{R}^N$ *with* $a=a(\rho)$ *and* $\mathbf{w}=a\nabla\log\rho$ *for some function* ρ *such that* $\partial_t\rho+\operatorname{div}(\rho\mathbf{v})=\rho g$ *Assume that* $(\mathcal{H})$ *is satisfied and that* $s>0$. *Then the following estimate holds true for all* $t\in[0,T]$ *and* $\alpha\in[0,1)$*:*

(3.17)

$$\|\mathbf{z}(t)\|_{\mathrm{H}^s}^2 \le C\biggl(\|\mathbf{z}_0\|_{\mathrm{H}^s}^2+\int_0^t\bigl(\|\mathbf{f}(\tau)\|_{\mathrm{H}^s}\|\mathbf{z}(\tau)\|_{\mathrm{H}^s}+A(\tau)\|\mathbf{z}(\tau)\|_{\mathrm{H}^s}^2\bigr)\,d\tau+\|D\rho(t)\|_{\mathrm{C}^{-\alpha}}^2\|\mathbf{z}(t)\|_{\mathrm{H}^{s-1+\alpha}}^2\biggr)$$

for some constant C *depending only on* N, α, s, $\underline{\rho}$ *and* $\widetilde{\rho}$, *and* $A(t)$ *equals*

$$\begin{array}{ll}\|g(t)\|_{\mathrm{L}^\infty}+\|D\rho(t)\|_{\mathrm{L}^\infty}+\|D\mathbf{v}(t)\|_{\mathrm{H}^{\frac N2}\cap\mathrm{L}^\infty}+\|D\mathbf{w}(t)\|_{\mathrm{H}^{\frac N2}\cap\mathrm{L}^\infty}+\|D^2\rho(t)\|_{\mathrm{H}^{\frac N2}\cap\mathrm{L}^\infty} & \text{if}\quad s<\frac N2+1,\\ \|g(t)\|_{\mathrm{L}^\infty}+\|D\rho(t)\|_{\mathrm{L}^\infty}+\|D\mathbf{v}(t)\|_{\mathrm{H}^{s-1}}+\|D\mathbf{w}(t)\|_{\mathrm{H}^{s-1}}+\|D^2\rho(t)\|_{\mathrm{H}^{s-1}} & \text{if}\quad s>\frac N2+1.\end{array}$$

If besides $\mathbf{v}=\operatorname{Re}\mathbf{z}$ *and* $\mathbf{w}=\operatorname{Im}\mathbf{z}$ *then inequality* (3.17) *holds with*

$$A(t)=1+\|g(t)\|_{\mathrm{L}^\infty}+\|D\mathbf{z}(t)\|_{\mathrm{L}^\infty} \tag{3.18}$$

provided $a_\sharp:=a\circ\mathcal{L}^{-1}$ *is in* $\mathrm{W}^{\sigma+2,\infty}$ *with* σ *the smallest integer such that* $\sigma\ge s$.

In the particular case where $g=0$ *and* $\mathbf{f}=q_\sharp(L)\mathbf{w}$ *for some* $q_\sharp$ *in* $\mathrm{W}^{\sigma+1,\infty}$, *we have*

$$\|\mathbf{z}(t)\|_{\mathrm{H}^s}^2\le C\biggl(\|\mathbf{z}_0\|_{\mathrm{H}^s}^2+\int_0^t(1+\|D\mathbf{z}\|_{\mathrm{L}^\infty})\|\mathbf{z}\|_{\mathrm{H}^s}^2\,d\tau+\|\mathbf{w}(t)\|_{\mathrm{C}^{-\alpha}}^2\|\mathbf{z}(t)\|_{\mathrm{H}^{s-1+\alpha}}^2\biggr). \tag{3.19}$$

Proof. Denoting $\psi_s:=a^{\frac s2}$ as in the potential case, the equation for $\psi_s\mathbf{Z}_s$ reduces to

$$\begin{aligned} D_t(\psi_s\mathbf{Z}_s)+i\nabla(a\operatorname{div}(\psi_s\mathbf{Z}_s)) &= \psi_s(\mathbf{F}_s+\mathbf{R}_s)+\tfrac12\left(D_t\log\psi_s^2+g-\operatorname{div}\mathbf{v}\right)\psi_s\mathbf{Z}_s\\ &+i\Bigl(\tfrac{\nabla(aD\phi_0)}{\phi_0}+\tfrac{\nabla(aD\psi_s)}{\psi_s}\Bigr)\cdot(\psi_s\mathbf{Z}_s)-ia\nabla_0(\psi_s\mathbf{Z}_s)\cdot\nabla\log\bigl(\phi_0\psi_s\bigr)+is\phi_0\psi_s\nabla\mathcal{P}\Lambda^s\mathbf{z}\cdot\nabla a.\end{aligned} \tag{3.20}$$

We observe that in the general case ($\operatorname{curl}\mathbf{z}\neq\mathbf{0}$) the last term in (3.20), $\nabla\mathcal{P}\Lambda^s\mathbf{z}\cdot\nabla a$ is responsible for the loss of one derivative. A second gauge function will be used to overcome this problem.

As a first step, we aim at getting a bound for $\|\mathcal{Q}(\psi_s\mathbf{Z}_s)\|_{\mathrm{L}^2}$. Since $\mathcal{Q}$ is a projector, this will be made by merely considering the inner product of the (3.20) with $\mathcal{Q}(\psi_s\mathbf{Z}_s)$. Indeed, the time derivative of $\|\mathcal{Q}(\psi_s\mathbf{Z}_s)\|_{\mathrm{L}^2}^2$ turns out to coincide with $2\operatorname{Re}\int\bigl(\mathcal{Q}(\psi_s\mathbf{Z}_s)\bigr)^*\cdot D_t(\psi_s\mathbf{Z}_s)\,dx$ up to zeroth order terms. This follows from successive integrations by parts as we show now. Recalling that $D_t=\partial_t+(\mathbf{v}^*\cdot\nabla)$, a first integration by parts yields

$$\begin{aligned}2\operatorname{Re}\int\bigl(\mathcal{Q}(\psi_s\mathbf{Z}_s)\bigr)^*\cdot D_t(\psi_s\mathbf{Z}_s)\,dx-\frac{d}{dt}\|\mathcal{Q}(\psi_s\mathbf{Z}_s)\|_{\mathrm{L}^2}^2&\\ =-2\operatorname{Re}\int(\operatorname{div}\mathbf{v})\bigl(\mathcal{Q}(\psi_s\mathbf{Z}_s)\bigr)^*\cdot(\psi_s\mathbf{Z}_s)\,dx&-2\operatorname{Re}\int\partial_j\overline{\bigl(\mathcal{Q}(\psi_s\mathbf{Z}_s)\bigr)}^k(\psi_s\mathbf{Z}_s^k)v^j\,dx.\end{aligned}$$

Since $\psi_s\mathbf{Z}_s=\mathcal{P}(\psi_s\mathbf{Z}_s)+\mathcal{Q}(\psi_s\mathbf{Z}_s)$, the last term can be rewritten as

$$\begin{aligned}\int\partial_j\overline{\bigl(\mathcal{Q}(\psi_s\mathbf{Z}_s)\bigr)}^k(\psi_s\mathbf{Z}_s^k)v^j\,dx&=\int\Bigl(\partial_j\overline{\bigl(\mathcal{Q}(\psi_s\mathbf{Z}_s)\bigr)}^k\bigl(\mathcal{Q}(\psi_s\mathbf{Z}_s)\bigr)^kv^j+\partial_j\overline{\bigl(\mathcal{Q}(\psi_s\mathbf{Z}_s)\bigr)}^k\bigl(\mathcal{P}(\psi_s\mathbf{Z}_s)\bigr)^kv^j\Bigr)dx,\\ &=-\frac12\int(\operatorname{div}\mathbf{v})\bigl|\mathcal{Q}(\psi_s\mathbf{Z}_s)\bigr|^2dx+\int\partial_k\overline{\bigl(\mathcal{Q}(\psi_s\mathbf{Z}_s)\bigr)}^j\bigl(\mathcal{P}(\psi_s\mathbf{Z}_s)\bigr)^kv^j\,dx,\end{aligned}$$

where we have integrated by parts the first term, and used the property $\operatorname{curl}\mathcal{Q}=0$ in the second term. An ultimate integration by parts in the latter combined with the property $\operatorname{div}\mathcal{P}=0$ eventually leads to

$$\begin{aligned}2\operatorname{Re}\int\bigl(\mathcal{Q}(\psi_s\mathbf{Z}_s)\bigr)^*\cdot D_t(\psi_s\mathbf{Z}_s)\,dx-\frac{d}{dt}\|\mathcal{Q}(\psi_s\mathbf{Z}_s)\|_{\mathrm{L}^2}^2=-\int(\operatorname{div}\mathbf{v})\bigl|\mathcal{Q}(\psi_s\mathbf{Z}_s)\bigr|^2dx&\\ -2\operatorname{Re}\int(\operatorname{div}\mathbf{v})\bigl(\mathcal{Q}(\psi_s\mathbf{Z}_s)\bigr)^*\cdot\mathcal{P}(\psi_s\mathbf{Z}_s)\,dx+2\operatorname{Re}\int\bigl(\mathcal{P}(\psi_s\mathbf{Z}_s)\bigr)^*\cdot\nabla\mathbf{v}\cdot\mathcal{Q}(\psi_s\mathbf{Z}_s)\,dx,&\end{aligned}$$

whence

$$(3.21)\quad \left|2\,\mathrm{Re}\int\big(\mathcal{Q}(\psi_s\mathbf{Z}_s)\big)^*\cdot D_t(\psi_s\mathbf{Z}_s)\,dx-\frac{d}{dt}\,\|\mathcal{Q}(\psi_s\mathbf{Z}_s)\|_{\mathrm{L}^2}^2\right|\leq C\,\|D\mathbf{v}\|_{\mathrm{L}^\infty}\,\|\mathbf{z}\|_{\mathrm{H}^s}\,\|\mathcal{Q}(\psi_s\mathbf{Z}_s)\|_{\mathrm{L}^2}\,.$$

Now, (3.20) implies that $\mathrm{Re}\int\big(\mathcal{Q}(\psi_s\mathbf{Z}_s)\big)^*\cdot D_t(\psi_s\mathbf{Z}_s)\,dx\;=$

$$\begin{aligned}&\mathrm{Re}\int\big(\mathcal{Q}(\psi_s\mathbf{Z}_s)\big)^*\cdot\Big(\psi_s(\mathbf{F}_s+\mathbf{R}_s)+\,(D_t\log\psi_s^2+g-\mathrm{div}\,\mathbf{v})\,\frac{\psi_s\mathbf{Z}_s}{2}\Big)\,dx\\&-\mathrm{Im}\int\big(\mathcal{Q}(\psi_s\mathbf{Z}_s)\big)^*\cdot\Big(\tfrac{\nabla(aD\phi_0)}{\phi_0}+\tfrac{\nabla(aD\psi_s)}{\psi_s}\Big)\cdot(\psi_s\mathbf{Z}_s)\,dx\\&+\mathrm{Im}\int\big(\mathcal{Q}(\psi_s\mathbf{Z}_s)\big)^*\cdot\Big(a\nabla_0(\psi_s\mathbf{Z}_s)\cdot\nabla\log(\phi_0\psi_s)-s\phi_0\psi_s\nabla\mathcal{P}\Lambda^s\mathbf{z}\cdot\nabla a\Big)\,dx\,.\end{aligned}$$

We have used here the property $\mathcal{Q}\nabla=\nabla$, which shows the second term in the left-hand side of (3.20) has no contribution. In the above equality, the first term in the right-hand side can be estimated exactly as in the potential case, and the second term is harmless: since a, ϕ_0 and ψ_s are functions of ρ, it is easy to show that

$$(3.22)\qquad \left|\int\big(\mathcal{Q}(\psi_s\mathbf{Z}_s)\big)^*\cdot\Big(\tfrac{\nabla(aD\phi_0)}{\phi_0}+\tfrac{\nabla(aD\psi_s)}{\psi_s}\Big)\cdot(\psi_s\mathbf{Z}_s)\,dx\right|\lesssim\left\|D^2\rho\right\|_{\mathrm{L}^\infty}\|\mathbf{z}\|_{\mathrm{H}^s}\,\|\mathcal{Q}(\psi_s\mathbf{Z}_s)\|_{\mathrm{L}^2}\,.$$

The last term requires more work. Its principal part will turn out to be

$$\frac12\;\mathrm{Im}\int\rho a^{s+1}\;\big(\mathcal{Q}\Lambda^s\mathbf{z}\big)^*\cdot\nabla\mathcal{P}\Lambda^s\mathbf{z}\cdot\nabla\log(\rho a^s)\Big)\,dx\,,$$

after several manipulations, integrations by parts, and commutator estimates. We first use Lemma 3.1 and rewrite

$$\mathrm{Im}\int a\big(\mathcal{Q}(\psi_s\mathbf{Z}_s)\big)^*\cdot\nabla_0(\psi_s\mathbf{Z}_s)\cdot\nabla\log(\phi_0\psi_s)\,dx=-\,\mathrm{Im}\int a\big(\mathcal{P}(\psi_s\mathbf{Z}_s)\big)^*\cdot\nabla_0(\psi_s\mathbf{Z}_s)\cdot\nabla\log(\phi_0\psi_s)\,dx\,.$$

Using again the property $\mathrm{div}\,\mathcal{P}=0$ and integrating by parts we see that

$$-\,\mathrm{Im}\int a\big(\mathcal{P}(\psi_s\mathbf{Z}_s)\big)^*\cdot\nabla(\psi_s\mathbf{Z}_s)\cdot\nabla\log(\phi_0\psi_s)\,dx=\;\mathrm{Im}\int\big(\mathcal{P}(\psi_s\mathbf{Z}_s)\big)^*\cdot D(a\nabla\log(\phi_0\psi_s))\cdot(\psi_s\mathbf{Z}_s)\,dx$$

is another harmless term. So the principal contribution of $\nabla_0(\psi_s\mathbf{Z}_s)$ comes from $\mathrm{div}\,(\psi_s\mathbf{Z}_s)\,\mathbf{I}_{\mathbb{C}^N}$. Recalling that $\psi_s=a^{\frac{s}{2}}$ and $\mathbf{Z}_s=\phi_0\,\Lambda^s\,\mathbf{z}$, this leads to

$$\left|\mathrm{Im}\int\Big(\big(\mathcal{Q}(\psi_s\mathbf{Z}_s)\big)^*\cdot\nabla_0(\psi_s\mathbf{Z}_s)\cdot\nabla\log(\phi_0\psi_s)-(\mathrm{div}\,\Lambda^s\mathbf{z})\,\big(\mathcal{P}(\psi_s\mathbf{Z}_s)\big)^*\cdot\nabla(\phi_0\psi_s)\Big)\;a\;dx\right|\lesssim\left\|D^2\rho\right\|_{\mathrm{L}^\infty}\|\mathbf{z}\|_{\mathrm{H}^s}^2$$

or, using once more the property $\mathrm{div}\,\mathcal{P}=0$ and integrating by parts,

$$\left|\mathrm{Im}\int\Big(\big(\mathcal{Q}(\psi_s\mathbf{Z}_s)\big)^*\cdot\nabla_0(\psi_s\mathbf{Z}_s)\cdot\nabla\log(\phi_0\psi_s)-\big(\mathcal{Q}\Lambda^s\mathbf{z}\big)^*\cdot\nabla\mathcal{P}(\psi_s\mathbf{Z}_s)\cdot\nabla(\phi_0\psi_s)\Big)a\,dx\right|\lesssim\left\|D^2\rho\right\|_{\mathrm{L}^\infty}\|\mathbf{z}\|_{\mathrm{H}^s}^2.$$

As $\nabla\mathcal{P}$ is a homogeneous Fourier multiplier of degree 1, Lemma A.4 insures that

$$(3.23)\qquad \|[\nabla\mathcal{P},\phi_0\psi_s]\Lambda^s\mathbf{z}\|_{\mathrm{L}^2}\lesssim\|D(\phi_0\psi_s)\|_{\mathrm{L}^\infty}\,\|\mathbf{z}\|_{\mathrm{H}^s}.$$

Therefore we have

$$(3.24)\quad \Big|\mathrm{Im}\int\Big(\big(\mathcal{Q}(\psi_s\mathbf{Z}_s)\big)^*\cdot\nabla_0(\psi_s\mathbf{Z}_s)\cdot\nabla\log(\phi_0\psi_s)-\phi_0\psi_s\big(\mathcal{Q}\Lambda^s\mathbf{z}\big)^*\cdot\nabla\mathcal{P}\Lambda^s\mathbf{z}\cdot\nabla(\phi_0\psi_s)\Big)a\,dx\Big| \lesssim \left\|D^2\rho\right\|_{\mathrm{L}^\infty}\|\mathbf{z}\|^2_{\mathrm{H}^s}.$$

In order to find the principal contribution of the last term in (3.20), we also integrate by parts and get

$$\Big|\mathrm{Im}\int\Big(\phi_0\psi_s\big(\mathcal{Q}(\psi_s\mathbf{Z}_s)\big)^*\cdot\nabla\mathcal{P}\Lambda^s\mathbf{z}\cdot\nabla a-\phi_0^2\psi_s^2\mathrm{div}\,\Lambda^s\mathbf{z}\,(\mathcal{P}\Lambda^s\mathbf{z})^*\cdot\nabla a\Big)\,dx\Big|\lesssim\left\|D^2\rho\right\|_{\mathrm{L}^\infty}\|\mathbf{z}\|^2_{\mathrm{H}^s}$$

or, thanks to a second integration by parts,

$$\Big|\mathrm{Im}\int\Big(\phi_0\psi_s\big(\mathcal{Q}(\psi_s\mathbf{Z}_s)\big)^*\cdot\nabla\mathcal{P}\Lambda_s\mathbf{z}\cdot\nabla a-\phi_0^2\psi_s^2\big(\mathcal{Q}\Lambda^s\mathbf{z}\big)^*\cdot\nabla\mathcal{P}\Lambda^s\mathbf{z}\cdot\nabla a\Big)\,dx\Big|\lesssim\left\|D^2\rho\right\|_{\mathrm{L}^\infty}\|\mathbf{z}\|^2_{\mathrm{H}^s}.$$

Plugging (3.21), (3.22), (3.24) and the above inequality in (3.20), we conclude that

$$(3.25)\quad \begin{aligned}\frac{d}{dt}\|\mathcal{Q}(\psi_s\mathbf{Z}_s)\|^2_{\mathrm{L}^2}&=2\,\mathrm{Re}\int\big(\mathcal{Q}(\psi_s\mathbf{Z}_s)\big)^*\cdot\psi_s\mathbf{F}_s\,dx\\&\quad+\mathrm{Im}\int\rho a^{s+1}(\mathcal{Q}\Lambda^s\mathbf{z})^*\cdot\nabla\mathcal{P}\Lambda^s\mathbf{z}\cdot\nabla\log\big(\tfrac{\rho}{a^s}\big)\,dx+\mathcal{R}_s,\end{aligned}$$

where the remainder term $\mathcal{R}_s$ may be bounded by taking advantage of the error bounds in (3.7), (3.8), (3.9), (3.10), (3.12), (3.13), (3.14), (3.21), (3.22) and of Proposition B.1. More precisely, one has

$$(3.26)\quad |\mathcal{R}_s|\lesssim\big(\|D\mathbf{v}\|_{\mathrm{H}^{\frac{N}{2}}\cap\mathrm{L}^\infty}+\|D\mathbf{w}\|_{\mathrm{H}^{\frac{N}{2}}\cap\mathrm{L}^\infty}+\|D^2\rho\|_{\mathrm{H}^{\frac{N}{2}}\cap\mathrm{L}^\infty}+\|g\|_{\mathrm{L}^\infty}\big)\|\mathbf{z}\|^2_{\mathrm{H}^s}\ \text{ if } s\in(-\tfrac{N}{2},\tfrac{N}{2}+1),$$

$$(3.27)\quad |\mathcal{R}_s|\lesssim\|(D\mathbf{v},D\mathbf{w},D^2\rho)\|_{\mathrm{H}^{s-1}}\|D\mathbf{z}\|_{\mathrm{L}^\infty}\|\mathbf{z}\|_{\mathrm{H}^s}+\left\|(D\mathbf{v},D\mathbf{w},D^2\rho,g)\right\|_{\mathrm{L}^\infty}\|\mathbf{z}\|^2_{\mathrm{H}^s}\ \text{ if } s>0.$$

In what follows, we denote by $\mathcal{R}_s$ any term which may be bounded as in (3.26), (3.27).

The bad first term in the second line of (3.25) is unlikely to vanish. This motivates us to look for a *second gauge* ϕ_s pertaining to the solenoidal part of $\mathbf{z}$. We have

$$\begin{aligned}D_t(\phi_s\mathbf{Z}_s)+i\nabla(a\,\mathrm{div}\,(\phi_s\mathbf{Z}_s))&=\phi_s(\mathbf{F}_s+\mathbf{R}_s)+(D_t\log\phi_s^2+g-\mathrm{div}\,\mathbf{v})\frac{\phi_s\mathbf{Z}_s}{2}\\&\quad+i\Big[\Big(\frac{\nabla(aD\phi_0)}{\phi_0}+\frac{\nabla(aD\phi_s)}{\phi_s}+a\nabla\log\phi_s\,D\log\Big(\frac{a^s}{\phi_s^2}\Big)\Big)\cdot(\phi_s\mathbf{Z}_s)\Big]\\&\quad+ia\Big(\nabla_0(\phi_s\mathbf{Z}_s)\cdot\nabla\log\big(\frac{\phi_s}{\phi_0a^s}\big)+\mathrm{div}\,(\phi_s\mathbf{Z}_s)\nabla\log\big(\frac{\phi_s^2}{a^s}\big)\Big)+is\phi_0\phi_s\nabla\mathcal{P}\Lambda^s\mathbf{z}\cdot\nabla a.\end{aligned}$$

In order to get an estimate for $\|\mathcal{P}(\phi_s\mathbf{Z}_s)\|_{\mathrm{L}^2}$, we take the real part of the inner product of the above equality with $\mathcal{P}(\phi_s\mathbf{Z}_s)$. For the first term, we get after several integration by parts,

$$\begin{aligned}2\,\mathrm{Re}\int\big(\mathcal{P}(\phi_s\mathbf{Z}_s)\big)^*\cdot D_t(\phi_s\mathbf{Z}_s)\,dx&=\frac{d}{dt}\|\mathcal{P}(\phi_s\mathbf{Z}_s)\|^2_{\mathrm{L}^2}\\&\quad-\int\mathrm{div}\,\mathbf{v}\,|\mathcal{P}(\phi_s\mathbf{Z}_s)|^2\,dx-2\,\mathrm{Re}\int\big(\mathcal{P}(\phi_s\mathbf{Z}_s)\big)^*\cdot\nabla\mathbf{v}\cdot\mathcal{Q}(\phi_s\mathbf{Z}_s)\,dx,\end{aligned}$$

hence

$$(3.28)\quad 2\,\mathrm{Re}\int\big(\mathcal{P}(\phi_s\mathbf{Z}_s)\big)^*\cdot D_t(\phi_s\mathbf{Z}_s)\,dx=\frac{d}{dt}\|\mathcal{P}(\phi_s\mathbf{Z}_s)\|^2_{\mathrm{L}^2}+\mathcal{R}_s.$$

The next term, namely $\nabla(a\,\mathrm{div}\,(\phi_s\mathbf{Z}_s))$, is a gradient, hence has no contribution. The remainder term $\mathbf{R}_s$ and the next two terms (which are of order zero) may be bounded as in (3.26) and (3.27). Now, by performing several integration by parts, one can easily check that

$$\int \big(\mathcal{P}(\phi_s\mathbf{Z}_s)\big)^*\cdot\bigg[a\nabla_0(\phi_s\mathbf{Z}_s)\cdot\nabla\log\Big(\frac{\phi_s}{\phi_0 a^s}\Big)+a\mathrm{div}\,(\phi_s\mathbf{Z}_s)\nabla\log\Big(\frac{\phi_s^2}{a^s}\Big)+s\phi_0\phi_s\nabla\mathcal{P}\Lambda^s\mathbf{z}\cdot\nabla a\bigg]\,dx$$
$$=\int a\,\mathrm{div}\,(\phi_s\mathbf{Z}_s)\,\big(\mathcal{P}(\phi_s\mathbf{Z}_s)\big)^*\cdot\nabla\log(\phi_0\phi_s)\,dx+\mathcal{R}_s.$$

Combining integration by parts and Lemma A.4, one can (up to a harmless remainder) replace $\mathrm{div}\,(\phi_s\mathbf{Z}_s)$ (resp. $\mathcal{P}(\phi_s\mathbf{Z}_s)$) by $\phi_0\phi_s\mathrm{div}\,\Lambda^s\mathbf{z}$ (resp. $\phi_0\phi_s\mathcal{P}\Lambda^s\mathbf{z}$). We end up with

$$\frac12\frac{d}{dt}\,\|\mathcal{P}(\phi_s\mathbf{Z}_s)\|_{\mathrm{L}^2}^2=\mathrm{Re}\int\big(\mathcal{P}(\phi_s\mathbf{Z}_s)\big)^*\cdot\phi_s\mathbf{F}_s\,dx-\mathrm{Im}\int\rho a\phi_s^2(\mathcal{Q}\Lambda^s\mathbf{z})^*\cdot\nabla\mathcal{P}\Lambda^s\mathbf{z}\cdot\nabla\log(\phi_0\phi_s)\,dx+\mathcal{R}_s.$$

Adding up inequality (3.25), we conclude that

$$\frac12\frac{d}{dt}\Big(\|\mathcal{P}(\phi_s\mathbf{Z}_s)\|_{\mathrm{L}^2}^2+\|\mathcal{Q}(\psi_s\mathbf{Z}_s)\|_{\mathrm{L}^2}^2\Big)=\mathrm{Re}\int\Big(\big(\mathcal{P}(\phi_s\mathbf{Z}_s)\big)^*\cdot\phi_s\mathbf{F}_s+\big(\mathcal{Q}(\psi_s\mathbf{Z}_s)\big)^*\cdot\psi_s\mathbf{F}_s\Big)\,dx$$
$$+\frac12\,\mathrm{Im}\int\rho a(\mathcal{Q}\Lambda^s\mathbf{z})^*\cdot\nabla\mathcal{P}\Lambda^s\mathbf{z}\cdot\Big(a^s\nabla\log\big(\tfrac{\rho}{a^s}\big)-\phi_s^2\nabla\log(\rho\phi_s^2)\Big)\,dx+\mathcal{R}_s.$$

It is now clear that ϕ_s has to be is chosen such that

$$a^s\nabla\log\big(\tfrac{\rho}{a^s}\big)-\phi_s^2\nabla\log(\rho\phi_s^2)=0.$$

Hence we set

$$\phi_s(\rho)=\sqrt{\frac{A_s(\rho)}{\rho}}\tag{3.29}$$

where A_s stands for a primitive of $a^s-\rho\frac{d}{d\rho}a^s$ which is *positive* on $[\underline{\rho},\widetilde{\rho}]$.

Let us write the equality we eventually get

$$\frac12\frac{d}{dt}\widetilde{\|\mathbf{z}\|}_s^2=\mathrm{Re}\int\Big(\big(\mathcal{P}(\phi_s\mathbf{Z}_s)\big)^*\cdot\phi_s\mathbf{F}_s+\big(\mathcal{Q}(\psi_s\mathbf{Z}_s)\big)^*\cdot\psi_s\mathbf{F}_s\Big)\,dx+\mathcal{R}_s,\tag{3.30}$$

with $\mathcal{R}_s$ satisfying (3.26), (3.27) and $\widetilde{\|\mathbf{z}\|}_s:=\sqrt{\|\mathcal{P}(\phi_s\mathbf{Z}_s)\|_{\mathrm{L}^2}^2+\|\mathcal{Q}(\psi_s\mathbf{Z}_s)\|_{\mathrm{L}^2}^2}\,.$

Let us focus on the proof of (3.17). Integrating (3.30) and appealing to (3.26) and (3.27) (and to the embedding $\mathrm{H}^s\hookrightarrow\mathrm{Lip}$ if $s>N/2+1$), we get

$$\forall t\in[0,T],\ \widetilde{\|\mathbf{z}(t)\|}_s^2\le\widetilde{\|\mathbf{z}_0\|}_s^2+2\int_0^t\widetilde{\|\mathbf{z}\|}_s\widetilde{\|\mathbf{f}\|}_s\,d\tau+C\int_0^t A\,\|\mathbf{z}\|_{\mathrm{H}^s}^2\,d\tau\tag{3.31}$$

where the function A has been defined in (3.17).

In order to bound $\|\mathbf{z}\|_{\mathrm{H}^s}$, we have to compare $\widetilde{\|\mathbf{z}\|}_s$ and $\|\mathbf{z}\|_{\mathrm{H}^s}$. On one hand, the assumptions in $(\mathcal{H})$ insure that ϕ_s and ψ_s are bounded, hence $\widetilde{\|\mathbf{z}\|}_s\lesssim\|\mathbf{z}\|_{\mathrm{H}^s}$. On the other hand, since ρ, ψ_s and ϕ_s are bounded away from zero, we have

$$\begin{aligned}\|\mathbf{z}\|_{\mathrm{H}^s}&\lesssim\big\|\sqrt{\rho}\phi_s\mathcal{P}\Lambda^s\mathbf{z}\big\|_{\mathrm{L}^2}+\big\|\sqrt{\rho}\psi_s\mathcal{Q}\Lambda^s\mathbf{z}\big\|_{\mathrm{L}^2}\,,\\&\lesssim\|\mathcal{P}(\phi_s\mathbf{Z}_s)\|_{\mathrm{L}^2}+\|\mathcal{Q}(\psi_s\mathbf{Z}_s)\|_{\mathrm{L}^2}+\big\|[\mathcal{P},\sqrt{\rho}\phi_s]\Lambda^s\mathbf{z}\big\|_{\mathrm{L}^2}+\big\|[\mathcal{Q},\sqrt{\rho}\psi_s]\Lambda^s\mathbf{z}\big\|_{\mathrm{L}^2}\,.\end{aligned}$$

Taking advantage of Lemma A.4 for bounding the two commutators, we conclude that

$$C^{-1}\widetilde{\|\mathbf{z}\|}_s\le\|\mathbf{z}\|_{\mathrm{H}^s}\le C\big(\widetilde{\|\mathbf{z}\|}_s+\|D\rho\|_{\mathrm{C}^{-\alpha}}\|\mathbf{z}\|_{\mathrm{H}^{s-1+\alpha}}\big)\tag{3.32}$$

whenever $\alpha\in[0,1)$ (with the convention that $\|D\rho\|_{\mathrm{C}^0}=\|D\rho\|_{\mathrm{L}^\infty}$ if $\alpha=0$).

Now, inequalities (3.32) and (3.31) enable us to conclude that for all $t \in [0,T]$, we have

$$\|\mathbf{z}(t)\|^2_{\mathrm{H}^s} \lesssim \|\mathbf{z}_0\|^2_{\mathrm{H}^s} + \int_0^t \|\mathbf{f}\|_{\mathrm{H}^s}\|\mathbf{z}\|_{\mathrm{H}^s}\,d\tau + \int_0^t A\,\|\mathbf{z}\|^2_{\mathrm{H}^s}\,d\tau + \|D\rho(t)\|^2_{\mathrm{C}^{-\alpha}}\|\mathbf{z}(t)\|^2_{\mathrm{H}^{s-1+\alpha}}.$$

The proof of (3.18) and (3.19) relies on (3.27) and on section B of the appendix for getting the desired expression for A in (3.31). The details are left to the reader. □

4 A fourth order approximate model

In order to solve the Euler-Korteweg model (1.2), we make use of a fourth order regularization of the extended formulation (ES). More precisely, for $\varepsilon \geq 0$, we introduce the following system:

$$(\mathrm{ES}_\varepsilon)\qquad \begin{cases} \partial_t L_\varepsilon + \mathbf{u}^*_\varepsilon\cdot\nabla L_\varepsilon + a_\sharp(L_\varepsilon)\mathrm{div}\,\mathbf{u}_\varepsilon + \varepsilon\Delta^2 L_\varepsilon = 0,\\ \partial_t\mathbf{z}_\varepsilon + (\mathbf{u}^*_\varepsilon\cdot\nabla)\,\mathbf{z}_\varepsilon + i\nabla\mathbf{z}_\varepsilon\cdot\mathbf{w}_\varepsilon + i\nabla(a_\sharp(L_\varepsilon)\mathrm{div}\,\mathbf{z}_\varepsilon) + \varepsilon\Delta^2\mathbf{z}_\varepsilon = q_\sharp(L_\varepsilon)\mathbf{w}_\varepsilon, \end{cases}$$

with $\mathbf{u}_\varepsilon = \mathrm{Re}\,\mathbf{z}_\varepsilon$ and $\mathbf{w}_\varepsilon = \mathrm{Im}\,\mathbf{z}_\varepsilon$. Remark that we *do not* impose any compatibility relation between L_ε and $\mathbf{z}_\varepsilon$.

4.1 Local existence

In the present section, we aim at proving the existence and uniqueness of local H^s solutions for $(\mathrm{ES}_\varepsilon)$ in the case $\varepsilon > 0$. Let us first define what we mean by *a* H^s *solution.*

Definition 4.1 *Let $\varepsilon \geq 0$. Let $L_0 \in \mathrm{L}^\infty$ be valued in $\mathbb{J}$ and such that $DL_0 \in \mathrm{H}^s$. Assume that $\mathbf{z}_0 \in \mathrm{H}^s$. The couple $(L_\varepsilon, \mathbf{z}_\varepsilon)$ of functions defined on $[0,T]\times\mathbb{R}^N$ is called a H^s solution of $(\mathrm{ES}_\varepsilon)$ if L_ε is valued in a compact subset of $\mathbb{J}$, $(L_\varepsilon,\mathbf{z}_\varepsilon)$ satisfies $(\mathrm{ES}_\varepsilon)$ with data $(L_0,\mathbf{z}_0)$ in the sense of distributions,*

$$(L_\varepsilon - L_0)\in\mathcal{C}([0,T];\mathrm{H}^{s+1}),\quad (DL_\varepsilon,\mathbf{z}_\varepsilon)\in\mathcal{C}([0,T];\mathrm{H}^s)\quad \textit{and}\quad \varepsilon(DL_\varepsilon,\mathbf{z}_\varepsilon)\in\mathrm{L}^2(0,T;\mathrm{H}^{s+2}).$$

Theorem 4.1 *Let $s > 1 + N/2$ and $\varepsilon > 0$. Let $L_0 \in \mathrm{L}^\infty$ be valued in a compact subset a $\mathbb{J}$ and such that $DL_0 \in \mathrm{H}^s$. Assume that $\mathbf{z}_0 \in \mathrm{H}^s$. There exists a positive T such that $(\mathrm{ES}_\varepsilon)$ has a unique H^s solution $(L_\varepsilon,\mathbf{z}_\varepsilon)$ on $[0,T]\times\mathbb{R}^N$.*

Proof. Take $\underline{L} < \widetilde{L}$ and $\eta > 0$ such that $[\underline{L},\widetilde{L}] \subset\subset \mathbb{J}$ and $\underline{L}+\eta \leq L_0 \leq \widetilde{L}-\eta$. In what follows we denote by $(S_t)_{t\geq 0}$ the analytic semi-group associated to the operator $-\Delta^2$ (hence $(S_{\varepsilon t})_{t\geq 0}$ is the semi-group for $-\varepsilon\Delta^2$), and use repeatedly the standard estimates

$$(4.1)\qquad \|S_{\varepsilon t}f_0\|_{\mathrm{L}^\infty_T(\mathrm{H}^s)} + \varepsilon^{\frac12}\|D^2 S_{\varepsilon t}f_0\|_{\mathrm{L}^2_T(\mathrm{H}^s)} \leq C\|f_0\|_{\mathrm{H}^s},$$

$$(4.2)\qquad \|\textstyle\int_0^t S_{\varepsilon(t-\tau)}f(\tau)\,d\tau\|_{\mathrm{L}^\infty_T(\mathrm{H}^s)} + \varepsilon^{\frac12}\|\textstyle\int_0^t S_{\varepsilon(t-\tau)}f(\tau)\,d\tau\|_{\mathrm{L}^2_T(\mathrm{H}^{s+2})} \leq C\varepsilon^{-\frac12}\mathrm{e}^{T\varepsilon}\|f\|_{L^2_T(\mathrm{H}^{s-2})}$$

for $t \leq T$, where the constant $C > 0$ is independent of $\varepsilon > 0$ and $T \geq 0$.

For convenience we introduce the shortcuts $L_\ell(t) := S_{\varepsilon t}L_0$ and $\mathbf{z}_\ell(t) := S_{\varepsilon t}\mathbf{z}_0$.

For $T > 0$, we define the Banach space

$$\mathrm{E}_T = \Big\{(\dot L,\dot{\mathbf{z}})\ \mid\ \dot L\in\mathcal{C}([0,T];\mathrm{H}^{s+1})\cap\mathrm{L}^2(0,T;\mathrm{H}^{s+3})\quad\text{and}\quad \dot{\mathbf{z}}\in\mathcal{C}([0,T];\mathrm{H}^s)\cap\mathrm{L}^2(0,T;\mathrm{H}^{s+2})\Big\}$$

endowed with the norm

$$\|(\dot L,\dot{\mathbf{z}})\|_{T,\varepsilon} := \|\dot L\|_{L^\infty_T(\mathrm{H}^{s+1})} + \|\dot{\mathbf{z}}\|_{L^\infty_T(\mathrm{H}^s)} + \varepsilon^{\frac12}\|D^2\dot L\|_{L^2_T(\mathrm{H}^{s+1})} + \varepsilon^{\frac12}\|D^2\dot{\mathbf{z}}\|_{L^2_T(\mathrm{H}^s)}.$$

We also consider the subset $\mathrm{E}_T^{\eta,R}$ of $(\dot L,\dot{\mathbf{z}})\in\mathrm{E}_T$ such that $|\dot L|\leq\eta/2$ and $\|(\dot L,\dot{\mathbf{z}})\|_{T,\varepsilon}\leq R$.

Finally, for $(\dot L,\dot{\mathbf{z}})\in\mathrm{E}_T^{\eta,R}$, we define $\Phi(\dot L,\dot{\mathbf{z}}):=(\Phi_1(\dot L,\dot{\mathbf{z}}),\Phi_2(\dot L,\dot{\mathbf{z}}))$ by

$$\Phi_1(\dot L,\dot{\mathbf{z}}):=-\int_0^t S_{\varepsilon(t-\tau)}\Big(\mathbf{u}^*\cdot\nabla L+a_\sharp(L)\mathrm{div}\,\mathbf{u}\Big)\,d\tau,$$

$$\Phi_2(\dot L,\dot{\mathbf{z}}):=\int_0^t S_{\varepsilon(t-\tau)}\Big(q_\sharp(L)\mathbf{w}-(\mathbf{u}^*\cdot\nabla)\mathbf{z}-i\nabla\mathbf{z}\cdot\mathbf{w}-i\nabla(a_\sharp(L)\mathrm{div}\,\mathbf{z})\Big)\,d\tau$$

where we denoted $L:=L_\ell+\dot L$, $\mathbf{z}:=\mathbf{z}_\ell+\dot{\mathbf{z}}$, $\mathbf{u}:=\mathrm{Re}\,\mathbf{z}$ and $\mathbf{w}:=\mathrm{Im}\,\mathbf{z}$.

Let $R_0:=\|\nabla L_0\|_{\mathrm{H}^s}+\|\mathbf{z}_0\|_{\mathrm{H}^s}$. We claim that Φ is well defined and has a fixed point $(\dot L,\dot{\mathbf{z}})$ in E_T^{η,R_0} for suitably small (positive) T. This will readily entail that $(L,\mathbf{z})$ satisfies the existence part of Theorem 4.1.

Boundedness of Φ

Remark that (4.1) insures that

$$\forall t\in\mathbb{R}^+,\ \|L_\ell(t)\|_{\mathrm{H}^{s+1}}+\|\mathbf{z}_\ell(t)\|_{\mathrm{H}^s}\lesssim R_0. \tag{4.3}$$

Next, by definition of L_ℓ, we have

$$L_\ell(t)-L_0=-\varepsilon\int_0^t\Delta\mathrm{div}\,(\nabla L_\ell)\,d\tau.$$

Hence, by virtue of Sobolev embeddings (remind that $s-1>\frac{N}{2}$) and Cauchy-Schwarz inequality,

$$\|L_\ell(t)-L_0\|_{\mathrm{L}^\infty}\leq C\|L_\ell(t)-L_0\|_{\mathrm{H}^{s-1}}\leq C\varepsilon T^{\frac12}\|D^2\nabla L_\ell\|_{L^2_T(\mathrm{H}^s)}.$$

Taking advantage of the inequality in (4.1) with $f_0:=\nabla L_0$, we conclude to the existence of a constant $C=C(s,N)$ such that $\|L_\ell(t)-L_0\|_{\mathrm{L}^\infty}\leq\eta/2$ whenever

$$C(\varepsilon T)^{\frac12}R_0\leq\eta. \tag{4.4}$$

From now on, assume that T has been chosen so that condition (4.4) is fulfilled. This ensures that Φ_1 and Φ_2 are well defined for any $(\dot L,\dot{\mathbf{z}})\in\mathrm{E}_T^{\eta,R_0}$.

By virtue of the inequality (4.2), we thus have

$$\|\Phi_1(\dot L,\dot{\mathbf{z}})\|_{L^\infty_T(\mathrm{H}^{s+1})}+\varepsilon^{\frac12}\|\Phi_1(\dot L,\dot{\mathbf{z}})\|_{L^2_T(\mathrm{H}^{s+3})}\lesssim\varepsilon^{-\frac12}\mathrm{e}^{T\varepsilon}\|\mathbf{u}^*\cdot\nabla L+a_\sharp(L)\mathrm{div}\,\mathbf{u}\|_{\mathrm{L}^2_T(\mathrm{H}^{s-1})},$$

$$\|\Phi_2(\dot L,\dot{\mathbf{z}})\|_{L^\infty_T(\mathrm{H}^s)}+\varepsilon^{\frac12}\|\Phi_2(\dot L,\dot{\mathbf{z}})\|_{L^2_T(\mathrm{H}^{s+2})}\lesssim\varepsilon^{-\frac12}\mathrm{e}^{T\varepsilon}\|q_\sharp(L)\mathbf{w}-(\mathbf{u}^*\cdot\nabla)\mathbf{z}-i\nabla\mathbf{z}\cdot\mathbf{w}-i\nabla(a_\sharp(L)\mathrm{div}\,\mathbf{z})\|_{\mathrm{L}^2_T(\mathrm{H}^{s-2})}.$$

The right-hand side of the first inequality above may be bounded by using Proposition B.2 and that H^{s-1} is an algebra. We get

$$\|\Phi_1(\dot L,\dot{\mathbf{z}})\|_{L^\infty_T(\mathrm{H}^{s+1})}+\varepsilon^{\frac12}\|\Phi_1(\dot L,\dot{\mathbf{z}})\|_{L^2_T(\mathrm{H}^{s+3})}\leq C\mathrm{e}^{T\varepsilon}\varepsilon^{-\frac12}\big(1+\|DL\|_{\mathrm{L}^\infty_T(\mathrm{H}^{s-1})}\big)\|\mathbf{u}\|_{\mathrm{L}^2_T(\mathrm{H}^s)} \tag{4.5}$$

for some constant C depending only on $\underline{L}$, $\widetilde{L}$, s, N and on the function $a_\sharp$.

Proposition B.2 combined with the fact that H^{s-1} is an algebra also yields

$$\|q_\sharp(L)\mathbf{w}\|_{\mathrm{H}^{s-2}}\leq\|q_\sharp(L)\mathbf{w}\|_{\mathrm{H}^{s-1}}\lesssim\big(1+\|DL\|_{\mathrm{H}^{s-2}}\big)\|\mathbf{w}\|_{\mathrm{H}^{s-1}},$$
$$\|(\mathbf{u}^*\cdot\nabla)\mathbf{z}\|_{\mathrm{H}^{s-2}}+\|\nabla\mathbf{z}\cdot\mathbf{w}\|_{\mathrm{H}^{s-2}}\lesssim\|\mathbf{z}\|^2_{\mathrm{H}^{s-1}},$$
$$\|\nabla(a_\sharp(L)\mathrm{div}\,\mathbf{z})\|_{\mathrm{H}^{s-2}}\leq\|a_\sharp(L)\mathrm{div}\,\mathbf{z}\|_{\mathrm{H}^{s-1}}\lesssim\big(1+\|DL\|_{\mathrm{H}^{s-2}}\big)\|\mathbf{z}\|_{\mathrm{H}^s},$$

whence,

$$\|\Phi_2(\dot L,\dot{\mathbf z})\|_{L^\infty_T(\mathrm H^s)}+\varepsilon^{\frac12}\|\Phi_2(\dot L,\dot{\mathbf z})\|_{L^2_T(\mathrm H^{s+2})}\leq C\mathrm e^{T\varepsilon}\varepsilon^{-\frac12}\bigl(1+\|\mathbf z\|_{\mathrm L^\infty_T(\mathrm H^{s-1})}\bigr)\|\mathbf z\|_{\mathrm L^2_T(\mathrm H^s)}.\tag{4.6}$$

Combining (4.3), (4.5), (4.6) and Cauchy-Schwarz inequality, we infer that there exists a constant C independent of ε, R_0 and T, and such that

$$\|\Phi(\dot L,\dot{\mathbf z})\|_{T,\varepsilon}\leq C\mathrm e^{T\varepsilon}\sqrt{\frac{T}{\varepsilon}}R_0(1+R_0)$$

whenever $(\dot L,\dot{\mathbf z})\in \mathrm E_T^{\eta,R_0}$ and condition (4.4) is satisfied.

Since $\Bigl\|\Phi_1(\dot L,\dot{\mathbf z})\Bigr\|_{\mathrm L^\infty}\leq C\|\Phi_1(\dot L,\dot{\mathbf z})\|_{\mathrm H^{s-1}}$, we conclude that Φ maps $\mathrm E_T^{\eta,R_0}$ in $\mathrm E_T^{\eta,R_0}$ provided the following two inequalities are fulfilled:

$$C\sqrt{T\varepsilon}R_0\leq\eta\quad\text{and}\quad C\mathrm e^{T\varepsilon}\sqrt{\frac{T}{\varepsilon}}R_0(1+R_0)\leq\min(\eta,R_0).\tag{4.7}$$

for some constant C independent of R_0, ε and T.

Contractivity of Φ

Let $(\dot L_1,\dot{\mathbf z}_1)$ and $(\dot L_2,\dot{\mathbf z}_2)$ be in $\mathrm E_T^{\eta,R_0}$ with T satisfying (4.7). We have

$$\bigl[\Phi_1(\dot L_2,\dot{\mathbf z}_2)-\Phi_1(\dot L_1,\dot{\mathbf z}_1)\bigr](t)=-\int_0^t S_{\varepsilon(t-\tau)}\Bigl(\delta\mathbf u^*\cdot\nabla L_2+\mathbf u_1^*\cdot\nabla\delta L+\delta a\,\mathrm{div}\,\mathbf u_2+a_\sharp(L_1)\mathrm{div}\,\delta\mathbf u\Bigr)\,d\tau$$

with the notation $\delta a:=a_\sharp(L_2)-a_\sharp(L_1)$, $\delta L:=\dot L_2-\dot L_1$, and $\delta\mathbf u:=\dot{\mathbf u}_2-\dot{\mathbf u}_1$.

Applying (4.2), we get

$$\begin{aligned}\|\Phi_1(\dot L_2,\dot{\mathbf z}_2)-\Phi_1(\dot L_1,\dot{\mathbf z}_1)\|_{\mathrm L^\infty_T(\mathrm H^{s+1})}+\varepsilon^{\frac12}\|\Phi_1(\dot L_2,\dot{\mathbf z}_2)-\Phi_1(\dot L_1,\dot{\mathbf z}_1)\|_{\mathrm L^2_T(\mathrm H^{s+3})}&\\ \leq C\mathrm e^{T\varepsilon}\varepsilon^{-\frac12}\Bigl(\|\delta\mathbf u^*\cdot\nabla L_2\|_{\mathrm L^2_T(\mathrm H^{s-1})}+\|\mathbf u_1^*\cdot\nabla\delta L\|_{\mathrm L^2_T(\mathrm H^{s-1})}&\\ +\|\delta a\,\mathrm{div}\,\mathbf u_2\|_{\mathrm L^2_T(\mathrm H^{s-1})}+\|a_\sharp(L_1)\mathrm{div}\,\delta\mathbf u\|_{\mathrm L^2_T(\mathrm H^{s-1})}\Bigr).&\end{aligned}$$

Since $\mathrm H^{s-1}$ is an algebra, we have by virtue of Cauchy-Schwarz inequality

$$\begin{aligned}\|\delta\mathbf u^*\cdot\nabla L_2\|_{\mathrm L^2_T(\mathrm H^{s-1})}&\leq CT^{\frac12}\|\delta\mathbf u\|_{L^\infty_T(\mathrm H^{s-1})}\|\nabla L_2\|_{L^\infty_T(\mathrm H^{s-1})},\\ \|\mathbf u_1^*\cdot\nabla\delta L\|_{\mathrm L^2_T(\mathrm H^{s-1})}&\leq CT^{\frac12}\|\mathbf u_1\|_{L^\infty_T(\mathrm H^{s-1})}\|\nabla\delta L\|_{L^\infty_T(\mathrm H^{s-1})}.\end{aligned}$$

Taking advantage of Corollary B.3 and of the embedding $\mathrm H^{s-1}\hookrightarrow\mathrm L^\infty$, we get

$$\|\delta a\,\mathrm{div}\,\mathbf u_2\|_{\mathrm H^{s-1}}\leq C\bigl(1+\|DL_1\|_{\mathrm H^{s-2}}+\|DL_2\|_{\mathrm H^{s-2}}\bigr)\|\mathrm{div}\,\mathbf u_2\|_{\mathrm H^{s-1}}\|\delta L\|_{\mathrm H^{s-1}}.$$

Applying Proposition B.2 yields

$$\|a_\sharp(L_1)\mathrm{div}\,\delta\mathbf u\|_{\mathrm H^{s-1}}\leq C\|\mathrm{div}\,\delta\mathbf u\|_{\mathrm H^{s-1}}\bigl(1+\|DL_1\|_{\mathrm H^{s-2}}\bigr).$$

Therefore, we eventually have

$$\begin{aligned}&\|\Phi_1(\dot L_2,\dot{\mathbf z}_2)-\Phi_1(\dot L_1,\dot{\mathbf z}_1)\|_{\mathrm L^\infty_T(\mathrm H^{s+1})}\\ &\quad+\varepsilon^{\frac12}\|\Phi_1(\dot L_2,\dot{\mathbf z}_2)-\Phi_1(\dot L_1,\dot{\mathbf z}_1)\|_{\mathrm L^2_T(\mathrm H^{s+3})}\leq C\mathrm e^{T\varepsilon}\sqrt{\tfrac{T}{\varepsilon}}(1+R_0)^2\|(\delta L,\delta\mathbf z)\|_{\mathrm E_T}.\end{aligned}\tag{4.8}$$

Similarly, we have

$$\big[\Phi_2(\dot L_2,\dot{\mathbf z}_2)-\Phi_2(\dot L_1,\dot{\mathbf z}_1)\big](t)=\int_0^t S_{\varepsilon(t-\tau)}\Big((q_\sharp(L_2)-q_\sharp(L_1))\mathbf w_2+q_\sharp(L_1)\delta\mathbf w-(\delta\mathbf u^*\cdot\nabla)\mathbf z_2$$
$$-(\mathbf u_1^*\cdot\nabla)\delta\mathbf z-i\nabla\mathbf z_2\cdot\delta\mathbf u-i\nabla\delta\mathbf z\cdot\mathbf u_1-i\nabla\big((a_\sharp(L_2)-a_\sharp(L_1))\operatorname{div}\mathbf u_2\big)-i\nabla\big(a_\sharp(L_1)\operatorname{div}\delta\mathbf u\big)\Big)\,d\tau.$$

By using the results of the appendix, we discover that $\Phi_2(\dot L_2,\dot{\mathbf z}_2)-\Phi_2(\dot L_1,\dot{\mathbf z}_1)$ may be bounded by the right-hand side of the inequality in (4.8). The details are left to the reader.

We conclude that there exists a constant C' independent of R_0, T and η, and such that

$$\|\Phi(\dot L_2,\dot{\mathbf z}_2)-\Phi(\dot L_1,\dot{\mathbf z}_1)\|_{T,\varepsilon}\le C'\mathrm e^{T\varepsilon}\sqrt{\frac{T}{\varepsilon}}(1+R_0)^2\|(\delta L,\delta\mathbf z)\|_{T,\varepsilon}.$$

It is now clear that if T has been chosen so that (4.7) holds true and $2C'\mathrm e^{T\varepsilon}\sqrt{\frac{T}{\varepsilon}}(1+R_0)^2\le 1$ then Φ is a contractive map on $\mathrm E_T^{\eta,R_0}$. Applying the contracting mapping theorem thus yields a fixed point $(\dot L,\dot{\mathbf z})\in\mathrm E_T^{\eta,R_0}$ for the function Φ.

Uniqueness

There is a very strong similarity between the proof of uniqueness for $(\mathrm{ES}_\varepsilon)$ and the proof of contractivity for Φ. Indeed, let $(L_1,\mathbf z_1)$ and $(L_2,\mathbf z_2)$ be two $\mathrm H^s$ solutions of $(\mathrm{ES}_\varepsilon)$ on $[0,T]\times\mathbb R^N$. Denote $\delta L:=L_2-L_1$ and $\delta\mathbf z:=\mathbf z_2-\mathbf z_1$. We have

$$\partial_t\delta L+\varepsilon\Delta^2\delta L=-\mathbf u_2^*\cdot\nabla\delta L-\delta\mathbf u^*\cdot\nabla L_1-\big(a_\sharp(L_2)-a_\sharp(L_1)\big)\operatorname{div}\mathbf u_2-a_\sharp(L_1)\operatorname{div}\delta\mathbf u.$$

By following the arguments used in the proof of the contractivity of Φ_1, we see that the $\mathrm L^2(0,t;\mathrm H^{s-1})$ norm of the right-hand side above may be bounded by

$$Ct^{\frac12}\Big(1+\|\nabla L_1\|_{\mathrm L_t^\infty(\mathrm H^s)}+\|\nabla L_2\|_{\mathrm L_t^\infty(\mathrm H^s)}\Big)\Big(\|\mathbf u_2\|_{\mathrm L_t^\infty(\mathrm H^s)}\|\delta L\|_{L_t^\infty(\mathrm H^s)}+\|\delta\mathbf u\|_{\mathrm L_t^\infty(\mathrm H^s)}\Big).$$

Hence, denoting

$$R:=\max_{i=1,2}\Big(\|\nabla L_i\|_{L_T^\infty(\mathrm H^s)}+\|\mathbf z_i\|_{L_T^\infty(\mathrm H^s)}\Big),$$

we end up with

$$\|\delta L\|_{L_t^\infty(\mathrm H^{s+1})}+\varepsilon^{\frac12}\|D^2\delta L\|_{L_t^2(\mathrm H^{s+1})}\le C\mathrm e^{t\varepsilon}\sqrt{\frac{t}{\varepsilon}}\,(1+R)^2\|(\delta L,\delta\mathbf z)\|_{t,\varepsilon}.$$

A similar bound holds for $\delta\mathbf z$. Hence we eventually get

$$\forall t\in[0,T],\ \|(\delta L,\delta\mathbf z)\|_{t,\varepsilon}\le C\sqrt{\frac{t}{\varepsilon}}\,(1+R)^2\|(\delta L,\delta\mathbf z)\|_{t,\varepsilon}.$$

This obviously entail $(\delta L,\delta\mathbf z)\equiv 0$ on $[0,t]$ whenever t satisfies $C\mathrm e^{t\varepsilon}\sqrt{\frac{t}{\varepsilon}}\,(1+R)^2<1$.

Arguing by induction, we conclude that $(\delta L,\delta\mathbf z)\equiv 0$ on the whole interval $[0,T]$. □

Remark that the proof of Theorem 4.1 supplies the lower bound $C\min(\varepsilon,\varepsilon^{-1})$ for the time of existence, with C depending only on the regularity parameters, on $\underline L$, $\widetilde L$, η and on the $\mathrm H^s$ norm of ∇L_0 and $\mathbf z_0$. By virtue of uniqueness, we thus easily get the following

Proposition 4.1 *Let $s>1+N/2$ and $(L,\mathbf z)$ be a $\mathrm H^s$ solution of $(\mathrm{ES}_\varepsilon)$ on $[0,T']\times\mathbb R^N$ for all $T'<T$, such that $L([0,T)\times\mathbb R^N)\subset\subset\mathbb J$ and $(DL,\mathbf z)\in\mathrm L^\infty(0,T;\mathrm H^s)$. Then $(L,\mathbf z)$ may be continued beyond T into a $\mathrm H^s$ solution of $(\mathrm{ES}_\varepsilon)$.*

The following corollary of Theorem 4.1 will help us to solve the Euler-Korteweg system.

Corollary 4.1 *Take* $s > 1 + \frac{N}{2}$ *and* $\varepsilon > 0$*. Let* $L_0 \in \mathrm{L}^\infty$ *be valued in a compact set of* $\mathbb{J}$ *and satisfy* $DL_0 \in \mathrm{H}^s$*. Assume that* $\mathbf{u}_0 \in \mathrm{H}^s$*. There exists a* $T > 0$ *such that system*

$$(\mathrm{EK}_\varepsilon) \qquad \begin{cases} \partial_t L + \varepsilon\Delta^2 L + \mathbf{u}^* \cdot \nabla L + a_\sharp(L)\mathrm{div}\,\mathbf{u} = 0, \\ \partial_t \mathbf{u} + \varepsilon\Delta^2\mathbf{u} + (\mathbf{u}^* \cdot \nabla)\mathbf{u} - \nabla^2 L \cdot \nabla L - \nabla(a_\sharp(L)\Delta L) = q_\sharp(L)\nabla L, \end{cases}$$

has a unique H^s *solution* $(L, \mathbf{u})$ *on* $[0,T]$*, with* L *valued in* $\mathbb{J}$*,* $(L - L_0) \in \mathcal{C}([0,T];\mathrm{H}^{s+1})$*,* $DL \in \mathrm{L}^2([0,T];\mathrm{H}^{s+2})$ *and* $\mathbf{u} \in \mathcal{C}([0,T];\mathrm{H}^s) \cap \mathrm{L}^2([0,T];\mathrm{H}^{s+2})$*. If besides* $\mathrm{curl}\,\mathbf{u}_0 = \mathbf{0}$ *then* $\mathrm{curl}\,\mathbf{u} \equiv \mathbf{0}$*.*

Proof. Let us denote $\mathbf{z}_0 := \mathbf{u}_0 + i\nabla L_0$. Applying Theorem 4.1 supplies a (unique) local H^s solution $(L, \mathbf{z})$ to $(\mathrm{ES}_\varepsilon)$ with data $(L_0, \mathbf{z}_0)$. Consider $\mathbf{w} := \mathrm{Im}\,\mathbf{z}$. A straightforward computation shows that

$$\partial_t(\mathbf{w} - \nabla L) + (\mathbf{u}^* \cdot \nabla)(\mathbf{w} - \nabla L) + \nabla\mathbf{u} \cdot (\mathbf{w} - \nabla L) + \varepsilon\Delta^2(\mathbf{w} - \nabla L) = \mathbf{0},$$

hence

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{w} - \nabla L\|_{\mathrm{L}^2}^2 \leq \frac{1}{2}\|\mathrm{div}\,\mathbf{u}\|_{\mathrm{L}^\infty}\|\mathbf{w} - \nabla L\|_{\mathrm{L}^2}^2 + \|\nabla\mathbf{u}\|_{\mathrm{L}^\infty}\|\mathbf{w} - \nabla L\|_{\mathrm{L}^2}^2.$$

Since $\mathbf{w}$ and ∇L coincide at time $t = 0$, Gronwall's lemma entails that $\mathbf{w} \equiv \nabla L$. We conclude that $(L, \mathbf{u})$ is a H^s solution to $(\mathrm{EK}_\varepsilon)$.

Uniqueness easily stems from Theorem 4.1. Indeed, given $(L_1, \mathbf{u}_1)$ and $(L_2, \mathbf{u}_2)$ two solutions of $(\mathrm{EK}_\varepsilon)$, we notice that $(L_1, \mathbf{u}_1 + i\nabla L_1)$ and $(L_2, \mathbf{u}_2 + i\nabla L_2)$ both solve $(\mathrm{ES}_\varepsilon)$ with the same data, and thus coincide.

Next, applying the curl operator to the second equation of $(\mathrm{EK}_\varepsilon)$ yields

$$\partial_t\mathrm{curl}\,\mathbf{u} + (\mathbf{u}^* \cdot \nabla)\mathrm{curl}\,\mathbf{u} + \varepsilon\Delta^2\mathrm{curl}\,\mathbf{u} + D\mathbf{u} \cdot \mathrm{curl}\,\mathbf{u} + \mathrm{curl}\,\mathbf{u} \cdot \nabla\mathbf{u} = 0.$$

A basic energy method gives

$$\frac{1}{2}\frac{d}{dt}\|\mathrm{curl}\,\mathbf{u}\|_{\mathrm{L}^2}^2 + \varepsilon\|\Delta\mathrm{curl}\,\mathbf{u}\|_{\mathrm{L}^2}^2 - \frac{1}{2}\int |\mathrm{curl}\,\mathbf{u}|^2\mathrm{div}\,\mathbf{u}\,dx + \int \mathrm{curl}\,\mathbf{u} : \big(D\mathbf{u}\cdot\mathrm{curl}\,\mathbf{u} + \mathrm{curl}\,\mathbf{u}\cdot\nabla\mathbf{u}\big)\,dx = 0,$$

whence

$$\frac{1}{2}\frac{d}{dt}\|\mathrm{curl}\,\mathbf{u}\|_{\mathrm{L}^2}^2 \leq \|\mathrm{curl}\,\mathbf{u}\|_{\mathrm{L}^2}^2\left(\frac{1}{2}\|\mathrm{div}\,\mathbf{u}\|_{\mathrm{L}^\infty} + 2\|D\mathbf{u}\|_{\mathrm{L}^\infty}\right).$$

It is now clear that $\mathrm{curl}\,\mathbf{u}_0 = \mathbf{0}$ entails $\mathrm{curl}\,\mathbf{u} \equiv \mathbf{0}$. □

4.2 Uniform a priori estimates

In the present section, we aim at proving a priori estimates *independent* of ε for the linearization of the fourth-order system $(\mathrm{ES}_\varepsilon)$.

Our main result is the following.

Proposition 4.2 *Take* $\varepsilon \geq 0$ *and* $s > 0$*. Let* $(L, \mathbf{z})$ *be a* H^s *solution of*

$$\begin{cases} \partial_t L + \mathbf{v}^* \cdot \nabla L + a_\sharp(L)\mathrm{div}\,\mathbf{v} + \varepsilon\Delta^2 L = 0, \\ \partial_t\mathbf{z} + (\mathbf{v}^* \cdot \nabla)\mathbf{z} + i\,\nabla\mathbf{z}\cdot\mathbf{w} + i\,\nabla(\,a_\sharp(L)\,\mathrm{div}\,\mathbf{z}\,) + \varepsilon\Delta^2\mathbf{z} = \mathbf{f} + \varepsilon\Delta\mathbf{h}. \end{cases}$$

on $[0,T]\times\mathbb{R}^N$. *Assume that* $\mathbf{w}=\nabla L$ *and that* $(\rho:=\mathcal{L}^{-1}(L),\, a:=a_\sharp(L))$ *satisfies* $(\mathcal{H})$. *Then the following estimates hold true for all* $t\in[0,T]$ *and* $\alpha\in[0,1)$*:*

$$\tag{4.9} \|\mathbf{z}(t)\|^2_{\mathrm{H}^s} +\varepsilon\int_0^t\|D^2\mathbf{z}\|^2_{\mathrm{H}^s}\,d\tau\leq C\biggl(\|\mathbf{z}_0\|^2_{\mathrm{H}^s} + \int_0^t\bigl(\|\mathbf{f}\|_{\mathrm{H}^s}\|\mathbf{z}\|_{\mathrm{H}^s}+A_\varepsilon\|\mathbf{z}\|^2_{\mathrm{H}^s}+\varepsilon\|\mathbf{h}\|^2_{\mathrm{H}^s}\bigr)\,d\tau+\|\mathbf{w}(t)\|^2_{\mathrm{C}^{-\alpha}}\|\mathbf{z}(t)\|^2_{\mathrm{H}^{s-1+\alpha}}\biggr),$$

$$\tag{4.10} \bigl\|(\sqrt{\rho}\mathbf{z})(t)\bigr\|^2_{\mathrm{L}^2}+\varepsilon\int_0^t\bigl\|D^2\mathbf{z}\bigr\|^2_{\mathrm{L}^2}\,d\tau\leq\bigl\|(\sqrt{\rho}\mathbf{z})(0)\bigr\|^2_{\mathrm{L}^2} +2\int_0^t\bigl\|\sqrt{\rho}\mathbf{z}\bigr\|_{\mathrm{L}^2}\bigl\|\sqrt{\rho}\mathbf{f}\bigr\|_{\mathrm{L}^2}\,d\tau+C\varepsilon\int_0^t\bigl(\|\mathbf{h}\|^2_{\mathrm{L}^2}+\|D^2L\|^2_{\mathrm{L}^\infty}\bigl\|\sqrt{\rho}\mathbf{z}\bigr\|^2_{\mathrm{L}^2}\bigr)\,d\tau,$$

for some constant C *depending only on* N, α, s, $\underline{a}$, $\widetilde{a}$, $\underline{\rho}$ *and* $\widetilde{\rho}$, *and* $A_\varepsilon(t)=\varepsilon\bigl\|D^2L(t)\bigr\|^2_{\mathrm{L}^\infty}+A(t)$ *with* A *defined as in Proposition 3.3 in the case* $g=0$.

Moreover, if $\mathbf{z}=\mathbf{v}+i\mathbf{w}$, $a_\sharp\in\mathrm{W}^{\sigma+2,\infty}$, $\mathbf{h}=\mathbf{0}$ *and* $\mathbf{f}=q_\sharp(L)\mathbf{w}$ *for some* $q_\sharp\in\mathrm{W}^{\sigma+1,\infty}$, *we have*

$$\tag{4.11} \|\mathbf{z}(t)\|^2_{\mathrm{H}^s}\leq C\biggl[\|\mathbf{z}_0\|^2_{\mathrm{H}^s}+\int_0^t(1+\|D\mathbf{z}\|_{\mathrm{L}^\infty}+\varepsilon\,\|D\mathbf{w}\|^2_{\mathrm{L}^\infty})\|\mathbf{z}\|^2_{\mathrm{H}^s}\,d\tau+\|\mathbf{w}(t)\|^2_{\mathrm{C}^{-\alpha}}\|\mathbf{z}(t)\|^2_{\mathrm{H}^{s-1+\alpha}}\biggr].$$

Proof. The proof relies on the gauge method introduced in Proposition 3.3. The only change is that we now have to include the fourth order term $\varepsilon\Delta^2\mathbf{z}$, which amounts to replacing $\mathbf{f}$ with $\mathbf{f}-\varepsilon\Delta^2\mathbf{z}$ in (LS) and to taking $g=-\varepsilon\frac{\Delta^2L}{a_\sharp(L)}$ in (T).

Denote by $\widetilde{\phi}_s$ (resp. $\widetilde{\psi}_s$) the "incompressible" (resp. "compressible") gauge[3]. Both gauges may be seen as functions of L so let us write an equation for $\Phi\Lambda^s\mathbf{z}$ with Φ an arbitrary suitably smooth function of L. Arguing as in the case $\varepsilon=0$, we get

$$\begin{aligned} D_t(\Phi\Lambda^s\mathbf{z})+i\nabla(a\mathrm{div}\,(\Phi\Lambda^s\mathbf{z}))+\varepsilon\Delta^2(\Phi\Lambda^s\mathbf{z})&=\Phi\bigl(\Lambda^s(\mathbf{f}+\varepsilon\Delta\mathbf{h})+\mathbf{R}_s\bigr)+\varepsilon\,\mathbf{Q}(\Phi,\mathbf{z})\\ -a\Phi'\Lambda^s\mathbf{z}\,\mathrm{div}\,\mathbf{v}&+i\Bigl[\Bigl(\frac{\nabla(aD\phi_0)}{\phi_0}+\frac{\nabla(aD\widetilde{\Phi})}{\widetilde{\Phi}}+a\nabla\log\widetilde{\Phi}\,D\log\Bigl(\frac{a^s}{\widetilde{\Phi}^2}\Bigr)\Bigr)\cdot(\Phi\Lambda^s\mathbf{z})\Bigr]\\ &+ia\Bigl(\nabla_0(\Phi\Lambda^s\mathbf{z})\cdot\nabla\log\Bigl(\frac{\Phi}{\phi_0^2a^s}\Bigr)+\mathrm{div}\,(\Phi\Lambda^s\mathbf{z})\nabla\log\Bigl(\frac{\Phi^2}{\phi_0^2a^s}\Bigr)\Bigr)+is\Phi\nabla\mathcal{P}\Lambda^s\mathbf{z}\cdot\nabla a\end{aligned}$$

with $\mathbf{R}_s$ defined in (3.2), $\mathbf{Q}(\Phi,\mathbf{z}):=\Delta^2(\Phi\Lambda^s\mathbf{z})-\Phi\Delta^2\Lambda^s\mathbf{z}-\Phi'\Delta^2L\,\Lambda^s\mathbf{z}$, $\phi_0:=\sqrt{\rho}$ and $\widetilde{\Phi}:=\Phi/\sqrt{\rho}$. In the equality hereabove Φ' stands for $d\Phi/dL$, and we have used that

$$D_t\Phi=-\bigl(a\mathrm{div}\,\mathbf{v}+\varepsilon\Delta^2L\bigr)\Phi'.$$

By going along the lines of the proof of (3.30), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\widetilde{\|\mathbf{z}\|}^2_s+\varepsilon\Bigl(\Bigl\|\mathcal{P}\Delta(\widetilde{\phi}_s\Lambda^s\mathbf{z})\Bigr\|^2_{\mathrm{L}^2}+\Bigl\|\mathcal{Q}\Delta(\widetilde{\psi}_s\Lambda^s\mathbf{z})\Bigr\|^2_{\mathrm{L}^2}\Bigr)=\mathcal{R}_s\\ &+\mathrm{Re}\int\bigl(\mathcal{P}(\widetilde{\phi}_s\Lambda^s\mathbf{z})\bigr)^*\cdot\bigl(\widetilde{\phi}_s\Lambda^s(\mathbf{f}+\varepsilon\Delta\mathbf{h})+\varepsilon\mathbf{Q}(\widetilde{\phi}_s,\mathbf{z})\bigr)+\bigl(\mathcal{Q}(\widetilde{\psi}_s\Lambda^s\mathbf{z})\bigr)^*\cdot\bigl(\widetilde{\psi}_s\Lambda^s(\mathbf{f}+\varepsilon\Delta\mathbf{h})+\varepsilon\mathbf{Q}(\widetilde{\psi}_s,\mathbf{z})\bigr)\,dx,\end{aligned}$$

where the remainder term $\mathcal{R}_s$ may be bounded according to (3.26) and (3.27).

In order to conclude to Proposition 4.2, we have to bound the terms pertaining to $\mathbf{Q}(\widetilde{\phi}_s,\mathbf{z})$ and $\mathbf{Q}(\widetilde{\psi}_s,\mathbf{z})$, and to compare $\Bigl\|\mathcal{P}\Delta(\widetilde{\phi}_s\Lambda^s\mathbf{z})\Bigr\|^2_{\mathrm{L}^2}+\Bigl\|\mathcal{Q}\Delta(\widetilde{\psi}_s\Lambda^s\mathbf{z})\Bigr\|^2_{\mathrm{L}^2}$ with $\|D^2\mathbf{z}\|^2_{\mathrm{H}^s}$. This is the purpose of lemmas 4.1 and 4.2 below.

[3] Remind that $\widetilde{\psi}_s=\sqrt{\rho}a^{\frac{s}{2}}$ and that $\widetilde{\phi}_s=\sqrt{A_s}$ with A_s defined in (3.29).

Lemma 4.1 *Let L be bounded and F be a $\mathrm{W}^{4,\infty}$ function of L. Let $\mathbf{Z}$ be a H^2 vectorfield. Let Π be a homogeneous Fourier multiplier of degree 0. Then for all $c>0$, there exists a $C>0$ depending only on c, F, Π and $\|L\|_{\mathrm{L}^\infty}$, and such that*

$$\left|\int \big(\Pi(F(L)\mathbf{Z})\big)^* \cdot \left[\Delta^2(F(L)\mathbf{Z})-F(L)\Delta^2\mathbf{Z}-F'(L)\Delta^2 L\,\mathbf{Z}\right] dx\right| \leq c\,\|\Delta\mathbf{Z}\|_{\mathrm{L}^2}^2+C\left\|D^2L\right\|_{\mathrm{L}^\infty}^2\|\mathbf{Z}\|_{\mathrm{L}^2}^2.$$

Proof. Easy (but cumbersome) computations yield

$$\begin{aligned}\Delta^2(F(L)\mathbf{Z}) - F(L)\Delta^2\mathbf{Z} - F'(L)\Delta^2 L\,\mathbf{Z} &= 4F''(L)DL\cdot\nabla\Delta L\,\mathbf{Z}\\ +F''(L)\big((\Delta L)^2+2\nabla^2L:\nabla^2L\big)\mathbf{Z} &+ F'''(L)(2\Delta L|\nabla L|^2+DL\cdot\nabla^2L\cdot\nabla L)\mathbf{Z}+F''''(L)|\nabla L|^4\mathbf{Z}\\ &+4D\mathbf{Z}\cdot\nabla\Delta F(L)+2\Delta\mathbf{Z}\Delta F(L)+4D^2\mathbf{Z}:D^2F(L)+4D\Delta\mathbf{Z}\cdot\nabla F(L).\end{aligned}$$

Let us denote

$$\begin{aligned}I_1 &:= \int \big(\Pi(F(L)\mathbf{Z})\big)^*\cdot\mathbf{Z}\,F''(L)DL\cdot\nabla\Delta L\,dx,\\ I_2 &:= \int \big(\Pi(F(L)\mathbf{Z})\big)^*\cdot\mathbf{Z}\,F''(L)\big((\Delta L)^2+2\nabla^2L:\nabla^2L\big)\,dx,\\ I_3 &:= \int \big(\Pi(F(L)\mathbf{Z})\big)^*\cdot\mathbf{Z}\,F'''(L)(2\Delta L|\nabla L|^2+DL\cdot\nabla^2L\cdot\nabla L)\,dx,\end{aligned}$$

$$\begin{aligned}I_4 &:= \int \big(\Pi(F(L)\mathbf{Z})\big)^*\cdot\mathbf{Z}\,F''''(L)|\nabla L|^4\,dx,\\ I_5 &:= \int \big(\Pi(F(L)\mathbf{Z})\big)^*\cdot D\mathbf{Z}\cdot\nabla\Delta F(L)\,dx,\\ I_6 &:= \int \big(\Pi(F(L)\mathbf{Z})\big)^*\cdot\big(\Delta\mathbf{Z}\,\Delta F(L)+2D^2\mathbf{Z}:D^2F(L)\big)\,dx,\\ I_7 &:= \int \big(\Pi(F(L)\mathbf{Z})\big)^*\cdot D\Delta\mathbf{Z}\cdot\nabla F(L)\,dx.\end{aligned}$$

Let us start with the study of I_1. Performing an integration by parts, we notice that

$$I_1 = -\int \big(\Pi(F(L)\mathbf{Z})\big)^*\cdot\mathbf{Z}\,\Delta L\,\mathrm{div}\,(F''(L)\nabla L)\,dx - \int D\Big(\big(\Pi(F(L)\mathbf{Z})\big)^*\cdot\mathbf{Z}\Big)\cdot\nabla L\,F''(L)\Delta L\,dx,$$

hence

$$|I_1| \lesssim \|\Delta L\|_{\mathrm{L}^\infty}\big(\|DL\|_{\mathrm{L}^\infty}^2+\|\Delta L\|_{\mathrm{L}^\infty}\big)\|\mathbf{Z}\|_{\mathrm{L}^2}^2+\|DL\|_{\mathrm{L}^\infty}\|\Delta L\|_{\mathrm{L}^\infty}\|\mathbf{Z}\|_{\mathrm{L}^2}\big(\|\mathbf{Z}\|_{\mathrm{L}^2}\|DL\|_{\mathrm{L}^\infty}+\|D\mathbf{Z}\|_{\mathrm{L}^2}\big).$$

Taking advantage of the inequalities (3.16),

$$\|DA\|_{\mathrm{L}^2}^2 \leq C\,\|A\|_{\mathrm{L}^2}\left\|D^2A\right\|_{\mathrm{L}^2}, \tag{4.12}$$

and of Young's inequality, we conclude that

$$|I_1| \leq C\left\|D^2L\right\|_{\mathrm{L}^\infty}^2\|\mathbf{Z}\|_{\mathrm{L}^2}^2+\frac{c}{4}\|\Delta\mathbf{Z}\|_{\mathrm{L}^2}^2.$$

By virtue of inequality (3.16). we readily have

$$|I_k| \leq C\left\|D^2L\right\|_{\mathrm{L}^\infty}^2\|\mathbf{Z}\|_{\mathrm{L}^2}^2 \quad \text{for} \quad k=2,3,4.$$

In order to bound I_5, we perform an integration by parts and rewrite

$$I_5 = -\int \big[\nabla\Pi(F(L)\mathbf{Z})\big]^*\cdot D\mathbf{Z}\,\Delta F(L)\,dx - \int \big(\Pi(F(L)\mathbf{Z})\big)^*\cdot\Delta\mathbf{Z}\,\Delta F(L)\,dx.$$

Appealing to (3.16), (4.12) and to Young's inequality, we easily get

$$|I_5| \le C \left\|D^2 L\right\|_{\mathrm{L}^\infty}^2 \|\mathbf{Z}\|_{\mathrm{L}^2}^2 + \frac{c}{4} \|\Delta \mathbf{Z}\|_{\mathrm{L}^2}^2 .$$

A direct use of (3.16) (with no integration by parts) yields the same inequality for I_6.

Finally, the term I_7 may be handled by making use of the following integration by parts

$$I_7 = -\int \big(\Pi(F(L)\mathbf{Z})\big)^* \cdot \Delta\mathbf{Z}\, \Delta F(L)\, dx - \int \big[\nabla \Pi(F(L)\mathbf{Z})\big]^* \cdot \Delta \mathbf{Z}\, \nabla F(L)\, dx.$$

□

Lemma 4.2 *Let* $L \in \mathrm{L}^\infty$, $F \in \mathrm{W}^{2,\infty}$ *and* $\mathbf{Z}$ *be a* H^2 *vectorfield. Let* Π *be a smooth homogeneous Fourier multiplier of degree* 0. *Then for all* $c > 0$, *there exists a positive constant* C *depending only on* c, F, Π *and* $\|L\|_{\mathrm{L}^\infty}$, *and such that*

$$\|[\Pi\Delta, F(L)]\mathbf{Z}\|_{\mathrm{L}^2}^2 \le c \|\Delta\mathbf{Z}\|_{\mathrm{L}^2}^2 + C \left\|D^2 L\right\|_{\mathrm{L}^\infty}^2 \|\mathbf{Z}\|_{\mathrm{L}^2}^2 .$$

Proof. We have

$$[\Pi\Delta, F(L)]\mathbf{Z} = \Pi\big(\mathbf{Z}\Delta F(L) + 2D(F(L)) \cdot \nabla \mathbf{Z}\big) + [\Pi, F(L)]\Delta\mathbf{Z}.$$

The last term may be bounded by mean of Lemma A.4 with $\eta = 0$. We get

$$\|[\Pi\Delta, F(L)]\mathbf{Z}\|_{\mathrm{L}^2} \lesssim (\|DL\|_{\mathrm{L}^\infty}^2 + \|\Delta L\|_{\mathrm{L}^\infty}) \|\mathbf{Z}\|_{\mathrm{L}^2} + \|DL\|_{\mathrm{L}^\infty} \|D\mathbf{Z}\|_{\mathrm{L}^2} + \|DL\|_{\mathrm{L}^\infty} \|\Delta\mathbf{Z}\|_{\mathrm{H}^{-1}}.$$

Using (3.16) and (4.12) completes the proof. □

Let us resume the proof of Proposition 4.2. Performing two integration by parts to handle the terms pertaining to $\mathbf{h}$, inserting the inequalities provided by lemmas 4.1 and 4.2 (with c suitably small) and taking advantage of (3.26) and (3.27), we discover that

$$\frac{1}{2}\frac{d}{dt} \widetilde{\|\mathbf{z}\|}_s^2 + c\varepsilon \|D^2\mathbf{z}\|_{\mathrm{H}^s}^2 \le \widetilde{\|\mathbf{z}\|}_s \widetilde{\|\mathbf{f}\|}_s + C A'_\varepsilon \|\mathbf{z}\|_{\mathrm{H}^s}^2 + C\varepsilon \|\mathbf{h}\|_{\mathrm{H}^s}^2 \tag{4.13}$$

for some constants c and C depending only on the usual parameters and A_ε defined according to the statement of Proposition 4.2. Arguing as in the case $\varepsilon = 0$, we easily get the inequality (4.9).

Let us now state the inequality (4.10). We have

$$\begin{aligned} D_t(\phi_0\mathbf{z}) + i\nabla(a\mathrm{div}\,(\phi_0\mathbf{z})) + \varepsilon\Delta^2(\phi_0\mathbf{z}) = \phi_0(\mathbf{f} + \varepsilon\Delta\mathbf{h}) - \frac{\mathrm{div}\,\mathbf{v}}{2}\phi_0\mathbf{z} \\ + i\nabla(aD\phi_0)\cdot\mathbf{z} - ia\nabla_0(\phi_0\mathbf{z})\cdot\nabla\log\phi_0 + \varepsilon\big(\Delta^2(\phi_0\mathbf{z}) - \phi_0\Delta^2\mathbf{z} - \Delta^2 L\phi_0'\mathbf{z}\big), \end{aligned}$$

with ϕ_0' denoting the derivative of ϕ_0 with respect to L. Multiplying by $\phi_0\mathbf{z}^*$, integrating over $\mathbb{R}^N$ and using lemmas 4.1 and 4.2 to handle the terms in ε, we eventually get

$$\frac{d}{dt}\|\phi_0\mathbf{z}\|_{\mathrm{L}^2}^2 + \varepsilon \left\|D^2\mathbf{z}\right\|_{\mathrm{L}^2}^2 \le 2\|\phi_0\mathbf{z}\|_{\mathrm{L}^2}\|\phi_0\mathbf{f}\|_{\mathrm{L}^2} + C\varepsilon\|D^2 L\|_{\mathrm{L}^\infty}^2 \|\phi_0\mathbf{z}\|_{\mathrm{L}^2}^2 + C\varepsilon\|\mathbf{h}\|_{\mathrm{L}^2}^2,$$

which obviously yields (4.10). □

Corollary 4.2 *Let* $(L, \mathbf{z})$ *satisfy the assumptions of Proposition 4.2 with* $\mathbf{h} = \mathbf{0}$. *Then there exists some constant* C *depending only on* N, s, $\underline{a}$, $\widetilde{a}$, $\underline{\rho}$ *and* $\widetilde{\rho}$ *such that*

$$\|\mathbf{z}\|_{L_T^\infty(\mathrm{H}^s)} + \sqrt{\varepsilon}\,\|D^2\mathbf{z}\|_{L_T^2(\mathrm{H}^s)} \le C\mathrm{e}^{C\int_0^T A_\varepsilon\,dt}\big(1 + \|\mathbf{w}\|_{\mathrm{L}_T^\infty(\mathrm{L}^\infty)}^{\max(1,s)}\big)\Big(\|\mathbf{z}_0\|_{\mathrm{H}^s} + \|\mathbf{f}\|_{L_T^1(\mathrm{H}^s)}\Big).$$

Proof. Remark that we have $\|\mathbf{z}\|_{\mathrm{H}^{s-1}} \leq \|\mathbf{z}\|_{\mathrm{L}^2}^{\frac{1}{s}} \|\mathbf{z}\|_{\mathrm{H}^s}^{\frac{s-1}{s}}$ if $s>1$ and $\|\mathbf{z}\|_{\mathrm{H}^{s-1}} \leq \|\mathbf{z}\|_{\mathrm{L}^2}$ otherwise. Hence applying (4.9) with $\alpha=0$ and appealing to Young's inequality, we get for all $t\in[0,T]$,

$$\|\mathbf{z}(t)\|_{\mathrm{H}^s}^2 + \varepsilon\int_0^t \|D^2\mathbf{z}\|_{\mathrm{H}^s}^2\,d\tau \leq C\biggl(\|\mathbf{z}_0\|_{\mathrm{H}^s}^2 + \int_0^t \bigl(\|\mathbf{f}\|_{\mathrm{H}^s}\|\mathbf{z}\|_{\mathrm{H}^s} + A_\varepsilon\|\mathbf{z}\|_{\mathrm{H}^s}^2\bigr)\,d\tau + \|\mathbf{w}(t)\|_{\mathrm{L}^\infty}^{2s_1^+}\|\mathbf{z}(t)\|_{\mathrm{L}^2}^2\biggr)$$

with $s_1^+ := \max(1,s)$.

Using Gronwall's type arguments yields

$$\|\mathbf{z}\|_{L_t^\infty(\mathrm{H}^s)} + \sqrt{\varepsilon}\,\|D^2\mathbf{z}\|_{L_t^2(\mathrm{H}^s)} \leq C\mathrm{e}^{C\int_0^t A_\varepsilon\,d\tau}\biggl(\|\mathbf{z}_0\|_{\mathrm{H}^s} + \int_0^t \|\mathbf{f}\|_{\mathrm{H}^s}\,d\tau + \sup_{\tau\in[0,t]}\|\mathbf{w}(t)\|_{\mathrm{L}^\infty}^{s_1^+}\|\mathbf{z}(t)\|_{\mathrm{L}^2}\biggr).$$

The term $\|\mathbf{z}(t)\|_{\mathrm{L}^2}$ may be bounded according to the inequality in (4.10). This completes the proof of Corollary 4.2. □

Remark 4.1 *In dimension $N=1$ and, more generally, for potential flows, the estimates are simpler since the sole gauge ψ_s suffices to close the estimates. Arguing as in Proposition 3.2, one can prove that the inequality* (4.9) *remains valid for all $s>-N/2$ and reduces to*

$$\|\mathbf{z}(t)\|_{\mathrm{H}^s} + \sqrt{\varepsilon}\|D^2\mathbf{z}\|_{L_t^2(\mathrm{H}^s)} \leq C\Bigl(\|\mathbf{z}_0\|_{\mathrm{H}^s} + \|\mathbf{f}\|_{L_t^1(\mathrm{H}^s)} + \sqrt{\varepsilon}\|\mathbf{h}\|_{L_t^2(\mathrm{H}^s)} + A_\varepsilon\|\mathbf{z}\|_{\mathrm{H}^s}\Bigr).$$

As for the inequality (4.11)*, it reduces to*

$$\|\mathbf{z}(t)\|_{\mathrm{H}^s} + \sqrt{\varepsilon}\|D^2\mathbf{z}\|_{L_t^2(\mathrm{H}^s)} \leq C\biggl(\|\mathbf{z}_0\|_{\mathrm{H}^s} + \int_0^t (1+\|D\mathbf{z}\|_{\mathrm{L}^\infty} + \varepsilon\,\|D\mathbf{w}\|_{\mathrm{L}^\infty}^2)\|\mathbf{z}\|_{\mathrm{H}^s}\,d\tau\biggr).$$

4.3 Local existence on an interval independent of ε

This section is devoted to the proof of the following result.

Proposition 4.3 *Take $\varepsilon_0>0$. Let $\mathbf{u}_0\in\mathrm{H}^s$ with $s>1+N/2$, and $L_0\in\mathrm{L}^\infty$ be valued in $\mathbb{K}\subset\subset\mathbb{J}$ and satisfy $DL_0\in\mathrm{H}^s$. There exist an exponent $\beta\geq 3$ depending only on s and N, a constant C depending only on $a_\sharp$, $q_\sharp$, $d(\mathbb{K},\mathbb{R}\setminus\mathbb{J})$, N and s, and a positive T depending (continuously) on ε_0, C, $\|DL_0\|_{\mathrm{H}^s}$ and $\|\mathbf{u}_0\|_{\mathrm{H}^s}$ such that for all $\varepsilon\in(0,\varepsilon_0]$ system* (EK_ε) *has a unique H^s solution $(L,\mathbf{u})$ on $[0,T]\times\mathbb{R}^N$ with besides*

$$\|\mathbf{z}\|_{L_T^\infty(\mathrm{H}^s)} + \|\mathbf{z}\|_{L_T^\infty(\mathrm{H}^{s-2})}^\beta + \sqrt{\varepsilon}\|D^2\mathbf{z}\|_{L_T^2(\mathrm{H}^s)} \leq C\bigl(\|\mathbf{z}_0\|_{\mathrm{H}^s} + \|\mathbf{z}_0\|_{\mathrm{H}^{s-2}}^\beta\bigr)\mathrm{e}^{CT} \tag{4.14}$$

where $\mathbf{z}:=\mathbf{u}+i\nabla L$ and $\mathbf{z}_0:=\mathbf{u}_0+i\nabla L_0$.

Proof. For all $\varepsilon>0$, Corollary 4.1 insures the existence of a unique H^s solution $(L,\mathbf{u})$ on some non trivial time interval. Let T^* be the lifespan of the maximal H^s solution to (EK_ε). Let us denote $\eta:=d(\mathbb{K},\mathbb{R}\setminus\mathbb{J})$. Fix a positive $T<T^*$ so that

$$\forall t\in[0,T],\forall x\in\mathbb{R}^N,\ J^- + \frac{\eta}{2} \leq L(t,x) \leq J^+ - \frac{\eta}{2}. \tag{4.15}$$

The continuity of L ensures the existence of such a T. We aim at finding a lower bound for T. In what follows, we assume that $s>\max(2,1+\frac{N}{2})$ (we shall explain afterwards how to handle the case $s\leq 2$). We have

$$\partial_t\mathbf{z} + (\mathbf{u}^*\cdot\nabla)\mathbf{z} + i\nabla\mathbf{z}\cdot\mathbf{w} + i\nabla(a_\sharp(L)\mathrm{div}\,\mathbf{z}) + \varepsilon\Delta^2\mathbf{z} = q_\sharp(L)\mathbf{w}. \tag{4.16}$$

Hence applying the inequality (4.13) and using Proposition B.1 for bounding $\|q_\sharp(L)\mathbf{w}\|_{\mathrm{H}^s}$, we get

$$\frac{1}{2}\frac{d}{dt}\widetilde{\|\mathbf{z}\|}_s^2+\varepsilon\|D^2\mathbf{z}\|_{\mathrm{H}^s}^2\le C\big(1+\|D\mathbf{z}\|_{\mathrm{L}^\infty}+\varepsilon\,\|D\mathbf{w}\|_{\mathrm{L}^\infty}^2\big)\|\mathbf{z}\|_{\mathrm{H}^s}^2. \tag{4.17}$$

In order to close the estimates, we now have to compare $\widetilde{\|\mathbf{z}\|}_s$ and $\|\mathbf{z}\|_{\mathrm{H}^s}$. For that, we introduce an $\alpha\in[0,1)$ such that $\frac{N}{2}-\alpha\le s-2$ (note that this is possible since $s>\frac{N}{2}+1$). Hence $\mathrm{H}^{s-2}\hookrightarrow \mathrm{C}^{-\alpha}$ thus, by virtue of (3.32),

$$\widetilde{\|\mathbf{z}\|}_s\lesssim\|\mathbf{z}\|_{\mathrm{H}^s}\lesssim\widetilde{\|\mathbf{z}\|}_s+\|\mathbf{w}\|_{\mathrm{H}^{s-2}}\|\mathbf{z}\|_{\mathrm{H}^{s-1+\alpha}}.$$

Now, interpolating between H^{s-2} and H^s and using Young's inequality, we easily conclude that there exists a constant C such that whenever (4.15) is satisfied, we have

$$C^{-1}\widetilde{\|\mathbf{z}\|}_s\le\|\mathbf{z}\|_{\mathrm{H}^s}\le C\big(\widetilde{\|\mathbf{z}\|}_s+\|\mathbf{z}\|_{\mathrm{H}^{s-2}}^\beta\big)\quad\text{with}\quad\beta:=\frac{3-\alpha}{1-\alpha}. \tag{4.18}$$

Hence we now have to get H^{s-2} bounds for $\mathbf{z}$. Applying Λ^{s-2} to (4.16), we discover that

$$D_t\Lambda^{s-2}\mathbf{z}+\varepsilon\Delta^2\Lambda^{s-2}\mathbf{z}=[\mathbf{u}^j,\Lambda^{s-2}]\partial_j\mathbf{z}+\Lambda^{s-2}\big(q_\sharp(L)\mathbf{w}-i\nabla\mathbf{z}\cdot\mathbf{w}-i\nabla(a\,\mathrm{div}\,\mathbf{z})\big).$$

The commutator may be handled like in (3.12) (remind that $s-2>0$). Then, applying a standard energy method, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{z}\|_{\mathrm{H}^{s-2}}^2+\varepsilon\|D^2\mathbf{z}\|_{\mathrm{H}^{s-2}}^2\lesssim\|\mathbf{z}\|_{\mathrm{H}^{s-2}}\big(&\|D\mathbf{z}\|_{\mathrm{L}^\infty}\|D\mathbf{u}\|_{\mathrm{H}^{s-3}}+\|D\mathbf{u}\|_{\mathrm{L}^\infty}\|D\mathbf{z}\|_{\mathrm{H}^{s-3}}\\&+\|\nabla\mathbf{z}\cdot\mathbf{w}\|_{\mathrm{H}^{s-2}}+\|a\,\mathrm{div}\,\mathbf{z}\|_{\mathrm{H}^{s-1}}+\|q_\sharp(L)\mathbf{w}\|_{\mathrm{H}^{s-2}}\big).\end{aligned}$$

The nonlinear terms may be bounded according to Lemma B.1 and Proposition B.1. We get

$$\|\nabla\mathbf{z}\cdot\mathbf{w}\|_{\mathrm{H}^{s-2}}+\|a\,\mathrm{div}\,\mathbf{z}\|_{\mathrm{H}^{s-1}}\lesssim\|\mathbf{z}\|_{\mathrm{H}^s}+\|D\mathbf{z}\|_{\mathrm{L}^\infty}\|\mathbf{w}\|_{\mathrm{H}^{s-2}}\quad\text{and}\quad\|q_\sharp(L)\mathbf{w}\|_{\mathrm{H}^{s-2}}\lesssim\|\mathbf{w}\|_{\mathrm{H}^{s-2}},$$

hence, combining with the inequality in (4.17),

$$\frac{d}{dt}\Big(\widetilde{\|\mathbf{z}\|}_s^2+\|\mathbf{z}\|_{\mathrm{H}^{s-2}}^{2\beta}\Big)\lesssim\big(1+\|D\mathbf{z}\|_{\mathrm{L}^\infty}+\varepsilon\,\|D\mathbf{w}\|_{\mathrm{L}^\infty}^2\big)\big(\widetilde{\|\mathbf{z}\|}_s^2+\|\mathbf{z}\|_{\mathrm{H}^{s-2}}^{2\beta}\big)+\|\mathbf{z}\|_{\mathrm{H}^{s-2}}^{2\beta-1}\|\mathbf{z}\|_{\mathrm{H}^s}.$$

Introduce the function $Z(t):=\sqrt{\widetilde{\|\mathbf{z}(t)\|}_s^2+\|\mathbf{z}(t)\|_{\mathrm{H}^{s-2}}^{2\beta}}$. Plugging (4.18) in the above equation and taking advantage of the embedding $\mathrm{H}^{s-1}\hookrightarrow\mathrm{L}^\infty$, it is easily found that Z satisfies the following differential inequality

$$\frac{1}{2}\frac{d}{dt}Z^2\le C(Z^2+Z^3+\varepsilon Z^4) \tag{4.19}$$

with C depending only on N, s, $a_\sharp$, $q_\sharp$ and η.

In order to pursue the computations, let us assume that

$$C\varepsilon_0\int_0^T Z^2(t)\le\log 2. \tag{4.20}$$

Then inequality (4.19) combined with Gronwall's lemma implies that

$$Z(t)\le 2Z(0)\mathrm{e}^{Ct}\mathrm{e}^{C\int_0^t Z(\tau)\,d\tau}.$$

Assuming that $T<\frac{1}{C}\log\Bigl(1+\frac{1}{2Z(0)}\Bigr)$, straightforward computations lead to

$$\mathrm{e}^{C\int_0^t Z(\tau)\,d\tau}\leq\frac{1}{1-2Z(0)(\mathrm{e}^{Ct}-1)}\quad\text{and}\quad Z(t)\leq\frac{2Z(0)\mathrm{e}^{Ct}}{1-2Z(0)(\mathrm{e}^{Ct}-1)}\quad\text{for}\quad t\in[0,T].\tag{4.21}$$

From now on, we assume that

$$T\leq\frac{1}{C}\log\biggl(1+\frac{1}{4Z(0)}\biggr)\tag{4.22}$$

so that the denominators in (4.21) are greater than $1/2$. Hence, if in addition we have

$$\mathrm{e}^{CT}\log\biggl(\frac{1}{1-2Z(0)(\mathrm{e}^{CT}-1)}\biggr)\leq\frac{\log 2}{4\varepsilon_0 Z(0)}\tag{4.23}$$

then condition (4.20) is satisfied as well.

By combining inequalities (4.18) and (4.17), we readily conclude that, under condition (4.15), we have (4.14).

We now have to find a condition which guarantees (4.15). For that, it suffices to find some $T>0$ such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^N}|L(t,x)-L_0(x)|\leq\frac{\eta}{2}.$$

We have

$$\|L(t)-L_0\|_{\mathrm{L}^\infty}\leq\|L(t)-S_{\varepsilon t}L_0\|_{\mathrm{L}^\infty}+\|S_{\varepsilon t}L_0-L_0\|_{\mathrm{L}^\infty}.$$

On one hand, the definition of S_t combined with Sobolev embeddings ensures that

$$\begin{aligned}\|L_0-S_{\varepsilon t}L_0\|_{\mathrm{L}^\infty}&\lesssim\varepsilon\textstyle\int_0^t\|\Delta^2 S_{\varepsilon\tau}L_0\|_{\mathrm{H}^{s-1}}\,d\tau,\\&\lesssim\sqrt{\varepsilon t}\|\Delta L_0\|_{\mathrm{H}^{s-1}}.\end{aligned}$$

On the other hand, as $\partial_t L+\varepsilon\Delta^2 L=-\mathbf{u}^*\cdot\nabla L-a_\sharp(L)\mathrm{div}\,\mathbf{u}$, Lemma B.2 and Proposition B.1 yield

$$\begin{aligned}\|L(t)-S_{\varepsilon t}L_0\|_{\mathrm{L}^\infty}&\lesssim\int_0^t\Bigl(\|\mathbf{u}^*\cdot\nabla L\|_{\mathrm{H}^{s-1}}+\|a_\sharp(L)\mathrm{div}\,\mathbf{u}\|_{\mathrm{H}^{s-1}}\Bigr)\,d\tau,\\&\lesssim\bigl(1+\|D^2L\|_{L^\infty_t(\mathrm{H}^{s-2})}\bigr)\|\mathbf{u}\|_{L^1_t(\mathrm{H}^s)}.\end{aligned}$$

Therefore, whenever (4.15) is fulfilled, we have

$$\max_{t\in[0,T]}\|L(t)-L_0\|_{\mathrm{L}^\infty}\leq C\Bigl(\sqrt{\varepsilon T}\|\Delta L_0\|_{\mathrm{H}^{s-1}}+\bigl(1+\|D^2L\|_{L^\infty_T(\mathrm{H}^{s-2})}\bigr)\|\mathbf{u}\|_{L^1_T(\mathrm{H}^s)}\Bigr).\tag{4.24}$$

Using (4.24) and arguing by induction, we conclude that (4.15) is satisfied provided (4.22),(4.23) are fulfilled and

$$-\bigl(1+Z(0)\mathrm{e}^{CT}\bigr)\log\Bigl(1-2Z(0)(\mathrm{e}^{CT}-1)\Bigr)\leq c\eta\quad\text{and}\quad\|\Delta L_0\|_{\mathrm{H}^{s-1}}\sqrt{\varepsilon_0 T}\leq c\eta\tag{4.25}$$

for some small enough positive constant c. This provides a uniform lower bound for T^* for $\varepsilon\in(0,\varepsilon_0]$. The proof of Proposition 4.3 is complete in the case $s>2$.

Remark 4.1 enables to treat the case $s\leq 2$. Indeed, this may occur only in the one-dimensional case, and one-dimensional flows are always potential. The proof is actually simpler since we need not to bound the H^{s-2} norm of $\mathbf{z}$. □

4.4 Lipschitz continuity with respect to the data

Proposition 4.4 *Let $s > 1 + N/2$ and $(L_0, \mathbf{u}_0)$ satisfy*

$$(DL_0, \mathbf{u}_0) \in \mathrm{H}^s \quad \textit{and} \quad L_0 \quad \textit{is valued in} \quad \mathbb{K} \subset\subset \mathbb{J}.$$

There exist a neighborhood $\mathcal{V}$ of $(0, \mathbf{0})$ in $\mathrm{H}^{s+1} \times \mathrm{H}^s$ and a positive T such that for all $(\widetilde{L}_0, \widetilde{\mathbf{u}}_0)$ such that $(\delta L_0, \delta\mathbf{u}_0) := (\widetilde{L}_0 - L_0, \widetilde{\mathbf{u}}_0 - \mathbf{u}_0) \in \mathcal{V}$ system $(\mathrm{EK}_\varepsilon)$ with data $(\widetilde{L}_0, \widetilde{\mathbf{u}}_0)$ has a unique solution $(\widetilde{L}, \widetilde{\mathbf{u}})$ with $\widetilde{L}$ valued in a compact subset $\mathbb{K}'$ of $\mathbb{J}$ and $(D\widetilde{L}, \widetilde{\mathbf{u}})$ uniformly bounded in

$$\mathcal{C}([0,T]; \mathrm{H}^s) \cap \mathrm{L}^2(0,T; \mathrm{H}^{s+2}).$$

Besides there exists a constant C depending only on $\mathcal{V}$, $d(\mathbb{K}, \mathbb{R} \setminus \mathbb{J})$, ε, s, N, $a_\sharp$ and $q_\sharp$ and such that

$$\|\delta L\|_{L^\infty_T(\mathrm{H}^{s+1})} + \|\delta\mathbf{u}\|_{L^\infty_T(\mathrm{H}^s)} \leq C\Big(\|\delta L_0\|_{\mathrm{H}^{s+1}} + \|\delta\mathbf{u}_0\|_{\mathrm{H}^s}\Big)$$

with $\delta L := \widetilde{L} - L$ and $\delta\mathbf{u} := \widetilde{u} - u$. Here $(L, \mathbf{u})$ stands for the solution with data $(L_0, \mathbf{u}_0)$.

Proof. According to Theorem 4.3, there exists a positive T and a neighborhood $\mathcal{V}$ of $(0, \mathbf{0})$ such that for all $(\widetilde{L}_0, \widetilde{\mathbf{u}}_0)$ satisfying $(\delta L_0, \delta\mathbf{u}_0) \in \mathcal{V}$, system $(\mathrm{EK}_\varepsilon)$ has a H^s solution $(\widetilde{L}, \widetilde{\mathbf{z}})$ on $[0,T] \times \mathbb{R}^N$ with $\widetilde{L}$ valued in a (fixed) compact subset of $\mathbb{J}$ and $(D\widetilde{L}, \widetilde{\mathbf{u}})$ uniformly bounded in $\mathrm{L}^\infty(0,T; \mathrm{H}^s) \cap \mathrm{L}^2(0,T; \mathrm{H}^{s+2})$ – with a bound depending on ε.

Let us first estimate the L^2 norm of δL. For doing so, we notice that

$$\partial_t \delta L + (\,\widetilde{\mathbf{u}}^* \cdot \nabla\,)\delta L + \varepsilon\Delta^2 \delta L = -(\,\delta\mathbf{u}^* \cdot \nabla)L - \delta a \,\mathrm{div}\, \widetilde{\mathbf{u}} - a_\sharp(L) \mathrm{div}\, \delta\mathbf{u} \quad \text{with} \quad \delta a := a_\sharp(\widetilde{L}) - a_\sharp(L).$$

Therefore, an obvious energy argument yields

$$\|\delta L(t)\|_{\mathrm{L}^2} \leq \|\delta L_0\|_{\mathrm{L}^2} + C \int_0^t \big(\|DL\|_{\mathrm{L}^\infty} \|\delta\mathbf{u}\|_{\mathrm{L}^2} + \|\delta L\|_{\mathrm{L}^2} \|\mathrm{div}\, \widetilde{\mathbf{u}}\|_{\mathrm{L}^\infty} + \|\mathrm{div}\, \delta\mathbf{u}\|_{\mathrm{L}^2}\big)\, d\tau. \tag{4.26}$$

In order to estimate $\|\delta\mathbf{u}\|_{\mathrm{H}^s}$, we introduce the complex valued functions $\widetilde{\mathbf{z}} := \widetilde{\mathbf{u}} + i\nabla\widetilde{L}$ and $\mathbf{z} := \mathbf{u} + i\nabla L$. Let $Q_\sharp$ be a primitive of $q_\sharp$. The function $\widetilde{\delta\mathbf{z}} := \widetilde{\mathbf{z}} - \mathbf{z}$ satisfies

$$\begin{aligned}\partial_t \delta\mathbf{z} + (\,\mathbf{u}^* \cdot \nabla\,)\delta\mathbf{z} + i\nabla\delta\mathbf{z} \cdot \mathbf{w} + i\nabla\big(a_\sharp(L)\mathrm{div}\, \delta\mathbf{z}\big) + \varepsilon\Delta^2 \delta\mathbf{z} \\ = \nabla\big(Q_\sharp(\widetilde{L}) - Q_\sharp(L)\big) - (\delta\mathbf{u}^* \cdot \nabla)\widetilde{\mathbf{z}} - i\nabla\widetilde{\mathbf{z}} \cdot \delta\mathbf{w} - i\nabla\big(\delta a\, \mathrm{div}\, \widetilde{\mathbf{z}}\big).\end{aligned}$$

By virtue of Corollary 4.2, we thus have

$$\begin{aligned}\|\delta\mathbf{z}(t)\|_{\mathrm{H}^s} \leq C\big(1 + \|\mathbf{w}\|^s_{\mathrm{L}^\infty_t(\mathrm{L}^\infty)}\big) \mathrm{e}^{C\int_0^t \left(\|D\mathbf{z}\|_{\mathrm{H}^{s-1}} + \varepsilon\|D\mathbf{w}\|^2_{\mathrm{L}^\infty}\right) d\tau} \Bigg(\|\delta\mathbf{z}_0\|_{\mathrm{H}^s} \\ + \int_0^t \Big(\|Q_\sharp(\widetilde{L}) - Q_\sharp(L)\|_{\mathrm{H}^{s+1}} + \|(\delta\mathbf{u}^* \cdot \nabla)\widetilde{\mathbf{z}}\|_{\mathrm{H}^s} + \|\nabla\widetilde{\mathbf{z}} \cdot \delta\mathbf{w}\|_{\mathrm{H}^s} + \|\delta a\, \mathrm{div}\, \widetilde{\mathbf{z}}\|_{\mathrm{H}^{s+1}}\Big)\, d\tau\Bigg).\end{aligned}$$

By taking advantage of Lemma B.1 and Corollary B.3, one can bound the integrand by

$$C\big(1 + \|D\widetilde{\mathbf{z}}\|_{\mathrm{H}^{s+1}}\big)\big(1 + \|D\mathbf{w}\|_{\mathrm{H}^{s-1}} + \|D\widetilde{\mathbf{w}}\|_{\mathrm{H}^{s-1}}\big)\big(\|\delta L\|_{\mathrm{L}^2} + \|\delta\mathbf{z}\|_{\mathrm{H}^s}\big).$$

Hence, adding up the inequality (4.26) then applying Gronwall's lemma, we end up with

$$\begin{aligned}\|\delta L(t)\|_{\mathrm{L}^2} + \|\delta\mathbf{z}(t)\|_{\mathrm{H}^s} \leq C\big(1 + \|\mathbf{w}\|^s_{\mathrm{L}^\infty_t(\mathrm{L}^\infty)}\big)\Big(\|\delta L_0\|_{\mathrm{L}^2} + \|\delta\mathbf{z}_0\|_{\mathrm{H}^s}\Big) \\ \times \mathrm{e}^{C\int_0^t \left[(1+\|D\widetilde{\mathbf{z}}\|_{\mathrm{H}^{s+1}})(1+\|D\mathbf{z}\|_{\mathrm{H}^{s-1}}+\|D\widetilde{\mathbf{w}}\|_{\mathrm{H}^{s-1}}) + \varepsilon\|D\mathbf{w}\|^2_{\mathrm{L}^\infty}\right] d\tau}\end{aligned} \tag{4.27}$$

for some constant C depending only on the usual parameters. The proof of Lemma 4.4 is complete. □

4.5 A blow-up criterion and more lower bounds on the lifespan

In this section, we aim at getting more informations on the lifespan of H^s solutions to $(\mathrm{EK}_\varepsilon)$. Let us first state a blow-up criterion.

Proposition 4.5 *Let L_0 be valued in a compact subset of $\mathbb{J}$ and satisfy $DL_0 \in \mathrm{H}^s$ for some $s > 1 + N/2$. Take $\mathbf{u}_0 \in \mathrm{H}^s$. Assume that the corresponding H^s solution $(L, \mathbf{u})$ of $(\mathrm{EK}_\varepsilon)$ is defined on $[0,T) \times \mathbb{R}^N$ and satisfies the following three conditions:*

$$L([0,T) \times \mathbb{R}^N) \subset\subset \mathbb{J}, \tag{4.28}$$

$$\textstyle\int_0^T \left(\left\|D^2 L(t)\right\|_{\mathrm{L}^\infty} + \|D\mathbf{u}(t)\|_{\mathrm{L}^\infty} + \varepsilon \left\|D^2 L\right\|_{\mathrm{L}^\infty}^2\right) dt < \infty, \tag{4.29}$$

$$\sup_{t\in[0,T]} \|L(t)\|_{\mathrm{C}^\alpha} < \infty \quad \textit{for some} \quad \alpha \in (0,1). \tag{4.30}$$

Then $(L, \mathbf{u})$ may be continued beyond T into a H^s solution of $(\mathrm{EK}_\varepsilon)$.

Proof. Interpolating between L^2 and H^s yields

$$\|L\|_{\mathrm{C}^\alpha}\|\mathbf{z}\|_{\mathrm{H}^{s-\alpha}} \le \eta\|\mathbf{z}\|_{\mathrm{H}^s} + \eta^{\frac{\alpha-s}{\alpha}}\|L\|_{\mathrm{C}^\alpha}^{\frac{s}{\alpha}}\|\mathbf{z}\|_{\mathrm{L}^2}$$

for all $\eta > 0$. Thus, applying the inequality in (4.11) to $\mathbf{z} := \mathbf{u} + i\nabla L$,

$$\|\mathbf{z}(t)\|_{\mathrm{H}^s}^2 \lesssim \|\mathbf{z}_0\|_{\mathrm{H}^s}^2 + \int_0^t (1 + \|D\mathbf{z}\|_{\mathrm{L}^\infty} + \varepsilon \left\|D^2 L\right\|_{\mathrm{L}^\infty})\|\mathbf{z}\|_{\mathrm{H}^s}^2\, d\tau + \|L(t)\|_{\mathrm{C}^\alpha}^{\frac{2s}{\alpha}}\|\mathbf{z}(t)\|_{\mathrm{L}^2}^2. \tag{4.31}$$

The term $\|\mathbf{z}(t)\|_{\mathrm{L}^2}$ may be bounded by appealing to the inequality in (4.10) and Gronwall's lemma. Hence the above inequality provides a bound in $\mathrm{L}^\infty(0,T;\mathrm{H}^s)$ for $\mathbf{z}$. By virtue of Proposition 4.1, we thus conclude that $(L, \mathbf{u})$ may be continued beyond T. □

Corollary 4.3 *Let $(L_0, \mathbf{u}_0)$ satisfy the assumptions of Proposition 4.5 and $s \ge s_1 > N/2 + 1$. Then the lifespan of a H^s solution to $(\mathrm{EK}_\varepsilon)$ with data $(L_0, \mathbf{u}_0)$ is the same as the lifespan of a H^{s_1} solution.*

Proof. Once noticed that $s_1 > 1 + N/2$ implies $\mathrm{H}^{s_1} \hookrightarrow \mathrm{Lip}$, we see that conditions (4.28), (4.29) and (4.30) are fulfilled. Hence Proposition 4.5 applies. □

5 Study of the Euler-Korteweg model

5.1 Local well-posedness

Let us state our main result.

Theorem 5.1 *Take $s > 1 + \frac{N}{2}$. Let $\rho_0 \in \mathrm{L}^\infty$ be valued in a compact subset of $\mathbb{J}_\rho$ and satisfy $D\rho_0 \in \mathrm{H}^s$. Let $\mathbf{u}_0$ be a vector-field with coefficients in H^s. There exists $T > 0$ such that (1.2) has a unique solution $(\rho, \mathbf{u})$ on $[0,T] \times \mathbb{R}^N$ satisfying*

$$(D\rho, \mathbf{u}) \in \mathcal{C}([0,T];\mathrm{H}^s) \cap \mathcal{C}^1([0,T];\mathrm{H}^{s-2}), \quad (\rho-\rho_0) \in \mathcal{C}([0,T];\mathrm{H}^{s+1}), \quad \rho([0,T]\times\mathbb{R}^N) \subset\subset \mathbb{J}_\rho. \tag{5.1}$$

Moreover, there exists a neighborhood $\mathcal{V}$ of $(\rho_0, \mathbf{u}_0)$ in $(\rho_0 + \mathrm{H}^{s+1}) \times \mathrm{H}^s$ such that for all $(\widetilde{\rho}_0, \widetilde{\mathbf{u}}_0) \in \mathcal{V}$, system (1.2) with data $(\widetilde{\rho}_0, \widetilde{\mathbf{u}}_0)$ has a unique solution $(\widetilde{\rho}, \widetilde{\mathbf{u}})$ on $[0,T] \times \mathbb{R}^N$ satisfying (5.1) uniformly, and the map

$$\left\{\begin{array}{ccl} \mathcal{V} & \longrightarrow & \mathcal{C}([0,T];\mathrm{H}^{s+1} \times \mathrm{H}^s) \cap \mathcal{C}^1([0,T];\mathrm{H}^{s-1} \times \mathrm{H}^{s-2}) \\ (\widetilde{\rho}_0, \widetilde{\mathbf{u}}_0) & \longmapsto & (\widetilde{\rho} - \widetilde{\rho}_0, \widetilde{\mathbf{u}}) \end{array}\right.$$

is continuous.

If besides $\mathrm{curl}\,\mathbf{u}_0 \equiv \mathbf{0}$ *then* $\mathrm{curl}\,\mathbf{u} = \mathbf{0}$ *on* $[0,T] \times \mathbb{R}^N$.

Proof. Using the unknown $L := \mathcal{L}(\rho)$, it suffices to prove the corresponding statement for

$$\text{(EK)}\qquad \begin{cases} \partial_t L + \mathbf{u}^* \cdot \nabla L + a_\sharp(L)\operatorname{div}\mathbf{u} = 0, \\ \partial_t \mathbf{u} + (\mathbf{u}^* \cdot \nabla)\mathbf{u} - \nabla^2 L \cdot \nabla L - \nabla(a_\sharp(L)\Delta L) = q_\sharp(L)\nabla L. \end{cases}$$

According to Corollary 4.1, if $(L, \mathbf{u})$ is a H^s solution to (EK), the assumption $\operatorname{curl}\mathbf{u}_0 = \mathbf{0}$ insures that $\operatorname{curl}\mathbf{u} = \mathbf{0}$ on $[0,T] \times \mathbb{R}^N$.

Now, let us briefly enumerate the main steps of the proof of well-posedness for (EK).

Step 1. Proof of uniqueness.

Step 2. Solving an approximate mollified problem.

In this step, we state that for small enough ε, then there exists a positive T such that (EK_ε) with data $L_{0,\varepsilon} := \chi_\varepsilon * L_0$, $\mathbf{u}_{0,\varepsilon} := \chi_\varepsilon * \mathbf{u}_0$ has a H^s solution on $[0,T] \times \mathbb{R}^N$ uniformly with respect to ε.

Here, the function χ_ε stands for the mollifier $\chi_\varepsilon := \varepsilon^{-\beta N}\chi(\varepsilon^{-\beta}\cdot)$ with χ a smooth function whose Fourier transform is compactly supported and is identically equal to one near the origin, and β is a small enough positive exponent to be specified hereafter.

Step 3. Convergence of the family $(L_\varepsilon, \mathbf{u}_\varepsilon)$ when ε goes to 0.

We show that for a convenient choice of β, the sequence $(L_\varepsilon, \mathbf{u}_\varepsilon)$ satisfies the *Cauchy* criterion (for ε going to 0^+) in the space

$$\Big(L_0 + \mathcal{C}([0,T]; \mathrm{H}^{s+1})\Big) \times \mathcal{C}([0,T]; \mathrm{H}^s)^N.$$

Step 4. Checking that the limit function $(L, \mathbf{u})$ is a solution to (EK).

Step 5. Proof of the continuity of the solution map.

Step 1 : uniqueness

The proof of uniqueness is a straightforward corollary of the following proposition.

Proposition 5.1 *Let $(L_1, \mathbf{z}_1)$ and $(L_2, \mathbf{z}_2)$ be two H^s solutions of (ES) on $[0,T] \times \mathbb{R}^N$, with $s > 1 + N/2$ and $s \neq 3 + N/2$. Assume in addition that L_i $(i = 1, 2)$ is valued in $\mathbb{K} \subset\subset \mathbb{J}$. Let us denote $\delta L := L_2 - L_1$ and $\delta\mathbf{z} := \mathbf{z}_2 - \mathbf{z}_1$. Then the following estimate holds true for all $t \in [0,T]$:*

$$\|(\delta L(t), \delta\mathbf{z}(t))\|_{\mathrm{H}^{s-2}} \leq C\big(1 + \|\mathbf{w}_1\|^{\max(1,s-2)}_{\mathrm{L}^\infty_t(\mathrm{L}^\infty)}\big)\|(\delta L(0), \delta\mathbf{z}(0))\|_{\mathrm{H}^{s-2}}$$
$$\times e^{C\int_0^t \left(1 + \|D\mathbf{z}_1\|_{\mathrm{H}^{s-1}} + (1 + \|D\mathbf{w}_1\|_{\mathrm{H}^{s-1}} + \|D\mathbf{w}_2\|_{\mathrm{H}^{s-1}})\|D\mathbf{z}_2\|_{\mathrm{H}^{s-1}}\right) d\tau}$$

where $\mathbf{w}_i := \operatorname{Im}\mathbf{z}_i$ and C depends only on $a_\sharp$, $q_\sharp$, s, N, $\mathbb{J}$ and $\mathbb{K}$.

Proof. Let $Q_\sharp$ stand for a primitive of $q_\sharp$. The equation satisfied by $\delta\mathbf{z}$ reads

$$\partial_t \delta\mathbf{z} + (\mathbf{u}_1^* \cdot \nabla)\delta\mathbf{z} + i\nabla\delta\mathbf{z} \cdot \mathbf{w}_1 + i\nabla\big(a_\sharp(L_1)\operatorname{div}\delta\mathbf{z}\big)$$
$$= \nabla\big(Q_\sharp(L_2) - Q_\sharp(L_1)\big) - (\delta\mathbf{u}^* \cdot \nabla)\mathbf{z}_2 - i\nabla\mathbf{z}_2 \cdot \delta\mathbf{w} - i\nabla\big(\delta a \operatorname{div}\mathbf{z}_2\big)$$

with $\delta a := a_\sharp(L_2) - a_\sharp(L_1)$, $\delta\mathbf{u} = \mathbf{u}_2 - \mathbf{u}_1$ and $\delta\mathbf{w} = \mathbf{w}_2 - \mathbf{w}_1$.

We notice that $\delta\mathbf{z}$ solves an equation of the type (LS). Besides, $\rho_1 := \mathcal{L}^{-1}(L_1)$ satisfies the mass conservation equation $\partial_t\rho_1 + \operatorname{div}(\rho_1\mathbf{u}_1) = 0$. Hence, if $s > 2$ and $s \neq 3 + N/2$, applying Corollary 4.2 with $\varepsilon = 0$ enables us to bound the H^{s-2} norm of $\delta\mathbf{z}$ as follows:

$$\begin{aligned}(5.2)\qquad \|\delta\mathbf{z}(t)\|_{\mathrm{H}^{s-2}} \lesssim \gamma(t)\mathrm{e}^{C\int_0^t(1+\|D\mathbf{z}_1\|_{\mathrm{H}^{s-1}})\,d\tau}\biggl(\|\delta\mathbf{z}(0)\|_{\mathrm{H}^{s-2}} + \int_0^t\Bigl(\|Q_\sharp(L_2) - Q_\sharp(L_1)\|_{\mathrm{H}^{s-1}}\\ +\|(\delta\mathbf{u}^*\cdot\nabla)\mathbf{z}_2\|_{\mathrm{H}^{s-2}} + \|\nabla\mathbf{z}_2\cdot\delta\mathbf{w}\|_{\mathrm{H}^{s-2}} + \|\delta a\operatorname{div}\mathbf{z}_2\|_{\mathrm{H}^{s-1}}\Bigr)\,dt\biggr)\end{aligned}$$

with $\gamma(t) := 1 + \|\mathbf{w}_1\|_{\mathrm{L}_t^\infty(\mathrm{L}^\infty)}^{\max(1,s-2)}$.

Note that if $s \leq 2$ (a case which may occur only if $N = 1$) remark 4.1 leads to the inequality (5.2) with $\gamma(t) = 1$.

We now have to estimate the integrand in the right-hand side of (5.2). According to Corollary B.3 with $k = 2$, we have

$$\|Q_\sharp(L_2) - Q_\sharp(L_1)\|_{\mathrm{H}^{s-1}} \lesssim (1 + \|D\mathbf{w}_1\|_{\mathrm{H}^{s-3}} + \|D\mathbf{w}_2\|_{\mathrm{H}^{s-3}})\|\delta L\|_{\mathrm{H}^{s-1}}.$$

Since for $s > 1 + N/2$, the usual product maps $\mathrm{H}^{s-1}\times\mathrm{H}^{s-2}$ in H^{s-2}, we have

$$\|\delta\mathbf{u}^*\cdot\nabla\mathbf{z}_2\|_{\mathrm{H}^{s-2}} + \|\nabla\mathbf{z}_2\cdot\delta\mathbf{w}\|_{\mathrm{H}^{s-2}} \lesssim \|D\mathbf{z}_2\|_{\mathrm{H}^{s-1}}\|\delta\mathbf{z}\|_{\mathrm{H}^{s-2}}.$$

Finally, since H^{s-1} is an algebra, we have, in view of Corollary B.3,

$$\|\delta a\operatorname{div}\mathbf{z}_2\|_{\mathrm{H}^{s-1}} \lesssim (1 + \|D\mathbf{w}_1\|_{\mathrm{H}^{s-3}} + \|D\mathbf{w}_2\|_{\mathrm{H}^{s-3}})\|\delta L\|_{\mathrm{H}^{s-1}}\|\operatorname{div}\mathbf{z}_2\|_{\mathrm{H}^{s-1}}.$$

Plugging all these inequalities in (5.2) and applying Gronwall's inequality, we end up with

$$\begin{aligned}(5.3)\qquad \|\delta\mathbf{z}(t)\|_{\mathrm{H}^{s-2}} \lesssim \gamma(t)\mathrm{e}^{C\int_0^t\left(1+\|D\mathbf{z}_1\|_{\mathrm{H}^{s-1}}+(1+\|D\mathbf{w}_1\|_{\mathrm{H}^{s-3}}+\|D\mathbf{w}_2\|_{\mathrm{H}^{s-3}})\|D\mathbf{z}_2\|_{\mathrm{H}^{s-1}}\right)\,d\tau}\\ \times\biggl(\|\delta\mathbf{z}(0)\|_{\mathrm{H}^{s-2}} + \int_0^t\bigl(1+\|D\mathbf{w}_1\|_{\mathrm{H}^{s-3}}+\|D\mathbf{w}_2\|_{\mathrm{H}^{s-3}}\bigr)\|D\mathbf{z}_2\|_{\mathrm{H}^{s-1}}\|\delta L\|_{\mathrm{H}^{s-2}}\,d\tau\biggr).\end{aligned}$$

In order to close the estimate, we now have to bound the term $\|\delta L\|_{\mathrm{H}^{s-2}}$ which appears in the right-hand side of (5.3). For doing so, we use the fact that δL satisfies

$$\partial_t\delta L + \mathbf{u}_2^*\cdot\nabla\delta L + \delta\mathbf{u}^*\cdot\nabla L_1 + \delta a\operatorname{div}\mathbf{u}_2 + \operatorname{div}(a_\sharp(L_1)\delta\mathbf{u}) - D(a_\sharp(L_1))\cdot\delta\mathbf{u} = 0,$$

whence

$$\partial_t\Lambda^{s-2}\delta L+\mathbf{u}_2^*\cdot\nabla\Lambda^{s-2}\delta L+\Lambda^{s-2}\Bigl(\delta\mathbf{u}^*\cdot\nabla L_1+\delta a\operatorname{div}\mathbf{u}_2+\operatorname{div}(a_\sharp(L_1)\delta\mathbf{u})-D(a_\sharp(L_1))\cdot\delta\mathbf{u}\Bigr)=[u_2^j,\Lambda^{s-2}]\partial_j\delta L.$$

Taking the L^2 inner product of the above equation with $\Lambda^{s-2}\delta L$, performing several integration by parts and using Lemma A.2 (which is allowed since $s-2 > -N/2$ and $s-2 \neq N/2+1$), we get

$$\begin{aligned}(5.4)\qquad \tfrac{1}{2}\tfrac{d}{dt}\|\delta L\|_{\mathrm{H}^{s-2}}^2 - C\|D\mathbf{u}_2\|_{\mathrm{H}^{s-1}}\|\delta L\|_{\mathrm{H}^{s-2}}^2 \leq \|\delta\mathbf{w}\|_{\mathrm{H}^{s-2}}\|a_\sharp(L_1)\delta\mathbf{u}\|_{\mathrm{H}^{s-2}}\\ +\|\delta L\|_{\mathrm{H}^{s-2}}\Bigl(\|\delta\mathbf{u}^*\cdot\nabla L_1\|_{\mathrm{H}^{s-2}} + \|\delta a\operatorname{div}\mathbf{u}_2\|_{\mathrm{H}^{s-2}} + \|D(a_\sharp(L_1))\cdot\delta\mathbf{u}\|_{\mathrm{H}^{s-2}}\Bigr).\end{aligned}$$

The right-hand side of the above inequality may be bounded by mean of lemmas B.2, B.3 and Corollary B.3. We get

$$\begin{aligned}\|a_\sharp(L_1)\delta\mathbf{u}\|_{\mathrm{H}^{s-2}} &\lesssim (1 + \|D\mathbf{w}_1\|_{\mathrm{H}^{s-1}})\|\delta\mathbf{u}\|_{\mathrm{H}^{s-2}},\\ \|\delta\mathbf{u}^*\cdot\nabla L_1\|_{\mathrm{H}^{s-2}} &\lesssim \bigl(\|\mathbf{w}_1\|_{\mathrm{L}^\infty} + \|D\mathbf{w}_1\|_{\mathrm{H}^{s-1}}\bigr)\|\delta\mathbf{u}\|_{\mathrm{H}^{s-2}},\\ \|\delta a\operatorname{div}\mathbf{u}_2\|_{\mathrm{H}^{s-2}} &\lesssim (1 + \|D\mathbf{w}_1\|_{\mathrm{H}^{s-1}} + \|D\mathbf{w}_2\|_{\mathrm{H}^{s-1}})\|\operatorname{div}\mathbf{u}_2\|_{\mathrm{H}^{s-1}}\|\delta L\|_{\mathrm{H}^{s-2}},\\ \|D(a_\sharp(L_1))\cdot\delta\mathbf{u}\|_{\mathrm{H}^{s-2}} &\lesssim \bigl(1 + \|D\mathbf{w}_1\|_{\mathrm{H}^{s-1}}\bigr)\|\delta\mathbf{u}\|_{\mathrm{H}^{s-2}}.\end{aligned}$$

Plugging these inequalities in (5.4), we get

$$\frac{1}{2}\frac{d}{dt}\|\delta L\|^2_{\mathrm{H}^{s-2}} \lesssim (1+\|D\mathbf{w}_1\|_{\mathrm{H}^{s-1}})\|\delta\mathbf{u}\|_{\mathrm{H}^{s-2}}\big(\|\delta\mathbf{w}\|_{\mathrm{H}^{s-2}}+\|\delta L\|_{\mathrm{H}^{s-2}}\big)$$
$$+(1+\|D\mathbf{w}_1\|_{\mathrm{H}^{s-1}}+\|D\mathbf{w}_2\|_{\mathrm{H}^{s-1}})\|D\mathbf{u}_2\|_{\mathrm{H}^{s-1}}\|\delta L\|^2_{\mathrm{H}^{s-2}},$$

hence, according to Gronwall's lemma,

$$\|\delta L(t)\|^2_{\mathrm{H}^{s-2}} \le \mathrm{e}^{C\int_0^t(1+\|D\mathbf{w}_1\|_{\mathrm{H}^{s-1}}+\|D\mathbf{w}_2\|_{\mathrm{H}^{s-1}})\|D\mathbf{u}_2\|_{\mathrm{H}^{s-1}}\,d\tau}\bigg(\|\delta L(0)\|^2_{\mathrm{H}^{s-2}}$$
$$+C\int_0^t\big(1+\|D\mathbf{w}_1\|_{\mathrm{H}^{s-1}}\big)\|\delta\mathbf{z}\|^2_{\mathrm{H}^{s-2}}\,d\tau\bigg).$$

Inserting (5.3), we get after a few computations

$$\|\delta L(t)\|^2_{\mathrm{H}^{s-2}} \le C\gamma^2(t)\mathrm{e}^{C\int_0^t A\,d\tau}\bigg[\|\delta L(0)\|^2_{\mathrm{H}^{s-2}}$$
$$+\bigg(\int_0^t A\mathrm{e}^{C\int_0^\tau A\,d\tau'}\,d\tau\bigg)\bigg(\|\delta\mathbf{z}(0)\|^2_{\mathrm{H}^{s-2}}+\int_0^t A\|\delta L\|_{\mathrm{H}^{s-2}}\,d\tau\bigg)^2\bigg]$$

with $A(t) := (1+\|D\mathbf{z}_1(t)\|_{\mathrm{H}^{s-1}}+(1+\|D\mathbf{w}_1(t)\|_{\mathrm{H}^{s-1}}+\|D\mathbf{w}_2(t)\|_{\mathrm{H}^{s-1}})\|D\mathbf{z}_2(t)\|_{\mathrm{H}^{s-1}}$.

Taking the square root and applying Gronwall's inequality, we easily conclude that

$$\|\delta L(t)\|_{\mathrm{H}^{s-2}} \le C\gamma(t)\mathrm{e}^{C\int_0^t A(\tau)\,d\tau}\big(\|\delta L(0)\|_{\mathrm{H}^{s-2}}+\|\delta\mathbf{z}(0)\|_{\mathrm{H}^{s-2}}\big). \tag{5.5}$$

Finally, plugging (5.5) in (5.3) yields the desired inequality. □

Step 2 : study of an approximate problem with smooth data

Remark that by virtue of Lemma (C.1) and Sobolev embeddings, we have for some constant C depending only on the choice of χ,

$$\|L_0-L_{0,\varepsilon}\|_{\mathrm{L}^\infty} \le C\varepsilon^{\beta(s+1-\frac{N}{2})}\|DL_0\|_{\mathrm{H}^s},\quad \|DL_{0,\varepsilon}\|_{\mathrm{H}^s}\le C\|DL_0\|_{\mathrm{H}^s}\quad\text{and}\quad \|\mathbf{u}_{0,\varepsilon}\|_{\mathrm{H}^s}\le C\|\mathbf{u}_0\|_{\mathrm{H}^s}.$$

Hence there exists an $\varepsilon_0>0$ such that for $\varepsilon\in(0,\varepsilon_0)$, the function $L_{0,\varepsilon}$ lies in a fixed compact subset $\mathbb{K}$ of $\mathbb{J}$. One can now apply Proposition 4.3 and Corollary 4.3 to get a positive time T (which may be bounded by below in terms of ε_0 and of the norm of the data) such that for all $\varepsilon\in(0,\varepsilon_0)$, system (EK$_\varepsilon$) has a unique H^∞ solution $(L_\varepsilon,\mathbf{u}_\varepsilon)$ on $[0,T]\times\mathbb{R}^N$ with besides

$$\|\mathbf{z}_\varepsilon\|_{L^\infty_T(\mathrm{H}^s)}+\varepsilon^{\frac12}\|\mathbf{z}_\varepsilon\|_{L^2_T(\mathrm{H}^{s+2})}+\|L_\varepsilon-L_{0,\varepsilon}\|_{L^\infty_T(\mathrm{H}^{s+1})}\le K \tag{5.6}$$

for some constant K independent of $\varepsilon\in(0,\varepsilon_0)$. As usual, it is understood that $\mathbf{z}_\varepsilon := \mathbf{u}_\varepsilon+i\nabla L_\varepsilon$.

Step 3 : the Cauchy criterion

From now on, we denote by C_K a generic constant depending only on K and on T.

We claim that for a convenient choice of β, the family $(L_\varepsilon-L_{0,\varepsilon},\mathbf{z}_\varepsilon)\in\mathcal{C}([0,T];\mathrm{H}^{s+1}\times\mathrm{H}^s)$ satisfies the Cauchy criterion in 0^+.

- H^{s-2} **estimate.**

  As a first step, we state the convergence of $(L_\varepsilon-L_{0,\varepsilon},\mathbf{z}_\varepsilon)$ in $\mathcal{C}([0,T];\mathrm{H}^{s-2})$. Let us denote $\delta L^\nu_\varepsilon := L_\varepsilon-L_\nu$ and $\delta\mathbf{z}^\nu_\varepsilon := \mathbf{z}_\varepsilon-\mathbf{z}_\nu$. Assuming that $\beta\in(0,1/4)$, we claim that

$$\|\delta L^\nu_\varepsilon\|_{\mathrm{L}^\infty([0,T];\mathrm{H}^{s-2})}+\|\delta\mathbf{z}^\nu_\varepsilon\|_{\mathrm{L}^\infty([0,T];\mathrm{H}^{s-2})}=o(\varepsilon^{2\beta}) \tag{5.7}$$

uniformly with respect to $0<\nu\leq\varepsilon<\varepsilon_0$

Let us first focus on the case $s\neq 3+N/2$. With obvious notation, we have

$$\partial_t\delta L^\nu_\varepsilon+\mathbf{u}^*_\nu\cdot\nabla\delta L^\nu_\varepsilon+\nu\Delta^2\delta L^\nu_\varepsilon=(\nu-\varepsilon)\Delta^2L_\varepsilon-(\delta\mathbf{u}^\nu_\varepsilon)^*\cdot\nabla L_\varepsilon-\delta a^\nu_\varepsilon\,\mathrm{div}\,\mathbf{u}_\varepsilon-\mathrm{div}\,(a_\nu\delta\mathbf{u}^\nu_\varepsilon)+Da_\nu\cdot\delta\mathbf{u}^\nu_\varepsilon.$$

Hence, arguing as for proving (5.4), we get

$$\frac12\frac{d}{dt}\|\delta L^\nu_\varepsilon\|^2_{\mathrm{H}^{s-2}}-C\|D\mathbf{u}_\nu\|_{\mathrm{H}^{s-1}}\|\delta L^\nu_\varepsilon\|^2_{\mathrm{H}^{s-2}}+\nu\|\Delta\delta L^\nu_\varepsilon\|^2_{\mathrm{H}^{s-2}}\leq\varepsilon\|\Delta\mathbf{w}_\varepsilon\|_{\mathrm{H}^{s-2}}\|\delta\mathbf{w}^\nu_\varepsilon\|_{\mathrm{H}^{s-2}}$$
$$+\|\delta L^\nu_\varepsilon\|_{\mathrm{H}^{s-2}}\big(\|(\delta\mathbf{u}^\nu_\varepsilon)^*\cdot\nabla L_\varepsilon\|_{\mathrm{H}^{s-2}}+\|\delta a^\nu_\varepsilon\,\mathrm{div}\,\mathbf{u}_\varepsilon\|_{\mathrm{H}^{s-2}}+\|Da_\nu\cdot\delta\mathbf{u}^\nu_\varepsilon\|_{\mathrm{H}^{s-2}}\big)+\|\delta\mathbf{w}^\nu_\varepsilon\|_{\mathrm{H}^{s-2}}\|a_\nu\delta\mathbf{u}^\nu_\varepsilon\|_{\mathrm{H}^{s-2}}.$$

Note that we performed an integration by parts to deal with the term $(\nu-\varepsilon)\Delta^2L_\varepsilon$.

Now, bounding the nonlinear terms as in the proof of Proposition 5.1, we end up with

$$\frac{d}{dt}\|\delta L^\nu_\varepsilon\|^2_{\mathrm{H}^{s-2}}\lesssim\big(1+\|D\mathbf{w}_\nu\|_{\mathrm{H}^{s-1}}+\|D\mathbf{w}_\varepsilon\|_{\mathrm{H}^{s-1}}\big)\big(1+\|D\mathbf{u}_\varepsilon\|_{\mathrm{H}^{s-1}}\big)\big(\|\delta L^\nu_\varepsilon\|^2_{\mathrm{H}^{s-2}}+\|\delta\mathbf{z}^\nu_\varepsilon\|^2_{\mathrm{H}^{s-2}}\big)$$
$$+\varepsilon\|\Delta\mathbf{w}_\varepsilon\|_{\mathrm{H}^{s-2}}\|\delta\mathbf{w}^\nu_\varepsilon\|_{\mathrm{H}^{s-2}}$$

whence, applying Gronwall's lemma, Young's inequality and taking advantage of the uniform bounds given by (5.6),

$$\|\delta L^\nu_\varepsilon(t)\|^2_{\mathrm{H}^{s-2}}\leq C_K\bigg(\|\delta L^\nu_\varepsilon(0)\|^2_{\mathrm{H}^{s-2}}+\int_0^t\Big(\|\delta\mathbf{z}^\nu_\varepsilon\|^2_{\mathrm{H}^{s-2}}+\varepsilon^2\|\Delta\mathbf{w}_\varepsilon\|^2_{\mathrm{H}^{s-2}}\,d\tau\Big)\bigg).\tag{5.8}$$

Let us now state an inequality for $\|\delta\mathbf{z}^\nu_\varepsilon\|_{\mathrm{H}^{s-2}}$. Since $\delta\mathbf{z}^\nu_\varepsilon$ solves

$$\partial_t\delta\mathbf{z}^\nu_\varepsilon+(\mathbf{u}^*_\nu\cdot\nabla)\delta\mathbf{z}^\nu_\varepsilon+\nu\Delta^2\delta\mathbf{z}^\nu_\varepsilon+i\nabla\delta\mathbf{z}^\nu_\varepsilon\cdot\mathbf{w}_\nu+i\nabla(a_\nu\,\mathrm{div}\,\delta\mathbf{z}^\nu_\varepsilon)$$
$$=\nabla\big(Q_\sharp(L_\varepsilon)-Q_\sharp(L_\nu)\big)-((\delta\mathbf{u}^\nu_\varepsilon)^*\cdot\nabla)\mathbf{z}_\varepsilon-i\nabla\mathbf{z}_\varepsilon\cdot\delta\mathbf{w}^\nu_\varepsilon+(\nu-\varepsilon)\Delta^2\mathbf{z}_\varepsilon+i\nabla(\delta a^\nu_\varepsilon\mathrm{div}\,\mathbf{z}_\varepsilon),$$

Corollary 4.2 insures (if $s>2$ and $s\neq 3+N/2$) that

$$\|\delta\mathbf{z}^\nu_\varepsilon\|_{L^\infty_t(\mathrm{H}^{s-2})}\leq C\gamma_\nu(t)\mathrm{e}^{C\int_0^t(\|D\mathbf{z}_\nu\|_{\mathrm{H}^{s-1}}+\nu\|D\mathbf{w}_\nu\|^2_{\mathrm{L}^\infty})\,d\tau}\bigg(\|\delta\mathbf{z}^\nu_\varepsilon(0)\|_{\mathrm{H}^{s-2}}+\int_0^t\Big(\varepsilon\|\Delta^2\mathbf{z}_\varepsilon\|_{\mathrm{H}^{s-2}}$$
$$+\|Q_\sharp(L_\varepsilon)-Q_\sharp(L_\nu)\|_{\mathrm{H}^{s-1}}+\|((\delta\mathbf{u}^\nu_\varepsilon)^*\cdot\nabla)\mathbf{z}_\varepsilon\|_{\mathrm{H}^{s-2}}+\|\nabla\mathbf{z}_\varepsilon\cdot\delta\mathbf{w}^\nu_\varepsilon\|_{\mathrm{H}^{s-2}}+\|\delta a^\nu_\varepsilon\mathrm{div}\,\mathbf{z}_\varepsilon\|_{\mathrm{H}^{s-1}}\Big)\,dt$$

with $\gamma_\nu(t):=1+\|\mathbf{w}_\nu\|^{\max(1,s-2)}_{\mathrm{L}^\infty_t(\mathrm{L}^\infty)}$.

Note that if $s\leq 2$, the above inequality holds true with $\gamma_\nu\equiv 1$ since Remark 4.1 applies.

All the nonlinear terms in the right-hand side may be bounded as in the proof of Proposition 5.1. Using the uniform bounds of the previous step, we get

$$\|\delta\mathbf{z}^\nu_\varepsilon(t)\|_{\mathrm{H}^{s-2}}\leq C_K\bigg(\|\delta\mathbf{z}^\nu_\varepsilon(0)\|_{\mathrm{H}^{s-2}}+\int_0^t\big(\|\delta L^\nu_\varepsilon\|_{\mathrm{H}^{s-2}}+\|\delta\mathbf{z}^\nu_\varepsilon\|_{\mathrm{H}^{s-2}}\big)\,d\tau+\varepsilon\int_0^t\|\Delta^2\mathbf{z}_\varepsilon\|_{\mathrm{H}^{s-2}}\,d\tau\bigg).$$

Combining with the inequality in (5.8) and using again Gronwall's lemma, we discover that

$$\|\delta L^\nu_\varepsilon(t)\|^2_{\mathrm{H}^{s-2}}+\|\delta\mathbf{z}^\nu_\varepsilon(t)\|^2_{\mathrm{H}^{s-2}}\leq C_K\bigg(\|\delta L^\nu_\varepsilon(0)\|^2_{\mathrm{H}^{s-2}}+\|\delta\mathbf{z}^\nu_\varepsilon(0)\|^2_{\mathrm{H}^{s-2}}$$
$$+\varepsilon^2\int_0^t\big(\|\Delta\mathbf{w}_\varepsilon\|^2_{\mathrm{H}^{s-2}}+\|\Delta^2\mathbf{z}_\varepsilon\|^2_{\mathrm{H}^{s-2}}\big)\,d\tau\bigg),$$

which, according to (5.6), leads to

$$\forall t\in[0,T],\ \|\delta L^\nu_\varepsilon(t)\|_{\mathrm{H}^{s-2}}+\|\delta\mathbf{z}^\nu_\varepsilon(t)\|_{\mathrm{H}^{s-2}}\le C_K\Big(\|\delta L^\nu_\varepsilon(0)\|_{\mathrm{H}^{s-2}}+\|\delta\mathbf{z}^\nu_\varepsilon(0)\|_{\mathrm{H}^{s-2}}+\sqrt{\varepsilon}\,\Big).$$

Now, Lemma C.1 enables us to complete the proof of (5.7) in the case $s\neq 3+N/2$.

If $s=3+N/2$, one can use the inequality (5.6) with $s-\eta$ for some $\eta\in(0,1)$ such that $(2+\eta)\beta<1/2$. Going along the lines of the previous computations, one gets

$$\|\delta L^\nu_\varepsilon(t)\|_{\mathrm{L}^\infty_T(\mathrm{H}^{s-2-\eta})}+\|\delta\mathbf{z}^\nu_\varepsilon(t)\|_{\mathrm{L}^\infty_T(\mathrm{H}^{s-2-\eta})}\le C_K\Big(\|\delta L^\nu_\varepsilon(0)\|_{\mathrm{H}^{s-2-\eta}}+\|\delta\mathbf{z}^\nu_\varepsilon(0)\|_{\mathrm{H}^{s-2-\eta}}+\sqrt{\varepsilon}\,\Big),$$

hence, according to Lemma C.1,

$$\|\delta L^\nu_\varepsilon\|_{\mathrm{L}^\infty([0,T];\mathrm{H}^{s-2-\eta})}+\|\delta\mathbf{z}^\nu_\varepsilon\|_{\mathrm{L}^\infty([0,T];\mathrm{H}^{s-2-\eta})}=o(\varepsilon^{(2+\eta)\beta}).$$

Interpolating with the uniform bounds in $\mathrm{L}^\infty([0,T];\mathrm{H}^s)$ supplied by (5.6), we conclude that (5.7) is still satisfied.

- H^s **estimates.**

Corollary 4.2 assures that

$$\begin{aligned}(5.9)\qquad \|\delta\mathbf{z}^\nu_\varepsilon(t)\|_{\mathrm{H}^s}\le C\big(1+\|\mathbf{w}_\nu\|^s_{\mathrm{L}^\infty_t(\mathrm{L}^\infty)}\big)\mathrm{e}^{C\int_0^t(\|D\mathbf{z}_\nu\|_{\mathrm{H}^{s-1}}+\varepsilon\|D\mathbf{w}_\nu\|^2_{\mathrm{L}^\infty})\,d\tau}\Big(\|\delta\mathbf{z}^\nu_\varepsilon(0)\|_{\mathrm{H}^s}\\ +\int_0^t\Big(\|Q_\sharp(L_\varepsilon)-Q_\sharp(L_\nu)\|_{\mathrm{H}^{s+1}}+\|((\delta\mathbf{u}^\nu_\varepsilon)^*\cdot\nabla)\mathbf{z}_\varepsilon\|_{\mathrm{H}^s}\\ +\|\nabla\mathbf{z}_\varepsilon\cdot\delta\mathbf{w}^\nu_\varepsilon\|_{\mathrm{H}^s}+\varepsilon\|\Delta^2\mathbf{z}_\varepsilon\|_{\mathrm{H}^s}+\|\delta a^\nu_\varepsilon\mathrm{div}\,\mathbf{z}_\varepsilon\|_{\mathrm{H}^{s+1}}\Big)\,d\tau\Big).\end{aligned}$$

According to (5.6), the exponential term may be uniformly bounded on $[0,T]$ for $0<\nu\le\varepsilon\le\varepsilon_0$. Next, the results of section B of the appendix yield

$$(5.10)\qquad \|Q_\sharp(L_\varepsilon)-Q_\sharp(L_\nu)\|_{\mathrm{H}^{s+1}}\le C\big(1+\|D\mathbf{w}_\varepsilon\|_{\mathrm{H}^{s-1}}+\|D\mathbf{w}_\nu\|_{\mathrm{H}^{s-1}}\big)\|\delta L^\nu_\varepsilon\|_{\mathrm{H}^{s+1}}.$$

Lemma B.2 and the embedding $\mathrm{H}^{s-2}\hookrightarrow\mathrm{C}^{-1}$ also ensure that

$$\|(\delta\mathbf{u}^\nu_\varepsilon)^*\cdot\nabla\mathbf{z}_\varepsilon\|_{\mathrm{H}^s}\lesssim\|D\mathbf{z}_\varepsilon\|_{\mathrm{L}^\infty}\,\|\delta\mathbf{u}^\nu_\varepsilon\|_{\mathrm{H}^s}+\|\delta\mathbf{u}^\nu_\varepsilon\|_{\mathrm{H}^{s-2}}\|D^2\mathbf{z}_\varepsilon\|_{\mathrm{H}^s},$$

hence, in view of (5.7) and of the uniform bounds of the previous step,

$$(5.11)\qquad \|((\delta\mathbf{u}^\nu_\varepsilon)^*\cdot\nabla)\mathbf{z}_\varepsilon\|_{\mathrm{H}^s}\le K\|\delta\mathbf{u}^\nu_\varepsilon\|_{\mathrm{H}^s}+o(\varepsilon^{2\beta})\|D^2\mathbf{z}_\varepsilon\|_{\mathrm{H}^s}$$

A similar argument leads to

$$(5.12)\qquad \|\nabla\mathbf{z}_\varepsilon\cdot\delta\mathbf{w}^\nu_\varepsilon\|_{\mathrm{H}^s}\le K\|\delta\mathbf{w}^\nu_\varepsilon\|_{\mathrm{H}^s}+o(\varepsilon^{2\beta})\|D^2\mathbf{z}_\varepsilon\|_{\mathrm{H}^s}.$$

For bounding the last term in (5.9), we use Lemma B.1 and Corollary B.3. We get

$$\begin{aligned}\|\delta a^\nu_\varepsilon\mathrm{div}\,\mathbf{z}_\varepsilon\|_{\mathrm{H}^{s+1}}&\lesssim\|\mathrm{div}\,\mathbf{z}_\varepsilon\|_{\mathrm{L}^\infty}\,\|\delta a^\nu_\varepsilon\|_{\mathrm{H}^{s+1}}+\|\delta a^\nu_\varepsilon\|_{\mathrm{L}^\infty}\,\|D^2\mathbf{z}_\varepsilon\|_{\mathrm{H}^s},\\ &\lesssim(1+\|D\mathbf{w}_\varepsilon\|_{\mathrm{H}^{s-1}}+\|D\mathbf{w}_\nu\|_{\mathrm{H}^{s-1}})\,\|\mathrm{div}\,\mathbf{z}_\varepsilon\|_{\mathrm{L}^\infty}\,\|\delta L^\nu_\varepsilon\|_{\mathrm{H}^{s+1}}+\|\delta L^\nu_\varepsilon\|_{\mathrm{L}^\infty}\,\|D^2\mathbf{z}_\varepsilon\|_{\mathrm{H}^s}\end{aligned}$$

whence, taking advantage of (5.6) and (5.7),

$$(5.13)\qquad \|\delta a^\nu_\varepsilon\mathrm{div}\,\mathbf{z}_\varepsilon\|_{\mathrm{H}^{s+1}}\le K\big(\|\delta L^\nu_\varepsilon\|_{\mathrm{H}^{s-2}}+\|\delta\mathbf{w}^\nu_\varepsilon\|_{\mathrm{H}^s}\big)+o(\varepsilon^{2\beta})\|D^2\mathbf{z}_\varepsilon\|_{\mathrm{H}^s}.$$

Plugging inequalities (5.10), (5.11), (5.12) and (5.13) in (5.9), and applying Gronwall's inequality, we conclude that for all $t\in[0,T]$,

$$(5.14)\quad \|\delta\mathbf{z}^\nu_\varepsilon(t)\|_{\mathrm{H}^s}\le C_K\Big(\|\delta\mathbf{z}^\nu_\varepsilon(0)\|_{\mathrm{H}^s}+\varepsilon\int_0^t\|\Delta^2\mathbf{z}_\varepsilon\|_{\mathrm{H}^s}\,d\tau\Big)+o(\varepsilon^{2\beta})\int_0^t\big(1+\|D^2\mathbf{z}_\varepsilon\|_{\mathrm{H}^s}\big)\,d\tau.$$

In order to conclude that the Cauchy criterion is indeed satisfied in H^s, we now have to bound $\varepsilon\|\Delta^2\mathbf{z}_\varepsilon\|_{L^1_T(\mathrm{H}^s)}$ and $\|D^2\mathbf{z}_\varepsilon\|_{L^1_T(\mathrm{H}^s)}$.

- **Bounds for $\mathbf{z}_\varepsilon$ in $\mathrm{L}^\infty(0,T;\mathrm{H}^{s+2})\cap\mathrm{L}^2(0,T;\mathrm{H}^{s+4})$.**

  Since $\mathbf{z}_\varepsilon\in\mathcal{C}([0,T];\mathrm{H}^\infty)$ satisfies

$$\partial_t\mathbf{z}_\varepsilon+(\mathbf{u}_\varepsilon^*\cdot\nabla)\mathbf{z}_\varepsilon+i\nabla\mathbf{z}_\varepsilon\cdot\mathbf{w}_\varepsilon+i\nabla(a_\sharp(L_\varepsilon)\mathrm{div}\,\mathbf{z}_\varepsilon)+\varepsilon\Delta^2\mathbf{z}_\varepsilon=q_\sharp(L_\varepsilon)\mathbf{w}_\varepsilon,$$

  we may apply Prop. 4.2 and Corollary 4.2 to bound $\mathbf{z}_\varepsilon$ in $\mathrm{L}^\infty(0,T;\mathrm{H}^{s+2})\cap\mathrm{L}^2(0,T;\mathrm{H}^{s+4})$. We get

$$\|\mathbf{z}_\varepsilon\|_{L^\infty_T(\mathrm{H}^{s+2})}+\varepsilon^{\frac12}\|D^2\mathbf{z}_\varepsilon\|_{L^2_T(\mathrm{H}^{s+2})}\leq C\big(1+\|\mathbf{w}_\varepsilon\|^{s+2}_{\mathrm{L}^\infty_t(\mathrm{L}^\infty)}\big)\|\mathbf{z}_{0,\varepsilon}\|_{\mathrm{H}^{s+2}}\mathrm{e}^{C\int_0^T\left(1+\|D\mathbf{z}_\varepsilon\|_{\mathrm{L}^\infty}+\varepsilon\|D\mathbf{w}_\varepsilon\|^2_{\mathrm{L}^\infty}\right)dt}.$$

  By virtue of Lemma C.1, we eventually conclude that

$$\|\mathbf{z}_\varepsilon\|_{L^\infty_T(\mathrm{H}^{s+2})}+\varepsilon^{\frac12}\|D^2\mathbf{z}_\varepsilon\|_{L^2_T(\mathrm{H}^{s+2})}\leq C_K\varepsilon^{-2\beta}. \tag{5.15}$$

- **Conclusion.**

  Plugging (5.15) in inequality (5.14) yields (remind that $\beta<1/4$)

$$\forall t\in[0,T],\ \|\delta\mathbf{z}_\varepsilon^\nu(t)\|_{\mathrm{H}^s}\leq C_K\|\delta\mathbf{z}_\varepsilon^\nu(0)\|_{\mathrm{H}^s}+o(1).$$

  Next, applying Lemma C.1 *(iii)* and combining with (5.7), one concludes that

$$\|\delta L_\varepsilon^\nu\|_{L^\infty_T(\mathrm{H}^{s+1})}+\|\delta\mathbf{u}_\varepsilon^\nu\|_{L^\infty_T(\mathrm{H}^s)}=o(1)$$

  uniformly with respect to $0<\nu\leq\varepsilon$.

  This insures that $(L_\varepsilon,\mathbf{u}_\varepsilon)$ satisfies the desired Cauchy criterion in 0^+.

Step 4 : Existence of a solution

Let $(L,\mathbf{z})\in\big(L_0+\mathcal{C}([0,T];\mathrm{H}^{s+1})\big)\times\mathcal{C}([0,T];\mathrm{H}^s)$ be the limit of $(L_\varepsilon,\mathbf{z}_\varepsilon)$ when ε goes to 0^+. Since $(L_\varepsilon-L_{0,\varepsilon},\mathbf{z}_\varepsilon)$ tends to $(L-L_0,\mathbf{z})$ in $\mathrm{L}^\infty(0,T;\mathrm{H}^s)$, proving that $(L,\mathbf{z})$ satisfies (ES) with data $(L_0,\mathbf{z}_0)$ is straightforward. The details are left to the reader.

Denoting $\rho:=\mathcal{L}^{-1}(L)$, it is now obvious that $(\rho,\mathbf{u}:=\mathrm{Re}\,\mathbf{z})$ is indeed a solution to (1.2). By making use of the results of the appendix, one can also show that $\partial_t\rho\in\mathcal{C}([0,T];\mathrm{H}^{s-1})$ and $\partial_t\mathbf{u}\in\mathcal{C}([0,T];\mathrm{H}^{s-2})$, whence $(L,\mathbf{u})\in\mathcal{C}^1\Big([0,T];\big(L_0+\mathrm{H}^{s-1}\big)\times\mathrm{H}^{s-2}\Big)$.

Step 5 : continuity of the solution map

Fix data $(L_0,\mathbf{u}_0)$ such that $L_0(\mathbb{R}^N)\subset\subset\mathbb{J}$ and $(DL_0,\mathbf{u}_0)$ belongs to H^s. Let $(L_0^n,\mathbf{u}_0^n)_{n\in\mathbb{N}}$ be a sequence of functions such that $L_0^n-L_0$ tends to 0 in H^{s+1} and $\mathbf{u}_0^n$ converges to $\mathbf{u}_0$ in H^s. One can assume with no loss of generality that there exists a compact set $\mathbb{K}\subset\subset\mathbb{J}$ such that

$$\forall n\in\mathbb{N},\ L_0^n(\mathbb{R}^N)\subset\mathbb{K}.$$

Hence, the previous steps of the proof supply a H^s solution $(L^n,\mathbf{u}^n)$ to (EK) with data $(L_0^n,\mathbf{u}_0^n)$ on some time inteval $[0,T]$ independent of n with, besides, $(L^n,\mathbf{u}^n)$ uniformly bounded in $\big(L_0+\mathcal{C}([0,T];\mathrm{H}^{s+1})\big)\times\mathcal{C}([0,T];\mathrm{H}^s)$. Of course one can arrange that system (EK) with data $(L_0,\mathbf{u}_0)$ also has a H^s solution $(L,\mathbf{u})$ on the same interval $[0,T]$.

Now, Proposition 5.1 entails that[4]

$$\lim_{n\to+\infty}(L^n,\mathbf{u}^n)=(L,\mathbf{u})\quad\text{in}\quad\big(L_0+\mathcal{C}([0,T];\mathrm{H}^{s-1})\big)\times\mathcal{C}([0,T];\mathrm{H}^{s-2}).$$

[4]If $s=3+\frac{N}{2}$ use Proposition 5.1 for getting estimates in H^{s-3} then interpolate with the uniform bounds in H^s.

In order to prove the continuity of the solution map in the space $\big(L_0 + \mathcal{C}([0,T];\mathrm{H}^{s+1})\big) \times \mathcal{C}([0,T];\mathrm{H}^s)$, we shall follow the method introduced by J. Bona and R. Smith for KdV in [6].

Let χ_ε be the mollifier defined in section 5.1. Let us denote $L^n_{0,\varepsilon} := \chi_\varepsilon * L^n_0$ and $\mathbf{u}^n_{0,\varepsilon} := \chi_\varepsilon * \mathbf{u}^n_0$. Taking advantage of Lemma C.1 and arguing as above, one can find an ε_0 *independent of* n such that for all $\varepsilon < \varepsilon_0$ and $n \in \mathbb{N}$, system (EK$_\varepsilon$) with data $(L^n_{0,\varepsilon}, \mathbf{u}^n_{0,\varepsilon})$ (resp. $(L_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon})$) has a H^s solution $(L^n_\varepsilon, \mathbf{u}^n_\varepsilon)$ (resp. $(L_\varepsilon, \mathbf{u}_\varepsilon)$) on $[0,T]$ which belongs to $\big(L_0+\mathcal{C}([0,T];\mathrm{H}^{s+1})\big)\times\mathcal{C}([0,T];\mathrm{H}^s)$ uniformly with respect to ε and n.

Next, introducing the complex notation $\mathbf{z}^n$, $\mathbf{z}^n_\varepsilon$, $\mathbf{z}$ and $\mathbf{z}_\varepsilon$ as before, we have

$$\|\mathbf{z}^n - \mathbf{z}\|_{L^\infty_T(\mathrm{H}^s)} \leq \|\mathbf{z}^n - \mathbf{z}^n_\varepsilon\|_{L^\infty_T(\mathrm{H}^s)} + \|\mathbf{z}^n_\varepsilon - \mathbf{z}_\varepsilon\|_{L^\infty_T(\mathrm{H}^s)} + \|\mathbf{z}_\varepsilon - \mathbf{z}\|_{L^\infty_T(\mathrm{H}^s)}. \tag{5.16}$$

In step 2, it has been shown that the last term tends to 0 when ε goes to 0. Moreover, in light of Lemma C.1 part *iii)* with $\sigma = 0$, one can show by going along the lines of step 2 that $\|\mathbf{z}^n - \mathbf{z}^n_\varepsilon\|_{L^\infty_T(\mathrm{H}^s)}$ tends to 0 *uniformly with respect to* n when ε goes to 0.

Therefore, for any $\gamma > 0$, there exists a $\varepsilon > 0$ such that the first and last terms in the right-hand side of (5.16) are bounded by $\gamma/3$ for all $n \in \mathbb{N}$.

Besides, it has been stated in Proposition 4.4 that

$$\|\mathbf{u}^n_\varepsilon - \mathbf{u}_\varepsilon\|_{L^\infty_T(\mathrm{H}^s)} + \|L^n_\varepsilon - L_\varepsilon\|_{L^\infty_T(\mathrm{H}^{s+1})} \leq C\Big(\|\mathbf{u}^n_{0,\varepsilon} - \mathbf{u}_{0,\varepsilon}\|_{\mathrm{H}^s} + \|L^n_{0,\varepsilon} - L_{0,\varepsilon}\|_{\mathrm{H}^{s+1}}\Big)$$

for some C which may depend on ε but does not depend on n.

Since, according to Lemma C.1 we have

$$\|\mathbf{u}^n_{0,\varepsilon} - \mathbf{u}_{0,\varepsilon}\|_{\mathrm{H}^s} + \|L^n_{0,\varepsilon} - L_{0,\varepsilon}\|_{\mathrm{H}^{s+1}} \lesssim \|\mathbf{u}^n_0 - \mathbf{u}_0\|_{\mathrm{H}^s} + \|L^n_0 - L_0\|_{\mathrm{H}^{s+1}},$$

it is now clear that for large enough n we also have

$$\|\mathbf{z}^n_\varepsilon - \mathbf{z}^n_\varepsilon\|_{L^\infty_T(\mathrm{H}^s)} \leq \frac{\gamma}{3}.$$

This completes the proof of the continuity of the solution map and of Theorem 5.1. □

5.2 Lower bounds on the lifespan and blow-up results

Let us first give a lower bound for the existence time of a H^s solution.

Proposition 5.2 *Under the assumptions of Theorem* 5.1, *we have the following lower bound for the existence time:*

$$T \geq \frac{1}{C}\log\Big(1 + \frac{c}{\widetilde{Z}_0 + \widetilde{Z}_0^2}\Big) \quad \textit{with} \quad \widetilde{Z}_0 := \|\mathbf{z}_0\|_{\mathrm{H}^s} + \|\mathbf{z}_0\|^\beta_{\mathrm{H}^{s-2}}, \quad \mathbf{z}_0 := \mathbf{u}_0 + i\nabla L_0$$

for some constants c and C depending only on the usual parameters, and $\beta \geq 3$ depending only on s and N.

Proof. It is only a matter of letting ε_0 goes to 0 in (4.23) and (4.25). □

Remark 5.1 *Note that for small data, we thus have* $T \gtrsim \log \|\mathbf{z}_0\|^{-1}_{\mathrm{H}^s}$.

Let us now state a blow-up criterion.

Proposition 5.3 *Let $s > 1 + N/2$ and $(\rho, \mathbf{u})$ be a H^s solution to system* (1.2) *on* $[0, T) \times \mathbb{R}^N$. *Assume that the following three conditions are satisfied:*

$$\rho([0, T) \times \mathbb{R}^N) \subset\subset \mathbb{J}_\rho, \tag{5.17}$$

$$\textstyle\int_0^T \big(\|\Delta\rho(t)\|_{\mathrm{L}^\infty} + \|\mathrm{curl}\,\mathbf{u}(t)\|_{\mathrm{L}^\infty} + \|\mathrm{div}\,\mathbf{u}(t)\|_{\mathrm{L}^\infty}\big)\,dt < \infty, \tag{5.18}$$

$$\textstyle\sup_{t\in[0,T]} \|\rho(t)\|_{\mathrm{C}^\alpha} < \infty \quad \textit{for some} \quad \alpha \in (0, 1). \tag{5.19}$$

Then $(\rho, \mathbf{u})$ may be continued beyond T into a H^s solution of (1.2).

If $\mathrm{curl}\,\mathbf{u} = \mathbf{0}$ *then condition* (5.19) *is not needed.*

Proof. By using the change of function $L := \mathcal{L}(\rho)$ and $\mathbf{z} := \mathbf{u} + i\nabla L$, it is easy to see that the above three conditions are equivalent to

$$L([0, T)\times\mathbb{R}^N) \subset\subset \mathbb{J}, \quad \int_0^T \big(\|\mathrm{curl}\,\mathbf{z}\|_{\mathrm{L}^\infty} + \|\mathrm{div}\,\mathbf{z}\|_{\mathrm{L}^\infty}\big)\,dt < \infty \ \text{ and } \ \sup_{t\in[0,T]} \|L(t)\|_{\mathrm{C}^\alpha} < \infty. \tag{5.20}$$

Combining the inequality in (4.31) with $\varepsilon = 0$ and proposition, and taking advantage of (5.19), we obtain the following inequality

$$\forall t \in [0, T],\ \|\mathbf{z}(t)\|^2_{\mathrm{H}^s} \le C\bigg(\|\mathbf{z}_0\|^2_{\mathrm{H}^s} + \int_0^t \big(1 + \|D\mathbf{z}\|_{\mathrm{L}^\infty}\big)\|\mathbf{z}\|^2_{\mathrm{H}^s}\,d\tau\bigg). \tag{5.21}$$

Of course, in the general case, the constant C depends on the data and on $\|L\|_{L^\infty_T(\mathrm{C}^\alpha)}$. If $\mathrm{curl}\,\mathbf{u} = \mathbf{0}$ however, the inequality (3.5) leads to (5.21) even if (5.19) is not assumed.

Now, a standard Gronwall argument would enable us to bound $\mathbf{z}$ in $\mathrm{L}^\infty(0, T; \mathrm{H}^s)$ is $D\mathbf{z}$ were assumed to be in $\mathrm{L}^1(0, T; L^\infty)$. It turns out that this may be somewhat relaxed by appealing to the following logarithmic interpolation inequality (see e.g inequality (2.2) in [12]):

$$\|D\mathbf{z}\|_{\mathrm{L}^\infty} \le C\big(1 + \|D\mathbf{z}\|_{\dot{\mathrm{B}}^0_{\infty,\infty}} \log(e + \|D\mathbf{z}\|_{\mathrm{H}^{s-1}})\big)$$

where $\dot{\mathrm{B}}^0_{\infty,\infty}$ is a *homogeneous Besov space* of regularity index 0 (for the precise definition, see for instance [17]), in which L^∞ is embedded.

Plugging this inequality in (5.21) and applying Gronwall's lemma, we get for some constant C_T which may depend on T,

$$\forall t \in [0, T),\ \log\big(e + \|\mathbf{z}(t)\|^2_{\mathrm{H}^s}\big) \le C_T \log\big(e + \|\mathbf{z}_0\|^2_{\mathrm{H}^s}\big) + \int_0^t \|D\mathbf{z}(\tau)\|_{\dot{\mathrm{B}}^0_{\infty,\infty}} \log\big(e + \|\mathbf{z}(\tau)\|^2_{\mathrm{H}^s}\big)\,d\tau,$$

which, after a second use of Gronwall's lemma supplies a bound for $\mathbf{z}$ in $\mathrm{L}^\infty(0, T; \mathrm{H}^s)$ provided $D\mathbf{z}$ belongs to $\mathrm{L}^1(0, T; \dot{\mathrm{B}}^0_{\infty,\infty})$ – this refinement really gains something since $\mathrm{L}^\infty \hookrightarrow \dot{\mathrm{B}}^0_{\infty,\infty}$ strictly.

Finally, we notice that

$$D\mathbf{z} = (-\Delta)^{-1} D\,\mathrm{div}\,\big(\mathrm{curl}\,\mathbf{z}\big) - (-\Delta)^{-1} D\nabla\big(\mathrm{div}\,\mathbf{z}\big).$$

Hence $D\mathbf{z}$ may be computed from $\mathrm{curl}\,\mathbf{z}$ and $\mathrm{div}\,\mathbf{z}$ through homogeneous operators of degree 0. Since such operators are continuous in $\dot{\mathrm{B}}^0_{\infty,\infty}$, we have

$$\begin{aligned} \|D\mathbf{z}\|_{\dot{\mathrm{B}}^0_{\infty,\infty}} &\lesssim \|\mathrm{curl}\,\mathbf{z}\|_{\dot{\mathrm{B}}^0_{\infty,\infty}} + \|\mathrm{div}\,\mathbf{z}\|_{\dot{\mathrm{B}}^0_{\infty,\infty}}, \\ &\lesssim \|\Delta L\|_{\mathrm{L}^\infty} + \|\mathrm{curl}\,\mathbf{u}\|_{\mathrm{L}^\infty} + \|\mathrm{div}\,\mathbf{u}\|_{\mathrm{L}^\infty}. \end{aligned}$$

Hence (5.18) implies that $D\mathbf{z}$ belongs to $\mathrm{L}^1(0, T; \dot{\mathrm{B}}^0_{\infty,\infty})$. □

Remark 5.2 *Up to condition* (5.19) *(which is of lower order and is not necessary if* $\mathbf{u}$ *is potential), the blow-up criterion stated in Proposition 5.3 is comparable to the celebrated Beale-Kato-Majda's criterion for incompressible Euler equations (see [2]). This is not a scoop that this kind of criterion may be generalized for most reasonable quasilinear hyperbolic PDE's. Note however that* (EK) *is not hyperbolic.*

Remark 5.3 *According to the proof of Proposition 5.3, condition* (5.18) *may be replaced by a weaker one, namely*

$$\int_0^T \big(\|\Delta L\|_{\dot{\mathrm{B}}^0_{\infty,\infty}} + \|\operatorname{curl}\mathbf{u}\|_{\dot{\mathrm{B}}^0_{\infty,\infty}} + \|\operatorname{div}\mathbf{u}\|_{\dot{\mathrm{B}}^0_{\infty,\infty}}\big)\, dt < \infty.$$

Starting from Proposition 5.3, it is easy to conclude to the following

Corollary 5.1 *Under the assumptions of Theorem 5.1, the lifespan of a* H^s *solution to* (1.2) *with data* $(\rho_0, \mathbf{u}_0)$ *is the same as the lifespan of a* H^{s_1} *solution with* $s_1 \leq s$*. In particular* H^∞ *data yield* H^∞ *solutions.*

6 Perturbation of capillary profiles

In the previous section, a local existence result has been stated for initial data $(\rho_0, \mathbf{u}_0)$ such that $D\rho_0$ and $\mathbf{u}_0$ belong to H^s. This statement is not completely satisfactory as it does not supply the existence of a solution for data which are small perturbations of a capillary profile. Indeed, a typical plane capillary profile is a smooth traveling wave solution with different left and right endstates and fast decay of derivatives at infinity (see e.g [3]). In general, the velocity of such a profile does not belong to any Sobolev space with nonnegative index.

More generally, if $(\underline{\rho}, \underline{\mathbf{u}})$ is a given smooth reference solution of (1.2) on $[0, \underline{T}] \times \mathbb{R}^N$, we address the question of local solvability of (1.2) for H^s perturbations of $(\underline{\rho}_0, \underline{\mathbf{u}}_0) := (\underline{\rho}(0), \underline{\mathbf{u}}(0))$. Our main result is the following

Theorem 6.1 *Take* $s > 1 + \frac{N}{2}$*. Let* $(\underline{\rho}, \underline{u})$ *be a solution to* (EK) *on* $[0, \underline{T}] \times \mathbb{R}^N$ *with*

$$\underline{\rho}([0, \underline{T}] \times \mathbb{R}^N) \subset\subset \mathbb{J} \quad and \quad (D^2\underline{\rho}, D\underline{\mathbf{u}}) \in \mathcal{C}([0, \underline{T}]; \mathrm{H}^{s+3}).$$

Assume that the data $(\rho_0, \mathbf{u}_0)$ *satisfy*

$$\rho_0(\mathbb{R}^N) \subset\subset \mathbb{J}, \quad \widetilde{\rho}_0 := \rho_0 - \underline{\rho}_0 \in \mathrm{H}^{s+1} \quad and \quad \widetilde{\mathbf{u}}_0 := \mathbf{u}_0 - \underline{\mathbf{u}}_0 \in \mathrm{H}^s.$$

There exists a positive $T \leq \underline{T}$ *such that* (EK) *has a unique solution* $(\rho, \mathbf{u})$ *on* $[0, T] \times \mathbb{R}^N$ *in*

$$\mathrm{A}^s_T := (\underline{\rho}, \underline{\mathbf{u}}) + \Big(\mathcal{C}([0, T]; \mathrm{H}^{s+1} \times \mathrm{H}^s) \cap \mathcal{C}^1([0, T]; \mathrm{H}^{s-1} \times \mathrm{H}^{s-2})\Big).$$

Besides the blow-up criterion stated in Proposition 5.3 remains valid and there exists a neighborhood $\mathcal{V}$ *of* $(\rho_0, \mathbf{u}_0)$ *in* $(\underline{\rho}_0, \underline{\mathbf{u}}_0) + (\mathrm{H}^{s+1} \times \mathrm{H}^s)$ *such that for all* $(\dot{\rho}_0, \dot{\mathbf{u}}_0) \in \mathcal{V}$*, system* (EK) *with data* $(\dot{\rho}_0, \dot{\mathbf{u}}_0)$ *has a unique solution* $(\dot{\rho}, \dot{\mathbf{u}})$ *on* $[0, T] \times \mathbb{R}^N$ *uniformly in* A^s_T*, and* $(\dot{\rho}_0, \dot{\mathbf{u}}_0) \mapsto (\dot{\rho}, \dot{\mathbf{u}})$ *maps continuously* $\mathcal{V}$ *in* A^s_T*.*

Finally, if $\operatorname{curl}\mathbf{u}_0 \equiv \mathbf{0}$ *then* $\operatorname{curl}\mathbf{u} \equiv \mathbf{0}$.

Proof. The main steps of the proof are the same as in Theorem 5.1. To simplify the computations, we perform the change of unknown $L := \mathcal{L}(\rho)$.

Step 1 : uniqueness

Let $(L_1, \mathbf{u}_1)$ and $(L_2, \mathbf{u}_2)$ belong to A^s_T and satisfy (EK) on $[0,T]\times\mathbb{R}^N$ with the same data. Obviously $(\mathbf{u}_2-\mathbf{u}_1)\in\mathcal{C}([0,T];\mathrm{H}^s)$ and $(L_2-L_1)\in\mathcal{C}([0,T];\mathrm{H}^{s+1})$. Besides, by virtue of the assumptions made on the reference solution $(\underline{L},\underline{\mathbf{u}})$, the functions D^2L_i and $D\mathbf{u}_i$ belong to $\mathcal{C}([0,T];\mathrm{H}^{s-1})$ for $i=1,2$. Hence Proposition 5.1 insures that $(L_1,\mathbf{u}_1)\equiv(L_2,\mathbf{u}_2)$ on $[0,T]\times\mathbb{R}^N$.

Step 2 : study of an approximate problem with smooth data

Denoting $\widetilde{L}:=L-\underline{L}$ and $\widetilde{\mathbf{u}}:=\mathbf{u}-\underline{\mathbf{u}}$, the system for $(\widetilde{L},\widetilde{\mathbf{u}})$ reads

$$(\widetilde{\mathrm{ES}})\qquad\begin{cases}\partial_t\widetilde{L}+\mathbf{u}^*\cdot\nabla\widetilde{L}+\widetilde{\mathbf{u}}^*\cdot\nabla\underline{L}+a\,\mathrm{div}\,\widetilde{\mathbf{u}}+\widetilde{a}\,\mathrm{div}\,\underline{\mathbf{u}}=0,\\ \partial_t\widetilde{\mathbf{u}}+\mathbf{u}^*\cdot\nabla\widetilde{\mathbf{u}}+\widetilde{\mathbf{u}}^*\cdot\nabla\underline{\mathbf{u}}-\nabla^2\widetilde{L}\cdot\nabla L-\nabla^2\underline{L}\cdot\widetilde{\nabla}L-\nabla(a\Delta\widetilde{L}+\widetilde{a}\Delta\underline{L})=\nabla(Q_\sharp(L)-Q_\sharp(\underline{L}))\end{cases}$$

with $a:=a_\sharp(L)$, $\underline{a}:=a_\sharp(\underline{L})$, $\widetilde{a}:=a-\underline{a}$ and $Q_\sharp$ a primitive of $q_\sharp$.

Let $\mathbf{w}:=\nabla L$, $\underline{\mathbf{w}}:=\nabla\underline{L}$ and $\widetilde{\mathbf{w}}:=\mathbf{w}-\underline{\mathbf{w}}$. It is easily found that $\widetilde{\mathbf{z}}:=\widetilde{\mathbf{u}}+i\widetilde{\mathbf{w}}$ satisfies

$$\partial_t\widetilde{\mathbf{z}}+\mathbf{u}^*\cdot\nabla\widetilde{\mathbf{z}}+\widetilde{\mathbf{u}}^*\cdot\nabla\underline{\mathbf{z}}+i\nabla\widetilde{\mathbf{z}}\cdot\mathbf{w}+i\nabla\underline{\mathbf{z}}\cdot\widetilde{\mathbf{w}}+i\nabla(a\,\mathrm{div}\widetilde{\mathbf{z}})+i\nabla(\widetilde{a}\,\mathrm{div}\underline{\mathbf{z}})=\nabla(Q_\sharp(L)-Q_\sharp(\underline{L})).$$

This induces us to solve the approximate mollified system

$$(\widetilde{\mathrm{ES}_\varepsilon})\qquad\begin{cases}\partial_t\widetilde{L}_\varepsilon+\varepsilon\Delta^2\widetilde{L}_\varepsilon=-\varepsilon\Delta^2\underline{L}-\mathbf{u}^*_\varepsilon\cdot\nabla\widetilde{L}_\varepsilon-\widetilde{\mathbf{u}}^*_\varepsilon\cdot\nabla\underline{L}-a_\varepsilon\mathrm{div}\,\widetilde{\mathbf{u}}_\varepsilon-\widetilde{a}_\varepsilon\mathrm{div}\,\underline{\mathbf{u}},\\ \partial_t\widetilde{\mathbf{z}}_\varepsilon+\varepsilon\Delta^2\widetilde{\mathbf{z}}_\varepsilon=-(\mathbf{u}^*_\varepsilon\cdot\nabla)\widetilde{\mathbf{z}}_\varepsilon-(\widetilde{\mathbf{u}}^*_\varepsilon\cdot\nabla)\underline{\mathbf{z}}-i\nabla\widetilde{\mathbf{z}}_\varepsilon\cdot\mathbf{w}_\varepsilon-i\nabla\underline{\mathbf{z}}\cdot\widetilde{\mathbf{w}}_\varepsilon\\ \qquad\qquad-i\nabla(a_\varepsilon\mathrm{div}\,\widetilde{\mathbf{z}}_\varepsilon)-i\nabla(\widetilde{a}_\varepsilon\mathrm{div}\,\underline{\mathbf{z}})+\nabla(Q_\sharp(L_\varepsilon)-Q_\sharp(\underline{L}))-\varepsilon\Delta^2\underline{\mathbf{z}}\end{cases}$$

with data $\widetilde{L}_{0,\varepsilon}:=\chi_\varepsilon*\big(L_0-\underline{L}_0\big)$, $\widetilde{\mathbf{z}}_{0,\varepsilon}:=\chi_\varepsilon*\big(\mathbf{z}_0-\underline{\mathbf{z}}_0\big)$ and χ_ε defined as in the proof of Theorem 5.1. Above, it is understood that $\mathbf{z}_\varepsilon:=\underline{\mathbf{z}}+\widetilde{\mathbf{z}}_\varepsilon$, $\mathbf{u}_\varepsilon:=\mathrm{Re}\,\mathbf{z}_\varepsilon$, $\mathbf{w}_\varepsilon:=\mathrm{Im}\,\mathbf{z}_\varepsilon$, $L_\varepsilon:=\underline{L}+\widetilde{L}_\varepsilon$, $a_\varepsilon:=a_\sharp(L_\varepsilon)$ and $\widetilde{a}_\varepsilon:=a_\varepsilon-\underline{a}$.

As in section 4, solving $(\widetilde{\mathrm{ES}_\varepsilon})$ in $\mathcal{C}([0,T];\mathrm{H}^{s+1}\times\mathrm{H}^s)\cap\mathrm{L}^2(0,T;\mathrm{H}^{s+3}\times\mathrm{H}^{s+2})$ for T suitably small stems from the contracting mapping theorem. Indeed, it suffices to find a fixed point for the functional $\widetilde{\Phi}:=(\widetilde{\Phi}_1,\widetilde{\Phi}_2)$ defined by

$$\begin{cases}\widetilde{\Phi}_1(\dot{L},\dot{\mathbf{z}})(t)=-\int_0^t S_{\varepsilon(t-\tau)}\big(\varepsilon\Delta^2\underline{L}+\mathbf{u}^*_\varepsilon\cdot\nabla\widetilde{L}_\varepsilon+\widetilde{\mathbf{u}}^*_\varepsilon\cdot\nabla\underline{L}+a_\varepsilon\mathrm{div}\,\widetilde{\mathbf{u}}_\varepsilon+\widetilde{a}_\varepsilon\mathrm{div}\,\underline{\mathbf{u}}\big)\,d\tau,\\ \widetilde{\Phi}_2(\dot{L},\dot{\mathbf{z}})(t)=-\int_0^t S_{\varepsilon(t-\tau)}\big((\mathbf{u}^*_\varepsilon\cdot\nabla)\widetilde{\mathbf{z}}_\varepsilon+(\widetilde{\mathbf{u}}^*_\varepsilon\cdot\nabla)\underline{\mathbf{z}}+i\nabla\widetilde{\mathbf{z}}_\varepsilon\cdot\mathbf{w}_\varepsilon+i\nabla\underline{\mathbf{z}}\cdot\widetilde{\mathbf{w}}_\varepsilon\\ \qquad\qquad+i\nabla(a_\varepsilon\mathrm{div}\,\widetilde{\mathbf{z}}_\varepsilon)+i\nabla(\widetilde{a}_\varepsilon\mathrm{div}\,\underline{\mathbf{z}})-\nabla(Q_\sharp(L_\varepsilon)-Q_\sharp(\underline{L}))+\varepsilon\Delta^2\underline{\mathbf{z}}\big)\,d\tau,\end{cases}$$

Above, we denoted $L_\varepsilon(t):=S_{\varepsilon t}\widetilde{L}_{0,\varepsilon}+\underline{L}(t)+\dot{L}(t)$, $\mathbf{z}_\varepsilon(t):=S_{\varepsilon t}\widetilde{\mathbf{z}}_{0,\varepsilon}+\underline{\mathbf{z}}(t)+\dot{\mathbf{z}}(t)$, $\mathbf{u}_\varepsilon:=\mathrm{Re}\,\mathbf{z}_\varepsilon$, $\mathbf{w}_\varepsilon:=\mathrm{Im}\,\mathbf{z}_\varepsilon$ and so on. Remark that

$$L_\varepsilon(t)-L_0=\dot{L}(t)+\big(\underline{L}(t)-\underline{L}_0\big)+\big(S_{\varepsilon t}(\chi_\varepsilon*\widetilde{L}_0)-\widetilde{L}_0\big).$$

Hence, since $\underline{L}$ is continuous, one can insure that $L_\varepsilon([0,T]\times\mathbb{R}^N)$ lies in a compact subset of $\mathbb{J}$ independent of ε provided T, ε and $\|\dot{L}\|_{\mathrm{L}^\infty_T(\mathrm{L}^\infty)}$ are small enough.

Now, going along the lines of the proof of Theorem 4.1 and using the results of the appendix for bounding the nonlinear terms, it is not difficult to prove that if $\|D\widetilde{L}_{0,\varepsilon}\|_{\mathrm{H}^s}+\|\widetilde{\mathbf{z}}_{0,\varepsilon}\|_{\mathrm{H}^s}\leq R_0$ then, denoting $\underline{R}:=\|D\underline{L}\|_{L^\infty_{\underline{T}}(\mathrm{H}^{s+2})}+\|D\underline{\mathbf{u}}\|_{L^\infty_{\underline{T}}(\mathrm{H}^{s+1})}$, we have

$$\|\widetilde{\Phi}(\dot{L},\dot{\mathbf{z}})\|_{T,\varepsilon}\leq C\mathrm{e}^{T\varepsilon}\sqrt{\frac{T}{\varepsilon}}\Big(\varepsilon\underline{R}+R_0(1+R_0+\underline{R})\Big)$$

whenever $(\dot{L},\dot{\mathbf{z}})$ belongs to the space E^{η,R_0}_T defined the proof of Theorem 4.1 and $T\leq\underline{T}$.

Besides, if $(\dot{L}_1,\dot{\mathbf{z}}_1)$ and $(\dot{L}_2,\dot{\mathbf{z}}_2)$ both belong to E_T^{η,R_0} then

$$\|\widetilde{\Phi}(\dot{L}_2,\dot{\mathbf{z}}_2)-\widetilde{\Phi}(\dot{L}_1,\dot{\mathbf{z}}_1)\|_{T,\varepsilon}\leq C\mathrm{e}^{T\varepsilon}\sqrt{\frac{T}{\varepsilon}}\Big(1+R_0+\underline{R}\Big)^2\|(\dot{L}_2-\dot{L}_1,\dot{\mathbf{z}}_2-\dot{\mathbf{z}}_1)\|_{T,\varepsilon}.$$

Hence the contracting mapping theorem yields a solution $(\dot{L}_\varepsilon,\dot{\mathbf{z}}_\varepsilon)$ in $\mathcal{C}([0,T_\varepsilon];\mathrm{H}^{s+1}\times\mathrm{H}^s)\cap\mathrm{L}^2(0,T_\varepsilon;\mathrm{H}^{s+3}\times\mathrm{H}^{s+2})$ (with besides $L_\varepsilon([0,T_\varepsilon]\times\mathbb{R}^N)\subset\subset\mathbb{J}$) for some small enough positive T_ε. Obviously $(L_\varepsilon,\mathbf{z}_\varepsilon)$ satisfies $\widetilde{\mathrm{ES}_\varepsilon}$ and has the desired regularity property.

Uniqueness relies on the same arguments as in Proposition 4.1. Moreover, by following Corollary 4.1, one can easily check that $\widetilde{\mathbf{w}}_{0,\varepsilon}=\nabla\widetilde{L}_{0,\varepsilon}$ and $\underline{\mathbf{w}}=\nabla\underline{L}$ imply $\widetilde{\mathbf{w}}_\varepsilon\equiv\nabla\widetilde{L}_\varepsilon$.

We now want to get a positive lower bound T for T_ε when ε goes 0. Remind that

$$\begin{aligned}(6.1)\qquad \partial_t\widetilde{\mathbf{z}}_\varepsilon+\;&(\mathbf{u}^*_\varepsilon\cdot\nabla)\widetilde{\mathbf{z}}_\varepsilon+i\nabla\widetilde{\mathbf{z}}_\varepsilon\cdot\mathbf{w}_\varepsilon+i\nabla(a_\varepsilon\mathrm{div}\,\widetilde{\mathbf{z}}_\varepsilon)+\varepsilon\Delta^2\widetilde{\mathbf{z}}_\varepsilon\\&=-(\widetilde{\mathbf{u}}^*_\varepsilon\cdot\nabla)\underline{\mathbf{z}}--i\nabla\underline{\mathbf{z}}\cdot\widetilde{\mathbf{w}}_\varepsilon-i\nabla(\widetilde{a}_\varepsilon\mathrm{div}\,\underline{\mathbf{z}})+\nabla\big(Q_\sharp(L_\varepsilon)-Q_\sharp(\underline{L})\big)+\varepsilon\Delta^2\underline{\mathbf{z}}\end{aligned}$$

and that $\mathbf{w}_\varepsilon=\nabla L_\varepsilon$ with $\partial_tL_\varepsilon+\mathbf{u}^*_\varepsilon\cdot\nabla L_\varepsilon+a_\varepsilon\mathrm{div}\,\mathbf{u}_\varepsilon+\varepsilon\Delta^2L_\varepsilon=0$.

Let us first assume that $s>2$. Applying the inequality (4.13) to the above equation, we get

$$\begin{aligned}\frac{d}{dt}\widetilde{\|\widetilde{\mathbf{z}}_\varepsilon\|}_s^2\lesssim\varepsilon\|\Delta\underline{\mathbf{z}}\|^2_{\mathrm{H}^s}+\big(\|D\mathbf{z}_\varepsilon\|_{\mathrm{H}^{s-1}}+\varepsilon\,\|D\mathbf{w}_\varepsilon\|^2_{\mathrm{L}^\infty}\big)\|\widetilde{\mathbf{z}}_\varepsilon\|^2_{\mathrm{H}^s}\\+\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^s}\big(\|(\widetilde{\mathbf{u}}^*_\varepsilon\cdot\nabla)\underline{\mathbf{z}}\|_{\mathrm{H}^s}+\|\nabla\underline{\mathbf{z}}\cdot\widetilde{\mathbf{w}}_\varepsilon\|_{\mathrm{H}^s}+\|\widetilde{a}_\varepsilon\mathrm{div}\,\underline{\mathbf{z}}\|_{\mathrm{H}^{s+1}}+\|Q_\sharp(L_\varepsilon)-Q_\sharp(\underline{L})\|_{\mathrm{H}^{s+1}}\big)\end{aligned}$$

where $\widetilde{\|\widetilde{\mathbf{z}}_\varepsilon\|}_s^2=\|\mathcal{P}(\sqrt{A_s(\rho_\varepsilon)}\Lambda^s\widetilde{\mathbf{z}}_\varepsilon)\|^2_{L^2}+\|\mathcal{Q}(\sqrt{\rho_\varepsilon a^s_\varepsilon}\Lambda^s\widetilde{\mathbf{z}}_\varepsilon)\|^2_{\mathrm{L}^2}$ with $\rho_\varepsilon:=\mathcal{L}^{-1}(L_\varepsilon)$ and A_s defined in (3.30).

In the following computations, we restrict ourselves on a time interval $[0,T]$ so small as

$$(6.2)\qquad \forall t\in[0,T],\ \forall x\in\mathbb{R}^N,\ J^-+\frac{\eta}{2}\leq L_\varepsilon(t,x)\leq J^+-\frac{\eta}{2}\quad\text{with}\quad \eta:=d(\mathbb{R}\setminus\mathbb{K},L_0(\mathbb{R}^N)).$$

The nonlinear terms in the right-hand side of the above inequality may be easily bounded by taking advantage of the results of the appendix. We eventually get

$$\begin{aligned}(6.3)\qquad \tfrac{d}{dt}\widetilde{\|\widetilde{\mathbf{z}}_\varepsilon\|}_s^2&\lesssim\varepsilon\|\Delta^2\underline{\mathbf{z}}\|^2_{\mathrm{H}^s}+\varepsilon\|\widetilde{\mathbf{z}}_\varepsilon\|^4_{\mathrm{H}^s}\\&+\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^s}\big(\|\widetilde{L}_\varepsilon\|_{\mathrm{L}^2}+\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^s}\big)\big(\varepsilon\,\|D\underline{\mathbf{w}}\|^2_{\mathrm{L}^\infty}+(1+\|D\underline{\mathbf{z}}\|_{\mathrm{H}^{s+1}})(1+\|D\underline{\mathbf{w}}\|_{\mathrm{H}^{s-1}}+\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^s})\big).\end{aligned}$$

Let us stress the fact that having $\widetilde{L}_\varepsilon$ in L^2 is needed for bounding $\|Q_\sharp(L_\varepsilon)-Q_\sharp(\underline{L})\|_{\mathrm{H}^{s+1}}$ and $\|\widetilde{a}_\varepsilon\mathrm{div}\,\underline{\mathbf{z}}\|_{\mathrm{H}^{s+1}}$. For instance, according to Proposition B.3, we have

$$\|\widetilde{a}_\varepsilon\mathrm{div}\,\underline{\mathbf{z}}\|_{\mathrm{H}^{s+1}}\lesssim\|\widetilde{a}_\varepsilon\|_{\mathrm{H}^{s+1}}\|\mathrm{div}\,\underline{\mathbf{z}}\|_{\mathrm{H}^{s+1}}\lesssim\|\widetilde{L}_\varepsilon\|_{\mathrm{H}^{s+1}}\|D\underline{\mathbf{z}}\|_{\mathrm{H}^{s+1}}(1+\|D\underline{\mathbf{w}}\|_{\mathrm{H}^{s-1}}+\|D\widetilde{\mathbf{w}}_\varepsilon\|_{\mathrm{H}^{s-1}}).$$

Let $\alpha\in[0,1)$ be such that $\frac{N}{2}-\alpha\leq s-2$ (note that such an α exists because $s>\frac{N}{2}+1$). Since $\mathrm{C}^{-\alpha}\hookrightarrow\mathrm{H}^{s-2}$, one can easily prove by arguing as in (4.18) that

$$(6.4)\qquad C^{-1}\widetilde{\|\widetilde{\mathbf{z}}_\varepsilon\|}_s\leq\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^s}\leq C\big(\widetilde{\|\widetilde{\mathbf{z}}_\varepsilon\|}_s+\|\underline{\mathbf{w}}\|^\gamma_{\mathrm{C}^{-\alpha}}\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^{s-2}}+\|\widetilde{\mathbf{z}}_\varepsilon\|^\delta_{\mathrm{H}^{s-2}}\big)$$

with $\gamma:=\frac{2}{1-\alpha}$ and $\delta:=\frac{3-\alpha}{1-\alpha}$.

Therefore, in order to close the estimates, bounds on $\|\widetilde{L}_\varepsilon\|_{\mathrm{L}^2}$ and on $\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^{s-2}}$ are needed. Getting the L^2 bounds is easy. Indeed, since

$$\partial_t\widetilde{L}_\varepsilon+\mathbf{u}^*_\varepsilon\cdot\nabla\widetilde{L}_\varepsilon+\widetilde{\mathbf{u}}^*_\varepsilon\cdot\nabla\underline{L}+a_\varepsilon\mathrm{div}\,\widetilde{\mathbf{u}}_\varepsilon+\widetilde{a}_\varepsilon\mathrm{div}\,\underline{\mathbf{u}}+\varepsilon\Delta^2\widetilde{L}_\varepsilon=-\varepsilon\Delta^2\underline{L},$$

a straightforward energy method yields

$$(6.5)\quad \frac{d}{dt}\|\widetilde{L}_\varepsilon\|^2_{\mathrm{L}^2}\lesssim(\|D\underline{\mathbf{u}}\|_{\mathrm{L}^\infty}+\|D\widetilde{\mathbf{u}}_\varepsilon\|_{\mathrm{L}^\infty})\|\widetilde{L}_\varepsilon\|^2_{\mathrm{L}^2}+\|\widetilde{L}_\varepsilon\|_{\mathrm{L}^2}\big(\|\underline{\mathbf{w}}\|_{\mathrm{L}^\infty}\|\widetilde{\mathbf{u}}_\varepsilon\|_{\mathrm{L}^2}+\|\widetilde{\mathbf{u}}_\varepsilon\|_{\mathrm{H}^1}\big)+\varepsilon\|\Delta\underline{L}\|^2_{\mathrm{L}^2}.$$

For getting H^{s-2} bounds, apply Λ^{s-2}:

$$\partial_t\Lambda^{s-2}\widetilde{\mathbf{z}}_\varepsilon+(\mathbf{u}_\varepsilon^*\cdot\nabla)\Lambda^{s-2}\widetilde{\mathbf{z}}_\varepsilon+\varepsilon\Delta^2\Lambda^{s-2}\widetilde{\mathbf{z}}_\varepsilon=[u_\varepsilon^j,\Lambda^{s-2}]\partial_j\widetilde{\mathbf{z}}_\varepsilon$$
$$-\Lambda^{s-2}\big(i\nabla\widetilde{\mathbf{z}}_\varepsilon\cdot\mathbf{w}_\varepsilon+i\nabla(a_\varepsilon\mathrm{div}\,\widetilde{\mathbf{z}}_\varepsilon)+(\widetilde{\mathbf{u}}_\varepsilon^*\cdot\nabla)\underline{\mathbf{z}}+i\nabla\underline{\mathbf{z}}\cdot\widetilde{\mathbf{w}}_\varepsilon+i\nabla(\widetilde{a}_\varepsilon\mathrm{div}\,\underline{\mathbf{z}})-\nabla(Q_\sharp(L_\varepsilon)-Q_\sharp(\underline{L}))-\varepsilon\Delta^2\underline{\mathbf{z}}.$$

Assuming that $s-2>0$ in the following computations, a standard energy method combined with the inequality (A.7) yields

$$\frac{d}{dt}\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^{s-2}}^2\lesssim\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^{s-2}}^2\big(\|D\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{L}^\infty}\|D\mathbf{u}_\varepsilon\|_{\mathrm{H}^{s-3}}+\|D\mathbf{u}_\varepsilon\|_{\mathrm{L}^\infty}\|\mathbf{z}_\varepsilon\|_{\mathrm{H}^{s-2}}\big)+\varepsilon\|\Delta\underline{\mathbf{z}}\|_{\mathrm{H}^{s-2}}^2$$
$$+\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^{s-2}}\big(\|\nabla\widetilde{\mathbf{z}}_\varepsilon\cdot\mathbf{w}_\varepsilon\|_{\mathrm{H}^{s-2}}+\|a_\varepsilon\mathrm{div}\,\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^{s-1}}+\|(\widetilde{\mathbf{u}}_\varepsilon^*\cdot\nabla)\underline{\mathbf{z}}\|_{\mathrm{H}^{s-2}}$$
$$+\|\nabla\underline{\mathbf{z}}\cdot\widetilde{\mathbf{w}}_\varepsilon\|_{\mathrm{H}^{s-2}}+\|\widetilde{a}_\varepsilon\mathrm{div}\,\underline{\mathbf{z}}\|_{\mathrm{H}^{s-1}}+\|Q_\sharp(L_\varepsilon)-Q_\sharp(\underline{L})\|_{\mathrm{H}^{s-1}}\big).$$

All the non linear terms appearing in the right-hand side may be bounded by appealing to lemmas B.1 and B.2, and to propositions B.1 and B.3. After a series of cumbersome computations, we end up with

$$\frac{d}{dt}\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^{s-2}}^2\lesssim\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^{s-2}}\Big(\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^s}\big(1+\|D\underline{\mathbf{z}}\|_{\mathrm{H}^{s-1}}+\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^{s-2}}\big)$$
$$+\big(\|\widetilde{L}_\varepsilon\|_{\mathrm{L}^2}+\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^{s-2}}\big)\big(1+\|D\underline{\mathbf{w}}\|_{\mathrm{H}^{s-1}}+\|\widetilde{\mathbf{w}}_\varepsilon\|_{\mathrm{H}^{s-2}}\big)\Big)+\varepsilon\|\Delta\underline{\mathbf{z}}\|_{\mathrm{H}^{s-2}}^2.$$

Hence, combining with (6.5) and (6.4),

$$\frac{d}{dt}\big(\|\widetilde{L}_\varepsilon\|_{\mathrm{L}^2}^2+\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^{s-2}}^2\big)\lesssim\varepsilon\|D\underline{\mathbf{z}}\|_{\mathrm{H}^{s-1}}^2$$
$$+\big(\|\widetilde{L}_\varepsilon\|_{\mathrm{L}^2}+\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^s}\big)\big(\|\widetilde{L}_\varepsilon\|_{\mathrm{L}^2}+\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^{s-2}}\big)\big(1+\|D\underline{\mathbf{z}}\|_{\mathrm{H}^{s-1}}+\|\widetilde{L}_\varepsilon\|_{\mathrm{L}^2}+\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^{s-2}}\big).\tag{6.6}$$

Multiplying this inequality by $\Big(\|\widetilde{L}_\varepsilon\|_{\mathrm{L}^2}^2+\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^{s-2}}^2\Big)^{\delta-1}$ and using Young's inequality, we also get

$$\frac{d}{dt}\big(\|\widetilde{L}_\varepsilon\|_{\mathrm{L}^2}^2+\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^{s-2}}^2\big)^\delta\lesssim\varepsilon\|D\underline{\mathbf{z}}\|_{\mathrm{H}^{s-1}}^{2\delta}+\varepsilon\big(\|\widetilde{L}_\varepsilon\|_{\mathrm{L}^2}^2+\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^{s-2}}^2\big)^\delta$$
$$+\big(\|\widetilde{L}_\varepsilon\|_{\mathrm{L}^2}+\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^s}\big)\big(\|\widetilde{L}_\varepsilon\|_{\mathrm{L}^2}^{2\delta-1}+\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^{s-2}}^{2\delta-1}\big)\big(1+\|D\underline{\mathbf{z}}\|_{\mathrm{H}^{s-1}}+\|\widetilde{L}_\varepsilon\|_{\mathrm{L}^2}+\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^{s-2}}\big).\tag{6.7}$$

It is now possible to close the estimates. Indeed, let $\widetilde{Z}_\varepsilon$ be defined by

$$\widetilde{Z}_\varepsilon^2(t):=\widetilde{\|\widetilde{\mathbf{z}}_\varepsilon(t)\|}_s^2+\|\widetilde{L}_\varepsilon(t)\|_{\mathrm{L}^2}^2+\|\widetilde{\mathbf{z}}_\varepsilon(t)\|_{\mathrm{H}^{s-2}}^2+\big(\|\widetilde{L}_\varepsilon(t)\|_{\mathrm{L}^2}^2+\|\widetilde{\mathbf{z}}_\varepsilon(t)\|_{\mathrm{H}^{s-2}}^2\big)^\delta.$$

Remark that (6.4) insures that

$$\|\widetilde{L}_\varepsilon\|_{\mathrm{L}^2}+\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^s}\lesssim\big(1+\|\underline{\mathbf{w}}\|_{\mathrm{C}^{-\alpha}}^\gamma\big)\widetilde{Z}_\varepsilon.$$

Hence adding up inequalities (6.3), (6.6) and (6.7), we get

$$\frac{d}{dt}\widetilde{Z}_\varepsilon^2\le\varepsilon\underline{P}+\underline{A}_\varepsilon\big(\varepsilon\widetilde{Z}_\varepsilon^4+\widetilde{Z}_\varepsilon^3+\widetilde{Z}_\varepsilon^2\big),$$

with $\underline{P}:=C\big(\|\Delta\underline{\mathbf{z}}\|_{\mathrm{H}^s}^2+\|D\underline{\mathbf{z}}\|_{\mathrm{H}^{s-1}}^2+\|D\underline{\mathbf{z}}\|_{\mathrm{H}^{s-1}}^{2\delta}\big)$ and

$$\underline{A}_\varepsilon:=C\big(1+\|\underline{\mathbf{w}}\|_{\mathrm{C}^{-\alpha}}^{4\gamma}\big)\big(\varepsilon\,\|D\underline{\mathbf{w}}\|_{\mathrm{L}^\infty}^2+(1+\|D\underline{\mathbf{w}}\|_{\mathrm{H}^{s-1}})(1+\|D\underline{\mathbf{z}}\|_{\mathrm{H}^{s+1}})\big).$$

Therefore, denoting $\widetilde{Z}_{\varepsilon,0}(t):=2\Big(\widetilde{Z}_\varepsilon^2(0)+\varepsilon\int_0^t\underline{P}\,d\tau\Big)^{\frac12}$ and assuming that

$$\mathrm{e}^{\frac{\varepsilon}{2}\int_0^T\underline{A}_\varepsilon(t)\widetilde{Z}_\varepsilon^2(t)\,dt}\le 2,\tag{6.8}$$

Gronwall's lemma leads to

$$\forall t\in[0,T],\ \widetilde Z_\varepsilon(t)\leq e^{\frac12\int_0^t \underline{A}_\varepsilon(\tau)\,d\tau}e^{\frac12\int_0^t \underline{A}_\varepsilon(\tau)\widetilde Z_\varepsilon(\tau)\,d\tau}\widetilde Z_{\varepsilon,0}(t).$$

Now, if in addition we have

$$\widetilde Z_{\varepsilon,0}(T)\Big(e^{\frac12\int_0^T \underline{A}_\varepsilon(t)\,dt}-1\Big)<1,\tag{6.9}$$

then for all $t\in[0,T]$,

$$e^{\frac12\int_0^t \underline{A}_\varepsilon\widetilde Z_\varepsilon\,d\tau}\leq\frac{1}{1-\widetilde Z_{\varepsilon,0}(t)\Big(e^{\frac12\int_0^t \underline{A}_\varepsilon\,d\tau}-1\Big)}\quad\text{and}\quad \widetilde Z_\varepsilon(t)\leq\frac{\widetilde Z_{\varepsilon,0}(t)e^{\frac12\int_0^t \underline{A}_\varepsilon\,d\tau}}{1-\widetilde Z_{\varepsilon,0}(t)\Big(e^{\frac12\int_0^t \underline{A}_\varepsilon\,d\tau}-1\Big)}.\tag{6.10}$$

In order to ensure condition (6.2), one can argue exactly as in the proof of Proposition 4.3 and apply (4.24) (indeed $(L_\varepsilon,\mathbf{u}_\varepsilon)$ satisfies $(\mathrm{EK}_\varepsilon)$). Hence for (6.2) to be satisfied, it suffices that

$$\sqrt{\varepsilon T}\|\mathrm{div}\,\mathbf{w}_0\|_{\mathrm{H}^{s-1}}+\big(1+\|\mathrm{div}\,\underline{\mathbf{w}}\|_{L^\infty_T(\mathrm{H}^{s-1})}+\|\mathrm{div}\,\widetilde{\mathbf{w}}_\varepsilon\|_{L^\infty_T(\mathrm{H}^{s-1})}\big)\big(\|\underline{\mathbf{u}}\|_{L^1_T(\mathrm{H}^s)}+\|\widetilde{\mathbf{u}}_\varepsilon\|_{L^1_T(\mathrm{H}^s)}\big)\leq c\eta$$

for some suitably small constant c.

A standard bootstrap argument shows that if $T\leq\underline{T}$ is so small as to satisfy

$$\begin{cases}\sqrt{\varepsilon T}\|\mathrm{div}\,\mathbf{w}_0\|_{\mathrm{H}^{s-1}}\leq\frac{c\eta}{2},\qquad \widetilde Z_{\varepsilon,0}(T)\Big(e^{\frac12\int_0^T \underline{A}_\varepsilon(t)\,dt}-1\Big)\leq\frac12,\\ -\varepsilon\widetilde Z_\varepsilon(0)e^{\frac12\int_0^T \underline{A}_\varepsilon(t)\,dt}\log\Big(1-\widetilde Z_{\varepsilon,0}(T)\Big(e^{\frac12\int_0^T \underline{A}_\varepsilon(t)\,dt}-1\Big)\Big)\leq\frac{\log 2}{2},\\ \Big((1+\|D\underline{\mathbf{w}}\|_{L^\infty_T(\mathrm{H}^{s-1})}+(1+\|\underline{\mathbf{w}}\|^\gamma_{L^\infty_T(\mathrm{C}^{-\alpha})})\widetilde Z_{\varepsilon,0}(T)e^{\frac12\int_0^T \underline{A}_\varepsilon(t)\,dt}\Big)\Big(\|\underline{\mathbf{u}}\|_{L^1_T(\mathrm{H}^s)}\\ \qquad\qquad -\log\Big(1-\widetilde Z_{\varepsilon,0}(T)\Big(e^{\frac12\int_0^T \underline{A}_\varepsilon(t)\,dt}-1\Big)\Big)\leq\frac{c\eta}{4}\end{cases}\tag{6.11}$$

then (6.10) holds true on $[0,T]$.

Taking advantage of Lemma C.1, we see that

$$\widetilde Z_{\varepsilon,0}(0)\lesssim\|L_0-\underline{L}_0\|_{\mathrm{H}^{s+1}}+\|L_0-\underline{L}_0\|^\delta_{\mathrm{H}^{s-1}}+\|\mathbf{u}_0-\underline{\mathbf{u}}_0\|_{\mathrm{H}^s}+\|\mathbf{u}_0-\underline{\mathbf{u}}_0\|^\delta_{\mathrm{H}^s}.$$

Moreover, the functions $Z_{\varepsilon,0}$ and $\underline{A}_\varepsilon$ are nondecreasing with respect to ε. Hence one can find some $T>0$ and $\varepsilon_0>0$ satisfying (6.11) for all $\varepsilon\leq\varepsilon_0$. Combining with (6.4) and (6.10), we eventually get uniform bounds in $\mathcal{C}([0,T];\mathrm{H}^{s+1}\times\mathrm{H}^s)$ for $(\widetilde L_\varepsilon,\widetilde{\mathbf{u}}_\varepsilon)$ when ε goes to 0. This achieves step 2 in the case $s>2$.

The case $s\leq 2$ (which may occur in dimension one only) is easier to handle. This is only a matter of applying Proposition 3.2 instead of Proposition 3.3. Since the auxiliary norm $\widetilde{\|\cdot\|}_{\mathrm{H}^s}$ is equivalent to the usual H^s norm, we need not estimate the H^{s-2} of $\widetilde{\mathbf{z}}_\varepsilon$. The details are left to the reader.

From now on, we denote by K a generic constant such that for all $\varepsilon\in(0,\varepsilon_0)$, we have

$$\|\widetilde{\mathbf{u}}_\varepsilon\|_{L^\infty_T(\mathrm{H}^s)}+\varepsilon^{\frac12}\|\widetilde{\mathbf{u}}_\varepsilon\|_{L^2_T(\mathrm{H}^{s+2})}+\|\widetilde L_\varepsilon\|_{L^\infty_T(\mathrm{H}^{s+1})}+\varepsilon^{\frac12}\|\widetilde L_\varepsilon\|_{L^2_T(\mathrm{H}^{s+3})}\leq K.$$

Step 3 : the Cauchy criterion

Since $\delta L^\nu_\varepsilon:=\widetilde L_\varepsilon-\widetilde L_\nu=L_\varepsilon-L_\nu$ and $\delta\mathbf{z}^\nu_\varepsilon:=\widetilde z_\varepsilon-\widetilde z_\nu=z_\varepsilon-z_\nu$, step 3 of the proof of Theorem 5.1 insures that

$$\|\delta L^\nu_\varepsilon\|_{\mathrm{L}^\infty_T(\mathrm{H}^{s+1})}+\|\delta\mathbf{u}^\nu_\varepsilon\|_{\mathrm{L}^\infty_T(\mathrm{H}^s)}=o(1)$$

holds true uniformly with respect to $0<\nu\leq\varepsilon<\varepsilon_0$, provided

$$L_\varepsilon(0,T\times\mathbb{R}^N)\subset\mathbb{K}\subset\subset\mathbb{J}\ \text{ uniformly in }\ \varepsilon, \tag{6.12}$$

$$D\mathbf{z}_\varepsilon\in \mathrm{L}^\infty(0,T;\mathrm{H}^{s-1})\ \text{ and }\ \sqrt{\varepsilon}\,D^3\mathbf{z}_\varepsilon\in\mathrm{L}^2(0,T;\mathrm{H}^{s-1})\ \text{ uniformly in }\ \varepsilon, \tag{6.13}$$

$$\|D^2\mathbf{z}_\varepsilon\|_{L^\infty_T(\mathrm{H}^s)}+\varepsilon^{\frac12}\|\Delta^2\mathbf{z}_\varepsilon\|_{L^2_T(\mathrm{H}^s)}=\mathcal{O}(\varepsilon^{-2\beta}). \tag{6.14}$$

Conditions (6.12) and (6.13) are insured by the previous step and by the assumption on the reference solution. Hence we are left with the proof of (6.14).

As $D^2\underline{\mathbf{z}}\in\mathrm{L}^\infty(0,T;\mathrm{H}^{s+2})$, it actually suffices to prove (6.14) for $\widetilde{\mathbf{z}}_\varepsilon$ instead of $\mathbf{z}_\varepsilon$. Starting with equation (6.1) and applying the inequality (4.13) with a slight modification (we mean that we do not use Sobolev embeddings for bounding the terms due to (3.27)), we end up with

$$\begin{aligned}\frac{d}{dt}\widetilde{\|}\widetilde{\mathbf{z}}_\varepsilon\|^2_{s+2}+\varepsilon\|D^2\widetilde{\mathbf{z}}_\varepsilon\|^2_{\mathrm{H}^{s+2}}\lesssim{}&\varepsilon\|\Delta\underline{\mathbf{z}}\|_{\mathrm{H}^{s+2}}\\ &+\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^{s+2}}\big(\|(\widetilde{\mathbf{u}}^*_\varepsilon\cdot\nabla)\underline{\mathbf{z}}\|_{\mathrm{H}^{s+2}}+\|\nabla\underline{\mathbf{z}}\cdot\widetilde{\mathbf{w}}_\varepsilon\|_{\mathrm{H}^{s+2}}+\|\widetilde{a}_\varepsilon\mathrm{div}\,\underline{\mathbf{z}}\|_{\mathrm{H}^{s+3}}+\|Q_\sharp(L_\varepsilon)-Q_\sharp(\underline{L})\|_{\mathrm{H}^{s+3}}\big)\\ &+\big(\varepsilon\,\|D\mathbf{w}_\varepsilon\|^2_{\mathrm{L}^\infty}+\|D\mathbf{z}_\varepsilon\|_{\mathrm{L}^\infty}\big)\|\widetilde{\mathbf{z}}_\varepsilon\|^2_{\mathrm{H}^{s+2}}+\|D\mathbf{z}_\varepsilon\|_{\mathrm{L}^\infty}\|D\mathbf{z}_\varepsilon\|_{\mathrm{H}^{s+1}}\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^{s+2}}.\end{aligned}$$

The nonlinear terms may be bounded thanks to Lemma B.1 and Corollary B.3. Hence,

$$\begin{aligned}\frac{d}{dt}\widetilde{\|}\widetilde{\mathbf{z}}_\varepsilon\|^2_{s+2}+\varepsilon\|D^2\widetilde{\mathbf{z}}_\varepsilon\|^2_{\mathrm{H}^{s+2}}\lesssim{}&\varepsilon\|\Delta\underline{\mathbf{z}}\|_{\mathrm{H}^{s+2}}+\big(1+\varepsilon\,\|D\mathbf{w}_\varepsilon\|^2_{\mathrm{L}^\infty}+\|D\mathbf{z}_\varepsilon\|_{\mathrm{L}^\infty}+\|D\underline{\mathbf{z}}\|_{\mathrm{H}^{s+2}}\big)\|\widetilde{\mathbf{z}}_\varepsilon\|^2_{\mathrm{H}^{s+2}}\\ &+\big(1+\|D\underline{\mathbf{w}}\|_{\mathrm{H}^{s+2}}\big)\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^{s+2}}\|\widetilde{L}_\varepsilon\|_{\mathrm{L}^2}+\|D\underline{\mathbf{z}}\|^2_{\mathrm{H}^{s+1}}\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^{s+2}}.\end{aligned}$$

Taking advantage of the uniform bounds supplied by the previous step and of Young's inequality, we conclude that, for some constant $C_{T,K}$ depending only on the usual parameters, on T and on the bound K,

$$\frac{d}{dt}\widetilde{\|}\widetilde{\mathbf{z}}_\varepsilon\|^2_{s+2}+\varepsilon\|D^2\widetilde{\mathbf{z}}_\varepsilon\|^2_{\mathrm{H}^{s+2}}\leq C_{T,K}\big(1+\|\widetilde{\mathbf{z}}_\varepsilon\|^2_{\mathrm{H}^{s+2}}\big),$$

whence, applying Gronwall's lemma,

$$\widetilde{\|}\widetilde{\mathbf{z}}_\varepsilon(t)\|^2_{s+2}+\varepsilon\int_0^t\|D^2\widetilde{\mathbf{z}}_\varepsilon\|^2_{\mathrm{H}^{s+2}}\,d\tau\leq \mathrm{e}^{tC_{T,K}}\big(\widetilde{\|}\widetilde{\mathbf{z}}_\varepsilon(0)\|^2_{s+2}+tC_{T,K}\big). \tag{6.15}$$

On one hand, the inequality (3.32) combined with a straightforward interpolation insures that

$$\widetilde{\|}\widetilde{\mathbf{z}}_\varepsilon\|_{s+2}\lesssim\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^{s+2}}\lesssim\widetilde{\|}\widetilde{\mathbf{z}}_\varepsilon\|_{s+2}+\|\widetilde{\mathbf{w}}_\varepsilon\|^2_{\mathrm{L}^\infty}\|\widetilde{\mathbf{z}}_\varepsilon\|_{\mathrm{H}^s}.$$

On the other hand, $\widetilde{\mathbf{z}}_\varepsilon$ is uniformly bounded in $\mathrm{L}^\infty(0,T;\mathrm{H}^s)$ and Lemma (C.1) insures that $\widetilde{\|}\widetilde{\mathbf{z}}_\varepsilon(0)\|_{s+2}\lesssim\varepsilon^{-2\beta}\|\mathbf{z}_0-\underline{\mathbf{z}}_0\|_{\mathrm{H}^s}$. Hence the inequality (6.15) entails (6.14).

Step 4 : Existence of a solution

Let $(\widetilde{L},\widetilde{\mathbf{u}})\in\mathcal{C}([0,T];\mathrm{H}^{s+1}\times\mathrm{H}^s)$ be the limit of $(\widetilde{L}_\varepsilon,\widetilde{\mathbf{z}}_\varepsilon)$ when ε goes to 0^+. Since convergence holds in a very strong sense, it is easy to show that $(\widetilde{L},\widetilde{\mathbf{u}})$ satisfies $(\widetilde{\mathrm{EK}})$.

Besides, by making use of the results of the appendix, one can state that $\partial_t\widetilde{L}\in\mathcal{C}([0,T];\mathrm{H}^{s-1})$ and $\partial_t\widetilde{\mathbf{u}}\in\mathcal{C}([0,T];\mathrm{H}^{s-2})$.

Step 5 : Continuity of the solution map

The proof relies on Proposition 4.4 which, after cosmetic changes may be adapted to the case where $L_0\in\underline{L}_0+\mathrm{H}^{s+1}$ and $\mathbf{u}_0\in\underline{\mathbf{u}}_0+\mathrm{H}^s$. Indeed, by looking at the inequality (4.27), one realizes that the constant C_K depends on $(L,\mathbf{u})$ only through $\|L\|_{\mathrm{L}^\infty}$, $\|D\mathbf{u}\|_{\mathrm{H}^{s-1}}$ and $\|D^2L\|_{\mathrm{H}^{s-1}}$. The details are left to the reader.

Step 6 : blow-up criterion

Under the assumptions of Theorem 6.1 on the reference solution $(\underline{\rho}, \underline{\mathbf{u}})$, one can show that $(\rho, \mathbf{u})$ may be continued beyond T provided

$$\begin{cases} \rho([0,T)\times\mathbb{R}^N) \subset\subset J_\rho, \\ \int_0^T \big(\|\Delta\rho(t)\|_{\mathrm{L}^\infty} + \|\mathrm{curl}\,\mathbf{u}(t)\|_{\mathrm{L}^\infty} + \|\mathrm{div}\,\mathbf{u}(t)\|_{\mathrm{L}^\infty}\big)\,dt < \infty, \\ \sup_{t\in[0,T]} \|\rho(t)\|_{\mathrm{C}^\alpha} < \infty \quad \text{for some} \quad \alpha\in(0,1) \quad (\text{no condition if } \ \mathrm{curl}\,\mathbf{u} = \mathbf{0}). \end{cases}$$

Starting from system $(\widetilde{\mathrm{EK}})$, the proof is an easy adaptation of the one of Proposition 5.3. □

Remark 6.1 *Passing to the limit ε goes to 0 in (6.11) yields a lower bound on the time of existence in Theorem 6.1. In the particular case where the reference solution $(\underline{\rho}, \underline{\mathbf{u}})$ is globally defined and has spatial norms which are time independent (e.g a traveling wave), we gather that initial data $(\rho_0, \mathbf{u}_0)$ such that $\|\rho_0 - \underline{\rho}_0\|_{\mathrm{H}^{s+1}} + \|\mathbf{u}_0 - \underline{\mathbf{u}}_0\|_{\mathrm{H}^s} \le \eta$ yield a solution $(\rho, \mathbf{u})$ with a lifespan of order (at least) $\log\eta^{-1}$.*

Appendix

A Commutator estimates

This section is devoted to the proof of estimates which have been used throughout the paper. For that, elementary paradifferential calculus (see [7] for the original presentation) based on a Littlewood-Paley decomposition, is needed.

According to a classical convention, $a(x, \mathrm{D})$ will stand for the operator of symbol $a(x,\xi)$ (for suitable functions a). This means D may be thought of as $\frac{1}{i}D$ with $D = (\partial_1, \cdots, \partial_N)$.

Let (χ, φ) be a couple of C_0^∞ functions such that

(*i*) χ is supported in $B(0, 4/3)$,

(*ii*) φ is supported in the annulus $C(0, 3/4, 8/3)$,

(*iii*) $\forall\xi\in\mathbb{R}^N,\ \chi(\xi) + \sum_{q\in\mathbb{N}} \varphi(2^{-q}\xi) = 1$.

Let us denote $S_q := \chi(2^{-q}\mathrm{D})$, $\Delta_q := \varphi(2^{-q}\mathrm{D})$ for $q\in\mathbb{N}$, and $\Delta_{-1} := S_0 = \chi(\mathrm{D})$. It is obvious that $S_q = \sum_{p=-1}^{q-1} \Delta_p$ and that $u = \sum_{q\ge -1} \Delta_q u$ whenever u is in $\mathcal{S}'(\mathbb{R}^N)$. Besides,

$$\text{(A.1)} \qquad |p-q| > 1 \Longrightarrow \Delta_q\Delta_p u = 0 \quad \text{and} \quad |p-q| > 4 \Longrightarrow \Delta_q(S_{p-1}u\Delta_p v) = 0.$$

The paraproduct of two temperate distributions u and v is defined by

$$T_u v := \sum_{q\in\mathbb{N}} S_{q-1}u\Delta_q v$$

and we have the following (formal) Bony's decomposition for the product of two distributions:

$$uv = T_u v + T_v u + R(u,v)$$

where the remainder $R(u,v)$ is defined by

$$R(u,v) := \sum_{q\ge -1} \Delta_q u\,\widetilde{\Delta}_q v \quad \text{with} \quad \widetilde{\Delta_q} := \Delta_{q-1} + \Delta_q + \Delta_{q+1}.$$

The following two estimates for the remainder and the paraproduct will be used repeatedly:

Lemma A.1 *There exists an absolute constant* C *such that for all* $m\in\mathbb{R}$, *we have*

$$\|R(f,g)\|_{\mathrm{L}^2} \leq C\|f\|_{\mathrm{H}^{-m}}\|g\|_{\mathrm{F}^m_{\infty,2}}, \tag{A.2}$$

$$\|T_f g\|_{\mathrm{H}^m} \leq C\,\|f\|_{\mathrm{L}^2}\,\|g\|_{\mathrm{F}^m_{\infty,2}}. \tag{A.3}$$

where

$$\|g\|_{\mathrm{F}^m_{\infty,2}} := \sup_{x\in\mathbb{R}^N}\Big(\sum_q 2^{2qm}|\Delta_q g(x)|^2\Big)^{\frac12}$$

stands for the norm in the Triebel-Lizorkin space $\mathrm{F}^m_{\infty,2}$.

Proof. We have

$$\|R(f,g)\|_{\mathrm{L}^2} = \sup_{\|v\|_{\mathrm{L}^2}=1}\int R(f,g)\,v\,dx.$$

Hence, taking advantage of (A.1), there exists $N_0\in\mathbb{N}$ such that

$$\|R(f,g)\|_{\mathrm{L}^2} = \sup_{\|v\|_{\mathrm{L}^2}=1}\underbrace{\int\sum_q \Delta_q f\,\widetilde{\Delta}_q g\,S_{q+N_0}v\,dx}_{I}.$$

Now, Cauchy-Schwarz and Hölder inequalities yield

$$\begin{aligned}|I| &\leq \int\Big(\sum_q 2^{2qm}|\widetilde{\Delta}_q g(x)|^2\Big)^{\frac12}\Big(\sum_q 2^{-2qm}|\Delta_q f(x)|^2\,|S_{q+N_0}v(x)|^2\Big)^{\frac12}dx,\\ &\leq \|g\|_{\mathrm{F}^m_{\infty,2}}\|f\|_{\mathrm{H}^{-m}}\Big(\int|\sup_q S_q v(x)|^2\,dx\Big)^{\frac12}.\end{aligned}$$

Thus, in order to complete the proof of (A.2), it suffices to state that

$$\Big(\int|\sup_q S_q v(x)|^2\,dx\Big)^{\frac12}\lesssim\|v\|_{\mathrm{L}^2}. \tag{A.4}$$

The above inequality is a mere consequence of the continuity in L^2 of the maximal function and of the following inequality

$$M_h v(x) := \sup_{\lambda>0}\lambda^N\int|h(\lambda y)|\,|v(x-y)|\,dy \leq CMv(x) \tag{A.5}$$

which holds true for all function h such that $y\mapsto(1+|y|)^K h(y)$ is bounded for some $K>N$.

Let us prove (A.5). Take $x\in\mathbb{R}^N$. Denoting $w(y):=v(x-y)$, it suffices to prove that

$$\sup_{\lambda>0}\lambda^N\int|h(\lambda y)|\,|w(y)|\,dy \leq C\sup_{r>0}r^{-N}\int_{|y|<r}|w(y)|\,dy.$$

This fact stems from the following inequalities

$$\begin{aligned}\int|h(\lambda y)|\,|w(y)|\,dy &= \int_{|y|<\lambda^{-1}}|h(\lambda y)|\,|w(y)|\,dy + \sum_{p\geq0}\int_{2^p\leq\lambda|y|<2^{p+1}}|h(\lambda y)|\,|w(y)|\,dy,\\ &\lesssim \lambda^{-N}\Big(\lambda^N\int_{|y|<\lambda^{-1}}|w(y)|\,dy + \sum_{p\geq0}2^{p(N-K)}\,(2^{p+1}\lambda^{-1})^{-N}\int_{\lambda|y|<2^{p+1}}|w(y)|\,dy\Big),\\ &\lesssim \lambda^{-N}Mw(0).\end{aligned}$$

The proof of (A.3) is similar. Indeed, in light of (A.1), we have

$$\|T_f g\|_{\mathrm{H}^m} = \sup_{\|v\|_{\mathrm{H}^{-m}}=1} \underbrace{\int S_{q-1}f\ 2^{qm}\Delta_q g\ 2^{-qm}\widetilde{\varphi}(2^{-q}\mathrm{D})v\,dx}_{J}$$

where $\widetilde{\varphi}$ stands for a C_0^∞ function equals to one on a wide enough annulus $C(0,r_1,r_2)$ and supported in an annulus $C(0,r_1',r_2')$.

Using Cauchy-Schwarz and Hölder inequality yields

$$|J| \le \int \sup_q \big|S_{q-1}f(x)\big| \Big(\sum_{q\ge 1} 2^{2qm}|\Delta_q g(x)|^2\Big)^{\frac12} \Big(\sum_q 2^{-2qm}|\widetilde{\varphi}(2^{-q}\mathrm{D})v(x)|^2\Big)^{\frac12} dx$$

whence, according to (A.4) and Cauchy-Schwarz inequality,

$$|J| \le \|g\|_{\mathrm{F}^m_{\infty,2}}\|v\|_{\mathrm{H}^{-m}}\,\|f\|_{\mathrm{L}^2}.$$

□

We can now prove the following lemma[5]:

Lemma A.2 *For all s such that $-N/2 < s < N/2+1$, the following inequality holds true for some constant $C = C_{s,N}$:*

(A.6) $$\|[a,\Lambda^s]u\|_{\mathrm{L}^2} \le C\|Da\|_{\mathrm{B}^{\frac{N}{2}}_{2,\infty}\cap \mathrm{L}^\infty}\|u\|_{\mathrm{H}^{s-1}}.$$

where $\|f\|_{\mathrm{B}^\beta_{2,\infty}} := \sup_q 2^{q\beta}\,\|\Delta_q f\|_{\mathrm{L}^2}$ stands for the norm in the Besov space $\mathrm{B}^\beta_{2,\infty}$.

Besides, for all positive s, there exists some $C = C_{s,N}$ such that

(A.7) $$\|[a,\Lambda^s]u\|_{\mathrm{L}^2} \le C\Big(\|u\|_{\mathrm{L}^\infty}\,\|Da\|_{\mathrm{H}^{s-1}} + \|Da\|_{\mathrm{L}^\infty}\,\|u\|_{\mathrm{H}^{s-1}}\Big).$$

Proof. Let us denote $\widetilde{a} := (\mathrm{Id}-\Delta_{-1})a$. For proving (A.6), we decompose

$$[a,\Lambda^s]u = \underbrace{[T_a,\Lambda^s]u}_{R^1} + \underbrace{T_{\Lambda^s u}a}_{R^2} - \underbrace{\Lambda^s T_u a}_{R^3} + \underbrace{R(\Lambda^s u,\widetilde{a})}_{R^4} - \underbrace{\Lambda^s R(u,\widetilde{a})}_{R^5} + \underbrace{R(\Delta_{-1}a,\Lambda^s u) - \Lambda^s R(\Delta_{-1}a,u)}_{R^6}.$$

Bounding R^1

Using (A.1) and the definition of R^1, we get

$$R^1 = \sum_{q\ge 1}\underbrace{S_{q-1}a\,\widetilde{\varphi}(2^{-q}\mathrm{D})\Lambda^s\,\Delta_q u - \widetilde{\varphi}(2^{-q}\mathrm{D})\Lambda^s(S_{q-1}a\Delta_q u)}_{R^1_q}$$

for some function $\widetilde{\varphi}\in C_0^\infty$ supported away from the origin and equals to one on a wide enough annulus.

Using first order Taylor's formula, the term R^1_q rewrites

$$R^1_q = \int_{\mathbb{R}^N}\int_0^1 h_{s,q}(y)\,y^*\cdot\nabla S_{q-1}a(x-\tau y)\Delta_q u(x-y)\,d\tau\,dy$$

[5] An inequality similar to (A.7) has been stated in e.g [10].

with $h_{s,q} := \mathcal{F}^{-1}(\widetilde{\varphi}(2^{-q}\cdot)\lambda^s)$ and $\lambda^s(\xi) := (1+|\xi|^2)^{\frac{s}{2}}$, hence

(A.8) $$\left\|R_q^1\right\|_{\mathrm{L}^2} \leq \||\cdot|h_{s,q}\|_{\mathrm{L}^1} \|\Delta_q u\|_{\mathrm{L}^2} \|Da\|_{\mathrm{L}^\infty}.$$

We claim that for all $k \in \mathbb{N}$, there exists $C_k > 0$ such that

(A.9) $$\forall x \in \mathbb{R}^N,\ |x|^{2k} 2^{-qN} |h_{s,q}(2^{-q}x)| \leq C_k 2^{qs}.$$

Indeed, we have

$$2^{-qN} h_{s,q}(2^{-q}x) = (2\pi)^{-N} 2^{qs} \int \mathrm{e}^{ix\cdot\eta} \widetilde{\varphi}(\eta)(|\eta|^2 + 2^{-2q})^{\frac{s}{2}}\, d\eta,$$

whence, performing integration by parts,

$$(-1)^k |x|^{2k} 2^{-qN} h_{s,q}(2^{-q}x) = (2\pi)^{-N} 2^{qs} \int \mathrm{e}^{ix\cdot\eta}\, \Delta^k \big(\widetilde{\varphi}(\eta)(|\eta|^2 + 2^{-2q})^{\frac{s}{2}}\big)\, d\eta.$$

In light of Leibniz formula, the integral decomposes into a sum of terms of the type

$$\int \mathrm{e}^{ix\cdot\eta}\, \partial^\alpha \widetilde{\varphi}(\eta)\, (|\eta|^2 + 2^{-2q})^{\frac{s}{2}+j-|\beta|} P_{\beta,j}(\eta)\, d\eta$$

with $|\alpha| + |\beta| = 2k$ and $P_{\beta,j}$ is a homogeneous polynomial of degree $|\beta| - 2j$ with coefficients *independent of* q.

Since $\widetilde{\varphi}$ is supported in an annulus centered at zero, each integral may be bounded by some constant independent of $q \in \mathbb{N}$, which completes the proof of (A.9).

Now, because

$$\int |y||h_{s,q}(y)|\, dy = 2^{-q} \int |x| 2^{-qN} |h_{s,q}(2^{-q}x)|\, dx,$$

the inequality in (A.9) enables us to get

$$\||\cdot|h_{s,q}\|_{\mathrm{L}^1} \leq C_{s,N} 2^{q(s-1)}.$$

Since $\left\|R^1\right\|_{\mathrm{L}^2}^2 \approx \sum_{q\geq 1} \left\|R_q^1\right\|_{\mathrm{L}^2}^2$, the inequality (A.8) thus entails that

$$\|R_1\|_{\mathrm{L}^2} \lesssim \|Da\|_{\mathrm{L}^\infty} \|u\|_{\mathrm{H}^{s-1}}.$$

Bounding R^2, R^3, R^4 and R^5

Standard results for the paraproduct combined with the fact that $T_{\Lambda^s u} a = T_{\Lambda^s u} \widetilde{a}$ yield

$$\|R_2\|_{\mathrm{L}^2} \lesssim \|\Lambda^s u\|_{\mathrm{H}^{-1}} \|\widetilde{a}\|_{B^1_{\infty,\infty}} \lesssim \|u\|_{\mathrm{H}^{s-1}} \|Da\|_{\mathrm{L}^\infty} \quad \text{for all} \quad s \in \mathbb{R}.$$

Since low frequencies of a are not involved in the definition of R^3, we have for all $s \in \mathbb{R}$,

$$\|R_3\|_{\mathrm{L}^2} \lesssim \|u\|_{\mathrm{L}^\infty} \|Da\|_{\mathrm{H}^{s-1}}.$$

Remark that if $s < 1 + N/2$, the following inequality is also available

$$\|R_3\|_{\mathrm{L}^2} \lesssim \|u\|_{\mathrm{H}^{s-1}} \|Da\|_{\mathrm{B}^{\frac{N}{2}}_{2,\infty}}.$$

Applying (A.2) with $m = 1$ yields

$$\|R_4\|_{\mathrm{L}^2} \lesssim \|\Lambda^s u\|_{\mathrm{H}^{-1}} \|\widetilde{a}\|_{\mathrm{F}^1_{\infty,2}}.$$

Since Lip $\hookrightarrow \mathrm{F}^1_{\infty,2}$ (see [17]) and $\widetilde{a}$ has no low frequencies, we conclude that

$$\|R_4\|_{\mathrm{L}^2} \lesssim \|u\|_{\mathrm{H}^{s-1}} \|Da\|_{\mathrm{L}^\infty}.$$

We have

$$\|R_5\|_{\mathrm{L}^2} \leq \|R(u,\widetilde{a})\|_{\mathrm{H}^s}.$$

Hence, if $s>0$,

$$\|R_5\|_{\mathrm{L}^2} \lesssim \|u\|_{\mathrm{L}^\infty} \|\widetilde{a}\|_{\mathrm{H}^s} \lesssim \|u\|_{\mathrm{L}^\infty} \|Da\|_{\mathrm{H}^{s-1}},$$

and if $s>-N/2$,

$$\|R_5\|_{\mathrm{L}^2} \lesssim \|u\|_{\mathrm{H}^{s-1}} \|\widetilde{a}\|_{\mathrm{B}^{\frac{N}{2}+1}_{2,\infty}} \lesssim \|u\|_{\mathrm{H}^{s-1}} \|Da\|_{\mathrm{B}^{\frac{N}{2}}_{2,\infty}}.$$

Bounding R^6

We argue as for R^1. Indeed, we have

$$R_6 = \sum_{q=-1}^{0} \underbrace{\Delta_q \Delta_{-1} a\, \Lambda^s \widetilde{\chi}(2^{-q}\mathrm{D})\, \widetilde{\Delta_q} u - \Lambda^s \widetilde{\chi}(2^{-q}\mathrm{D})(\Delta_q \Delta_{-1} a \widetilde{\Delta_q} u)}_{R_q^6},$$

with $\widetilde{\chi} \in C_0^\infty$ equals to one on a wide enough ball centered at the origin.

Denoting $\widetilde{h}_{s,q} := \mathcal{F}^{-1}(\widetilde{\chi}(2^{-q}\cdot)\lambda^s)$ and applying first order Taylor's formula, we get

$$R_q^6 = \int_{\mathbb{R}^N}\int_0^1 \widetilde{h}_{s,q}(y)\, y^* \cdot \nabla \Delta_q \Delta_{-1} a(x-\tau y) \widetilde{\Delta_q} u(x-y)\, dy\, d\tau$$

As $\widetilde{h}_{s,q}$ is in L^1 and R_6 has a finite number of terms, (A.1) entails that

$$\|R_6\|_{\mathrm{L}^2} \lesssim \|D\Delta_{-1} a\|_{\mathrm{L}^\infty} \|S_3 u\|_{\mathrm{L}^2} \lesssim \|Da\|_{\mathrm{L}^\infty} \|u\|_{\mathrm{H}^{s-1}}.$$

The proof of (A.6) and (A.7) is complete. □

Let us now state a more accurate result (which is a variation on an exercise in [1], page 179, see also [13] for similar inequalities pertaining to more general pseudo-differential operators). We recall that the Poisson bracket of two symbols a and b is defined by

$$\{a,b\} = \sum_j \frac{\partial a}{\partial \xi_j} \frac{\partial b}{\partial x_j} - \frac{\partial a}{\partial x_j} \frac{\partial b}{\partial \xi_j}.$$

Lemma A.3 *Take $m \in (-N/2, 1+N/2)$ and $s \in (-N/2-m, N/2+2-m)$. There exists $C = C_{s,m,N}$ such that*

$$\|[a,\Lambda^s]u - \tfrac{1}{i}\{a,\lambda^s\}(\mathrm{D})u\|_{\mathrm{H}^m} \leq C \|\nabla^2 a\|_{\mathrm{B}^{\frac{N}{2}}_{2,\infty} \cap \mathrm{L}^\infty} \|u\|_{\mathrm{H}^{s-2+m}}.$$

Besides, if ($m \in [0,1]$ and $s+m>0$) or ($m>1$ and $s>1$) then we have

$$\|[a,\Lambda^s]u - \tfrac{1}{i}\{a,\lambda^s\}(\mathrm{D})u\|_{\mathrm{H}^m} \leq C\Big(\|u\|_{\mathrm{L}^\infty} \|\nabla^2 a\|_{\mathrm{H}^{s-2+m}} + \|\nabla^2 a\|_{\mathrm{L}^\infty} \|u\|_{\mathrm{H}^{s-2+m}}\Big).$$

Proof. Observe that $\{a,\lambda^s\} = -(\partial_j a)\,(\partial^j\lambda^s)$. (In order to avoid confusion, ∂^j stands for $\partial/\partial\xi_j$, whereas ∂_j means $\partial/\partial x_j$.) Therefore, using Bony's decomposition,

$$[a,\Lambda^s]u-\frac{1}{i}\{a,\lambda^s\}(\mathrm{D})u=\underbrace{[T_a,\Lambda^s]u-i\,T_{\partial_j a}(\partial^j\lambda^s)(\mathrm{D})u}_{R^1}+\underbrace{T_{\Lambda^s u}a}_{R^2}-\underbrace{\Lambda^s T_u a}_{R^3}$$
$$-i\,\underbrace{T_{(\partial^j\lambda^s)(\mathrm{D})u}\partial_j a}_{R^4}+\underbrace{R(\widetilde{a},\Lambda^s u)}_{R^5}-\underbrace{\Lambda^s R(\widetilde{a},u)}_{R^6}-i\,\underbrace{R(\partial_j\widetilde{a},(\partial^j\lambda^s)(\mathrm{D})u)}_{R^7}$$
$$+\underbrace{R(\Delta_{-1}a,\Lambda^s u)-\Lambda^s R(\Delta_{-1}a,u)-iR(\partial_j\Delta_{-1}a,(\partial^j\lambda^s)(\mathrm{D})u)}_{R^8}$$

with $\widetilde{a}:=a-\Delta_{-1}a$.

Bounding R^1

Using the definition of the paraproduct and (A.1), we get $R^1=\sum_{q\geq 0}R^1_q$ with

$$R^1_q:=S_{q-1}a\,\widetilde{\varphi}(2^{-q}\mathrm{D})\Lambda^s\,\Delta_q u-\widetilde{\varphi}(2^{-q}\mathrm{D})\Lambda^s\big(S_{q-1}a\Delta_q u\big)-iS_{q-1}\partial_j a\,\widetilde{\varphi}(2^{-q}\mathrm{D})(\partial^j\lambda^s)(\mathrm{D})\Delta_q u,$$

and $\widetilde{\varphi}\in C_0^\infty$ supported away from the origin and equals to one on a wide enough annulus $C(0,r_1,r_2)$. Using second order Taylor's formula, we gather

$$R^1_q=-\int_{\mathbb{R}^N}\int_0^1 h_{s,q}(y)y^*\cdot D^2S_{q-1}a(x-\tau y)\cdot y\,\Delta_q u(x-y)(1-\tau)\,d\tau\,dy,$$

with $h_{s,q}:=\mathcal{F}^{-1}(\lambda^s\widetilde{\varphi}(2^{-q}\cdot))$.

Therefore,

$$\big\|R^1_q\big\|_{\mathrm{L}^2}\leq\big\||\cdot|^2h_{s,q}\big\|_{\mathrm{L}^1}\big\|D^2S_{q-1}a\big\|_{\mathrm{L}^\infty}\|\Delta_q u\|_{\mathrm{L}^2}.$$

Taking advantage of inequality (A.9) to bound $\big\||\cdot|^2h_{s,q}\big\|_{\mathrm{L}^1}$, we end up with

$$\|R^1\|_{\mathrm{H}^m}\lesssim\big\|\nabla^2a\big\|_{\mathrm{L}^\infty}\|u\|_{\mathrm{H}^{s-2+m}}.$$

Bounding R^2

Standard continuity results for the paraproduct combined with the fact that low frequencies of a are not involved in the definition of R^2 yield

$$\|R^2\|_{\mathrm{H}^m}\lesssim\|\Lambda^s u\|_{\mathrm{H}^{m-2}}\|\widetilde{a}\|_{\mathrm{B}^2_{\infty,\infty}}\lesssim\|u\|_{\mathrm{H}^{s+m-2}}\big\|\nabla^2a\big\|_{\mathrm{L}^\infty}\qquad\text{if}\qquad m<2,$$
$$\|R^2\|_{\mathrm{H}^m}\lesssim\|\Lambda^s u\|_{\mathrm{B}^{-s}_{\infty,\infty}}\|\widetilde{a}\|_{\mathrm{H}^{s+m}}\lesssim\|u\|_{\mathrm{L}^\infty}\|\nabla^2a\|_{\mathrm{H}^{s+m-2}}\qquad\text{if}\qquad s>0.$$

Note that for general (m,s) such that $m-2<N/2$, the following inequality is available:

$$\|R^2\|_{\mathrm{H}^m}\lesssim\|\Lambda^s u\|_{\mathrm{H}^{m-2}}\|\widetilde{a}\|_{\mathrm{B}^{\frac{N}{2}+2}_{2,\infty}}\lesssim\|u\|_{\mathrm{H}^{m+s-2}}\|\nabla^2a\|_{\mathrm{B}^{\frac{N}{2}}_{2,\infty}}.$$

Bounding R^3

Since Λ^s maps H^{s+m} in H^m, and low frequencies of a are not involved in the definition of R^3, we have for all $s\in\mathbb{R}$,

$$\|R_3\|_{\mathrm{H}^m}\lesssim\|u\|_{\mathrm{L}^\infty}\|\nabla^2a\|_{\mathrm{H}^{s-2+m}}.$$

Note that if $s-2+m<N/2$, one also has

$$\|R_3\|_{\mathrm{H}^m}\lesssim\|u\|_{\mathrm{H}^{s-2+m}}\|\nabla^2a\|_{\mathrm{B}^{\frac{N}{2}}_{2,\infty}}.$$

Bounding R^4

Standard continuity results for the paraproduct combined with the fact that $(\partial^j\lambda^s)(\mathrm{D})$ is a pseudo-differential operator of degree $s-1$ yield

$$\|R^4\|_{\mathrm{H}^m} \lesssim \|(\partial^j\lambda^s)(\mathrm{D})u\|_{\mathrm{H}^{m-1}}\|D\widetilde{a}\|_{\mathrm{W}^{1,\infty}} \lesssim \|u\|_{\mathrm{H}^{s+m-2}}\left\|\nabla^2 a\right\|_{\mathrm{L}^\infty} \qquad \text{if} \qquad m<1,$$
$$\|R^4\|_{\mathrm{H}^m} \lesssim \|(\partial^j\lambda^s)(\mathrm{D})u\|_{\mathrm{C}^{1-s}}\|D\widetilde{a}\|_{\mathrm{H}^{s+m-1}} \lesssim \|u\|_{\mathrm{L}^\infty}\|\nabla^2 a\|_{\mathrm{H}^{s+m-2}} \qquad \text{if} \qquad s>1.$$

Actually, if $m=1$, the inequality (A.3) combined with the embedding $\mathrm{Lip} \hookrightarrow \mathrm{F}^1_{\infty,2}$ insures that

$$\|R^4\|_{\mathrm{H}^m} \lesssim \left\|(\partial^j\lambda^s)(\mathrm{D})u\right\|_{\mathrm{L}^2}\|\widetilde{a}\|_{\mathrm{F}^1_{\infty,2}} \lesssim \|u\|_{\mathrm{H}^{s-1}}\left\|\nabla^2 a\right\|_{\mathrm{L}^\infty}.$$

Of course, the following inequality is also available as soon as $m<1+N/2$:

$$\|R^4\|_{\mathrm{H}^m} \lesssim \|(\partial^j\lambda^s)(\mathrm{D})u\|_{\mathrm{H}^{m-1}}\|\widetilde{a}\|_{\mathrm{B}^{2+\frac{N}{2}}_{2,\infty}} \lesssim \|u\|_{\mathrm{H}^{s+m-2}}\|\nabla^2 a\|_{\mathrm{B}^{\frac{N}{2}}_{2,\infty}}.$$

Bounding R^5

Basic results of continuity for the remainder insure that for $s\in\mathbb{R}$ and $m>0$, we have

$$\|R^5\|_{\mathrm{H}^m} \lesssim \|\widetilde{a}\|_{\mathrm{H}^{s+m}}\|\Lambda^s u\|_{\mathrm{B}^{-s}_{\infty,\infty}} \lesssim \|\nabla^2 a\|_{\mathrm{H}^{s+m-2}}\|u\|_{\mathrm{L}^\infty}.$$

Note that since Λ^s maps L^∞ in $\mathrm{F}^{-s}_{\infty,2}$, (A.2) insures that we also have

$$\left\|R^5\right\|_{\mathrm{L}^2} \lesssim \|\nabla^2 a\|_{\mathrm{H}^{s-2+m}}\|u\|_{\mathrm{L}^\infty}.$$

Of course, if $m>-N/2$, one has for all $s\in\mathbb{R}$,

$$\|R^5\|_{\mathrm{H}^m} \lesssim \|u\|_{\mathrm{H}^{s+m-2}}\|\nabla^2 a\|_{\mathrm{B}^{\frac{N}{2}}_{2,\infty}}.$$

Bounding R^6

If $s+m>0$, we have

$$\|R^6\|_{\mathrm{H}^m} \lesssim \|u\|_{\mathrm{H}^{s-2+m}}\left\|\nabla^2 a\right\|_{\mathrm{L}^\infty},$$

whereas if $s+m+N/2>0$,

$$\|R^6\|_{\mathrm{H}^m} \lesssim \|u\|_{\mathrm{H}^{s-2+m}}\|\nabla^2 a\|_{\mathrm{B}^{\frac{N}{2}}_{2,\infty}}.$$

Bounding R^7

Since $(\partial^j\lambda^s)$ is a homogeneous multiplier of degree $s-1$, we easily get for $s\in\mathbb{R}$ and $m>0$:

$$\|R^7\|_{\mathrm{H}^m} \lesssim \|D\widetilde{a}\|_{\mathrm{H}^{s-1+m}}\|(\partial^j\lambda^s)(\mathrm{D})u\|_{\mathrm{C}^{1-s}} \lesssim \|\nabla^2 a\|_{\mathrm{H}^{s-2+m}}\|u\|_{\mathrm{L}^\infty}$$

Note that (A.2) also insures that

$$\left\|R^7\right\|_{\mathrm{L}^2} \lesssim \|\nabla^2 a\|_{\mathrm{H}^{s-2}}\|u\|_{\mathrm{L}^\infty}.$$

Finally, if $m>-N/2$, standard results of continuity for the remainder give

$$\|R^7\|_{\mathrm{H}^m} \lesssim \|u\|_{\mathrm{H}^{s-2+m}}\|\nabla^2 a\|_{\mathrm{B}^{\frac{N}{2}}_{2,\infty}}.$$

Bounding R^8

According to (A.1), we have for a convenient $\widetilde{\chi}\in C_0^\infty$,

$$R^8=\sum_{q\leq 0}\underbrace{\Delta_{-1}\Delta_q a\Lambda^s\widetilde{\chi}(\mathrm{D})\widetilde{\Delta}_q u-\Lambda^s\widetilde{\chi}(\mathrm{D})\big(\Delta_{-1}\Delta_q a\widetilde{\Delta}_q u\big)-i\Delta_{-1}\Delta_q\partial_j a\,\widetilde{\chi}(\mathrm{D})(\partial^j\lambda^s)(\mathrm{D})\widetilde{\Delta}_q u}_{R_q^8}.$$

Now, arguing as for bounding $\|R_q^1\|_{\mathrm{L}^2}$, we get

$$R_q^8(x)=\iint_0^1\big(\mathcal{F}^{-1}(\lambda^s\widetilde{\chi})\big)(y)y^*\cdot D^2\Delta_{-1}\Delta_q a(x-\tau y)\cdot y\,\widetilde{\Delta_q}u(x-y)\tau\,d\tau\,dy.$$

Since there are only a finite number of R_q^8 each of them being bounded by

$$C\left\||\cdot|^2\mathcal{F}^{-1}(\lambda^s\widetilde{\chi})\right\|_{\mathrm{L}^1}\left\|D^2\Delta_{-1}a\right\|_{\mathrm{L}^\infty}\left\|\widetilde{\Delta_q}u\right\|_{\mathrm{L}^2},$$

we easily conclude that

$$\|R^8\|_{\mathrm{H}^m}\lesssim\left\|D^2\Delta_{-1}a\right\|_{\mathrm{L}^\infty}\|S_3u\|_{\mathrm{L}^2}\lesssim\left\|\nabla^2 a\right\|_{\mathrm{L}^\infty}\|u\|_{\mathrm{H}^{s-2+m}}.$$

□

Let us state an ultimate commutator estimate.

Lemma A.4 *Let* $\eta\in[0,1)$ *and* $A\in\mathcal{C}^\infty(\mathbb{R}^N\setminus\{0\})$ *be a homogeneous function of degree* $m\in[0,1-\eta]$. *Then there exists a constant* C *depending only on* A, η, m *and* N *such that*

$$\|[A(\mathrm{D}),b]u\|_{\mathrm{L}^2}\leq C\|Db\|_{\mathrm{C}^{-\eta}}\|b\|_{\mathrm{H}^{m-1+\eta}}.$$

with the convention $\|Db\|_{\mathrm{C}^0}:=\|Db\|_{\mathrm{L}^\infty}$.

Proof. Let us denote $\widetilde{A}(\mathrm{D}):=(\mathrm{Id}-\Delta_{-1})A(\mathrm{D})$. We have

$$\begin{aligned}[A(\mathrm{D}),b]u=[\widetilde{A}(\mathrm{D}),T_b]u+\Delta_{-1}A(\mathrm{D})T_bu-T_b\Delta_{-1}A(\mathrm{D})u&\\ +A(\mathrm{D})T_ub-T_{A(\mathrm{D})u}b+A(\mathrm{D})R(u,b)-R(A(\mathrm{D})u,b).&\end{aligned}$$

By using first order Taylor's formula and that the function $A(1-\chi)$ is smooth and homogeneous of degree m outside a small ball centered at the origin and $\eta\geq 0$, it is easy to show that

$$\left\|[\widetilde{A}(\mathrm{D}),T_b]u\right\|_{\mathrm{L}^2}\leq C\|Db\|_{\mathrm{C}^{-\eta}}\|u\|_{\mathrm{H}^{m-1+\eta}}.$$

Since only low frequencies are involved in the operator $A(\mathrm{D})\Delta_{-1}$, we obviously have

$$\begin{aligned}\|\Delta_{-1}A(\mathrm{D})T_bu\|_{\mathrm{L}^2}&\lesssim\|T_bu\|_{\mathrm{H}^{m-1+\eta}}&&\lesssim\|b\|_{\mathrm{L}^\infty}\|u\|_{\mathrm{H}^{m-1+\eta}},\\ \|T_b\Delta_{-1}A(\mathrm{D})u\|_{\mathrm{L}^2}&\lesssim\|b\|_{\mathrm{L}^\infty}\|\Delta_{-1}A(\mathrm{D})u\|_{\mathrm{L}^2}&&\lesssim\|b\|_{\mathrm{L}^\infty}\|u\|_{\mathrm{H}^{m-1+\eta}}.\end{aligned}$$

Using (A.2), (A.3) and standard results of continuity for the paraproduct and the remainder, we also have (provided $\eta<1$ and $m\in[0,1-\eta]$)

$$\begin{aligned}\|A(\mathrm{D})T_ub\|_{\mathrm{L}^2}&\lesssim\|T_ub\|_{\mathrm{H}^m}&&\lesssim\|u\|_{\mathrm{H}^{m-1+\eta}}\|b\|_{\mathrm{C}^{1-\eta}},\\ \left\|T_{A(\mathrm{D})u}b\right\|_{\mathrm{L}^2}&\lesssim\|A(\mathrm{D})u\|_{\mathrm{H}^{-1+\eta}}\|b\|_{\mathrm{C}^{1-\eta}}&&\lesssim\|u\|_{\mathrm{H}^{m-1+\eta}}\|b\|_{\mathrm{C}^{1-\eta}},\\ \|A(\mathrm{D})R(u,b)\|_{\mathrm{L}^2}&\lesssim\|R(u,b)\|_{\mathrm{H}^m}&&\lesssim\|u\|_{\mathrm{H}^{m-1+\eta}}\|b\|_{\mathrm{C}^{1-\eta}},\\ \|R(A(\mathrm{D})u,b)\|_{\mathrm{L}^2}&\lesssim\|A(\mathrm{D})u\|_{\mathrm{H}^{-1+\eta}}\|b\|_{\mathrm{C}^{1-\eta}}&&\lesssim\|u\|_{\mathrm{H}^{m-1+\eta}}\|b\|_{\mathrm{C}^{1-\eta}},\end{aligned}$$

whence

$$\|[A(\mathrm{D}),b]u\|_{\mathrm{L}^2}\leq C(\|b\|_{\mathrm{L}^\infty}+\|Db\|_{\mathrm{C}^{-\eta}})\|u\|_{\mathrm{H}^{m-1+\eta}}.$$

In order to eliminate the term $\|b\|_{\mathrm{L}^\infty}$, it suffices to apply the above inequality to $b_\lambda:=b(\lambda\cdot)$ and $u_\lambda:=u(\lambda\cdot)$ and to make λ tend to infinity. □

B Tame estimates for the product or composition of functions

This section is devoted to the proof of various tame estimates which have been used repeatedly throughout the paper. Let us first state estimates for the product of two functions.

Lemma B.1 *For all $s \geq 0$ and $k \in \mathbb{N}$, there exists a constant C depending only on s, k and N and such that*

$$\|uv\|_{\mathrm{H}^s} \leq C\Big(\|u\|_{\mathrm{L}^\infty}\|v\|_{\mathrm{H}^s} + \|v\|_{\mathrm{L}^\infty}\|D^k u\|_{\mathrm{H}^{s-k}}\Big).$$

Proof. The case $k = 0$ is standard (see e.g [1]). We rule out the case $s = 0$ which is trivial. For proving the general case $k \geq 1$ and $s > 0$, one can take advantage of Bony's decomposition

$$uv = T_u v + R(u,v) + T_v u. \tag{B.10}$$

It is well known that for all $s > 0$, we have

$$\|T_u v\|_{\mathrm{H}^s} + \|R(u,v)\|_{\mathrm{H}^s} \lesssim \|u\|_{\mathrm{L}^\infty}\|v\|_{\mathrm{H}^s}.$$

In order to bound the last term, we notice that $T_v u = T_v(\mathrm{Id} - \chi(\mathrm{D}))u$ where the low frequency cut-off χ has been defined in the previous section. Hence

$$\|T_v u\|_{\mathrm{H}^s} \lesssim \|v\|_{\mathrm{L}^\infty}\|(\mathrm{Id} - \chi(\mathrm{D}))u\|_{\mathrm{H}^s}.$$

Now, since

$$\|(\mathrm{Id} - \chi(\mathrm{D}))u\|_{\mathrm{H}^s}^2 = \int (1-\chi(\xi))^2\left(\frac{1+|\xi|^2}{|\xi|^2}\right)^k (1+|\xi|^2)^{s-k}|\widehat{D^k u}(\xi)|^2\,d\xi,$$

we readily have $\|(\mathrm{Id} - \chi(\mathrm{D}))u\|_{\mathrm{H}^s} \lesssim \|D^k u\|_{\mathrm{H}^{s-k}}$. □

The following variation on Lemma B.1 is also needed.

Lemma B.2 *For all $s \geq 0$ and $k \in \mathbb{N}$, there exists a constant C depending only on s, k and N and such that*

$$\|uv\|_{\mathrm{H}^s} \leq C\Big(\|u\|_{\mathrm{L}^\infty}\|v\|_{\mathrm{H}^s} + \|v\|_{\mathrm{C}^{-1}}\|D^k u\|_{\mathrm{H}^{s+1-k}}\Big).$$

Proof. Decompose uv as in (B.10) and bound the first two terms as in Lemma B.1. For the third one, one uses that

$$\|T_v u\|_{\mathrm{H}^s} \lesssim \|v\|_{\mathrm{C}^{-1}}\|u\|_{\mathrm{H}^{s+1}}.$$

Of course, as $T_v u = T_v(\mathrm{Id} - \chi(\mathrm{D}))u$, the term $\|u\|_{\mathrm{H}^{s+1}}$ may be replaced by $\|D^k u\|_{\mathrm{H}^{s+1-k}}$. □

Lemma B.3 *Take $s < 0$. There exists a constant C depending only on s and N, and such that*

$$\|uv\|_{\mathrm{H}^s} \leq C\|u\|_{\mathrm{C}^{|s|}}\|v\|_{\mathrm{H}^s}.$$

Proof. The proof still relies on Bony decomposition for uv. Since $s < 0$, standard results of continuity for the paraproduct yield

$$\|T_u v\|_{\mathrm{H}^s} + \|T_v u\|_{\mathrm{H}^s} \leq C\,\|u\|_{\mathrm{L}^\infty}\|v\|_{\mathrm{H}^s}.$$

For bounding the remainder, we use that $\mathrm{L}^2 \hookrightarrow \mathrm{H}^s$ and inequality (A.2). Since $\mathrm{C}^{|s|} \hookrightarrow \mathrm{F}^{|s|}_{\infty,2}$, we discover that

$$\|R(u,v)\|_{\mathrm{H}^s} \leq C\|u\|_{\mathrm{C}^{|s|}}\|v\|_{\mathrm{H}^s},$$

which completes the proof of (B.3). □

We now turn to the proof of composition estimates. Let us first recall the following lemma (the proof of which may be found in e.g [1]).

Lemma B.4 *Let I, J be two intervals of $\mathbb{R}$ with $J \subset\subset I$ and I open. Let $s \geq 0$ and σ be the smallest integer such that $\sigma \geq s$. Let F be in $\mathrm{W}^{\sigma+1,\infty}(\mathbb{R}; I)$ and satisfy $F(0) = 0$. Assume that $v \in \mathrm{H}^s$ has values in J. There exists a constant $C = C_{s,I,J,N}$ such that*

$$\|F(v)\|_{\mathrm{H}^s} \leq C\big(1+\|v\|_{\mathrm{L}^\infty}\big)^\sigma \|F'\|_{\mathrm{W}^{\sigma,\infty}(I)} \|v\|_{\mathrm{H}^s}.$$

Since we often manipulate bounded functions which need not be in Sobolev spaces but whose gradient does belong to a Sobolev space, the following improvement of Lemma B.4 is very useful.

Proposition B.1 *Let I, J be two open intervals of $\mathbb{R}$ with $J \subset\subset I$. Take $m \in \mathbb{N}^*$, $s > -m$ and let σ be the smallest integer such that $\sigma \geq s$. Take $F \in \mathrm{W}^{\sigma+m+1,\infty}_{loc}$ and let v be valued in J and such that $D^m v \in \mathrm{H}^s$. There exists a constant $C = C_{s,I,J,N}$ such that*

$$\|D^m(F(v))\|_{\mathrm{H}^s} \leq C\big(1+\|v\|_{\mathrm{L}^\infty}\big)^{\sigma+m} \|F'\|_{\mathrm{W}^{\sigma+m,\infty}(I)} \|D^m v\|_{\mathrm{H}^s}.$$

Proof.

1. As a warm-up, let us estimate $D^k F(v)$ for $k \in \mathbb{N}^*$.

   For any multi-index α of length k, Faá-di-Bruno's formula yields

   $$\partial^\alpha\big(F(v)\big) = \sum_{j=1}^{k} \sum_{\alpha_1+\cdots+\alpha_j=\alpha} \left(c_{\alpha_1,\cdots,\alpha_j} F^{(j)}(v) \prod_{i=1}^{j} \partial_{\alpha_i} v \right)$$

   where the coefficients $c_{\alpha_1,\cdots,\alpha_j}$ are positive integers whose value does not matter here.

   Combining Hölder and Gagliardo-Nirenberg inequalities, we get

   $$\begin{aligned} \left\| F^{(j)}(v) \textstyle\prod_{i=1}^{j} \partial_{\alpha_i} v \right\|_{\mathrm{L}^2} &\leq \left\|F^{(j)}(v)\right\|_{\mathrm{L}^\infty} \textstyle\prod_{i=1}^{j} \|\partial_{\alpha_i} v\|_{\mathrm{L}^{\frac{2k}{|\alpha_i|}}}, \\ &\leq \left\|F^{(j)}(v)\right\|_{\mathrm{L}^\infty} \textstyle\prod_{i=1}^{j} \|v\|_{\mathrm{L}^\infty}^{1-\frac{|\alpha_i|}{k}} \left\|D^k v\right\|_{\mathrm{L}^2}^{\frac{|\alpha_i|}{k}}, \end{aligned}$$

   whence

   $$\|D^k\big(F(v)\big)\|_{\mathrm{L}^2} \leq C_k \left\|D^k v\right\|_{\mathrm{L}^2} \sum_{j=0}^{k-1} \|v\|_{\mathrm{L}^\infty}^j \left\|F^{(j+1)}(v)\right\|_{\mathrm{L}^\infty}. \tag{B.11}$$

2. Assume that s is a nonnegative integer. Since

   $$\|D^m F(v)\|_{\mathrm{H}^s} \leq \sum_{k=m}^{s+m} \|D^k\big(F(v)\big)\|_{\mathrm{L}^2},$$

   inequality (B.11) readily yields the estimate in Proposition B.1.

3. We now have to prove Proposition B.1 for general $s > -1$. This is actually an easy variation on the proof of Lemma B.4 based on Meyer's first linearization method.

   Of course, one can change F for a function $\widetilde{F} \in \mathrm{W}^{\sigma+m+1,\infty}_{loc}(\mathbb{R})$ compactly supported in I and such that $\widetilde{F} \equiv F$ on a neighborhood of J. In what follows, we denote $\widetilde{F}$ by F.

   Decompose $F(v)$ into

   $$F(v) = F(S_0 v) + \sum_{p\geq 0} F(S_{p+1}) - F(S_p v).$$

According to first order Taylor's formula, we have

$$F(S_{p+1})-F(S_pv)=m_p\Delta_pv \quad \text{with} \quad m_p:=\int_0^1 F'(S_pv+\tau\Delta_pv)\,d\tau.$$

One can easily prove that the m_p's are Meyer multipliers, namely

$$\text{(B.12)} \qquad \forall k\in\{0,\cdots,\sigma+m\},\ \left\|D^km_p\right\|_{\mathrm{L}^\infty}\leq C_k2^{pk}(1+\|v\|_{\mathrm{L}^\infty})^k\|F'\|_{\mathrm{W}^{k,\infty}}.$$

Take $q\geq -1$. According to the above equality, $\Delta_q\big(F(v)-F(S_0v)\big)$ decomposes into

$$\Delta_q\big(F(v)-F(S_0v)\big)=\underbrace{\sum_{0\leq p\leq q}\Delta_q\big(m_p\,\Delta_pv\big)}_{\Delta_q^1}+\underbrace{\sum_{p\geq q+1}\Delta_q\big(m_p\,\Delta_pv\big)}_{\Delta_q^2}.$$

Mimicking the proof given in [1] and taking advantage of the Bernstein's inequality:

$$\exists C>0,\ \forall p\in\mathbb{N},\ \|\Delta_pv\|_{\mathrm{L}^2}\leq C2^{-pm}\left\|\Delta_pD^mv\right\|_{\mathrm{L}^2},$$

we easily get

$$\begin{aligned}
&2^{q(s+m)}\left\|\Delta_q^1\right\|_{\mathrm{L}^2}\leq C\textstyle\sum_{0\leq p\leq q}\big(2^{-p(\sigma+m)}\left\|D^{\sigma+m}m_p\right\|_{\mathrm{L}^\infty}\big)\big(2^{ps}\left\|D^m\Delta_pv\right\|_{\mathrm{L}^2}\big)\,2^{(p-q)(\sigma-s)},\\
&2^{q(s+m)}\left\|\Delta_q^2\right\|_{\mathrm{L}^2}\leq C\textstyle\sum_{p\geq q+1}\|m_p\|_{\mathrm{L}^\infty}\big(2^{ps}\left\|D^m\Delta_pv\right\|_{\mathrm{L}^2}\big)\,2^{(q-p)(s+m)}
\end{aligned}$$

which, in view of (B.12) eventually[6] leads to

$$\text{(B.13)} \qquad \begin{aligned}\|D^m\big(F(v)-F(S_0v)\big)\|_{\mathrm{H}^s}&\leq \|F(v)-F(S_0v)\|_{\mathrm{H}^{s+m}},\\ &\lesssim (1+\|v\|_{\mathrm{L}^\infty})^{\sigma+m}\|F'\|_{\mathrm{W}^{\sigma+m,\infty}}\|D^mv\|_{\mathrm{H}^s}.\end{aligned}$$

In order to bound the "low frequency" part $D\big(F(S_0v)\big)$, we use the previous step of the proof with S_0v and the integer σ. This yields

$$\|D^m\big(F(S_0v)\big)\|_{\mathrm{H}^s}\leq\|D^m\big(F(S_0v)\big)\|_{\mathrm{H}^\sigma}\leq C(1+\|v\|_{\mathrm{L}^\infty})^{\sigma+m-1}\|F'\|_{\mathrm{W}^{\sigma+m-1,\infty}}\|D^mS_0v\|_{\mathrm{H}^\sigma},$$

whence the desired inequality since $\|D^mS_0v\|_{\mathrm{H}^\sigma}\lesssim\|D^mv\|_{\mathrm{H}^s}$.

□

Corollary B.1 *Let v, I and F satisfy the assumptions of Proposition B.1 with $s>0$ and $m=1$. Let σ be the smallest integer such that $\sigma\geq s$. Assume that w is bounded and that $Dw\in\mathrm{H}^s$. There exists a constant $C=C_{s,I,J,N}$ such that*

$$\|F(v)Dw\|_{\mathrm{H}^s}\leq C\Big(\|F(v)\|_{\mathrm{L}^\infty}\|Dw\|_{\mathrm{H}^s}+\|w\|_{\mathrm{L}^\infty}\big(1+\|v\|_{\mathrm{L}^\infty}\big)^{\sigma+1}\|F'\|_{\mathrm{W}^{\sigma+1,\infty}(I)}\|Dv\|_{\mathrm{H}^s}\Big).$$

Proof. According to Lemma B.2, we have

$$\|F(v)Dw\|_{\mathrm{H}^s}\lesssim\|F(v)\|_{\mathrm{L}^\infty}\|Dw\|_{\mathrm{H}^s}+\|Dw\|_{\mathrm{C}^{-1}}\|DF(v)\|_{\mathrm{H}^s}.$$

Using that $\|Dw\|_{\mathrm{C}^{-1}}\lesssim\|w\|_{\mathrm{L}^\infty}$ and applying Proposition B.1 yields the desired inequality. □

The following variation on Corollary B.1 and Proposition B.1 will prove to be also very useful.

[6] remind that $\sigma>s$ and $s>-m$

Corollary B.2 *Take $k\in\mathbb{N}$ and $s\geq 0$. Let F be as in Proposition B.1. We have*

$$\|F(v)w\|_{\mathrm{H}^s}\leq C\Big(\|F(v)\|_{\mathrm{L}^\infty}\,\|w\|_{\mathrm{H}^s}+\|w\|_{\mathrm{L}^\infty}\,(1+\|v\|_{\mathrm{L}^\infty})^{\sigma+k}\|F'\|_{\mathrm{W}^{\sigma+k,\infty}(I)}\|D^kv\|_{\mathrm{H}^{s-k}}\Big).$$

Proof. Combine Lemma B.1 and Proposition B.1. □

Corollary B.3 *Let I,J be two intervals of $\mathbb{R}$ with $J\subset\subset I$, and v and w be two J-valued functions. The following a priori estimates hold true.*

- *If $s>0$ and $F\in\mathrm{W}^{\sigma+2,\infty}(I)$ where σ is the smallest integer such that $\sigma\geq s$ then*

  $$\|F(w)-F(v)\|_{\mathrm{H}^s}\leq C\Big(\|F'\|_{\mathrm{L}^\infty(J)}\|w-v\|_{\mathrm{H}^s}$$
  $$+\|w-v\|_{\mathrm{L}^\infty}\big(1+\sup_{\tau\in[0,1]}\|v+\tau(w-v)\|_{\mathrm{L}^\infty}\big)^{\sigma+k}\|F'\|_{\mathrm{W}^{\sigma+k,\infty}(I)}\big(\|D^kv\|_{\mathrm{H}^{s-k}}+\|D^k(w-v)\|_{\mathrm{H}^{s-k}}\big)\Big)$$

  for some $C=C_{s,k,I,J,N}$.

- *If $F\in\mathrm{W}^{1,\infty}(J)$ then $\|F(w)-F(v)\|_{\mathrm{L}^2}\leq\|F'\|_{\mathrm{L}^\infty(J)}\|w-v\|_{\mathrm{L}^2}$.*

Proof. According to first order Taylor's formula, we have

$$F(w)-F(v)=\int_0^1(w-v)F'(v+\tau(w-v))\,d\tau.$$

Therefore,

$$\|F(w)-F(v)\|_{\mathrm{H}^s}\leq\int_0^1\|(w-v)F'(v+\tau(w-v))\|_{\mathrm{H}^s}\,d\tau,$$

which implies the desired result if $s=0$. The case $s>0$ readily stems from Corollary B.2. □

C Mollifiers

The following lemma which is a straightforward extension of Lemma 5 in [6] to the multidimensional case is used repeatedly in the regularization process of system (EK).

Lemma C.1 *Let $\chi\in\mathcal{S}(\mathbb{R}^N)$ be such that $\widehat{\chi}$ is compactly supported and equals to 1 in a neighborhood of 0. For $\eta>0$, denote $\chi_\eta:=\eta^{-N}\chi(\eta^{-1}\cdot)$. Then we have the following results.*

(i) There exists a constant C such that for all $s\in\mathbb{R}$ and $f\in\mathrm{H}^s(\mathbb{R}^N)$, we have

$$\|\chi_\eta*f\|_{\mathrm{H}^{s+\sigma}}\leq C\eta^{-\sigma}\|f\|_{\mathrm{H}^s}\quad\textit{for all}\quad\sigma\geq 0\quad\textit{and}\quad\eta\in(0,1).$$

(ii) There exists a constant C such that for all $s\in\mathbb{R}$ and $f\in\mathrm{H}^s(\mathbb{R}^N)$, we have

$$\|f-\chi_\eta*f\|_{\mathrm{H}^{s-\sigma}}\leq C\eta^{\sigma}\|f\|_{\mathrm{H}^s}\quad\textit{for all}\quad\sigma\geq 0\quad\textit{and}\quad\eta\in(0,1).$$

(iii) For all $s\in\mathbb{R}$, $\sigma\geq 0$ and $f\in\mathrm{H}^s(\mathbb{R}^N)$, we have

$$\|f-\chi_\eta*f\|_{\mathrm{H}^{s-\sigma}}=o(\eta^\sigma)\quad\textit{when}\quad\eta\quad\textit{goes to}\quad 0.$$

Besides, if $(f^n)_{n\in\mathbb{N}}$ tends to f in H^s then for all $\sigma\geq 0$,

$$\eta^{-\sigma}\|f^n-\chi_\eta*f^n\|_{\mathrm{H}^{s-\sigma}}\to 0$$

uniformly for $n\in\mathbb{N}$ when η goes to zero.

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Index of symbols

• **Miscellaneous (greek)**