Title: Major and Descent Statistics for the even-signed Permutation Group

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ABSTRACT

We introduce and study three new statistics on the even-signed permutation group D_n. We show that two of these are Mahonian, i.e. are equidistributed with length, and that a pair of them gives a generalization of Carlitz's identity on the Euler-Mahonian distribution of the descent number and major index over S_n.

Key words: major index, descent number, Coxeter Groups, Mahonian statistics.

Major and Descent Statistics for the even-signed Permutation Group [1]

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Abstract

We introduce and study three new statistics on the even-signed permutation group D_n. We show that two of these are Mahonian, i.e. are equidistributed with length, and that a pair of them gives a generalization of Carlitz's identity on the Euler-Mahonian distribution of the descent number and major index over S_n.

1 Introduction

A well known classical result due to MacMahon (see [14]) asserts that the inversion number and the major index are equidistributed on the symmetric group. The joint distribution of major index and descent number was studied by Carlitz [6] and others. Several results of this nature have been generalized to the hyperoctahedral group B_n (see, e.g., [5],[12]) and many candidates for a major index for B_n have been suggested (see, e.g., [7],[8],[9],[11],[17]), but no generalizations of MacMahon's result have been found until the discovery of the flag major index in the recent paper [1]. After that, Foata posed the problem of finding a "descent statistic" that, together with the flag major index, allows the generalization to B_n of the well known Carlitz's identity on the Euler-Mahonian distribution of descent number and major index over S_n. In [2] Adin, Brenti and Roichman give two answers to Foata's question. Now it's natural to wonder if some of these statistics and results can be generalized to the even-signed permutation group D_n.

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The goal of this paper is to show that this is the case. More precisely, we introduce and study three new statistics on D_n; the D-negative descent number ($ddes$), the D-negative major index ($dmaj$) and the D-flag major index ($fmaj_D$). When restricted to S_n, $ddes$ reduces to descent number and $dmaj$ to the major index. The two major indices on D_n are equidistributed with length, and the pair $(ddes, dmaj)$ gives a generalization of Carlitz's identity to D_n.

The organization of the paper is as follows. In the next section we collect some definitions, notation and results that are needed in the rest of the work. In §3 we introduce a new "descent set" and hence in a very natural way new definitions of "descent number" and "major index" on D_n. It's shown that $dmaj$ is equidistributed with length and that $(ddes, dmaj)$ gives a generalization of Carlitz's identity. In §4 we define, in terms of Coxeter elements, the D-flag major index for D_n and we show that it's equidistributed with length. Furthermore, we describe a combinatorial algorithm to compute it. Finally, in §5 we discuss some open problems arising from our work.

2 Notation, Definitions and Preliminaries

In this section we give some definitions, notation and results that will be used in the rest of this work. We let $\mathbf{P} := \{1,2,3, \ldots\}$, $\mathbf{N} := \mathbf{P} \cup \{0\}$, $\mathbf{Z}$ be the set of integers and $\mathbf{C}$ be the set of complex numbers; for $a \in \mathbf{N}$ we let $[a] := \{1, 2, \ldots, a\}$ (where $[0] := \emptyset$). Given $n, m \in \mathbf{Z}$, $n \leq m$, we let $[n, m] := \{n, n+1, \ldots, m\}$. The cardinality of a set A will be denoted by $|A|$ and we let $\binom{[n]}{2} := \{S \subseteq [n] : |S| = 2\}$. More generally, given a multiset $M = \{1^{a_1}, 2^{a_2}, \ldots, r^{a_r}\}$ we denote by $|M|$ its cardinality, so $|M| = \sum_{i=1}^{r} a_i$. Given a variable q and a commutative ring R we denote by $R[q]$ (respectively, $R[[q]]$) the ring of polynomials (respectively, formal power series) in q with coefficient in R. For $i \in \mathbf{N}$ we let, as customary, $[i]_q := 1 + q + q^2 + \ldots + q^{i-1}$ (so $[0]_q = 0$).

Given a sequence $\sigma = (a_1, \ldots, a_n) \in \mathbf{Z}^n$ we say that $(i, j) \in [n] \times [n]$ is an *inversion* of σ if $i < j$ and $a_i > a_j$. We say that $i \in [n-1]$ is a *descent* of σ if $a_i > a_{i+1}$. We denote by $Inv(\sigma)$ and $Des(\sigma)$ the set of inversions and the set of descents of σ and by $inv(\sigma)$ and $des(\sigma)$ their cardinalities, respectively. We also let

$$maj(\sigma) := \sum_{i \in Des(\sigma)} i \tag{1}$$

and call it the *major index* of σ.

Let S_n be the set of all bijections $\sigma : [n] \rightarrow [n]$. If $\sigma \in S_n$ then we write $\sigma = \sigma_1 \ldots \sigma_n$ to mean that $\sigma(i) = \sigma_i$, for $i = 1, \ldots, n$. If $\sigma \in S_n$ then we may also write σ in *disjoint cycle form* (see, e.g., [15, p.17]) and we will usually omit to write the 1-cycles of σ. For example, if $\sigma = 64175823$ then we also write $\sigma = (2,4,7)(1,6,8,3)$. Given $\sigma, \tau \in S_n$ we let $\sigma\tau := \sigma \circ \tau$ (composition of functions) so that, for example, $(1,2)(2,3) = (1,2,3)$.

We denote by B_n the group of all bijections π of the set $[-n, n] \setminus \{0\}$ onto itself such that

$$\pi(-a) = -\pi(a)$$

for all $a \in [-n, n] \setminus \{0\}$, with composition as the group operation. This group is usually known as the group of *signed permutations* on $[n]$, or as the *hyperoctahedral group* of rank n. We identify S_n as a subgroup of B_n, and B_n as a subgroup of S_{2n}, in the natural ways.

If $\pi \in B_n$ then we write $\pi = [a_1, \ldots, a_n]$ to mean that $\pi(i) = a_i$ for $i = 1, \ldots, n$, we call this the *window* notation of w, and we let

$$\begin{aligned} inv(\pi) &:= inv(a_1, \ldots, a_n), \\ des(\pi) &:= des(a_1, \ldots, a_n), \\ maj(\pi) &:= maj(a_1, \ldots, a_n), \\ Neg(\pi) &:= \{i \in [n] : a_i < 0\}, \\ N_1(\pi) &:= |Neg(\pi)|, \end{aligned} \tag{2}$$

and

$$N_2(\pi) := |\{\{i, j\} \in \binom{[n]}{2} : a_i + a_j < 0\}|. \tag{3}$$

We denote by D_n the subgroup of B_n consisting of all the signed permutations having an even number of negative entries in their window notation, more precisely

$$D_n := \{\pi \in B_n : N_1(\pi) \equiv 0 \ (\ mod\ 2\ )\}.$$

Obviously, the definitions in (2) and (3) are still valid for $\pi \in D_n$.
It is well known (see, e.g., [4, §8.2]) that D_n is a Coxeter group with respect to the generating set $S := \{s_0, s_1, \ldots, s_{n-1}\}$ where

$$s_0 := [-2, -1, 3, \ldots, n]$$

and

$$s_i := [1, 2, \ldots, i-1, i+1, i, i+2, \ldots, n]$$

for $i = 1, \ldots, n-1$. This gives rise to another natural statistic on D_n, the *length* (similarly definable for any Coxeter group), namely

$$\ell(\pi) := min\{r \in \mathbf{N} : \pi = s_{i_1} \ldots s_{i_r} \text{ for some } i_1, \ldots, i_r \in [0, n-1]\}.$$

There is a well known direct combinatorial way to compute this statistic for $\pi \in D_n$ (see, e.g., [4, §8.2]), namely

$$\ell(\pi) = inv(\pi) - \sum_{i \in Neg(\pi)} \pi(i) - N_1(\pi). \tag{4}$$

It's not hard to prove that for all $\pi \in B_n$ (and so also for $\pi \in D_n$),

$$N_1(\pi) + N_2(\pi) = -\sum_{i \in Neg(\pi)} \pi(i), \tag{5}$$

so equivalently (4) becomes

$$\ell(\pi) = inv(\pi) + N_2(\pi). \tag{6}$$

For example, if $\pi := [-4, 1, 3, -5, -2, -6] \in D_6$ then $inv(\pi) = 10$, $des(\pi) = 2$, $maj(\pi) = 8$, $N_1(\pi) = 4$, $N_2(\pi) = 13$ and $\ell(\pi) = 23$.

We follow [4] for general Coxeter group notation and terminology. In particular, let (W, S) be a Coxeter system, for $J \subseteq S$ we let W_J be the subgroup of W generated by J, and

$$W^J := \{w \in W : \ell(ws) > \ell(w) \text{ for all } s \in J\}.$$

We call W_J the *parabolic subgroup* generated by J and W^J the *set of minimal left coset representatives* of W_J or the *quotient*. The quotient W^J is a poset according to the Bruhat order (see, e.g., [4] or [13]).
The following is well known (see, e.g., [4] or [13]).

Proposition 2.1 *Let $J \subseteq S$. Then:*

i) *Every $w \in W$ has a unique factorization $w = w^J w_J$ such that $w^J \in W^J$ and $w_J \in W_J$.*

ii) *For this factorization* $\ell(w) = \ell(w^J) + \ell(w_J)$.

Now we let

$$T := \{\pi \in D_n : des(\pi) = 0\}. \tag{7}$$

It is well known, and easy to see, that

$$D_n = \biguplus_{u \in S_n} \{\pi u : \pi \in T\}, \tag{8}$$

where $\biguplus$ denotes disjoint union. We will often use this decomposition in this paper. Note that (8) is one case of the multiplicative decomposition of a Coxeter group into a parabolic subgroup and its minimal coset representatives (see Proposition 2.1), more precisely T is the quotient corresponding to the maximal parabolic subgroup generated by $J := S \setminus \{s_0\}$. In the next section we will analyze this issue in more detail.

For $n \in \mathbf{P}$ we let

$$A_n(t,q) := \sum_{\sigma \in S_n} t^{des(\sigma)} q^{maj(\sigma)},$$

and $A_0(t,q) := 1$. For example, $A_3(t,q) = 1 + 2tq^2 + 2tq + t^2q^3$. The following result is due to Carlitz, and we refer the reader to [6] for its proof.

Theorem 2.2 *Let* $n \in \mathbf{P}$. *Then*

$$\sum_{r \geq 0} [r+1]_q^n t^r = \frac{A_n(t,q)}{\prod_{i=0}^{n}(1-tq^i)} \tag{9}$$

in $\mathbf{Z}[q][[t]]$.

3 The "Negative" Statistics

In this section we define a new "descent set" for elements of D_n. This gives rise, in a very natural way, to the definitions of "major index" and "descent number" for D_n. We then show that these two statistics give a generalization of Carlitz's identity to D_n, and that the former is equidistributed with length.

3.1 The D-Negative Descent Multiset

For $\pi \in D_n$ let

$$Des(\pi) := \{i \in [n-1] \ : \ \pi(i) > \pi(i+1)\},$$

we define the *D-negative descent multiset*

$$DDes(\pi) := Des(\pi) \biguplus \{-\pi(i) - 1 \ : \ i \in Neg(\pi)\} \setminus \{0\}, \tag{10}$$

where $Neg(\pi)$ is the set of positions of negative entries in π, defined in (2).
For example, if $\pi = [-4, 1, 3, -5, -2, -6] \in D_6$ then $Des(\pi) = \{3, 5\}$ and $DDes(\pi) = \{1, 3^2, 4, 5^2\}$.

Note that if $\pi \in S_n$ then $DDes(\pi)$ is a set and coincides with the usual descent set of π. Also, note that $DDes(\pi)$ can be defined rather naturally also in purely Coxeter group theoretic terms. In fact, for $i \in [n-1]$ let $\xi_i \in D_n$ be defined by

$$\xi_i := [-1, 2, \ldots, i, -i-1, i+2, \ldots, n].$$

Then $\xi_1, \ldots, \xi_{n-1}$ are reflections (in the Coxeter group sense, see e.g., [4] or [13]) of D_n and it is clear from (4) that

$$DDes(\pi) := \{i \in [n-1] : \ell(\pi s_i) < \ell(\pi)\} \biguplus \{i \in [n-1] : \ell(\pi^{-1}\xi_i) < \ell(\pi^{-1})\}.$$

These considerations explain why it is natural to think of $DDes(\pi)$ as a "descent set", so the following definitions are natural.
For $\pi \in D_n$ we let

$$ddes(\pi) := |DDes(\pi)|$$

and

$$dmaj(\pi) := \sum_{i \in DDes(\pi)} i.$$

For example if $\pi = [-4, 1, 3, -5, -2, -6] \in D_6$ then $ddes(\pi) = 6$, and $dmaj(\pi) = 21$.
Note that from (10) there follows that

$$dmaj(\pi) = maj(\pi) - \sum_{i \in Neg(\pi)} \pi(i) - N_1(\pi) = maj(\pi) + N_2(\pi). \tag{11}$$

This formula is also one of the motivations behind our definition of $dmaj(\pi)$, because of the corresponding formulas (4) and (6) (see also [2]) .
Also note that

$$ddes(\pi) = des(\pi) + N_1(\pi) + \epsilon(\pi), \tag{12}$$

where

$$\epsilon(\pi) := \begin{cases} -1 & \text{if } 1 \notin \pi([n]) \\ 0 & \text{if } 1 \in \pi([n]). \end{cases}$$

3.2 Equidistribution

Our first result shows that $dmaj$ and ℓ are equidistributed in D_n.

Proposition 3.1 *Let $n \in \mathbf{P}$. Then*

$$\sum_{\pi \in D_n} q^{dmaj(\pi)} = \sum_{\pi \in D_n} q^{\ell(\pi)}.$$

Proof. Let T be defined by (7). It is clear from our definitions that for all $u \in S_n$ and $\sigma \in T$,

$$maj(\sigma u) = maj(u), \qquad inv(\sigma u) = inv(u), \qquad N_2(\sigma u) = N_2(\sigma). \tag{13}$$

Therefore, from (6), (8), (11) and the corresponding result for S_n (see, e.g., [10] or [14, Chapter VI]), we conclude that

$$\begin{aligned}
\sum_{\pi \in D_n} q^{dmaj(\pi)} &= \sum_{\sigma \in T} \sum_{u \in S_n} q^{dmaj(\sigma u)} \\
&= \sum_{\sigma \in T} \sum_{u \in S_n} q^{maj(\sigma u)+N_2(\sigma u)} \\
&= \sum_{\sigma \in T} q^{N_2(\sigma)} \sum_{u \in S_n} q^{maj(u)} \\
&= \sum_{\sigma \in T} q^{N_2(\sigma)} \sum_{u \in S_n} q^{inv(u)} \\
&= \sum_{\sigma \in T} \sum_{u \in S_n} q^{inv(\sigma u)+N_2(\sigma u)} \\
&= \sum_{\pi \in D_n} q^{\ell(\pi)},
\end{aligned}$$

as desired. □

3.3 Generalization of Carlitz's Identity

We start with some notation and terminology concerning partitions (see [16, §7.2]). By an (integer) *strict partition* we mean a sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $\lambda_1 > \lambda_2 > \ldots > \lambda_k$. We denote by $|\lambda| := \sum_i \lambda_i$. We denote by $\mathcal{P}_S$ the set of all (integer) strict partitions. For any $\mu, \lambda \in \mathcal{P}_S$ we define $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all i. Given $n \in \mathbf{P}$ we let

$$\mathcal{P}_S(n) := \{\lambda \in \mathcal{P}_S : \lambda \subseteq (n, n-1, \ldots, 2, 1)\}.$$

As before, let $T = \{\pi \in D_n : des(\pi) = 0\}$ so

$$T = \{\pi \in D_n : \pi(1) < \pi(2) < \ldots < \pi(n)\}.$$

Therefore, given $\pi \in T, \pi \neq e$, there is a unique even $k \in [n]$ such that

$$\pi(k) < 0 < \pi(k+1).$$

Given $\pi \in T$ we associate to π the strict partition

$$\Lambda(\pi) := (-\pi(1) - 1, -\pi(2) - 1, \ldots, -\pi(k) - 1). \tag{14}$$

Note that the last entry of $\Lambda(\pi)$ is equal to 0 if $\pi(k) = -1$.
The following is known (see, e.g., [3]).

Proposition 3.2 *The map Λ defined by (14) is a bijection between T and $\mathcal{P}_S(n-1)$. Furthermore $\ell(\pi) = |\Lambda(\pi)|$ and $\pi \leq \sigma$ in T if and only if $\Lambda(\pi) \subseteq \Lambda(\sigma)$, for all $\pi, \sigma \in T$.*

We will find it convenient to identify a strict partition $\lambda \in \mathcal{P}_S(n)$ with a subset of $[n]$. In fact we have an inclusion preserving obvious bijection ϕ between $\mathcal{P}_S(n)$ and $\wp(n) := \{S : S \subseteq [n]\}$ given by:

$$(\lambda_1, \lambda_2, \ldots, \lambda_n) \stackrel{\phi}{\longleftrightarrow} \{\lambda_1, \lambda_2, \ldots, \lambda_n\}.$$

We begin with the following lemma.

Lemma 3.3 *Let $n \in \mathbf{P}$. Then*

$$\sum_{\sigma \in T} t^{N_1(\sigma)+\epsilon(\sigma)} q^{N_2(\sigma)} = \sum_{S \subseteq [n-1]} t^{|S|} q^{\sum_{i \in S} i} = \prod_{i=1}^{n-1} (1 + tq^i).$$

Proof. From (6) we have that $N_2(\sigma) = \ell(\sigma)$, for all $\sigma \in T$. By Proposition 3.2 we have $\ell(\sigma) = |\Lambda(\sigma)|$ and by definition of ϕ that $|\Lambda(\sigma)| = \sum_{i \in \phi(\Lambda(\sigma))} i$. Therefore $N_2(\sigma) = \sum_{i \in \phi(\Lambda(\sigma))} i$.
Let $\sigma \in T$. Suppose first that $1 \in \sigma([n])$, then $|\phi(\Lambda(\sigma))| = N_1(\sigma)$. On the other hand, if $1 \notin \sigma([n])$, we have that $|\phi(\Lambda(\sigma))| = N_1(\sigma) - 1$. Hence $|\phi(\Lambda(\sigma))| = N_1(\sigma) + \epsilon(\sigma)$, and if we let $S = \phi(\Lambda(\sigma))$ the result follows. □

We are now ready to prove the main result of this work, namely that the pair of statistics $(ddes, dmaj)$ solves Foata's problem for the group of even-signed permutations D_n.

Theorem 3.4 *Let $n \in \mathbf{P}$. Then*

$$\sum_{r \geq 0} [r+1]_q^n t^r = \frac{\sum_{\pi \in D_n} t^{ddes(\pi)} q^{dmaj(\pi)}}{(1-t)(1-tq^n)\prod_{i=1}^{n-1}(1-t^2q^{2i})} \tag{15}$$

in $\mathbf{Z}[q][[t]]$.

Proof. Let T be defined by (7). Then it is clear from our definitions that

$$des(\sigma u) = des(u) \ , \ N_1(\sigma u) = N_1(\sigma) \ , \ \epsilon(\sigma u) = \epsilon(\sigma)$$

and

$$\sum_{i \in Neg(\sigma u)} \sigma u(i) = \sum_{i \in Neg(\sigma)} \sigma(i),$$

for all $\sigma \in T$ and $u \in S_n$. Therefore we have from (8), (11), (12), (13) and Lemma 3.3 that

$$\begin{aligned}
\sum_{\pi \in D_n} t^{ddes(\pi)} q^{dmaj(\pi)} &= \sum_{\sigma \in T} \sum_{u \in S_n} t^{des(\sigma u)+N_1(\sigma u)+\epsilon(\sigma u)} q^{maj(\sigma u)+N_2(\sigma u)} \\
&= \sum_{\sigma \in T} t^{N_1(\sigma)+\epsilon(\sigma)} q^{N_2(\sigma)} \sum_{u \in S_n} t^{des(u)} q^{maj(u)} \\
&= \prod_{i=1}^{n-1}(1+tq^i) \sum_{u \in S_n} t^{des(u)} q^{maj(u)}
\end{aligned}$$

and the result follows from Theorem 2.2. □

Note that, as in (9) for S_n and in ([2, Theorem 3.2]) for B_n, the powers of q in the denominator of formula (15) are the Coxeter degrees of D_n (see [13, p.59]).

4 The Flag Major Index for D_n

In this section we introduce another new "major index" statistic for D_n. This is an analogue of the flag major index introduced in [1]. We show that this statistic is equidistributed with length and we give a combinatorial algorithm to compute it.

4.1 The D-Flag Major Index

For $i = 0, \ldots, n-1$ we define

$$t_i := s_i s_{i-1} \cdots s_0, \tag{16}$$

explicitly for all $i \in [n-1]$

$$t_i = [-1, -i-1, 2, 3, \dots, i, i+2, \dots, n], \tag{17}$$

and for $i = 0$

$$t_0 = [-2, -1, 3, \dots, n] = s_0. \tag{18}$$

These are Coxeter elements (see e.g., [13, §3.16]), in a distinguished flag of parabolic subgroups

$$1 < G_1 < G_2 < \dots < G_n = D_n$$

where $G_i \simeq D_i$ ($i \geq 2$) is the parabolic subgroup of D_n generated by $s_0, s_1, \dots, s_{i-1}$. The family $\{t_i\}_i$ is a new set of generators for D_n, and we have the following proposition.

Proposition 4.1 *For every $\pi \in D_n$ there exists a unique representation*

$$\pi = t_0^{h_{n-1}} t_{n-1}^{k_{n-1}} t_0^{h_{n-2}} t_{n-2}^{k_{n-2}} \cdots t_0^{h_1} t_1^{k_1} \tag{19}$$

with $0 \leq h_r \leq 1$, $0 \leq k_r \leq 2r-1$ and

$$k_r \in \{2r-1, r-1\} \text{ if } h_r = 1 \tag{20}$$

for all $r = 1, \dots, n-1$.

Proof. We proceed by induction on n. For $n = 2$ the result is clear, so suppose $n \geq 3$. We define

$$D_{n,*} := \{t_{n-1}^{k_{n-1}} w \ : \ k_{n-1} \in [0, 2n-3] \ , \ w \in D_{n-1}\},$$

$$D_{n,1} := \{t_0 t_{n-1}^{2n-3} w \ : \ w \in D_{n-1}\},$$

$$D_{n,-1} := \{t_0 t_{n-1}^{n-2} w \ : \ w \in D_{n-1}\}.$$

It is not hard to see that $|D_{n,1}| = |D_{n,-1}| = |D_{n-1}|$ and that $D_{n,1} \cap D_{n,-1} = \emptyset$ as $\pi(n) = 1$ and $\sigma(n) = -1$, for all $\pi \in D_{n,1}$ and $\sigma \in D_{n,-1}$.
On the other hand if $t_{n-1}^r w_1 = t_{n-1}^s w_2$ with $w_1, w_2 \in D_{n-1}$ and $r, s \in [0, 2n-3]$, it is easy to see that $r = s$ and $w_1 = w_2$, hence $|D_{n,*}| = (2n-2)|D_{n-1}|$. Moreover the elements $\pi \in D_{n,*}$ satisfy $\pi(n) \neq \pm 1$. Therefore we have the following decomposition of D_n

$$D_n = D_{n,*} \biguplus D_{n,1} \biguplus D_{n,-1},$$

and so the result follows by induction. □

Note that the representation (19) is not unique if we drop the condition (20). For example consider $\pi = [-2, 4, 1, -3] \in D_4$, then π has two different representations of type (19), namely, $\pi = t_3^3 t_0 t_2 t_0 t_1$ and $\pi = t_0 t_3^3 t_2 t_0 t_1$. The representation of Proposition 4.1 is the first one.

Let $\pi \in D_n$, then we define the *D-flag major index* of π by

$$fmaj_D(\pi) := \sum_{i=1}^{n-1} k_i + \sum_{i=1}^{n-1} h_i. \tag{21}$$

4.2 Equidistribution

For $0 \leq m \leq 2n-1$ we define $r_{n,m} \in D_n$ as follows: for $n = 2$,

$$r_{2,m} := \begin{cases} e & \text{if } m = 0 \\ s_1 & \text{if } m = 1 \\ s_1 s_0 & \text{if } m = 2 \\ s_0 & \text{if } m = 3 \end{cases}$$

and for $n > 2$,

$$r_{n,m} := \begin{cases} e & \text{if } m = 0 \\ s_{n-m} s_{n-m+1} \cdots s_{n-1} & \text{if } 0 < m < n \\ s_{m-n+1} s_{m-n} \cdots s_0 s_2 s_3 \cdots s_{n-1} & \text{if } n \leq m < 2n-1 \\ s_0 s_2 s_3 \cdots s_{n-1} & \text{if } m = 2n-1 \end{cases}$$

The set $\{r_{n,m} \ : \ 0 \leq m < 2n\}$ forms a complete set of representatives of minimal length for the left cosets of D_{n-1} in D_n. Moreover this is still valid for every $i \in [3, n]$, namely, $r_{i,m} \in D_i^{J_i}$ for all $m \in [0, 2i-1]$, where $J_i := S \setminus \{s_{n-1}, \ldots, s_{i-1}\}$. Hence we have the following decomposition

$$D_n = D_n^{J_n} D_{n-1}^{J_{n-1}} \cdots D_2.$$

Note that the length of $r_{i,m}$ is $\bar{m}$, where

$$\bar{m} := \begin{cases} m & \text{if } 0 \leq m \leq 2i-2 \\ i-1 & \text{if } m = 2i-1. \end{cases}$$

From $i)$ of Proposition 2.1 we know that each element $\pi \in D_n$ has a unique representation as a product

$$\pi = \prod_{k=1}^{n-1} r_{n+1-k, m_{n+1-k}} \tag{22}$$

where $0 \leq m_j < 2j$ for all j. From $ii)$ of Proposition 2.1 it follows that

$$\ell(\pi) = \sum_{j=2}^{n} \bar{m}_j. \tag{23}$$

Thanks to the unique representation (22) we define a map $\phi : D_n \to D_n$ in the following way:

$$\phi(\prod_{k=1}^{n-1} r_{n+1-k, m_{n+1-k}}) := \prod_{k=1}^{n-1} \phi(r_{n+1-k, m_{n+1-k}}),$$

where for $i \neq 2$,

$$\phi(r_{i,m}) := \begin{cases} t_{i-1}^{m} & \text{if } m < 2i-2 \\ t_0 t_{i-1}^{\bar{m}-1} & \text{if } 2i-2 \leq m \leq 2i-1, \end{cases}$$

and for $i = 2$,

$$\phi(r_{2,m}) := \begin{cases} e & \text{if } m = 0 \\ t_1 & \text{if } m = 1 \\ t_0 t_1 & \text{if } m = 2 \\ t_0 & \text{if } m = 3. \end{cases}$$

The definition of ϕ, together with Proposition 4.1 and (22), imply the following result.

Proposition 4.2 *The map $\phi : D_n \to D_n$ is a bijection.* □

Now we are ready to state the main result of this section, namely that the D-flag major index is equidistributed with the length in D_n.

Theorem 4.3 *Let $n \in \mathbf{P}$. Then*

$$\sum_{\pi \in D_n} q^{fmaj_D(\pi)} = \sum_{\pi \in D_n} q^{\ell(\pi)}.$$

Proof. From Proposition 4.2 and the definition of $\bar{m}$, the map ϕ is a bijection which sends the length function in the D-flag major index. $\square$

Note that the B-flag major index ($fmaj$) defined on B_n (see [1]) does not work on D_n. Namely if we consider $\pi \in D_n$ as an element of B_n, then $fmaj(\pi)$ is not equidistributed with length on D_n. For example, if $\pi = [-2,-1]$ then $fmaj_D(\pi) = 1$ while $fmaj(\pi) = 4$, and in D_2 there is no element of length 4.
Note also that $fmaj_D$ restricted to S_n is not the major index and it's not equidistributed with length. It seems to be a new statistic on S_n. It's easy to see that for each $\pi \in S_n$, $fmaj_D(\pi)$ is always even and that $fmaj_D(\pi) \geq maj(\pi)$. If we let $E_n(q) := \sum_{\pi \in S_n} q^{fmaj_D(\pi)}$, for $n \leq 4$ we have $E_1(q) = 1$, $E_2(q) = 1 + q^2$, $E_3(q) = 1 + 3q^2 + q^4 + q^6$ and $E_4(q) = 1 + 5q^2 + 6q^4 + 7q^6 + 3q^8 + q^{10} + q^{12}$.

4.3 A combinatorial algorithm

In this section we describe a combinatorial algorithm that allows us to compute the D-flag major index $fmaj_D$, without using the representation of Proposition 4.1.

Let $\sigma = (a_1, \ldots, a_n) \in \mathbf{Z}^n$ and $i \geq 1$. We use this *split-notation*

$$\sigma = [a_1][a_2, \ldots, a_{i+1}][a_{i+2}, \ldots, a_n].$$

Sometimes it will be useful to denote the first part by A and the second by C_i where i represents the number of its elements.
We define the following operations on $\sigma \in \mathbf{Z}^n$:

$$\overset{\rightarrow_i^0}{\sigma} := [-a_2][-a_1, a_3, \ldots, a_{i+1}][a_{i+2}, \ldots, a_n],$$

and

$$\overset{\rightarrow_i^1}{\sigma} := [-a_1][-a_{i+1}, a_2, \ldots, a_i][a_{i+2}, \ldots, a_n].$$

In these cases we will write $\overset{\rightarrow_i^0}{\sigma} = (A^0, C_i^0, [a_{i+2}, \ldots, a_n])$ and $\overset{\rightarrow_i^1}{\sigma} = (\overset{\rightarrow_1}{A}, \overset{\rightarrow_1}{C}_i, [a_{i+2}, \ldots, a_n])$.
Moreover for all $n \in \mathbf{P}$ we define

$$\rightarrow_i^n := \rightarrow_i^1 \circ \cdots \circ \rightarrow_i^1 \quad n\text{-times}. \tag{24}$$

Note that for every $\sigma \in \mathbf{Z}^n$ and $i \geq 1$, $\overset{\rightarrow_i^{2i}}{\sigma} = \sigma$.
For example, if $\pi \in D_5$, $\pi = [-2][1, 3, -4, 5] = (A, C_4)$, then

$$\overset{\rightarrow_4^0}{\pi} = [-1][2, 3, -4, 5] = (A^0, C_4^0),$$

$$\stackrel{\rightarrow 5}{\pi}{}^{4} = [2][5,-1,-3,4] = (\stackrel{\rightarrow 5}{A}, \stackrel{\rightarrow 5}{C_4}),$$

and

$$\stackrel{\rightarrow 2}{\pi}{}^{3} = [-2][-3,4,1][5] = (\stackrel{\rightarrow 2}{A}, \stackrel{\rightarrow 2}{C_3}, [5]).$$

These are the two technical properties that we will use in the algorithm. Fix $i \in [n-1]$, let t_i be as in (17),

$$t_i = [-1][-i-1, 2, 3, \ldots, i][i+2, \ldots n].$$

It's easy to see that for all $i \in [n-1]$ we have

$$t_i^2 = t_i t_i = \stackrel{\rightarrow 1}{t_i}{}^{i}, \tag{25}$$

and by (24) that for $k \in \mathbf{P}$

$$t_i^k = \stackrel{\rightarrow k-1}{t_i}{}^{i}. \tag{26}$$

Now consider $t_{i-1} = [-1][-i, 2, \ldots, i-1][i+1, \ldots, n]$. As before it is not hard to see that

$$t_i t_{i-1} = \stackrel{\rightarrow 1}{t_i}{}^{i-1}. \tag{27}$$

Now we are able to state the algorithm to compute the unique representation of π as in Proposition 4.1, namely

$$\pi = f_{n-1} \cdots f_1$$

where for all $r \in [n-1]$, $f_r = t_0^{h_r} t_r^{k_r}$ with $h_r \in [0,1]$ and $k_r \in [0, 2r-1]$.
We construct a sequence $e_0, \ldots, e_{n-1}$ of elements of D_n such that

$i)$ $e_0 = e$, $e_{n-1} = \pi$;

$ii)$ $e_i = f_{n-1} \cdots f_{n-i}$, for all $i \in [1, n-1]$;

$iii)$ $\pi(j) = e_i(j)$, for all $j > n-i$.

From $iii)$ there immediately follows that $e_{n-1} = \pi$.

We need to do $n-1$ steps. From now on to avoid confusion, we put on A an index corresponding to the number of steps. We begin with $e_0 = [1][2, \ldots, n]$. Assume that e_{n-i} has been constructed, and we will construct $e_{n-i+1} = e_{n-i} f_{i-1}$. Then by $iii)$,

$$e_{n-i} = (A_{n-i}, C_{i-1}, [\pi(i+1), \ldots \pi(n)]).$$

For simplicity, we define $p(i)$ and $p(-i)$ to be the positions of $\pi(i)$ and $-\pi(i)$ in C_{i-1} or C^0_{i-1}respectively. There are four cases to consider.

1) $\pi(i) \in C_{i-1}$

Then we let $k_{i-1} = i-1-p(i)$ and $h_{i-1} = 0$. Hence $f_{i-1} = t_{i-1}^{i-1-p(i)}$.

2) $-\pi(i) \in C_{i-1}$

Then we let $k_{i-1} = 2i-2-p(-i)$ and $h_{i-1} = 0$. Hence $f_{i-1} = t_{i-1}^{2i-2-p(-i)}$.

3) $\pi(i) \in A_{n-i}$

Then $-\pi(i) \in C^0_{i-1}$ and in particular $p(-i) = 1$. We let $k_{i-1} = 2i-3$ and $h_{i-1} = 1$. Hence $f_{i-1} = t_0 t_{i-1}^{2i-3}$.

4) $-\pi(i) \in A_{n-i}$

Then $\pi(i) \in C^0_{i-1}$ and $p(i) = 1$. We let $k_{i-1} = i-2$ and $h_{i-1} = 1$. Hence $f_{i-1} = t_0 t_{i-1}^{i-2}$.

We have determined the factor f_{i-1}. By $ii)$ we let $e_{n-i+1} = e_{n-1} f_{i-1}$. From (24) and (26) it follows that $e_{n-i+1}(i) = \pi(i)$ and by (27) $iii)$ again holds.
Therefore

$$e_{n-i+1} = (A_{n-i+1}, C_{i-2}, [\pi(i), \ldots, \pi(n)]),$$

where in cases 1) and 2),

$$A_{n-i+1} := \overset{\rightarrow k_{i-1}}{A_{n-i}} \;,\;\; C_{i-2} := \overset{\rightarrow k_{i-1}}{C_{i-1}} \setminus [\pi(i)],$$

while in cases 3) and 4),

$$A_{n-i+1} := \overset{\rightarrow k_{i-1}}{A^0_{n-i}} \;,\;\; C_{i-2} := \overset{\rightarrow k_{i-1}}{C^0_{i-1}} \setminus [\pi(i)].$$

Observe that in the first step $p(n) = \pi(n) - 1$ and $p(-n) = -\pi(n) - 1$. These can be used for the computation of e_1.
We finish this section by illustrating the procedure with an example.
Let $\pi = [5, 3, -4, 1, -2] \in D_5$. We start from

$$e = e_0 = [1][2, 3, 4, 5] = (A_0, C_4).$$

$1^{st} - step)$ $\quad i = 5 \;,\; -\pi(5) = 2 \in C_4$

We are in case 2) and $p(-5)=1$, so $k_4=7$, $h_4=0$ and $f_4=t_4^7$.
It follows that $A_1 = \overset{\to 7}{A_0} = [-1]$ and $C_3 = \overset{\to 7}{C_4} \setminus [-2] = [3,4,5]$. Hence,

$$e_1 = [-1][3,4,5][-2].$$

$2^{nd}-step)$ $\quad i=4\ ,\ -\pi(4)=-1\in A_1$

We are in case 4) so $k_3=2$, $h_3=1$ and $f_3=t_0t_3^2$.
It follows that $A_2 = \overset{\to 2}{A_1^0} = [-3]$ and $C_2 = \overset{\to 2}{C_3^0} \setminus [1] = [-4,-5]$. Hence,

$$e_2 = [-3][-4,-5][1,-2].$$

$3^{rd}-step)$ $\quad i=3\ ,\ \pi(3)=-4\in C_2$

We are in case 1) and $p(3)=1$, so $k_2=1$, $h_2=0$ and $f_2=t_2$.
It follows that $A_3 = \overset{\to 1}{A_2} = [3]$ and $C_1 = \overset{\to 1}{C_2} \setminus [-4] = [5]$. Hence,

$$e_3 = [3][5][-4,1,-2].$$

$4^{th}-step)$ $\quad i=2\ ,\ \pi(2)=3\in A_3$

We are in case 3) so $k_1=1$, $h_1=1$ and $f_1=t_0t_1$.
It follows that $A_4 = \overset{\to 1}{A_3^0} = [5]$ and $C_0=\emptyset$. Hence,

$$e_4 = [5][3,-4,1,-2] = \pi,$$

and we are done. Finally $\pi = t_4^7 t_0 t_3^2 t_2 t_0 t_1$ and $fmaj_D(\pi)=12$.

5 Open Problems

We close this paper with a few open questions that we find interesting.

(5.1) In [1] Adin and Roichman define the flag major index statistic ($fmaj$), and show that it's equidistributed with length in the hyperoctahedral group B_n. Moreover it appears naturally in the Hilbert series of the diagonal action invariant algebra. More precisely, let $\mathbf{P}_n := \mathbf{C}[x_1,\ldots,x_n]$ be the algebra of polynomials in n indeterminates. There is a natural action of G ($G=A_{n-1},\ B_n,\ D_n$) on $\mathbf{P}_n$, $\varphi : G \longrightarrow Aut(\mathbf{P}_n)$ defined on the generators by $\sigma(x_i) := x_{\sigma(i)}$ for $\sigma\in G$, where $x_{-j} := -x_j$ for all $j\in[n]$. Consider now the tensor power $\mathbf{P}_n^{\otimes t} := \mathbf{P}_n \otimes \cdots \otimes \mathbf{P}_n$ (t-times), with the natural *tensor action* φ_T of $G^t := G \times \cdots \times G$ (t-times). The diagonal embedding $d : G \hookrightarrow G^t$ defined by $g \mapsto (g, \ldots, g)$ defines the *diagonal action* of G on $\mathbf{P}_n^{\otimes t}$, $\varphi_D := \varphi_T \circ d$. The tensor invariant algebra

$$\mathrm{TIA} := \{\bar{p} \in \mathbf{P}_n^{\otimes t} : \varphi_T(\bar{g})\bar{p} = \bar{p} \ \ \forall\, \bar{g} \in G^t\}$$

is a subalgebra of the diagonal invariant algebra

$$\mathrm{DIA} := \{\bar{p} \in \mathbf{P}_n^{\otimes t} : \varphi_D(g)\bar{p} = \bar{p} \ \ \forall\, g \in G\}.$$

Let $F_D(\bar{q})$, be the Hilbert series for the dimensions of the homogeneous components in DIA,

$$F_D(\bar{q}) := \sum_{n_1,\ldots,n_t} \dim_{\mathbf{C}}(\mathrm{DIA}_{n_1,\ldots,n_t}) q_1^{n_1} \cdots q_t^{n_t},$$

where $\mathrm{DIA}_{n_1,\ldots,n_t}$ is the homogeneous piece of multi-degree $(n_1, \ldots, n_t)$ in DIA, and $F_T(\bar{q})$ similarly defined for TIA. They show the following ([1]):

Theorem 5.1 *For all $n, t \geq 1$*

$$\frac{F_D(q)}{F_T(q)} = \sum_{\pi_1 \cdots \pi_t = e} \prod_{i=1}^{t} q_i^{fmaj(\pi_i)}$$

where the sum extends over all t-uples $(\pi_1, \ldots, \pi_t)$ of elements in $G = B_n$ such that the product $\pi_1 \cdots \pi_t$ is equal to the identity.

It would be interesting to investigate if in the case of $G = D_n$ the analogue of Theorem 5.1 is still valid.

(5.2) In [2] Adin, Brenti and Roichman define a new descent statistic, the flag descent on B_n. This and $fmaj$ extend Carlitz's result to B_n, answering Foata's question. Can, a similar flag descent statistic ($fdes_D$?), be defined for D_n so that the pair of statistic $(fdes_D, fmaj_D)$ gives a generalization of Carlitz's identity to D_n ?

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