GENERATION OF HIGH-ORDER POLYNOMIAL BASES OF NÉDÉLEC H(CURL) FINITE ELEMENTS FOR MAXWELL'S EQUATIONS

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Abstract

The purpose of this study is the construction of a vectorial polynomial base for the Nédélec mixed finite element [3]. We also aim to build automatically a code written in fortran90 for the elementary mass ans stiffness matrices. To this end, it is essential to use a symbolic calculus tool (here, Maple), so that the input data for the development of such a finite element of order k, are the number k itself and the k-order mesh made of triangles or tetrahedra. In particular the main motive is to generate automatically finite elements, and never to mention the expression of the polynomial basis functions, which is attached to the symbolic calculus: **there is no practical interest to the representation of basis functions**.

Introduction

This article is concerned with the need to increase the order of finite element methods in electromagnetism [1],[2],[5]. The aim is here to determine polynomial basis functions of triangular and tetrahedral elements for any order k. For this purpose, we need first to express the polynomial space of start R^k for any order k, then to give the mathematical expression of degrees of freedom, and finally to solve the unisolvance system. For all these steps, we use a symbolic mathematical program.

1 Higher-order finite element of order of $H(curl)$ Finite element of order k in $\mathbb{R}^3$ on tetrahedra

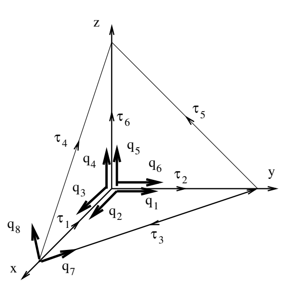

Figure 1: Reference tetrahedron

We recall notations of some polynomial spaces used to define Nédélec finite elements:
$\mathbb{P}_k$ linear space of polynomials of degree $\leq k$,
$\tilde{\mathbb{P}}_k$ linear space of homogeneous polynomials of degree k,
$S^k = \{u \in (\tilde{\mathbb{P}}_k)^3, u_1x + u_2y + u_3z = 0\}$,
$R^k = (\mathbb{P}_{k-1})^3 \oplus S^k$ linear space of polynomials for the finite element of order k and class $H(curl)$.

2 Space of polynomials and degrees of freedom of k-order $H(curl)$ finite element

Our interest is the k-order finite element built on a tetrahedron (see figure 1). In the sequel we use classical notations of finite element: $\hat{K}$ is the reference tetrahedron with nodes $(0,0,0), (1,0,0), (0,1,0), (0,0,1)$, K is any tetrahedron of the mesh, $\hat{f}$ a face of $\hat{K}$, f a face of K, etc.

2.1 *Characterizing and determining S^k*

To built R^k, we first need to determine explicitly S^k. We prove that S^k is entirely describe by the following polynomials:

$$\text{for } m+n \leq k : \begin{pmatrix} \hat{x}^{m-1}\hat{y}^n\hat{z}^{k-m-n+1} \\ 0 \\ -\hat{x}^m\hat{y}^n\hat{z}^{k-m-n} \end{pmatrix}, \quad (1)$$

$$\text{for } m+n = k+1 : \begin{pmatrix} \hat{x}^{m-1}\hat{y}^n \\ -\hat{x}^m\hat{y}^{n-1} \\ 0 \end{pmatrix}, \quad (2)$$

$$\text{for } m+n \leq k : \begin{pmatrix} 0 \\ \hat{x}^m\hat{y}^{n-1}\hat{z}^{k-m-n+1} \\ -\hat{x}^m\hat{y}^n\hat{z}^{k-m-n} \end{pmatrix}. \quad (3)$$

2.2 *Definition of the degrees of freedom*

We put:

$$\overrightarrow{\hat{p}} = \begin{pmatrix} \hat{p}_1(\hat{x},\hat{y},\hat{z}) \\ \hat{p}_2(\hat{x},\hat{y},\hat{z}) \\ \hat{p}_3(\hat{x},\hat{y},\hat{z}) \end{pmatrix}.$$

Degrees of freedom of edge type $\int_\Gamma \overrightarrow{p} \cdot \overrightarrow{\tau} d\gamma$

If k is the order of the element, the number of edge degrees of freedom is $n_e = 6k$ and they are:
for $m = 1, k$

$$\int_0^1 -\hat{p}_1(1-\hat{x},0,0)\hat{x}^{m-1}d\hat{x} \quad (4)$$

$$\int_0^1 \hat{p}_2(0,\hat{y},0)\hat{y}^{m-1}d\hat{y} \tag{5}$$

$$\int_0^1 (\hat{p}_1(\hat{x},1-\hat{x},0)-\hat{p}_2(\hat{x},1-\hat{x},0))\hat{x}^{m-1}d\hat{x} \tag{6}$$

$$\int_0^1 (-\hat{p}_1(1-\hat{x},0,\hat{x})+\hat{p}_3(1-\hat{x},0,\hat{x}))\hat{x}^{m-1}d\hat{x} \tag{7}$$

$$\int_0^1 (\hat{p}_2(0,\hat{y},1-\hat{y})-\hat{p}_3(0,\hat{y},1-\hat{y}))\hat{y}^{m-1}d\hat{y} \tag{8}$$

$$\int_0^1 \hat{p}_3(0,0,\hat{z})\hat{z}^{m-1}d\hat{z} \tag{9}$$

Degrees of freedom of volume type

If k is the order of the element, the number of volume degrees of freedom is $n_v = k(k-1)(k-2)/2$ and they are:
for $m+n+l \leq k-3$,

$$\int_0^1\int_0^{1-\hat{x}}\int_0^{1-\hat{x}-\hat{y}} \hat{p}_1(\hat{x},\hat{y},\hat{z})\hat{x}^m\hat{y}^n\hat{z}^l d\hat{x}d\hat{y}d\hat{z} \tag{10}$$

$$\int_0^1\int_0^{1-\hat{x}}\int_0^{1-\hat{x}-\hat{y}} \hat{p}_2(\hat{x},\hat{y},\hat{z})\hat{x}^m\hat{y}^n\hat{z}^l d\hat{x}d\hat{y}d\hat{z} \tag{11}$$

$$\int_0^1\int_0^{1-\hat{x}}\int_0^{1-\hat{x}-\hat{y}} \hat{p}_3(\hat{x},\hat{y},\hat{z})\hat{x}^m\hat{y}^n\hat{z}^l d\hat{x}d\hat{y}d\hat{z} \tag{12}$$

Degrees of freedom of face type

To define them, we use the following eight vectors:

$$\hat{q}_1 = \hat{q}_6 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \hat{q}_2 = \hat{q}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\hat{q}_4 = \hat{q}_5 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \hat{q}_7 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \hat{q}_8 = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}.$$

If k is the order of the element, the number of faces degrees of freedom is $n_f = 4k(k-1)$ and they are:
for $m+n \leq k-2$,

$$\int_0^1\int_0^{1-\hat{x}} ((\overrightarrow{\hat{p}} \wedge \overrightarrow{\hat{n}}) \cdot \overrightarrow{\hat{q}_1})(\hat{x},\hat{y},0)\hat{x}^m\hat{y}^n d\hat{x}d\hat{y} \tag{13}$$

$$\int_0^1\int_0^{1-\hat{x}} ((\overrightarrow{\hat{p}} \wedge \overrightarrow{\hat{n}}) \cdot \overrightarrow{\hat{q}_2})(\hat{x},\hat{y},0)\hat{x}^m\hat{y}^n d\hat{x}d\hat{y} \tag{14}$$

$$\int_0^1\int_0^{1-\hat{x}} ((\overrightarrow{\hat{p}} \wedge \overrightarrow{\hat{n}}) \cdot \overrightarrow{\hat{q}_3})(\hat{x},0,\hat{z})\hat{x}^m\hat{z}^n d\hat{x}d\hat{z} \tag{15}$$

$$\int_0^1\int_0^{1-\hat{x}} ((\overrightarrow{\hat{p}} \wedge \overrightarrow{\hat{n}}) \cdot \overrightarrow{\hat{q}_4})(\hat{x},0,\hat{z})\hat{x}^m\hat{z}^n d\hat{x}d\hat{z} \tag{16}$$

$$\int_0^1\int_0^{1-\hat{y}} ((\overrightarrow{\hat{p}} \wedge \overrightarrow{\hat{n}}) \cdot \overrightarrow{\hat{q}_5})(0,\hat{y},\hat{z})\hat{y}^m\hat{z}^n d\hat{y}d\hat{z} \tag{17}$$

$$\int_0^1\int_0^{1-\hat{y}} ((\overrightarrow{\hat{p}} \wedge \overrightarrow{\hat{n}}) \cdot \overrightarrow{\hat{q}_6})(0,\hat{y},\hat{z})\hat{y}^m\hat{z}^n d\hat{y}d\hat{z} \tag{18}$$

$$\int_0^1\int_0^{1-\hat{x}} ((\overrightarrow{\hat{p}} \wedge \overrightarrow{\hat{n}}) \cdot \overrightarrow{\hat{q}_7})(\hat{x},\hat{y},1-\hat{x}-\hat{y})\hat{x}^m\hat{y}^n d\hat{x}d\hat{y} \tag{19}$$

$$\int_0^1\int_0^{1-\hat{x}} ((\overrightarrow{\hat{p}} \wedge \overrightarrow{\hat{n}}) \cdot \overrightarrow{\hat{q}_8})(\hat{x},\hat{y},1-\hat{x}-\hat{y})\hat{x}^m\hat{y}^n d\hat{x}d\hat{y} \tag{20}$$

3 Change of bases for the degrees of freedom to preserve $H(curl)$ continuity

As the $H(curl)$ finite elements are invariant by affine transformation if we use - for a given (3×3) matrice B:

$$\overrightarrow{p} = B^{*-1}\overrightarrow{\hat{p}}$$

we can define affine equivalent finite elements [2],[3].

3.1 Change of bases for edge degree of freedom

For all number i of an edge $\hat{\Gamma}$, we impose to find $\overrightarrow{\hat{p}}_j$, such that $\hat{\sigma}_i(\overrightarrow{\hat{p}}_j) = \delta_{ij}$, where δ_{ij} is the kronecker symbol and where for these degrees of freedom:

$$\hat{\sigma}_i(\overrightarrow{\hat{p}}_j) = \int_{\hat{\Gamma}} \overrightarrow{\hat{p}_j} \cdot \overrightarrow{\tau}\, \hat{s}^i d\hat{\gamma}$$

We define:

$$\overrightarrow{\tilde{p}_j} = B^{*-1}\overrightarrow{\hat{p}_j}$$

We suppose that the degrees of freedom for each edge Γ of a tetrahedron K are, for $i = 1, k$:

$$\tilde{\sigma}_i(\overrightarrow{\tilde{p}}_j) = \int_{\Gamma} \overrightarrow{\tilde{p}_j} \cdot \overrightarrow{\tau} s^{i-1} d\gamma$$

Then $\tilde{\sigma}_i$ is verifying: $\tilde{\sigma}_i(\overrightarrow{\tilde{p}}_j) = \hat{\sigma}_i(\overrightarrow{\hat{p}}_j) = \delta_{ij}$.
We define: $\sigma_i(\overrightarrow{p}) = \tilde{\sigma}_i(\overrightarrow{p})$ when the edge Γ is covered in the same way as $\hat{\Gamma}$. If this edge is covered in the opposite way, we define, refering to a parametrisation of Γ on the interval $[0,1]$:

$$\sigma_i(\overrightarrow{p}) = -\int_{\Gamma} \overrightarrow{p} \cdot \overrightarrow{\tau}(1-s)^{i-1} d\gamma$$

Then we formally define the matrix: $A_e = (a^e_{il})$, where $a^e_{il} = (-1)^{l+1}\begin{pmatrix} i \\ l \end{pmatrix}$, if $i \geq l$ and $a^e_{il} = 0$ if not. We have:

$$\begin{pmatrix} \sigma_1 \\ ... \\ \sigma_i \\ ... \\ \sigma_k \end{pmatrix} = A_e \begin{pmatrix} \tilde{\sigma}_1 \\ ... \\ \tilde{\sigma}_i \\ ... \\ \tilde{\sigma}_k \end{pmatrix} \tag{21}$$

Then we must choose as $H(curl)$ compatible basis $\text{Span}\{\overrightarrow{p}_i\}$ related to edges:

$$\begin{pmatrix} \overrightarrow{p}_1 \\ \cdots \\ \overrightarrow{p}_i \\ \cdots \\ \overrightarrow{p}_k \end{pmatrix} = {}^t A_e \begin{pmatrix} \overrightarrow{\tilde{p}}_1 \\ \cdots \\ \overrightarrow{\tilde{p}}_i \\ \cdots \\ \overrightarrow{\tilde{p}}_k \end{pmatrix}. \tag{22}$$

3.2 Change of bases for faces degree of freedom

As well as for the edges, the $H(curl)$ continuity at faces interfaces has to be formulated. We restrict this study to the 2nd-order case; the k-order general case will be presented in a paper to appear.

For all number i of a face $\hat{f}$, we impose to find $\overrightarrow{\hat{p}}_j$, such that $\hat{\sigma}_i(\overrightarrow{\hat{p}}_j) = \delta_{ij}$ where for these degrees of freedom:

$$\hat{\sigma}_i(\overrightarrow{\hat{p}}_j) = \frac{1}{|\hat{f}|}\int_{\hat{f}} (\overrightarrow{\hat{p}}_j \wedge \overrightarrow{\hat{n}}) \cdot \overrightarrow{\hat{q}_i}\, d\hat{\gamma}$$

So we have:

$$\hat{\sigma}_i(\overrightarrow{\hat{p}}_j) = \tilde{\sigma}_i(\overrightarrow{\hat{p}}_j) = \frac{1}{|f|}\int_f (\overrightarrow{\tilde{p}_j} \wedge \overrightarrow{n}) \cdot \overrightarrow{u_i}\, d\gamma$$

where: $\overrightarrow{\tilde{p}_j} = B^{*-1}\overrightarrow{\hat{p}}_j$ and $\overrightarrow{u_i} = B(\overrightarrow{\hat{n}} \wedge \overrightarrow{\hat{q}_i}) \wedge \overrightarrow{n}$. We define the degree of freedom for a face f in the mesh by:

$$\sigma_i(\overrightarrow{p}) = \frac{1}{|f|}\int_f (\overrightarrow{p} \wedge \overrightarrow{n}) \cdot \overrightarrow{q_i}\, d\gamma$$

where $\overrightarrow{q}_i = \frac{B\overrightarrow{\hat{q}_i}}{|B\overrightarrow{\hat{q}_i}|}$ and $\overrightarrow{q_{i+1}} = \overrightarrow{q_i}^{\perp}, \overrightarrow{q_{i+1}} \subset f$, for $i \in \{1, 3, 5, 7\}$. Then we look for basis functions $\overrightarrow{p}_j$ related to faces verifying:

$$\sigma_i(\overrightarrow{p_j}) = \tilde{\sigma}_i(B^{*-1}\overrightarrow{\hat{p}}_j) = \delta_{ij}$$

To this end, we decompose - following the figure - the $\overrightarrow{u}_i$ on each orthonormal vector base face $(\overrightarrow{q}_i, \overrightarrow{q}_{i+1})$, so that:

$$\overrightarrow{u}_i = \alpha_i \overrightarrow{q}_i + \beta_i \overrightarrow{q}_{i+1}$$

$$\overrightarrow{u}_{i+1} = \alpha_{i+1} \overrightarrow{q}_i + \beta_{i+1} \overrightarrow{q}_{i+1}$$

For the degrees of freedom, we have:

$$\tilde{\sigma}_i = \alpha_i \sigma_i + \beta_i \sigma_{i+1}$$

$$\tilde{\sigma}_{i+1} = \alpha_{i+1} \sigma_i + \beta_{i+1} \sigma_{i+1}$$

or with the (2×2) matrix $A_f = \begin{pmatrix} \alpha_i & \beta_i \\ \alpha_{i+1} & \beta_{i+1} \end{pmatrix}$.

$$\begin{pmatrix} \sigma_i \\ \sigma_{i+1} \end{pmatrix} = A_f^{-1} \begin{pmatrix} \tilde{\sigma}_i \\ \tilde{\sigma}_{i+1} \end{pmatrix} \tag{23}$$

finally the effective vectorial basis on face f is taken to be:

$$\begin{pmatrix} \overrightarrow{p}_j \\ \overrightarrow{p}_{j+1} \end{pmatrix} = {}^t A_f \begin{pmatrix} \overrightarrow{\tilde{p}}_j \\ \overrightarrow{\tilde{p}}_{j+1} \end{pmatrix}. \tag{24}$$

4 Obtention of mass and stiffness matrices

The next step of our study is the determination of the mass and stiffness matrices:

$$M = (m_{ij})$$

$$K = (k_{ij})$$

where

$$m_{ij} = \int_K \overrightarrow{p}_i \cdot \overrightarrow{p}_j \, dxdydz \tag{25}$$

$$k_{ij} = \int_K curl\overrightarrow{p}_i \cdot curl\overrightarrow{p}_j \, dxdydz \tag{26}$$

For this purpose, we use symbolic program to make an exact integration of these integrals with a simplification by the 6 factors of the symetric matrix B^*B.

Conclusion

The equations (1) to (20), (25) and (26), are solved with symbolic calculus. Although polynomial vector bases are effectively produced by these equations, their contents never appear. The main effort will then be devoted to the $H(curl)$ compatibility equations (21) to (24) which are to be treated with a small but not easy to handle program written in fortran 90.

This method can be extended to the $H(div)$ conforming finite elements [4], and more generally for any mixed finite elements using such degrees of freedom.

References

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