HIGHER-ORDER OPTIMAL EDGE FINITE ELEMENT FOR HYBRID MESHES

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We propose $H(div)$ and $H(curl)$ finite elements on hexahedra, prisms and pyramids based on Nédélec's first family, providing an optimal rate of the convergence in $H(div)$ and $H(curl)$-norm respectively. A comparison with other existing elements found in the literature is performed. Numerical results show the good behaviour of these new finite elements.

We study $H(curl)$ and $H(div)$ elements of Nédélec's first family which are well-known in the case of tetrahedra, prisms, and hexahedra (see for example [5]). When non-affine elements are considered, neither $H(curl)$ nor $H(div)$ elements of the Nédélec's first family are providing an optimal rate of the convergence of the numerical solution toward the solution of the exact problem in $H(curl)$-norm and $H(div)$-norm respectively. Following the principle set in [1] for quadrilateral elements, we propose new finite element spaces for pyramids, prisms, and hexahedra to recover the optimal convergence at any order of approximation.

For the $H(curl)$-approximation, the same space as in [2] has been found for $r = 1$ for hexahedral elements. The prismatic space can also be found as a combination of optimal quadrilateral and triangular elements. The pyramidal optimal finite element space $\hat{P}_r(\hat{K})$ is the following

$$\hat{P}_r(\hat{K}) = \left(\mathbb{P}_r(x,y,z) \oplus \sum_{0\leq k\leq r-1} \mathbb{P}_k(x,y) \left(\frac{xy}{1-z}\right)^{r-k}\right)^3 \oplus \left\{ \begin{bmatrix} \frac{\hat{x}^p\,\hat{y}^{p+1}}{(1-\hat{z})^{p+1}} \\ \frac{\hat{x}^{p+1}\,\hat{y}^{p}}{(1-\hat{z})^{p+1}} \\ \frac{\hat{x}^{p+1}\,\hat{y}^{p+1}}{(1-\hat{z})^{p+2}} \end{bmatrix} \quad 0 \leq p \leq r-1 \right\}$$
$$\oplus \left\{ \begin{bmatrix} \frac{\hat{x}^m\,\hat{y}^{n+2}}{(1-\hat{z})^{m+1}} \\ 0 \\ \frac{\hat{x}^{m+1}\hat{y}^{n+2}}{(1-\hat{z})^{m+2}} \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \frac{\hat{x}^{n+2}\,\hat{y}^{m}}{(1-\hat{z})^{m+1}} \\ \frac{\hat{x}^{n+2}\hat{y}^{m+1}}{(1-\hat{z})^{m+2}} \end{bmatrix} \atop 0 \leq m \leq n \leq r-2 \right\} \oplus \left\{ \begin{bmatrix} \frac{\hat{x}^p\,\hat{y}^{q}}{(1-\hat{z})^{p+q-r}} \\ 0 \\ \frac{\hat{x}^{p+1}\,\hat{y}^{q}}{(1-\hat{z})^{p+q+1-r}} \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \frac{\hat{x}^{q}\,\hat{y}^{p}}{(1-\hat{z})^{p+q-r}} \\ \frac{\hat{x}^{q}\,\hat{y}^{p+1}}{(1-\hat{z})^{p+q+1-r}} \end{bmatrix} \atop 0 \leq p \leq r-1,\ 0 \leq q \leq r+1 \right\}$$

For $H(div)$ elements, we have constructed super-optimal finite elements space satisfying the principles of [2], and also optimal finite elements such that the finite element space contains $D_r = \mathbb{P}^3_{r-1} + [x,y,z]^t\,\tilde{\mathbb{P}}_{r-1}$. The latter spaces are simpler to use, and their dimension is very close to super-optimal spaces.

"Nodal" basis functions and hierarchical basis functions have been constructed for all these spaces. A pyramidal finite element space compatible with classical hexahedral and prismatic elements of the Nédélec's first family in $H(curl)$ and $H(div)$ can also be constructed.

A comparison of the dispersion error obtained with a mesh made of a repeated cell composed of non-affine pyramids and affine pyramids for the finite element constructed for this space with other existing elements found in the literature ([3], [4], [6], [7]) is performed (see Fig. 1).

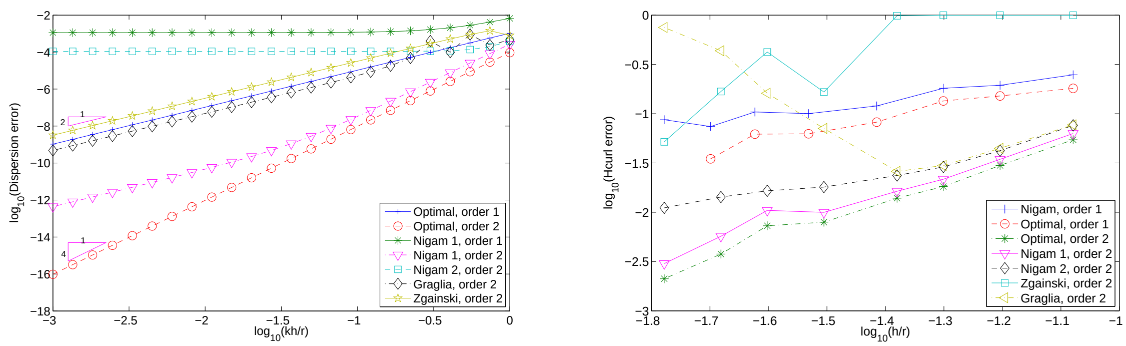

Figure 1: Dispersion error in log-log scale (left) and $H(curl)$-error in log-log scale with a Gaussian source inside a cubic cavity for Maxwell's equations (right).

Our finite elements have been tested on general hybrid meshes for Maxwell's equations and Helmholtz equation and give accurate results as expected.

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