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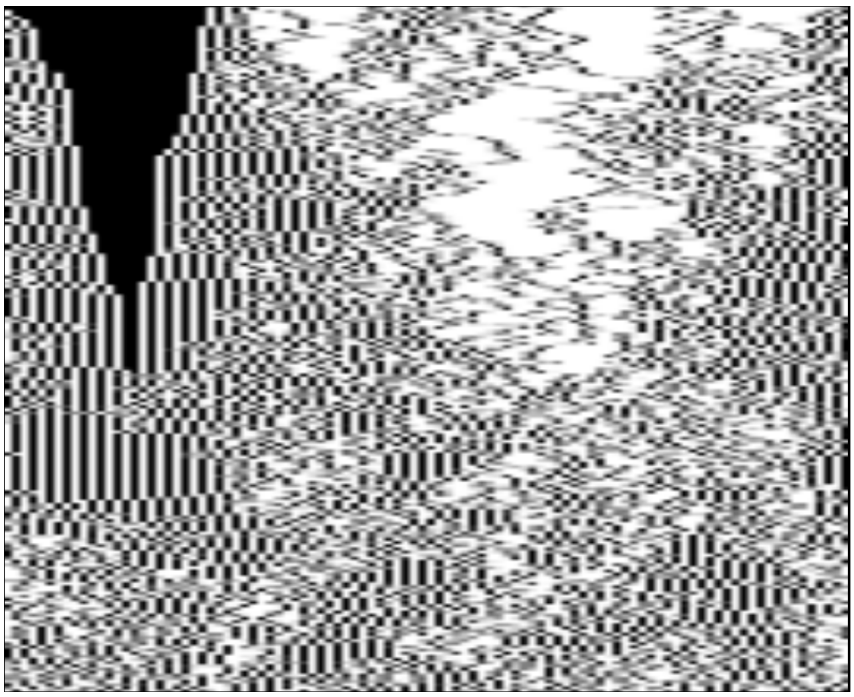
Dynamiques aléatoires avec taux dégénérés

Systèmes de particules avec contraintes cinétiques, limites hydrodynamiques, marches aléatoires en milieu aléatoire



Oriane Blondel

Habilitation à diriger les recherches

Université Claude Bernard Lyon 1

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Oriane Blondel

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Introduction en français

Je présente dans ce manuscrit des travaux réalisés depuis ma thèse, principalement en tant que chargée de recherche à l'ICJ depuis 2014. Comme la suite le montrera, ces travaux touchent à des domaines a priori distants des probabilités. Leur juxtaposition n'est pas pour autant fortuite. Leur point de départ est une famille de modèles de physique statistique : les systèmes de particules avec contraintes cinétiques (KCM). Ce sont les modèles sur lesquels je m'étais concentrée pendant la thèse et que je présente au début de la section 1.1. Outre leur intérêt intrinsèque, ils fournissent des exemples naturels de systèmes dans lesquels des hypothèses classiques (typiquement, d'ellipticité ou de monotonie) ne sont pas vérifiées. C'est avec ces exemples en tête que j'ai commencé à explorer les thématiques des limites hydrodynamiques et des marches aléatoires en milieu aléatoire dynamique (RWDRE). Le présent texte est structuré suivant ces trois thèmes : KCM en chapitre 1, limites hydrodynamiques en chapitre 2 et RWDRE en chapitre 3. Inévitablement, il y a une part d'arbitraire dans le découpage effectué, et certains problèmes et discussions auraient pu se retrouver naturellement dans une partie différente.

Le premier chapitre concerne donc les KCM. Il s'agit de systèmes de particules sur réseaux dont la dynamique est aléatoire, et qui ont la particularité de n'évoluer que si une certaine contrainte locale – garantissant un certain espace disponible – est satisfaite. Ces processus ont été introduits avec une motivation physique précise : l'étude de la transition vitreuse. Cette origine oriente les travaux de ce domaine dans des directions particulières. J'ai regroupé dans le chapitre 1 les travaux présentés pour cette HDR dans lesquels le point de vue me semblait vraiment spécifique aux KCM.

La section 1.1 est une introduction aux modèles avec contraintes cinétiques. Il contient d'abord un rappel des motivations physiques qui ont mené à leur formulation (Section 1.1.1). Viennent ensuite (en Section 1.1.2) une définition assez générale et des exemples illustratifs. J'y rappelle en particulier la distinction KCSM (Kinetically Constrained Spin Models, dynamiques de type Glauber avec création/destruction de particules) et KCLG (Kinetically Constrained Lattice Gases, dynamiques de type Kawasaki avec déplacement de particules). J'énonce ensuite certaines propriétés de base des KCM, et d'abord que tous ces systèmes sont réversibles par rapport à une mesure produit. J'explique ensuite comment identifier à quelles densités les KCSM sont ergodiques, ce qui dépend exclusivement de la présence ou non d'*amas bloqués*, une notion issue de la percolation bootstrap. On voit enfin qu'en général le trou spectral des KCSM est non nul dans la région d'ergodicité, ce qui implique un mélange exponentiel de la dynamique à l'équilibre.

On entre ensuite dans le vif du sujet avec une section 1.2 donnant un récapitulatif (partiel !) des résultats concernant la dynamique à l'équilibre et à basse température (ou à l'approche du seuil critique, au-delà duquel le système cesse d'être ergodique). Le but des études présentées ici est de comparer le comportement des KCM à celui des matériaux vitreux, et en particulier si les KCM présentent un ralentissement de la dynamique similaire

à celui des systèmes physiques. Il y a différentes façons de quantifier la rapidité de la dynamique, ou la fluidité du système. On s'intéresse en général principalement au temps de relaxation (inverse du trou spectral), qui est étroitement lié au temps de persistance, mais le coefficient de diffusion d'un traceur peut constituer une mesure alternative de la mobilité du système. On se demande donc dans cette section comment ces quantités divergent à l'approche du seuil critique. Concernant le trou spectral, les résultats sont nombreux (et particulièrement complets en dimension 2 pour les KCSM, en particulier grâce à une approche systématique inspirée par celle développée pour la percolation bootstrap). Je ne les évoque que partiellement - ceux qui sont abordés sont regroupés en Section 1.2.1. En section 1.2.2 j'évoque les résultats plus partiels connus sur le coefficient de diffusion d'un traceur. Je décris en particulier le premier résultat présenté pour cette HDR [B3], écrit avec Cristina Toninelli : il n'y a pas de transition de phase pour la diffusion d'une particule marquée dans le modèle de Kob-Andersen (KA). Ceci contredit des prédictions issues de la littérature physique et basées sur des simulations numériques. L'erreur d'interprétation peut s'expliquer par la convergence très rapide vers 0 du coefficient de diffusion.

La section 1.3 conclut ce chapitre avec un recensement (à la fois plus court et plus complet) des résultats connus sur la dynamique hors équilibre des KCSM. La question générale est la suivante : lorsque la configuration initiale est choisie suivant une mesure qui n'est pas la mesure d'équilibre, la loi du système converge-t-elle vers la mesure d'équilibre, et si oui comment ? Ce problème est en fait abordé sous deux angles un peu différents. Le cas le plus naturel est celui où la mesure initiale est produit, mais avec une densité différente de celle de la mesure d'équilibre. Dans le même ordre d'idée, on peut considérer différentes mesures initiales, avec un comportement assez homogène en espace. On s'attend au moins à ce qu'en l'absence d'amas bloqués au temps initial, et pour des densités d'équilibre inférieures au seuil critique, il y ait convergence vers la mesure d'équilibre. Ce résultat minimal n'est pourtant connu que dans des cas très particuliers. Un deuxième type de problème se présente lorsqu'on choisit la mesure initiale de sorte qu'une fraction de l'espace soit initialement occupée. On s'intéresse alors à la frontière entre la région des points où une mise à jour a été possible et son complémentaire. Dans des cas très particuliers (dont l'un est étudié dans [B4], avec Aurélia Deshayes et Cristina Toninelli), on sait montrer que cette frontière progresse linéairement en temps, avec des fluctuations gaussiennes en dimension 1. Pour les deux questions, les résultats sont très partiels et laissent à penser qu'il manque encore des outils pour résoudre le problème. Un point crucial dans les travaux présentés et qui demande à être mieux compris est la capacité d'un KCSM à générer des espaces vides (permettant de satisfaire les contraintes) – ou du moins à les conserver.

Dans le deuxième chapitre de ce manuscrit, je présente mes travaux liés aux limites hydrodynamiques. Les questions étudiées sont les incontournables de ce domaine : on considère un système de particules avec dynamique conservative, et on s'interroge sur l'évolution du profil de densité avec le temps. Plus précisément, on cherche d'une part quelle équation aux dérivées partielles (EDP) décrit cette évolution au premier ordre, et d'autre part comment décrire les fluctuations autour de ce profil limite. J'étudie ici des dynamiques où le comportement de chaque particule est de type marche aléatoire symétrique, ce qui induit une évolution du système sur une échelle de temps diffusive (une région de taille N évoluera en temps d'ordre N^2), et des EDP limites où la dérivée en temps est donnée par le Laplacien d'une certaine fonction de la densité. Après un bref rappel des questions posées et des méthodes classiques, en Section 2.1, je m'intéresse à des modèles particuliers.

Le premier est le processus d'exclusion facilité (FEP) ; les résultats le concernant sont

décrits en Section 2.2. Dans ce modèle de type exclusion, chaque particule saute vers un voisin vide à condition d'avoir une autre particule comme voisine. Les résultats disponibles jusqu'ici sont restreints à la dimension 1. Dans ce cas, il est en effet possible d'associer au FEP un processus de type zero-range dont l'analyse est plus simple. Dans une paire de travaux en collaboration avec Clément Erignoux, Makiko Sasada et Marielle Simon [B5, B6], nous montrons que la limite hydrodynamique de ce système est décrite par une équation de type problème de Stefan (donc problème à frontière libre)

$$\partial_t \rho = \Delta \left(\frac{2\rho - 1}{\rho} \mathbf{1}_{\rho > 1/2} \right). \tag{1}$$

Dans [B5], nous montrons d'abord le résultat de limite hydrodynamique dans le régime surcritique où la densité est uniformément supérieure à $1/2$. Pour appliquer la méthode d'entropie, l'étape principale consiste à montrer qu'en temps sous-diffusif, le système atteint une configuration active (dans laquelle les sites vides sont isolés). Dans [B6], nous identifions les mesures stationnaires invariantes par translation du FEP, ce qui nous permet de montrer la limite hydrodynamique pour un profil initial quelconque. Le deuxième résultat principal de l'article est de montrer que se forme en temps sous-diffusif une frontière entre phases active/gelée au niveau microscopique. Cette frontière coïncide nécessairement avec la frontière macroscopique définie par le problème de Stefan limite.

Le deuxième modèle considéré est celui des milieux poreux, traité en Section 2.3. Dans ce modèle, une particule en x saute en $x \pm 1$ à un taux donné par le nombre de particules présentes sur les deux sites voisins de la paire $\{x, x \pm 1\}$. Il était connu [53] que la limite hydrodynamique est donnée par l'équation des milieux poreux $\partial_t u = \Delta u^2$. Dans [B7], avec Patricia Gonçalves et Marielle Simon, nous étudions les fluctuations de ce système à l'équilibre (donc autour d'un profil constant) dans le cadre faiblement asymétrique, où chaque particule a un biais d'ordre $N^{-\gamma}$ vers la droite. Nous montrons que, pour $\gamma > 1$, le champ de fluctuations converge dans la limite diffusive vers un processus d'Ornstein-Uhlenbeck $d\mathcal{Y}_t = 2\rho\Delta\mathcal{Y}_t dt + \sqrt{2\rho^2(1-\rho)}\nabla d\mathcal{W}_t$. Dans le cas plus intéressant $\gamma = 1/2$, c'est l'équation de Burgers stochastique qui émerge à la limite :

$$d\mathcal{Y}_t = 2\rho\Delta\mathcal{Y}_t - 4\nabla(\mathcal{Y}_t^2)dt + \sqrt{2\rho^2(1-\rho)}\nabla d\mathcal{W}_t.$$

L'ingrédient principal pour montrer ce résultat est la preuve d'un principe de Boltzmann-Gibbs à l'ordre 2. Les propriétés d'ergodicité du processus ne sont pas suffisantes pour suivre la preuve standard de ce principe (en particulier, une configuration dans laquelle les particules sont toutes à distance supérieure à 3 est gelée). Nous pallions ce problème en utilisant des arguments de chemin.

Pour terminer, le chapitre 3 de ce manuscrit rassemble mes travaux effectués dans le domaine des marches aléatoires en milieu aléatoire (dynamique). Je commence par rappeler en Sections 3.1.1 et 3.1.2 certains résultats classiques basés sur des arguments d'analyse spectrale qui peuvent s'utiliser lorsque l'environnement vu de la particule est stationnaire. On peut considérer que cette partie de la théorie est bien comprise. Lorsque l'environnement vu de la particule n'est plus stationnaire, les résultats sont beaucoup plus partiels. En Section 3.1.3, je rappelle certains résultats disponibles dans ce cas, sous des hypothèses de mélange fortes sur l'environnement. Mes travaux visent en particulier à obtenir des théorèmes limites (loi des grands nombres et théorème central limite) sous des hypothèses de mélange *non-uniforme* – un des exemples d'applications que je garde en tête étant les KCM.

En Section 3.2, je décris les travaux [B1, B2] écrits avec Luca Avena et Alessandra Faggionato. Dans [B1], nous considérons des environnements stationnaires satisfaisant une inégalité de Poincaré (par exemple comme beaucoup de KCSM dans le régime ergodique). Les marches aléatoires que nous considérons sont des perturbations de marches "stationnaires" : l'environnement vu de la marche non perturbée reste stationnaire, et la taille de la perturbation est contrôlée par le trou spectral de l'environnement. Ces hypothèses nous permettent d'écrire des développements en série de quantités qui caractérisent l'environnement vu de la marche perturbée, et à terme de montrer loi des grands nombres et TCL. Une des difficultés est qu'on doit changer d'espace : si l'environnement est naturellement décrit par un semi-groupe dans $L^2(\mu)$, l'environnement vu de la marche perturbé se comporte mieux dans $L^2(\hat{\mu})$, où $\hat{\mu}$ est sa mesure stationnaire. Dans [B2], nous explorons quelques conséquences des résultats de [B1] dans des cas particuliers, par exemple lorsque l'environnement est donné par le modèle Est – l'un des KCSM les plus emblématiques. Un résultat intriguant porte sur la ε–marche aléatoire, qui saute à droite avec proba $1+\varepsilon$ sur une particule, $1-\varepsilon$ sur un site vide. Dans le régime où les résultats de [B1] sont valides (ε assez petit), la vitesse asymptotique est la même (et non nulle), que l'environnement soit le modèle Est ou son image par une symétrie miroir.

Dans les travaux suivants, je travaille avec des hypothèses de mélange moins rapide (non exponentiel) sur l'environnement, au prix d'une perte de généralité soit sur l'environnement soit sur la marche aléatoire.

En Section 3.3, je considère un environnement donné par un nuage poissonnien de marches aléatoires simples indépendantes. Dans [62], les auteurs montrent qu'à forte densité, la ε–marche aléatoire satisfait LGN et TCL. Dans [B8, B10], avec Marcelo Hilário, Renato dos Santos, Vladas Sidoravicius and Augusto Teixeira, nous étendons ces résultats dans plusieurs directions. D'une part, toujours à haute densité mais en dimension plus grande ; d'autre part à basse densité ou avec fort biais sur les particules en dimension 1. La preuve repose sur un argument de renormalisation pour montrer une propriété de ballisticité de la marche aléatoire. C'est principalement cette étape de la preuve qui demande des idées nouvelles par rapport au cadre de [62]. Une fois la ballisticité établie, la connaissance fine que nous avons de l'environnement choisi permet de définir des temps de renouvellement, et donc de découper la trajectoire de la marche en incréments iid.

Le dernier travail présenté ici ([B9], avec Marcelo Hilário et Augusto Teixeira, Section 3.4) porte sur des environnements assez généraux, mais uniquement en dimension 1. L'autre restriction forte porte sur les marches aléatoires qui peuvent être considérées : nous n'autorisons que des marches faisant des sauts de ± 1. En revanche, notre cadre de travail n'utilise pas de propriété de Markov, et nous pouvons considérer des marches en milieu aléatoire avec une interaction plus générale que celle couramment utilisée. L'hypothèse principale que nous faisons sur l'environnement demande une décroissance des covariances polynomiale en temps, avec un exposant suffisamment élevé. En particulier, cette hypothèse ne demande pas d'uniformité dans la propriété de mélange, et permet (en dimension 1 et pour des marches aux plus proches voisins) d'étendre les résultats de [B1] hors du régime perturbatif. L'hypothèse de mélange étant aveugle à l'inversion du temps, nous pouvons aussi traiter le cas par exemple du processus de contact rétrograde, beaucoup moins bien compris que le processus de contact standard.

Introduction in English

In this manuscript, I present work I did after my PhD, mainly as a junior researcher placed at ICJ since 2014. As we will see, though these works concern a priori distant domains in probability, they are not placed together by chance. At the beginning of the story lies a family of models: kinetically constrained models, or KCM. These are the systems on which I did my PhD and I present them at the beginning of Section 1.1. Aside from their intrinsic interest, they offer natural examples of systems in which classical hypotheses fail (typically, ellipticity, monotonicity). It is therefore with these examples in mind that I started to explore hydrodynamic limits (Chapter 2) and random walks in random environment (RWDRE). The present manuscript is organized along these three themes: KCM in Chapter 1, hydrodynamic limits in Chapter 2 and RWDRE in Chapter 3. Inevitably, this segmentation is partly arbitrary and some problems and discussions would be appropriate in a different section.

The first chapter is about KCM. Those are interacting particle systems on lattices, with stochastic synamics, with the specificity that they only evolve if a certain local constraint – guaranteeing a certain amount of free space – is satisfied. These processes were introduced with a precise physical motivation: the study of the glass transition. This origin guides the works in this domain towards specific directions. I collected in Chapter 1 the works presented for this habilitation in which the point of view seemed most specific to KCM.

Section 1.1 is an introduction to kinetically constrained models. It contains first a reminder of the physical motivations that led to their formulation (Section 1.1.1). Next (in Section 1.1.2) we find a general definition and illustrative examples. I recall in particular the distinction between KCSM (Kinetically Constrained Spin Models, with Glauber-type dynamics, i.e. creation/destruction of particles) and KCLG (Kinetically Constrained Lattice Gases, with Kawasaki-type dynamics in which particles jump around). I then state some basic properties of KCM, first that they are reversible w.r.t. a product measure. I then explain how to identify at which densities KCSM are ergodic, which depends exclusively on the presence of *blocked clusters*, a notion coming from bootstrap percolation. Finally, we see that in general the spectral gap of KCSM is non-zero in the ergodicity region, which implies exponential mixing of the equilibrium dynamics.

Then we enter the core of the subject, with a Section 1.2 giving a (partial!) recap of results concerning equilibrium and low temperature dynamics. The goal is to compare the behavior of KCM to that of glassy materials, and in particular check whether KCM present a dynamical slowdown similar to that of physical systems. There are several ways to quantify the speed of the dynamics, or the system's fluidity. Generally, one considers mostly the relaxation time (inverse of the spectral gap), which is closely related to the persistence time. However, the diffusion coefficient of a tracer can be an alternative to measure the mobility of the system. In this section, we examine how those quantities diverge as the critical threshold is approached. Concerning the spectral gap, there are numerous results

(and particularly complete in dimension 2 for KCSM, thanks to a systematic approach inspired by that developed for bootstrap percolation). I only mention them partially - those that do appear are gathered in Section 1.2.1. In Section 1.2.2, I mention the sparser results available on the diffusion coefficient of a tracer. I describe in particular the first result presented for this habilitation [B3], written with Cristina Toninelli: there is no phase transition for the diffusion of a tagged particle in the Kob-Andersen model (KA). This contradicts predictions coming from the physics literature and based on numerical simulations. The interpretation error may be explained by the very fast convergence to 0 of the diffusion coefficient.

Section 1.3 concludes this chapter with a tour (both shorter and more complete) of known results on non-equilibrium dynamics of KCSM. The general question is the following: when the initial configuration is chosen with a distribution that is not the equilibrium measure, does the law of the system converge to the equilibrium measure, and if so, how? This problem is in fact attacked from two different angles. The most natural case is the one in which the initial measure is product, but with a density slightly different from that of the equilibrium measure. Similarly, one can consider different initial measures that are more or less homogeneous in space. We expect that, unless there is initially a blocked cluster, for equilibrium densities below the critical threshold, there is convergence to equilibrium. However, this result is only known in very specific cases. A second type of problem arises when one chooses the initial configuration so that a whole fraction of space (e.g. a half-space) is initially occupied. We may then consider the frontier between the region in which updates are possible and its complement. In very specific cases (one of which is presented in [B4]), we can show that this frontier advances linearly in time, with gaussian fluctuations in dimension 1. For both types of questions, the results are very partial and we think that we are still lacking appropriate tools to handle the problem. A crucial point in the works presented here, and which begs to be better understood, is the capacity of a KCM to generate empty spots (which allow the constraint to be satisfied) – or at least to preserve them.

In the second chapter of this manuscript, I present my works in connection to hydrodynamic limits. The questions I study are the fundamental ones: consider an interacting particle system with conservative dynamics. How does its density profile evolve? More precisely, what PDE describes this evolution at first order, and how can the fluctuations around the limit profile be described. I study dynamics in which the behavior of each particle is similar to a symmetric simple random walk, which means the system evolves on a diffusive time scale (a region of size N evolves in a time of order N^2). Also, the time derivative in the limit PDE is given by the Laplacian of some function of the density. After a short reminder of standard questions and methods in Section 2.1, I consider specific models.

The first one is the facilitated exclusion process (FEP). The results about it are presented in Section 2.2. In this exclusion-type model, each particle jumps to an empty neighbor provided it has another particle as a neighbor. So far the only available results are restricted to dimension 1. Indeed, in that case it is possible to associate with the FEP a zero-range process which is simpler to analyze. In two works in collaboration with Clément Erignoux, Makiko Sasada and Marielle Simon [B5, B6], we show that the hydrodynamic limit of this system is described by a Stefan problem (i.e. by a PDE with a free boundary problem)

$$\partial_t \rho = \Delta \left(\frac{2\rho - 1}{\rho} \mathbf{1}_{\rho > 1/2} \right). \tag{2}$$

In [B5], we first show the hydrodynamic result in the supercritical regime in which the density is uniformly above 1/2. To apply the entropy method, the main step consists in showing that the system reaches an active configuration (in which empty sites are isolated) in subdiffusive time. In [B6], we identify the translation invariant stationary measures of the FEP, which allows us to show the hydrodynamic limit for a general initial density profile. The second main result of the paper is to show that a separation between active/frozen region appears in subdiffusive time. In the hydrodynamic limit, this frontier necessarily coincides with that which appears in the Stefan problem.

The second model I consider, in Section 2.3, is the porous medium model. In this system, a particle at x jumps to $x \pm 1$ at a rate given by the number of particles present on the two neighboring sites of the pair $\{x, x \pm 1\}$. It was already known [53] that the hydrodynamic limit is given by the porous medium equation $\partial_t u = \Delta u^2$. In [B7], with Patricia Gonçalves and Marielle Simon, we study the fluctuations of this system at equilibrium (i.e. around a constant profile), in the weakly asymmetric setting where every particle has a bias of order $N^{-\gamma}$ to the right. We show that, for $\gamma > 1$, the fluctuation field converges in the diffusive limit to an Ornstein-Uhlenbeck process $d\mathcal{Y}_t = 2\rho\Delta\mathcal{Y}_t dt + \sqrt{2\rho^2(1-\rho)}\nabla d\mathcal{W}_t$. In the more interesting case $\gamma = 1/2$, it is the stochastic Burgers equation that we find in the limit:

$$d\mathcal{Y}_t = 2\rho\Delta\mathcal{Y}_t - 4\nabla(\mathcal{Y}_t^2)dt + \sqrt{2\rho^2(1-\rho)}\nabla d\mathcal{W}_t.$$

The main ingredient to show this result is the proof of a second-order Boltzmann-Gibbs principle. The ergodicity properties of the process are not sufficient to follow the standard method to obtain this principle (in particular, a configuration in which all particles are pairwise at a distance larger than 3 is frozen). We remedy this issue by using path arguments.

The last chapter gathers my works on random walks in (dynamic) random environment. In Sections 3.1.1 and 3.1.2, I start by recalling some standard results based on spectral arguments that can be used when the environment seen from the particle is stationary. We may consider this part of the theory well-understood. When the environment seen from the particle is no longer stationary, results are much sparser. In Section 3.1.3, I recall some available results in this setting, with strong mixing hypotheses on the environment. My works aim mostly to obtain limit theorems (law of large numbers and central limit theorem) under *non-uniform* mixing hypotheses – a class of applications that I keep in mind being the KCM.

In Section 3.2, I describe [B1, B2], written with Luca Avena and Alessandra Faggionato. In [B1], we consider stationary environment satisfying a Poincaré inequality (like e.g. most KCSM in the ergodic regime). The random walks we consider are perturbations of "stationary" walks: the environment seen from the unperturbed walk is still stationary, and the perturbation size is controlled by the spectral gap of the environment. These hypotheses allow us to write series expansions of quantities that characterize the environment seen from the perturbed random walk. Eventually, this allows to show LLN and CLT. One of the difficulties is that we need to change space: if the environment is naturally described by a semigroup in $L^2(\mu)$, the environment seen from the perturbed walk behaves better in $L^2(\hat{\mu})$, where $\hat{\mu}$ is its stationary measure. In [B2], we explore some consequences of the results of [B1] in specific cases, e.g. when the environment is given by the East model – one of the most emblematic KCSM. We have an intriguing result on the ε–random walk, which jumps right with probability $1 + \varepsilon$ on a particle, $1 - \varepsilon$ on an empty site. In the regime where the results of [B1] apply (small ε), the asymptotic speed is that same (and non-zero), whether the environment is the East model or its image under mirror symmetry.

In the next works, I work with slower (non exponential) mixing hypotheses on the environment. The cost is a loss of generality either on the environment or the random walker.

In Section 3.3, the environment is given by a poissonian cloud of independent simple random walks. In [62] the authors show that, at high density, the ε–random walk satisfies LLN and CLT. In [B8, B10], with Marcelo Hilário, Renato dos Santos, Vladas Sidoravicius and Augusto Teixeira, we extend this result in several directions. On the one hand, still at high density but in higher dimension; on the other hand at low density or strong drift on particles, in dimension 1. The proof relies on a renormalization argument to show a ballisticity property of the random walker. It is mostly this step of the proof that requires new ideas w.r.t. [62]. Once ballisticity is established, the precise understanding we have of the environment allows us to build regeneration times, and thus to slice the random walker trajectory into iid increments.

The last work presented here ([B9], with Marcelo Hilário and Augusto Teixeira, Section 3.4) concerns rather general environments, but only in dimension 1. The other strong restriction is on the random walkers that may be considered: we only allow walkers that perform ± 1. On the other hand, our setting does not rely on the Markov property, and we may consider walkers with an interaction with their environment more general than the one commonly used. The main hypothesis we have on the environment requires a polynomial decay of covariances in time, with an exponent high enough. In particular, this hypothesis requires no uniformity in the mixing property and allows (in dimension 1 and for nearest-neighbor random walkers) to extend the results of [B1] out of the perturbative regime. The mixing hypothesis is also insensitive to time-reversal, so we may for instance treat the case of the backward contact process, which is generally much less understood than the forward version.

Chapter 1

Kinetically constrained models

1.1 Generalities on KCM

1.1.1 Physical motivations

Kinetically constrained models (KCM) have been introduced in the physics literature, originally by H. G. Fredrickson and H. C. Andersen[1] [46, 45]. Their works are part of an ongoing investigation into the physics of glasses. Glassy materials present indeed two features whose juxtaposition is puzzling: they are amorphous, yet solid. Microscopically, there is no discernible order in the molecules that constitute them. Macroscopically, at room temperature, glass is rigid enough that is can be used routinely in containers or windows [103]. Glassy materials are often described as *amorphous solids*.

There are several phenomena that are known to play a role in the particular dynamics of glasses (see [30, 5] for reviews). Let me focus on one that specifically motivated the definition of KCM: the *cage effect*. In order to produce glass, one heats up an appropriate material (e.g. silica) until it starts being malleable. It is then cooled down very rapidly (quenched), in order to avoid the transition to crystal. One thing that happens during this cooldown is that –schematically– molecules are pressed together. This creates a crowding effect, meaning that molecules become trapped by their neighbors, which effectively form a cage. Note that in order to open a cage by removing a neighbor, one needs the cage of the latter to be open, and so on. This gives rise to the idea of models in which motion is forbidden unless there is enough space around. Kinetically constrained models with conservation of the number of particles (KCLG – typically the Kob-Andersen model [72], see below) fit this picture reasonably well.

It is a slightly different phenomenon, maybe more subtle, that gave rise to the formulation of the Fredrickson-Andersen (FA) model and their cousins the kinetically constrained spin models (KCSM). It occurs on a coarse-grained scale rather than on the molecular level. There, one can observe that the system is split between regions which are densely packed and practically frozen, and more active regions. The latter are rare, being energetically costly, but their presence helps rearrange neighboring regions which in turn can become active and so on. This phenomenon is called dynamical facilitation. In the model proposed in [46], spins represent coarse-grained regions, and the authors impose the constraint that regions can flip between active and frozen only if a neighboring region is active.

In view of their origin, the study of KCM has mostly been focused on their dynamical properties. The goal is to identify which of the landmarks of glassy behavior are well reproduced by these models whose thermodynamic behavior (equilibrium distribution) is trivial. Features of interest include rigidification at low temperature, super-Arrhenius divergence of the viscosity at the approach of the transition, heterogeneous dynamics, aging... [88, 98] As we will see, KCM successfully reproduce a number of these phenomena, up to a certain degree. For appropriate choices of constraints, they can display an ergodic phase transition corresponding to rigidification below a certain temperature (see Section 1.1.3). Super-Arrhenius divergence of the relaxation

[1] Not to be confused with the author of *The Little Mermaid*.

time is a feature commonly shared by most KCM, while Arrhenius behavior (which occurs in strong glasses) is also possible (see Section 1.2.1). Dynamical heterogeneity has also been identified rigorously in some cases [31, 24], as well as aging phenomena [42, 43, 99].

More recently, KCM have showed up in a different context [90]. In a system of qubits with strong conservative noise, the second moment of operators is described by the FA-1f model (see below), and the results of [B4] describe the spreading of the support of local operators.

1.1.2 Definitions and examples

Kinetically constrained models are Markov processes on $\{0,1\}^V$, where V is a priori the countable vertex set of a graph G. In this manuscript, we will always consider G to be either $\mathbb{Z}^d$ (with its canonical graph structure) for some integer d, or a subset $\Lambda \subset \mathbb{Z}^d$. $(e_1, \dots, e_d)$ denotes the canonical basis of $\mathbb{Z}^d$. I will write $x \sim y$ to say that x and y are connected by an edge. Let $\Omega_\Lambda := \{0,1\}^\Lambda$ for $\Lambda \subset \mathbb{Z}^d$. When $\Lambda = \mathbb{Z}^d$ the subscript can be omitted. KCM will be denoted by $(\eta_t)_{t\geq 0}$, with $\eta_t \in \Omega_\Lambda$. For a configuration $\eta \in \Omega_\Lambda$, $\eta(x) \in \{0,1\}$ will denote the state of η at site $x \in \Lambda$. I will refer to sites with $\eta(x) = 1$ as occupied by a particle, and those with $\eta(x) = 0$ as holes, empty sites or zeros. For $\Lambda' \subset \Lambda$, $\eta_{|\Lambda'}$ denotes the restriction of η to Λ'. For $x, y \in \Lambda$, $\eta \in \Omega_\Lambda$, we define moreover η^x and $\eta^{x,y}$ the configurations satisfying respectively

$$\eta^x(z) = \begin{cases} \eta(z) & \text{if } z \neq x \\ 1-\eta(x) & \text{if } z = x \end{cases}, \quad \eta^{x,y}(z) = \begin{cases} \eta(z) & \text{if } z \notin \{x,y\} \\ \eta(y) & \text{if } z = x \\ \eta(x) & \text{if } z = y. \end{cases} \tag{1.1}$$

A local function $f : \Omega_\Lambda \to \mathbb{R}$ is a function that depends on a finite number of coordinates, i.e. there exists a finite set $\Lambda' \subset \Lambda$ such that for all $\eta, \eta' \in \Omega_\Lambda$, $\eta_{|\Lambda'} = \eta'_{\Lambda'}$ implies $f(\eta) = f(\eta')$.

In this Chapter 1, for a parameter $p \in [0,1]$ we will write $q := 1-p$, and denote by μ_p the product Bernoulli measure with parameter p on Ω or Ω_Λ. When possible, p may be omitted from the notation. We also write Var_{μ_p}, or simply Var_p for the variance associated with μ_p.

We use the following standard notation for asymptotic comparisons: for two functions $f, g : (0,1] \to \mathbb{R}_+^*$, we write
$$\begin{cases} f = O(g) & \text{if } \limsup_{q\to 0} f(q)/g(q) < \infty, \\ f = o(g) & \text{if } f(q)/g(q) \to_{q\to 0} 0, \\ f = \Omega(g) & \text{if } \liminf_{q\to 0} f(q)/g(q) > 0, \\ f = \Theta(g) & \text{if } f = \Omega(g) \text{ and } f = O(g). \end{cases}$$

In order to define KCSM and KCLG, let us first introduce the notion of *update family*: it is a finite family $\mathcal{U} = \{U_1, \dots, U_n\}$ of finite subsets of $\mathbb{Z}^d \setminus \{0\}$. With a given update family is associated a family of *constraints* $(c_x(\eta))_{x\in\mathbb{Z}^d, \eta\in\Omega}$ which takes values in $\{0,1\}$ and is defined as

$$c_x(\eta) = \begin{cases} 1 & \text{if } \exists k \in \{1,\dots,n\} \text{ s.t. } \eta_{|x+U_k} \equiv 0, \\ 0 & \text{else.} \end{cases} \tag{1.2}$$

Definition 1. *The KCSM with update family $\mathcal{U} = \{U_1, \dots, U_n\}$ and density $p \in [0,1]$ is the Markov process $(\eta_t)_{t\geq 0}$ with generator*

$$Lf(\eta) = \sum_{x\in\mathbb{Z}^d} c_x(\eta)[p(1-\eta(x)) + q\eta(x)]\,[f(\eta^x) - f(\eta)]\,, \tag{1.3}$$

defined on local functions. We will sometimes abbreviate $r_x(\eta) = p(1-\eta(x)) + q\eta(x)$.

A version of this process can be constructed more explicitly in a standard way. Attach to each site $x \in \mathbb{Z}^d$ independent Poisson point processes of parameter 1 and think of them as clock rings, signalling times for possible updates. When a clock rings at x, check if the constraint is satisfied. If that is the case, sample independently a Bernoulli variable with parameter p and replace the value of the configuration at x by this variable. If the constraint is not satisfied, the configuration is not updated. A standard argument allows to justify that this construction is

well-defined [59]. The associated semi-group (acting on $f \in L^2(\mu_p)$) will be denoted by $P_t f(\cdot)$ or $\mathbb{E}_\cdot[f(\eta_t)]$.

We can also define KCSM in a restricted region of space $\Lambda \subset \mathbb{Z}^d$. In that case, we fix a configuration $\omega \in \Omega$ which acts as a boundary condition and define

$$c_x^\Lambda(\eta) = c_x(\eta_{|\Lambda} \cdot \omega_{|\Lambda^c}), \tag{1.4}$$

where the dot indicates concatenation. The KCSM in Λ with boundary condition ω is then defined as (1.3), restricting the sum to $x \in \Lambda$ and replacing c_x by c_x^Λ. A graphical construction similar to the one above is possible.

Let me introduce some of the most emblematic KCSM. The East and FA-1f models in dimension 1 are illustrated in Figure 1.1.

FA-jf The Fredrickson-Andersen j-spin facilitated model has the constraint that at least j nearest neighbors are empty. In the above framework, that means that the U_i's are the $\binom{2d}{j}$ subsets of nearest neighbors of the origin with cardinality j. In particular, we need $j \leq 2d$ in order to possibly satisfy the constraint somewhere. $j = 0$ is just the empty constraint, in which case we are left with the dynamics known as Independent Spin Flips.

East The East model was originally introduced in one dimension [66], with the constraint that the neighbor on the East side is empty. On $\mathbb{Z}$, this means that $\mathcal{U} = \{\{1\}\}$. In higher dimension, this can be generalized to $\mathcal{U} = \{\{e_1\}, \dots, \{e_d\}\}$, so the constraint requires at least one empty neighbor in the positive direction.

North-East This is another possible generalization of the East model in higher dimension. The constraint is stronger, in that it requires *all* neighbors in the positive direction to be empty, i.e. $\mathcal{U} = \{\{e_1, \dots, e_d\}\}$.

Let us now define Kinetically Constrained Lattice Gases (KCLG). In general, one could be interested in dynamics that allow long-range jumps (or exchanges). For the purpose of this manuscript, it is enough to consider only nearest-neighbor jumps. We can then define KCLG through a family of constraints $(c_{x,y}(\eta))_{x\sim y\in\mathbb{Z}^d,\eta\in\Omega}$ that should be satisfied in order to allow an exchange between the configurations at x and y. The constraint should not involve x, y in that it should be satisfied both before and after the exchange. It should also be local and translation invariant.

Definition 2. *The KCLG with constraints $(c_{x,y}(\eta))_{x\sim y\in\mathbb{Z}^d,\eta\in\Omega}$ is the Markov process with generator*

$$Lf(\eta) = \sum_{x\in\mathbb{Z}^d} \sum_{y\sim x} c_{x,y}(\eta)\left[f(\eta^{x,y}) - f(\eta)\right]. \tag{1.5}$$

Just as for KCSM, one can give a graphical construction for these processes, in which Poisson clocks are attached to edges and signal times at which states might be exchanged at an edge's ends. We use the same notations as for KCSM. KCLG restricted to a subset Λ can also be defined, and one then has to decide what happens at the boundary. One possibility is to fix a boundary condition and allow exchanges only between sites in Λ; another is to allow creation/destruction of particles at the boundary, which then acts as a reservoir.

Let us now give the examples of KCLG that we will consider here.

KA-jf The Kob-Andersen models [72] are the conservative version of the FA-jf models. In the KA model with parameter j, $c_{x,y}$ requires that both x and y have at least $j-1$ empty neighbors in $\mathbb{Z}^d \setminus \{x, y\}$. For $j = 1$, we recover the simple symmetric exclusion process (SSEP).

PMM The porous medium model has been studied in [13, 53]. Section 2.3 will explain why this name is starting to stick. In its simplest version, $c_{x,y}$ requires that either x has an empty neighbor before and after the exchange, or y does. In Section 2.3 we will see a more complicated version of this.

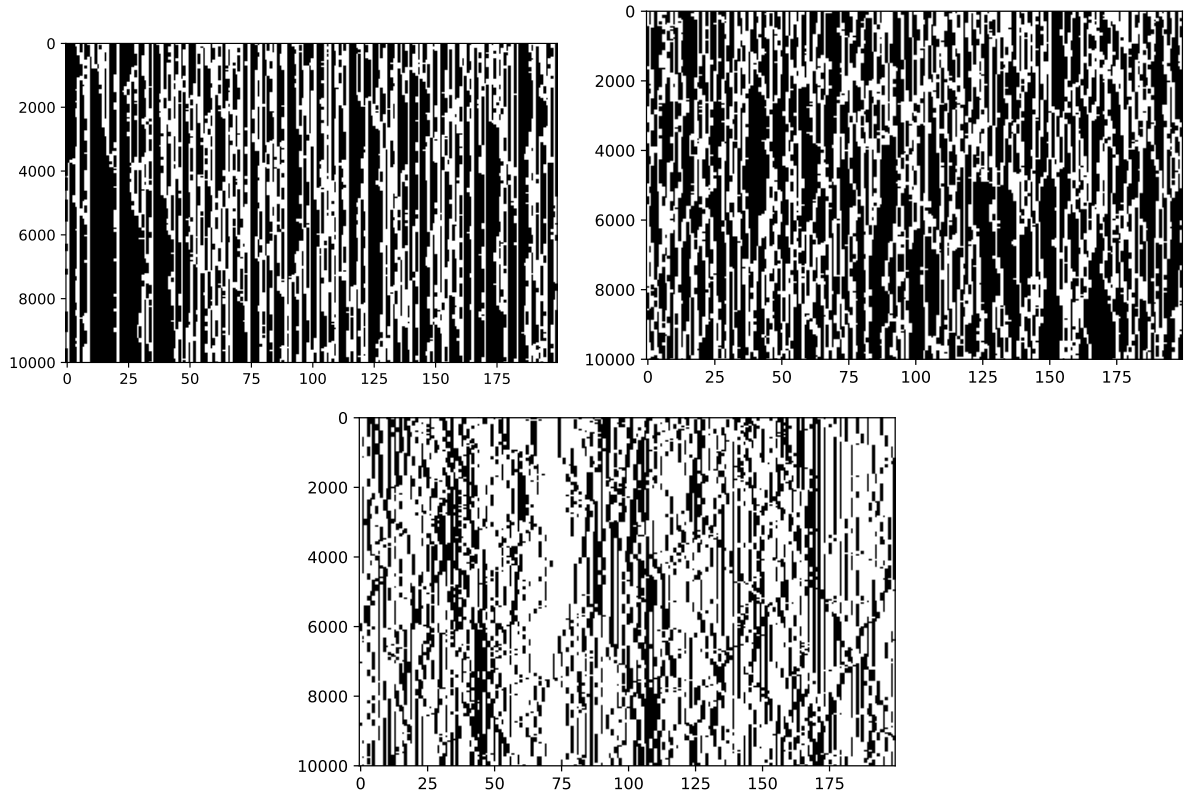

Figure 1.1: Top, the dynamics of the East and FA-1f models in dimension 1 with density $p = 0.4$; bottom, the PMM model. Particles are in black, holes in white. The initial configuration is chosen as the product Bernoulli measure with parameter 0.4. The time direction is vertical and downwards (but the latter precision is superfluous, see Property 1).

1.1.3 Basic properties: reversibility and ergodicity

The first notable property that is shared by KCSM and KCLG is that they admit a very simple equilibrium measure[2].

Property 1. *Let $p \in [0,1]$. KCSM with density p and KCLG are reversible w.r.t. μ_p. μ_p is called equilibrium distribution.*

This can be easily checked by noticing that the detailed balance condition is satisfied. If one removes the constraints from (1.3) and (1.5) this is a well-known fact. But adding the constraint does not affect detailed balance, since the constraint is the same to perform an update and reverse it.

Remark 1. *If one wants to write the probability of a configuration σ as $\exp(-\beta H(\sigma))$, with $H(\sigma) = \sum_x (1 - \sigma(x))$ a Hamiltonian without interaction terms, it is natural to express the inverse temperature β as $\log(p/q)$.*

This gives us in particular one invariant measure for KCSM and a family of invariant measures for KCLG. It is however not true in general that it is the only one (it is not even the only reversible measure in general). In fact, we have few answers to the following question.

Question 1. *What are the reversible or invariant measures for a given KCM?*

An easier question, directly related to the physical motivation for KCMs, is whether they can be qualified as rigid at some density, meaning that some updates would never occur.

Question 2.
1. *Fix a KCSM with density $p \in (0,1)$. Starting from μ_p, is it true that with probability 1 any given site will switch state during the dynamics (i.e. the dynamics is* ergodic*)?*
2. *Fix a KCLG and a density $p \in (0,1)$. Starting from μ_p, is it true that with probability 1 any pair of neighboring sites will exchange their states during the dynamics?*

There is a satisfying answer to this question in the case of KCSM, as I detail below (see Proposition 1). For KCLG however a general strategy to answer this question has yet to be found. Before moving to the answer in the case of KCSM, let us examine some illustrative examples.

The first one is the case of initial density 1: when every site is initially occupied by a particle, unless the constraint is moot (e.g. a KCSM with $\mathcal{U} = \emptyset$), the system remains frozen in its initial state. This is a trivial observation, but already signals that KCSM admit at least two reversible measures: μ_p and $\delta_{\mathbf{1}}$, where $\mathbf{1}$ denotes the entirely filled configuration.

Let us examine the case of FA-3f in dimension $d = 2$. There, it is readily checked that a 2×2 initially occupied square will remain occupied throughout the dynamics, since the four particles collaborate in preventing their respective constraints from being satisfied. In particular, if the initial configuration is sampled with law μ, there are infinitely many such finite structures that remain frozen. This argument can be generalized to any FA-jf with $j > d$, and from now on we will only consider the cases $j \leq d$. Similarly, KA models will always have parameter $j \leq d$.

Let us now move to FA-2f. There it can be checked that no finite structure is frozen on its own. However, a double infinite line of particles can never be updated. One could also cook up more complicated structures that remain frozen throughout the dynamics.

It appears that one obstruction to answering Question 2 positively is the existence of a *blocked cluster*, which is defined below. It turns out that it is in fact the only one. In order to state this properly, let us define the *bootstrap* procedure associated with the update family $\mathcal{U}$. It takes an initial configuration $\eta \in \Omega$, removes particles at every site where the constraint is satisfied and iterates infinitely many times (this is similar to running the KCSM dynamics in discrete time with parameter $p = 0$). In the limit, we obtain a configuration $[\eta]$ in which no occupied site satisfies the constraint. We say that the remaining particles form blocked clusters.

[2] This is in fact a built-in property, see e.g. [46, 88].

Proposition 1. *[27] The KCSM dynamics at density p is ergodic iff $\mu_p([\eta]_0 = 0) = 1$. This defines a critical density $p_c \in [0,1]$ such that the KCSM is ergodic for all $p < p_c$ and non-ergodic for $p > p_c$.*

In particular, East and FA-jf ($j \leq d$) are ergodic at any density $p < 1$; FA-jf ($j > d$) is ergodic at no density $p > 0$; North-East is ergodic for $p \leq p_c$, where p_c is the critical parameter for site oriented percolation (every remaining particles in $[\eta]$ is the starting point of an infinite up-right path of occupied sites).

For KCLG, there is no known systematic procedure that would play the role of bootstrap percolation. It is however possible to show the existence of a critical density p_c. It is characterized ([29, Proposition 2.16]) as the highest density below which, with probability one, it is possible (through arbitrarily many allowed exchanges) to exchange the values at the extremities of any edge. For KA with parameter $j \leq d$ and the PMM model, $p_c = 1$ [101]. Note that this is obvious in the case of the PMM: if initially there are somewhere in the system two empty sites at distance 1 or 2, it is possible to move them around the space, regardless of the configuration of particles, and thereby bring them where they are needed to perform an exchange. Systems presenting this feature (a finite group of empty sites can move freely throughout space) are known as *non-cooperative*. KA models on the other hand are *cooperative* (except for $j = 1$), which makes their study significantly more involved.

Question 1 is harder to answer, even in the case of KCSM. It is already clear that in the non-ergodic cases, there are several reversible measures. This can also happens in ergodic cases: in the East model, any product measure in which the density is 1 to the right of a certain x, 0 at x and p to the left of x is reversible (they are in fact the only invariant measures, along with the limit cases $x = \pm\infty$ corresponding to μ and $\delta_{\mathbf{1}}$). Thanks to Jan Swart, we recently discovered that [84] can be adapted to show that the only invariant measures for the FA-1f model in dimension 1 are μ and $\delta_{\mathbf{1}}$ (Appendix A). This is also expected to hold in higher dimension. Aside from that, unfortunately, little is known.

1.2 Equilibrium and low temperature dynamics

In this section, we focus on equilibrium dynamics of KCMs: we consider their dynamical properties when the initial configuration is sampled from the equilibrium measure μ_p. The goal is to understand how fast the system relaxes, that is, how long it needs to wait before most information on the initial configuration is erased. Of course we need to give a precise meaning to this. A general result [27] shows that for a large class of KCSM, in the ergodic regime, the process relaxes to equilibrium exponentially fast in $L^2(\mu_p)$, i.e. $\mathrm{Var}_p(P_t) \leq e^{-t/T}$ for some $T = T(p) \in \mathbb{R}_+^*$. The main object of interest here is the (best) constant $T(p)$, and specifically how it behaves as p approaches the threshold for ergodicity. On this question, tremendous progress has been made in the last few years, of which I can give but an unfair account. The interested reader would do well to check out Ivailo Hartarsky's thesis manuscript [60] for a more accurate picture.

1.2.1 Relaxation time

A central tool in the study of the dynamics of KCMs is the inverse of the *spectral gap* of L, or *relaxation time*. This quantity can be defined through spectral notions, but we will mostly use the following variational characterization.

Definition 3. *The relaxation time T_{rel} of a KCM is given by*

$$T_{\mathrm{rel}}^{-1} := \inf_{f\colon \mathrm{Var}_p(f)\neq 0} \frac{D(f)}{\mathrm{Var}_{\mu_p}}, \tag{1.6}$$

where the infimum is taken over $f \in L^2(\mu_p)$ satisfying $\mathrm{Var}_p(f) \neq 0$, and $D(f)$ is the Dirichlet form which can be computed as

$$D(f) = -\mu_p(fLf) = \frac{1}{2}\sum_{x\in\mathbb{Z}^d} \mu_p\left(c_x(\eta) r_x(\eta)\left[f(\eta^x) - f(\eta)\right]^2\right). \tag{1.7}$$

It is a standard exercise to show the following, which quantifies the decorrelation properties of the system in terms of T_{rel}.

Property 2. *For any $f, g \in L^2(\mu_p)$ with $\mu_p(g) = 0$,*

$$\mathrm{Var}_p(P_t f) \leq \mathrm{Var}_p(f) e^{-2t/T_{\mathrm{rel}}}, \quad |\mu_p(f P_t g)| \leq \sqrt{\mathrm{Var}_p(f)\mathrm{Var}_p(g)} e^{-t/T_{\mathrm{rel}}}. \tag{1.8}$$

In view of the previous proposition, it is natural to ask whether T_{rel} is a finite quantity.

Question 3. *In which cases do we have*

$$T_{\mathrm{rel}} < \infty?$$

It is immediately clear that this cannot be the case for KCSM in the non-ergodic regime. Theorem 3.3 in [27] gives a (not too restrictive) condition on KCSM which implies $T_{rel}(p) < \infty$ in the ergodic regime $p < p_c$. In all known cases, $T_{rel}(p) \to \infty$ as $p \nearrow p_c$.

For KCLG on $\mathbb{Z}^d$, as for any conservative system, $T_{rel} = \infty$ at every density. A more interesting question is to consider the dynamics restricted to a finite volume, say $\Lambda_L := [-L, L]^d$, and ask whether $T_{\mathrm{rel},\Lambda_L}/L^2$ has a finite limit (or at least liminf and limsup) as $L \nearrow \infty$. To have a non-trivial answer, one should further restrict to an ergodic component of the dynamics. This can be achieved either by replacing μ_{p,Λ_L} by the measure conditioned on an irreducible component of the Markov chain [83], or by adding sources in the vicinity of the boundary [13, 101, 29, 79].

Before we move to the understanding of T_{rel} as p approaches p_c, let us state another natural question.

Question 4. *Is $p \mapsto T_{rel}(p)$ non-decreasing? Increasing in the ergodic regime?*

It would seem intuitively clear that, as density increases, the system should take longer to relax, since fewer constraints are satisfied. The monotonicity of T_{rel} in the density is however still an open question.

It is now time to consider the most important question concerning T_{rel}. This quantity (or closely related ones) can be measured experimentally in glassy materials, and its anomalous behavior as the system freezes is one of the landmark of glassiness. Being able to quantify the divergence of T_{rel} at the approach of p_c is therefore critical to assess the relevance of KCSM in the study of glassy materials.

Question 5. *In KCSM, how does $T_{rel}(p)$ diverge as $p \nearrow p_c$?*

For a comprehensive review of the current state of the art on this question, the reader is advised to check [60]. Here, I will merely collect the results concerning the examples of models introduced in Section 1.1.2. But I do want to point out that available results have reached an impressive degree of generality, especially in dimension 2. In that case, for every model with $p_c = 1$, the leading behavior of T_{rel} can be predicted on the basis of simple geometric properties of the update family.

Theorem 1. *We have the following estimates on $T_{rel}(p)$ as $p \nearrow p_c$ (recall that $q = 1 - p$).*

FA-1f [27, 92]

$$\exists 0 < c < C < \infty \text{ s.t. } \begin{array}{ll} T_{\mathrm{rel}} = \Theta(1/q^3) & \text{if } d = 1 \\ c/q^2 \leq T_{\mathrm{rel}} \leq C \log(1/q)/q^2 & \text{if } d = 2 \\ T_{\mathrm{rel}} = \Theta(1/q^2) & \text{if } d \geq 3. \end{array}$$

FA-2f [61] In dimension $d = 2$,

$$\exp\left(\frac{\pi^2}{9q}(1 - O(\sqrt{q}))\right) \leq T_{\mathrm{rel}} \leq \exp\left(\frac{\pi^2}{9q}(1 + \sqrt{q}(\log 1/q)^{O(1)})\right)$$

East [1, 27, 35, 23, 34] Let $n := \lceil \log_2(1/q) \rceil$. *In dimension* $d = 1$,

$$T_{\mathrm{rel}} = (1/q)^{O(1)} \frac{n!}{q^n 2^{\binom{n}{2}}} \text{ and } T_{\mathrm{rel}} = q^{O(1)} \frac{n!}{q^n 2^{\binom{n}{2}}}, \text{ so that } T_{\mathrm{rel}} = e^{\frac{\log(1/q)^2}{2\log 2}(1+o(1))}.$$

In dimension $d \geq 2$,

$$T_{\mathrm{rel},\mathbb{Z}^d} = (T_{\mathrm{rel},\mathbb{Z}})^{\frac{1}{d}(1+o(1))}.$$

The results above display a variety of possible behaviors for T_{rel}. The divergence for FA-1f (a power of q) is called Arrhenius behavior and corresponds to an exponential in the inverse temperature. The divergence for East (an exponential in the square of the inverse temperature) corresponds to a fit that has been proposed for fragile glasses [88].

Question 6. *In KCLG, how do* $\liminf$ *or* $\limsup T_{\mathrm{rel},\Lambda_L}/L^2$ *diverge as* $p \nearrow p_c$*?*

For KA, the question is solved by the following result. Here, a source is a point on the boundary which is updated by creating (resp. destroying) a particle at rate p (resp. q). Adding these sources to the dynamics on Λ_L fixes the equilibrium density at p.

Theorem 2. *[79] For the KA-2f model with sources at the boundary of* Λ_L*, there exist positive constants* c, c' *such that*

$$\exp\left(c\frac{1}{q^{1/(d-1)}}\right) \leq \liminf T_{\mathrm{rel},\Lambda_L}/L^2 \leq \limsup T_{\mathrm{rel},\Lambda_L}/L^2 \leq \exp\left(c'\frac{\log(q)^2}{q^{1/(d-1)}}\right).$$

For the KA-jf model with sources at the boundary of Λ_L*, there exist positive constants* c, c' *such that*

$$\exp_{(j-1)}\left(c\frac{1}{q^{1/(d-1)}}\right) \leq \liminf T_{\mathrm{rel},\Lambda_L}/L^2 \leq \limsup T_{\mathrm{rel},\Lambda_L}/L^2 \leq \exp_{(j-1)}\left(c'\frac{1}{q^{1/(d-1)}}\right),$$

where $\exp_k$ *denotes the exponential iterated* k *times (e.g.* $\exp_{(2)}(x) = e^{e^x}$*).*

[13, 83] gives the existence of the lim sup for the PMM model with sources or restricted to an ergodic component with fixed number of particles. The divergence of the optimal bounds as $q \to 0$ is still unsettled ([83] gives an upper bound q^{-4}).

1.2.2 Tracer diffusion

1.2.2.1 Diffusion coefficient

Another way to measure the mobility of the system consists in probing it with an extra particle (as in Brown's observation of trapped particles in pollen grains). This extra particle can be described by a Brownian motion with a certain diffusion coefficient which is related to the dynamics of the surrounding fluid. A celebrated property of homogeneous liquids, is the Stokes-Einstein relation, which states that the product between diffusion coefficient D and relaxation time is proportional to the temperature [58]:

$$DT_{\mathrm{rel}} \propto \beta^{-1}. \tag{1.9}$$

In heterogeneous media, this relation is no longer true, which is interpreted as the fact that the extra particle only needs local rearrangements to diffuse and can restrict its trajectory to more favorable regions of space, whereas relaxation time by nature has to consider equally regions with different mobilities. In experiments, (1.9) has to be corrected to $D \approx T_{\mathrm{rel}}^{-\xi}$ with $\xi \neq 1$.

Let us define what we mean by a tracer in a KCM. In KCLG, it is simply a tagged particle started from the origin. If we denote L the generator of the KCLG, the generator of the joint process of the tracer and its environment $(X_t, \eta_t)_{t\geq 0}$ is given by

$$L_T f(x,\eta) = Lf(x,\cdot)(\eta) + \sum_{y\sim x} c_{x,y}(\eta)\eta(x)(1-\eta(y))\left[f(y,\eta^{x,y}) - f(x,\eta)\right]. \tag{1.10}$$

In KCSM, a tracer is an extra particle that is not felt by the surrounding material and can only move between empty sites. The generator of the joint process is given by

$$L_T f(x,\eta) = Lf(x,\cdot)(\eta) + \sum_{y\sim x}(1-\eta(x))(1-\eta(y))\left[f(y,\eta)-f(x,\eta)\right]. \tag{1.11}$$

Note that in both cases, the trajectory of the tracer $(X_t)_{t\geq 0}$ is not a Markov process. The process with generator L_T is a bit annoying to study, given the interdependence of its coordinates, its non-compact state space etc. A more pleasant object is the *environment seen from the particle*, which we will meet again in Part 3. It is simply the process defined on Ω by $\sigma_\cdot := (\tau_{X_t}\eta_t)_{t\geq 0}$. A first nice feature is that it is markovian, with generator

$$\begin{aligned}\mathcal{L}f(\sigma) =& \sum_{y\sim 0} c_{0,y}(\sigma)(1-\sigma(y))\left[f(\tau_y(\sigma^{0,y}))-f(\sigma)\right] \\ &+ \sum_{x\sim y\in\mathbb{Z}^d\setminus\{0\}} c_{x,y}(\sigma)\left[f(\sigma^{x,y})-f(\sigma)\right] \quad (KCLG) \end{aligned}\tag{1.12}$$

$$\begin{aligned}\mathcal{L}f(\sigma) =& \sum_{y\sim 0}(1-\sigma(0))(1-\sigma(y))\left[f(\tau_y\sigma)-f(\sigma)\right] \\ &+ \sum_{x\in\mathbb{Z}^d} c_x(\sigma)\left[f(\sigma^x)-f(\sigma)\right] \quad (KCSM). \end{aligned}\tag{1.13}$$

Also, several nice properties of the environment are transferred to this process.

Property 3. *1. The process with generator $\mathcal{L}$ is reversible w.r.t. μ_p in the KCSM case, $\bar{\mu}_p := \mu_p(\cdot|\sigma(0)=1)$ in the KCLG case.*

2. If the process with generator L is ergodic w.r.t. μ_p, the process with generator $\mathcal{L}$ is ergodic w.r.t. the previous reversible measure.

In addition to having nice properties, the environment seen from the particle is useful, in that its knowledge allows to reconstruct the trajectory $(X_t)_{t\geq 0}$. We will only need this to be true in a very weak sense. Precisely,

$$X_t = \int_0^t j(\sigma_s)ds + M_t, \tag{1.14}$$

where M is a martingale w.r.t. the natural filtration and j is the instantaneous drift of the tracer, i.e.

$$j(\sigma) = \begin{cases} \sum_{y\sim 0} y(1-\sigma(0))(1-\sigma(y)) & \text{(KCSM)} \\ \sum_{y\sim 0} y c_{0,y}(\sigma)(1-\sigma(y)) & \text{(KCLG).} \end{cases} \tag{1.15}$$

(1.14) is very useful, given the nice properties of the process $\sigma_\cdot$ that appears in the integral. Indeed, as I detail in Chapter 3, standard results [70, 37, 95] allow to deduce the following limit theorem for $(X_t)_{t\geq 0}$.

Proposition 2. *When the tracer starts from the origin and the environment is chosen initially with law μ_p for KCSM (resp. $\bar{\mu}_p$ for KCLG) in the ergodic regime:*

$$\varepsilon X_{t/\varepsilon^2} \Longrightarrow_{\varepsilon\to 0} \sqrt{2D}B_t, \tag{1.16}$$

where B is a standard Brownian motion and the convergence holds in the sense of weak convergence of path measures in the Skorokhod space $D(\mathbb{R}_+,\mathbb{R}^d)$.

Moreover, the diffusion matrix D is characterized by the following variational formula: for $u\in\mathbb{R}^d$, for KCSM

$$u\cdot Du = \inf_f \left\{ D(f) + \sum_{x\sim 0}\mu_p\left((1-\sigma(0))(1-\sigma(x))\left[u\cdot x + f(\tau_x\sigma)-f(\sigma)\right]^2\right)\right\}, \tag{1.17}$$

while for KCLG,

$$u \cdot Du = \inf_f \left\{ \sum_{x \sim y \in \mathbb{Z}^d \setminus \{0\}} \bar{\mu}_p \left(c_{x,y}(\sigma) \left[f(\sigma^{x,y}) - f(\sigma) \right]^2 \right) + \sum_{x \sim 0} \bar{\mu}_p \left(c_{0,x}(1 - \sigma(x)) \left[u \cdot x + f(\tau_x(\sigma^{0,x})) - f(\sigma) \right]^2 \right) \right\}. \quad (1.18)$$

In both cases, the infimum is taken over local functions f.

1.2.2.2 Positivity of D

The convergence (1.16) is only meaningful if $D > 0$[3]. It is therefore natural to aks when this holds.

Question 7. *When do we have $D > 0$?*

For KCSM, $D > 0$ holds as soon as the relaxation time is finite (which in most cases covers the ergodic regime, except maybe at the critical density). This is consistent with the idea that, if the system can rearrange, it can also allow the tracer to diffuse.

Proposition 3. *[23] For KCSM, there exists $c > 0$ such that*

$$DT_{\text{rel}} \geq cq^2.$$

For KCLG as usual the answer is less general. [13] shows that $D > 0$ for the PMM model, but relies a lot on the non-cooperative nature of this model (a finite group of empty sites can move freely throughout space). With Cristina Toninelli, we showed [B3] that $D > 0$ for KA-jf models with $j \leq d$, suggesting a strategy of proof that could apply to other cooperative models. After collecting the above results, let me explain the strategy with some detail.

Theorem 3. *[13, B3]At any density $p < 1$, for the PMM and KA-jf ($j \leq d$) models, $D > 0$.*

Strategy of proof for KA. The main tool that we use is the one that was developed to show ergodicity of the models, namely the notion *frameability*. A box $\Lambda_L = [-L, L]^d$ is frameable if it is possible to rearrange the configuration inside it (without looking outside) in such a way that its boundary becomes empty. It is known [101, 29] that, as $L \to \infty$, the probability of being frameable goes to 1.

The initial idea [100] to show $D > 0$ was to couple the tagged particle with a random walk in a random environment given by the superposition of frameable boxes. This turned out to be difficult to track down, as the motion of the tagged particle had itself an effect on its environment. What we did instead was to compare D with the diffusion coefficient of an auxiliary walk, which could only jump on a coarse-grained lattice $(L+2)\mathbb{Z}^d$, with L large enough.

It is easier to describe this auxiliary walk in dimension 2 (and $j = 2$). The interested reader can check [B3] for the generalization to higher dimensions. We split the coarse-grained lattice as indicated in Figure 1.2.2.2. We have square boxes of side length L, rectangles of dimension $2 \times L$ or $L \times 2$, and small 2×2 squares. The auxiliary walk jumps between down-left corners of small squares, but only if a certain condition is satisfied by the environment around these positions. We want that the 6 big squares in Figure 1.2.2.2 are frameable, and that the 7 rectangles each have an extra zero. Rotating the picture gives the condition required to jump between x and $x + (L+2)e_2$. This auxiliary walk satisfies two conditions, which put together yield the theorem.

1. For L large enough, the auxiliary walk is a random walk on a supercritical percolation cluster, and its diffusion coefficient D_{aux} is therefore positive [37][4].

[3] If $D = 0$, we have the wrong scaling. This happens for instance in the KA-1f model (aka SSEP) in dimension 1[95].

[4] There is a minor issue with the fact that the percolation structure underlying the auxiliary walk has a finite range of dependence, but this is easily solved.

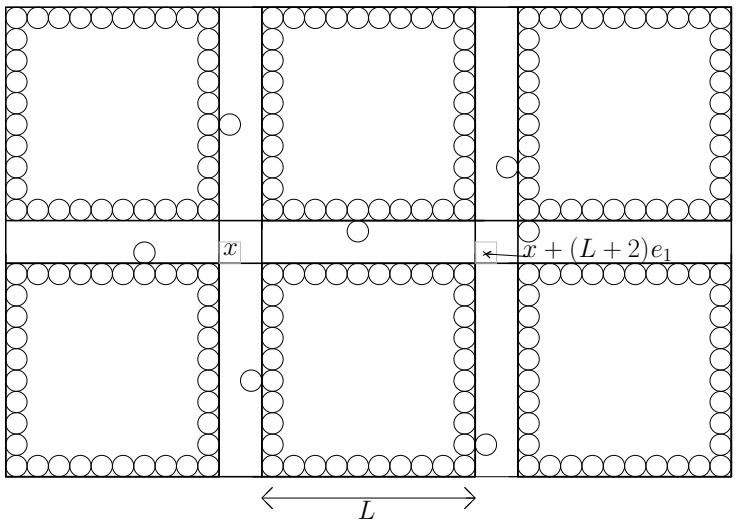

Figure 1.2: The condition that has to be satisfied by the environment in order for the auxiliary walk to be allowed to jump between x and $x+(L+2)e_1$. Frameable boxes are depicted as already framed (i.e. with empty boundary).

2. If the conditions are satisfied for the auxiliary walk to jump between x and $x + (L + 2)e_1$, it is possible for the KA dynamics to bring a particle from x to $x + (L + 2)e_1$ in a number of (allowed) moves bounded by some constant depending on L (more on that later). This means that the terms appearing in the variational formula for D_{aux} (similar to (1.17)) can be bounded (thanks to Cauchy-Schwarz inequality) by a sum of terms that appear in (1.17). In particular, $D \geq c(L)D_{aux}$ with $c(L) > 0$.

To conclude, we just need to take L large enough but finite.

□

Let me conclude this section with an intriguing question. In [51], the authors consider the spiral model in 3 dimensions (a KCSM with special rules that we do not need to define for the discussion) and argue that its critical density coincides with that of oriented site percolation (which is false and does not even fit their simulations). More interestingly, they also observe in their simulations a regime of densities above the critical one (i.e. where the KCSM is not ergodic) in which a tracer seems to have positive diffusion coefficient. The intuitive explanation is that just above the critical density, the first blocked clusters that develop are essentially one-dimensional. In particular their presence should not restrain the tracer to a finite region in space, and it would be able to move through their complement, a bit like a bird in a room encumbered by columns. As the density increases further, the blocked clusters start forming a percolating surface and the diffusion coefficient vanishes.

Question 8. *Is it possible to have diffusion in the non-ergodic regime? Can the convergence* (1.16) *still hold in this regime and can the diffusion coefficient have a transition that does not coincide with the ergodic one?*

1.2.2.3 Low temperature estimates

The next natural question concerns the behavior of D at the approach of the critical density. The picture is less complete than for the relaxation time and (apart from Proposition 3) concerns only special models.

Theorem 4. *[23] For the FA-1f model, $D = \Theta(q^2)$.*

For the East model, $q^{O(1)} \leq DT_{\text{rel}} \leq (1/q)^{O(1)}$. In particular, the exponent ξ mentioned below (1.9) *can only be* 1 *in this case.*

What about other KCSM? For models where T_{rel} is a power of $1/q$, we expect $D = q^{\alpha+o(1)}$ and that the methods in [23] should allow to compute α for each example (though no general proof is available). For other models, the lower bound of Proposition 3 always holds, and the q^2 correction does not matter: at first order, D is lower bounded by T_{rel}. On the other side, one typically needs to find a good test function to plug into (1.17). Following the idea of [79], one can find an upper bound in terms of the infection time of the origin in bootstrap percolation. This sometimes matches the lower bound, but for models with a discrepancy one would need to cook something more precise.

Let us now turn to KCLG. For the PMM, [13] indicates that $cq^4 \leq D \leq Cq^2$ for some constants $C > c > 0$. For KA models, [B3] left the question of the correct behavior open. We concluded the proof of Theorem 3 by saying the length L appearing in the construction should be well chosen. In fact, as $q \to 0$, the length at which a box starts to become frameable with good probability diverges with q. Let us call it L_q. The bounds obtained in [B3] only showed $D \geq e^{-cL_q}$, while the natural (and correct guess) would be $D \approx 1/L_q$. This has been showed, along with (almost) matching upper bounds by Assaf Shapira and my student Anatole Ertul in [41] thanks to the understanding developed in [79].

Theorem 5. *[41] For the KA-jf model, for some positive constants c, c',*

$$\exp\left(-\frac{c\log(1/q)^2}{q^{1/(d-1)}}\right) \leq D \leq \exp\left(-\frac{c'}{q^{1/(d-1)}}\right) \quad \text{if } k = 2 \tag{1.19}$$

$$\exp_{(j-1)}\left(-\frac{c}{q^{1/(d-1)}}\right) \leq D \leq \exp_{(j-1)}\left(-\frac{c'}{q^{1/(d-1)}}\right) \quad \text{if } k \geq 3. \tag{1.20}$$

Explaining this would take us too far for the purpose of the manuscript; the interested reader can check [41] for more details. Let us just note that it is no accident that this resembles closely Theorem 2.

1.3 Non-equilibrium dynamics

So far, we have only considered KCM at equilibrium, in the sense that the initial configuration was always chosen with distribution μ_p. In this section, we examine what happens when this is not the case. As we will see, results in this direction are still very partial. In this section, we will in fact only discuss KCSM. We discuss some results on KCLG in Chapter 2, particularly in Section 2.3.

The first question we examine – in Section 1.3.1 – is that of return to equilibrium. Pick ν a distribution on Ω and let $(\eta_t)_{t\geq 0}$ be a KCSM with $\eta_0 \sim \nu$. We ask whether it is true that the law of η_t is close to μ_p for large t, and in which sense.

Question 9. *Consider the random variables $\mathbb{E}_\eta[f(\eta_t)]$, where f belongs to some class of test functions[5] and $\eta \sim \nu$. Does the following holds for some notion of convergence:*

$$\mathbb{E}_\eta[f(\eta_t)] \underset{t\to\infty}{\longrightarrow} \mu_p(f)? \tag{1.21}$$

The typical example to have in mind for the choice of ν is $\mu_{p'}$, for some $p' \neq p$. Physically, this would correspond to a quench in temperature of the system.

In Section 1.3.2, we investigate how excitations propagate through the system. We choose the initial configuration in such a way that part of the system (say, a half space) is initially occupied. We want to elucidate if and how empty spaces eventually invade the whole space. A follow-up question is whether this propagation of zeros entails relaxation to equilibrium; this is in fact very connected to the issues treated in Section 1.3.1.

[5] Typically, local functions, but in some cases it will be possible and useful to make the support of f dependent on t.

1.3.1 Non-equilibrium relaxation

When we choose the initial distribution as μ_p, where p is the equilibrium density of the KCSM, we are in a reversible setting. As we saw in Section 1.2, an implication of this fact is that we have at our disposal quantities that can be easily manipulated (mainly through variational formulas) to understand the relaxation of the system. There is a quantity which plays a role analog to that of the spectral gap in the non-equilibrium setting. It is the log-Sobolev constant, which has a variational characterization similar to (1.6):

$$\alpha = \sup_{f\colon \mathrm{Var}_{\mu_p}(f)\neq 0} \frac{2\mathrm{Ent}(f^2)}{D(f)}, \tag{1.22}$$

where $\mathrm{Ent}(f) = \mu_p(f\log f) - \mu_p(f)\log\mu_p(f)$. When this quantity if finite, (1.21) holds for every initial η. For us however, $\alpha = \infty$ and it is simply not true that (1.21) holds regardless of the initial configuration (just consider $\eta \equiv 1$). In fact, the log-Sobolev constant for the dynamics restricted to a finite set Λ typically scales like the volume $|\Lambda|$ [43, 20].

Another tool one could wish to use are coupling arguments. For instance, it may seem natural that, if ν initially contains more empty sites than μ_p, there should be no significant obstacle to (1.21). Similarly, it would seem that if (1.21) holds for some $p > 0$, it should also hold for $p' < p$. Sadly, we have no way of affirming this rigorously. The reason is that KCSM are by nature non-attractive: in the standard coupling, it is easy to find transitions that do not respect monotonicity. In particular, we cannot use censoring inequalities, which have proved a very efficient tool in other contexts.

So what do we have? Positivity of the spectral gap, and not much more. Consequently, we also have only very partial results, in three directions: 1) perturbative (ν is close to μ_p), 2) on the East model (for which a very specific feature allows a rather complete study), 3) on FA-1f and other supercritical models, at low enough density.

1.3.1.1 A perturbative result

A natural idea, since everything we know is about equilibrium dynamics, is to replace the initial configuration by one sampled from μ. On a finite volume, this has a cost that can be estimated by the infinite norm of the Radon-Nicodym derivative of ν w.r.t. μ_p. If this cost is small enough, equilibrium relaxation kicks in and we get (1.21). If we deal with infinite volume dynamics, by finite speed propagation we can reduce the problem to that of a segment with diameter growing linearly with time. Since the decay given by the spectral gap is exponential in time, the resulting estimate goes to zero as long as the Radon-Nicodym derivative is at most $e^{c\ell/T_{\mathrm{rel}}}$ on $[-\ell,\ell]$ for some $c > 0$ small enough.

Proposition 4. *[28] Assume $T_{\mathrm{rel}} < \infty$ and $d = 1$. There exists $c > 0$ (depending on the range of the update family) such that, if $\sup_\ell e^{-c\ell/T_{\mathrm{rel}}} \left\| \frac{d\nu_{|[-\ell,\ell]}}{d(\mu_p)_{|[-\ell,\ell]}} \right\|_\infty < \infty$, then for any local function f there exist $c' = c'(p) > 0$ and $C = C(f) < \infty$ such that*

$$|\nu\left(\mathbb{E}_\eta[f(\eta_t)]\right) - \mu_p(f)| \leq Ce^{-c't}.$$

Note that the condition holds if for instance $\nu = \mu_{p'}$ with p' sufficiently close to p. On the other hand, even in this simple case, it would not be true in higher dimension.

1.3.1.2 The East model

The East model in dimension 1 is certainly the best understood KCSM, and this is mainly due to a very specific feature, identified in [1] and refined in [28, 21, 49]. Essentially, the distinguished zero is a particular trajectory on $\mathbb{Z}$ which acts as a dynamic empty boundary condition for the system on its left (see Section 3.4.2 for a proper definition). Its most important property is that, if the system on its left is initially at equilibrium, it remains so at a later time. The idea is then the same as before: by paying an appropriate price we can pretend that the initial distribution

is μ_p and then use the relaxation provided by the spectral gap. The advantage here is that the region on which we need to change the measure is potentially much smaller than in the previous setting.

Theorem 6. *[28, 21, 22, 49] Let f a local function with $\mathrm{Supp}(f) = [a, b]$ and assume η has a zero at some position $z > b$. Then for the East process on $\mathbb{Z}$, there exist $c = c(p) > 0$ and $C = C(p) < \infty$ such that*

$$|\mathbb{E}_\eta[f(\eta_t)] - \mu_p(f)| \leq \|f\|_\infty (z - a) C^{\Delta(\eta)} e^{-ct}, \tag{1.23}$$

where $\Delta(\eta)$ measures the maximum distance between zeros in $[a, z]$.

In higher dimension, the distinguished zero is a bit less powerful but still very useful. Using it, [33] was able to establish a stretched exponential decay, later reinforced by Marêché.

Theorem 7. *[77] Assume ν satisfies $\nu(\eta_{|[0,\ell]^d} \equiv 1) \leq Ce^{-c\ell}$ for some $0 < c < C < \infty$. Then there exist $c' > 0, C' < \infty$ such that for all $t > 0$, for all functions f with support in $[-C't^{1/d}, 0]^d \setminus \{0\}$,*

$$\nu\left(|\mathbb{E}_\eta[f(\eta_t)] - \mu_p(f)|\right) \leq \|f\|_\infty e^{-c't}/c'. \tag{1.24}$$

1.3.1.3 The FA-1f model

Let us now turn to the FA-1f model. One might hope that, the constraint being easier to satisfy w.r.t. the East model, relaxation to equilibrium should occur with greater facility. This may be true, but we have so far no way of actually showing this, and results on the FA-1f model are limited to small enough densities.

Recall that Theorem A1 shows that, in dimension 1, the only invariant measures for the FA-1f model at density p are $\delta_{\mathbf{1}}$ and μ_p. By compactness of the set of probability measures on Ω, this implies that in order to show (1.21), it is enough to prove that no subsequence of νP_t converges to $\delta_{\mathbf{1}}$. This should hold as soon as $\nu \neq \delta_{\mathbf{1}}$, but so far we are only able to show this for small enough p (see [84]). The game is essentially to ensure that zeros cannot escape to infinity, that there is some persistence of zeros around their initial position. This issue haunts the study of non-equilibrium dynamics for KCSM. In the East model, it was (more or less) easily resolved via the use of the distinguished zero. In the FA-1f model, the question is more delicate.

The strategies follow essentially two directions. One [20] is rather in the spirit of the above paragraph, trying to bring the non-equilibrium dynamics into a framework where we can exploit the positivity of the spectral gap. This is rather efficient in one dimension, where it gives exponential decay for a rather wide set of densities; less so in higher dimension where it only gives stretched exponential decay for $p < 1/2$. The persistence of zeros is controlled by showing that the distance of the closest zero to the origin is dominated by a biased random walk – which requires $p < 1/2$. The second strategy [81] consists in a percolation-type argument saying that, if the equilibrium density is low enough, the density of updates where the constraint is satisfied is very high. This has the inconvenient of restricting rather arbitrarily the set of densities covered by the result, but does give the correct exponential decay in any dimension, and generalises to every supercritical model in dimension 2 [78].

With the first strategy, we get the following results. The first statement is a more quantitative result in dimension 1, useful for considering supports of the test functions growing with t. Note that while (1.23) allowed to choose a support growing exponentially in t, the condition here is a bit more restrictive.

Theorem 8. *[20],[B4, Corollary 3.3]*

1. *Fix $\alpha < 1/2$, $d = 1$. If $p < 1/2$, there exists $c > 0$ such that, for all $t \geq 0$, if $K \leq e^{t^\alpha}$, for all f with support in $[-K, K]$, for all σ whose longest sequence of ones is shorter than $\sqrt{t}$,*

$$|\mathbb{E}_\nu[f(\eta_t)] - \mu_p(f)| \leq c^{-1} \|f\|_\infty e^{-c\sqrt{t}}. \tag{1.25}$$

2. *For $x \in \mathbb{Z}^d$, $\eta \in \Omega$, denote $\xi^x(\eta)$ the smallest distance between x and a zero of η. Assume there exists $\theta > 1$ such that $\sup_{x\in\mathbb{Z}^d} \nu\left(\theta^{\xi^x}\right) < \infty$ and $p < 1/2$. Then there exist $c > 0$ such that for every local function f, there exists $C < \infty$ such that*

$$|\mathbb{E}_\nu[f(\eta_t)] - \mu_p(f)| \leq \begin{cases} Ce^{-ct} & \text{if } d = 1, \\ Ce^{-c(t/\log t)^{1/d}} & \text{if } d \geq 2. \end{cases} \tag{1.26}$$

The second strategy on the other hand yields the following.

Theorem 9. *[81] Assume there exists $z \in \mathbb{Z}^d$ such that $\nu(\theta^{\xi^z}) < \infty$ for some $\theta > 1$. There exists $p_0 > 0$ such that for all $p < p_0$ there exists $c > 0$ such that for all local function f, there exists $C < \infty$ such that*

$$|\mathbb{E}_\nu[f(\eta_t)] - \mu_p(f)| \leq C(f)e^{-ct}. \tag{1.27}$$

1.3.2 Front progression

1.3.2.1 Results in one dimension

One way of interpreting the first point in Theorem 8 is to say that if initially in a given region of space, there are no zeros across a distance L, this will be resolved by time at most $O(L^2)$. This is merely an upper bound, but does raise the question of understanding how initially distant zeros end up at the distance $O(1)$ predicted by the equilibrium distribution.

Let us start in dimension 1 and consider an initial distribution such that, with probability 1, the negative integers are all occupied by particles, and the origin is empty. We would like to understand how the occupied space is invaded by zeros. Let us denote X_t the position of the left-most zero ($X_0 = 0$).

Question 10. *Does $X_t \to -\infty$ as $t \to \infty$? Does X_t/t converge to a negative constant? If so, can we quantify the fluctuations around the limit?*

It is somewhat remarkable that we can basically answer either all or none of the above questions, depending on the model and density we consider.

Theorem 10. *In the East model at any density $p \in [0, 1)$ [21, 49], or in the FA-1f model [B4] at density $p < \bar{p}$ with $\bar{p} \approx 0.24$, there exists $v > 0$, $\sigma \geq 0$ such that, for every distribution ν such that $\nu(\eta(x) = 1) = 1$ for all $x < 0$ and $\nu(\eta(0) = 0) = 1$,*

$$\frac{X_t}{t} \underset{t\to\infty}{\longrightarrow} -v \quad \mathbb{P}_\nu\text{–a.s.}, \tag{1.28}$$

$$\frac{X_t + vt}{\sqrt{t}} \underset{t\to\infty}{\Longrightarrow} \mathcal{N}(0, \sigma^2) \quad \text{under } \mathbb{P}_\nu. \tag{1.29}$$

Let us make a few remarks about the statement. First, while it may not be surprising to have a law of large numbers, it is in fact highly non-trivial to prove. The reason is once again the lack of monotonicity of KCSMs. If the systems considered were attractive, $(X_t)_{t\geq 0}$ would be a subadditive process and Kingman's subadditive theorem would give (1.28) almost for free (though some work would still be required to check $v > 0$).

Note that the threshold for validity of the result in the FA-1f model is even lower than in Theorem 8 for instance. The reason is that we need to guarantee an a priori "spreading of zeros" even stronger than for the non-equilibrium results. $\bar{p}$ can in fact be defined in terms of the critical parameter for the contact process λ_c: $\bar{p} = (1 + 2\lambda_c)^{-1}$. This is not the optimal value that our proof allows: in principle we could take $\bar{p} = (1 + \lambda_c^T)^{-1}$, with λ_c^T the critical parameter of the *threshold* contact process, defined below.

Also notice that in order to fully solve Question 10 in the regime covered by the theorem, one would additionnally need to show that $\sigma > 0$. This is so far only known in the East model for small enough densities [49]. Finally, note that the speed v and variance σ^2 depend only on the model and density – not the initial distribution. This should become clear with Theorem 11 below: Theorem 10 is a consequence of the (fast) convergence of the process seen from the front to an invariant measure.

Theorem 11. *Define $\omega_t := \tau_{X_t}\eta_t$. Denote by $\tilde{\mu}_t^\nu$ the distribution of ω_t under $\mathbb{P}_\nu$, and by $\|\pi - \pi'\|_\Lambda$ the total variation distance between the restrictions to Ω_Λ of π, π' two probability measures on Ω.*

1. *[49] In the East model, for any $p \in [0,1)$, there exist $\tilde{\mu}$ a probability measure on Ω, $d_* > 0$, $\alpha \in (0,1)$, $C \in \mathbb{R}_+$ s.t.*
$$\|\tilde{\mu}_t^\nu - \tilde{\mu}\|_{[0,d_* t]} \leq Ce^{-t^\alpha}. \tag{1.30}$$

2. *[B4] In the FA-1f model, for any $p < \bar{p}$, there exist $\tilde{\mu}$ a probability measure on Ω, $d_* > 0$, $c > 0$, $C \in \mathbb{R}_+$ s.t.*
$$\|\tilde{\mu}_t^\nu - \tilde{\mu}\|_{[0,d_* t]} \leq Ce^{-ce^{(\log t)^{1/4}}}. \tag{1.31}$$

Remark 2. *It is possible to view the front as a random walk in random environment (RWRE – see Chapter 3). With this point of view, Theorem 11 is a very natural statement, saying that the environment seen from the particle is ergodic and quantifying how fast it approches its invariant measure.*

The framework here differs however from the standard one, since the random walk is inserted in the environment, e.g. in the sense that its graphical construction uses the same Poisson clocks and Bernoulli variables. Another standard assumption in works on RWRE is that the law of the environment is translation invariant. This is obviously not the case here – except in a few specific situations.

Consider for instance the East model in dimension 1. *Since the dynamics on x depend only on the system restricted to $[x, \infty]$, it is not difficult to check that the front can be artificially considered as a random walk in* stationary *East environment (and using the same set of Poisson clocks and Bernoulli variables). This allows to apply the results of [B9] (see Section 3.4).*

Once again it is instructive to compare how far we can go in the FA-1f and East model. The slower decay (still faster than any polynomial) in FA-1f is ultimately inherited from the slower decay in (1.25) w.r.t. (1.23).

Let us conclude this section by going back to the original problem: Theorem 10 (and its proof) allows us to quantify even more precisely the return to equilibrium studied in Section 1.3.1. Let us this time consider the East or FA-1f process in finite volume $[1, L]$, with a boundary condition that ensures ergodicity (empty boundary condition at $L+1$ for the East model, at 0 and/or $L+1$ for FA-1f). General finite state Markov chain results ensure that the processes converge exponentially fast to equilibrium. But we can be a lot more precise: the understanding we have gained from the study of the front allows to establish *cutoff*.

Theorem 12. *[49, 40] Denote by $\mu_{t,L}^\sigma$ the distribution of the process started from σ at time t, and $d(t) := \sup_{\sigma \in \Omega_{[1,L]}} \|\mu_{t,L}^\sigma - \mu_p\|_{TV}$ the maximum total variation distance between the law of the process at time t and the equilibrium measure μ_p. Under the conditions of Theorem 10, for all $\varepsilon > 0$ there exists $\alpha \in \mathbb{R}_+$ independent of L s.t. for L large enough*

- *For the East model with zero boundary condition on $L+1$, or FA-1f with mixed boundary condition (zero on $L+1$, one on 0 or vice-versa),*
$$d\left(\frac{L}{v} - \alpha\sqrt{L}\right) \geq 1 - \varepsilon, \quad d\left(\frac{L}{v} + \alpha\sqrt{L}\right) \leq \varepsilon, \tag{1.32}$$
where v is the speed appearing in Theorem 10.

- *For the FA-1f model with zero boundary condition at 0 and $L+1$,*
$$d\left(\frac{L}{2v} - \alpha\sqrt{L}\right) \geq 1 - \varepsilon, \quad d\left(\frac{L}{2v} + \alpha\sqrt{L}\right) \leq \varepsilon. \tag{1.33}$$

1.3.2.2 Sketch of proof of Theorem 10

Most of the work consists in establishing Theorem 11. With the quantitative convergence to the invariant measure in our pocket, we can consider $X_t = \sum_{k=1}^{\lfloor t \rfloor}(X_{k+1} - X_k) + X_t - X_{\lfloor t \rfloor}$ as a sum of random variables with good mixing properties (i.e. $X_{k+1} - X_k$ and $X_{k'+1} - X_{k'}$ are only weakly correlated if $|k - k'|$ is large). In a stationary setting, it is a well-known problem to establish limit theorems for such sums [25]. Here, since we are close to stationarity by Theorem 11, it is also possible to follow the same kind of strategy.

So how do we prove Theorem 11? Let us assume we have a way to guarantee that zeros do not wander too far: if at a time $s < t$ there is a zero at a certain position x, then with high probability at time t there will still be a zero in a small window around x. Then this should give a push to the front leftwards, and also ensure that the space it goes through remains populated by zeros. Let us state this as an informal lemma – precise statements are [B4, Lemma 4.4], [49, Theorem 3.3] or [21, Corollary 4.2].

Lemma 1. *Under the hypotheses of Theorem 10, if ℓ, L, t are large and the initial configuration satisfies some condition depending on M, with high probability the longest sequence of ones in $(\omega_t)_{|[L,L+M]}$ is shorter than ℓ.*

For the East model, the desired property is easily established using the distinguished zero. For the FA-1f, it is more tricky, and the way we follow in [B4] is to use a coupling with an auxiliary *threshold contact process*. This is where the threshold $\bar{p}$ appears. The auxiliary process $(\sigma_t)_{t\geq 0}$ uses the same clocks and Bernoulli variables as the graphical construction of the FA-1f process (see below Definition 1). The difference is that when the Bernoulli variable is 1, the threshold contact does not need to satisfy the constraint to update the system: it is always allowed to create particles. It is not difficult to check that the coupling is order-preserving: if $\sigma_0 \geq \eta_0$, for all $t \geq 0$, $\sigma_t \geq \eta_t$. The interest of introducing σ is that we fall in the realm of attractive systems. In particular it is now standard that below a certain density threshold[6], with positive probability, starting from a unique infection, the set of infections grows linearly with time (survival event). Since there are more zeros in FA-1f than infections in the contact process, this gives us the property we want. It just remain to notice that, on the event of extinction, we can restart the process from one of the extra zeros of FA-1f (see Figure 1.3.2.2).

It is unclear that this lemma can generalize to higher densities in the FA-1f model. In the high density regime, the zeros look like they form a branching/coalescing structure (Figure 1.3.2.2), though there is yet no formalization of this intuition.

With Lemma 1, we are in a good position to apply the results of Section 1.3.1. The issue of course is that while we try to show that we relax to equilibrium in $[X_t, X_t + M]$, the front is moving, so that we always fall short of grasping what is happening immediately near the front. The workaround is to consider somewhat separately a region "near the front", which is allowed only a very short relaxation time and will only manage to behave well with some (small) positive probability, and a larger region "behind the front", which is allowed the necessary time to be brought close to equilibrium. We thus build a coupling between the processes seen from the front started from two different configurations and show that with high probability they eventually coincide on $[0, d_* t]$.

1.3.2.3 Open problems

Question 10 has natural generalizations in higher dimensions. Rather than generalizations of X_t, it is convenient to consider A_t the set of sites that have been updated by time t, i.e. sites where there was a clock ring before t at a time when the constraint was satisfied. In dimension 1, A_t is an interval and it is not difficult to show (under the hypotheses of Theorem 10) that its leftmost extremity is close to X_t. Now in order to formulate a relevant question, we need to

[6] In the contact process literature, one usually uses the vocabulary of infections; in our setting, infections correspond to zeros and a low density for FA-1f means a high infection rate in the contact process.

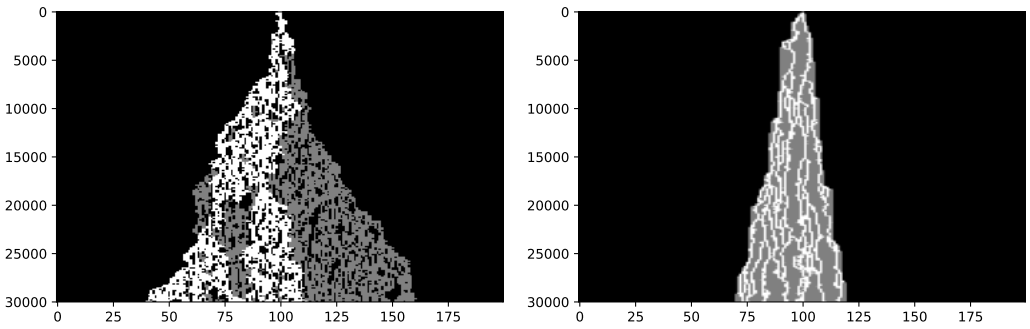

Figure 1.3: Two simulations of the FA-1f process on $\mathbb{Z}$ started from a single zero. Space is horizontal and time goes downwards. Left, density is .3; particles are depicted in black, zeros present in the FA-1f and contact process are in white, zeros present in the FA-1f but not the contact process in gray. Right, density is .7; initial particles are in black, zeros in white, occupied sites that have been updated at least once are in gray.

chose an initial configuration that allows A_t (at least in principle) to invade the entire space. This will depend on the model. Rather than a long discussion, a few examples are displayed in Figure 1.3.2.3.

Question 11. *Does the process $(A_t)_{t\geq 0}$ have a limit shape, i.e. A s.t.*

$$(1-\varepsilon)tA \subset A_t \subset (1+\varepsilon)tA \tag{1.34}$$

for any $\varepsilon > 0$ and t large enough?

This is so far only known in the case $p = 0$ (and then only for specific examples, but there should be no significant difficulty to extend the arguments), using subadditivity. Although it brings us rather far from the initial questions of Section 1.1.1, $p = 0$ is in fact an interesting case to consider. Indeed, the FA-1f model with $p = 0$ is nothing but First Passage Percolation (FPP) [6] with exponentially distributed i.i.d. passage times (also known as the Richardson model). The North-East model with $p = 0$ corresponds to Last Passage Percolation [67], again with i.i.d. exponential passage times. The East model in dimension 2 with $p = 0$ corresponds to oriented FPP [50]. KCM at $p = 0$ give a natural generalization of these well-studied models.

In the case $p > 0$, only very partial results are available. [32] shows that the mixing time of oriented models in $[0, L]$ with empty boundary condition is $O(L \log L)$ as soon as the spectral gap is positive. If there is a limit shape in that case, we expect instead that the mixing time should be of order L.

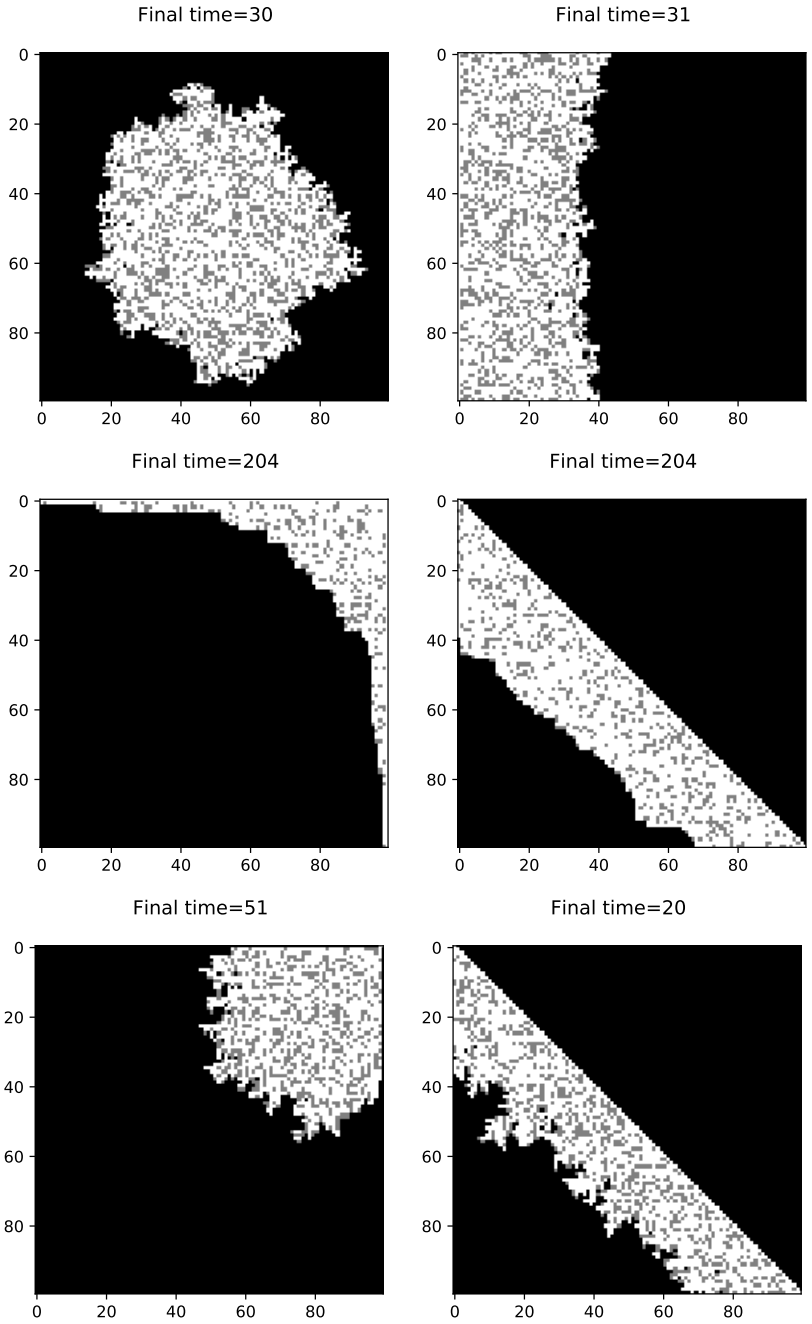

Figure 1.4: Simulations of (top to bottom) the FA-1f, North-East and East model in dimension 2, at densities $0.3, 0.2, 0.3$ respectively. Zeros are in white, particles that have been updated at least once (i.e. in A_t) in gray, other particles in black. Simulations of FA-1f start from a single zero in the middle (left) or with empty boundary condition on the left and periodic top/bottom b.c. (right). North-East is sampled with empty top-right boundary condition (left), or starting from an empty diagonal (right). East is sampled with empty boundary condition in the top-right corner (left) and starting from an empty diagonal (right).

Chapter 2

Scaling limits of constrained models

2.1 Scaling limits of conservative particle systems

The evolution of many physical systems is traditionally described through differential equations. This is often very efficient and gives good predictions for the behavior of the system considered. However, in the microscopic world, the relevant structures are discrete in nature (atoms, molecules, cells, etc). It is a longstanding endeavour to reconcile these two visions of physical systems. Mathematically, this is generally achieved by taking the scaling limit of a discrete system.

We consider a Markov process $(\eta_t)_{t\geq 0}$. For simplicity, we consider the state space to be $\{0,1\}^{\mathbb{Z}}$, or $\{0,1\}^{\mathbb{T}_N}$, where $\mathbb{T}_N := \mathbb{Z}/N\mathbb{Z}$. In principle, a lot of the discussion here would apply to more general state spaces, e.g. $S^{\mathbb{Z}^d}$ for some countable set S. N, whether present in the state space or not, is a integer parameter whose destiny is to go to infinity and fixes the space scale ($\mathbb{T}_N$ should be thought of as embedded in $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, with points separated by distance $1/N$, and similarly $\mathbb{Z}$ embedded in $\mathbb{R}$). We will consider conservative systems with nearest neighbor jumps. This differs from the examples of Section 1.3, and implies that we will have to wait for a time at least of order N before the evolution of the system can be felt on the macroscopic scale.

The easiest example that we can keep in mind for comparison purposes is the Simple Symmetric Exclusion Process (SSEP), in which particles jump left/right with equal probability, provided their landing site is empty. Its generator is given by

$$\mathcal{L}_{SSEP} f(\eta) = \sum_{x\in\mathbb{Z}} [f(\eta^{x,x+1}) - f(\eta)]. \tag{2.1}$$

2.1.1 Hydrodynamic limits of symmetric gradient systems

The type of questions we are interested in in this chapter is the following. We want to start our process from a measure μ^N associated with a density profile $\rho_0 : \mathbb{T} \to [0,1]$. A standard way of doing that is to choose $\mu^N = \nu^N_{\rho_0(\cdot)}$, the product measure on $\{0,1\}^{\mathbb{T}_N}$ satisfying

$$\nu^N_{\rho_0(\cdot)}(\eta(x) = 1) = \rho_0(x/N). \tag{2.2}$$

When $\rho_0 \equiv \alpha \in [0,1]$, we just write ν^N_α.

There are several ways to allow for more general initial distributions. The one that will appear in the following discussions involves the relative entropy

$$H(\mu|\nu) = \begin{cases} \int \frac{d\mu}{d\nu} \log \frac{d\mu}{d\nu}\, d\nu & \text{if } \mu \ll \nu \\ \infty & \text{else} \end{cases} \tag{2.3}$$

$$= \sup_f \left\{ \int f d\mu - \int e^f d\nu \right\}, \tag{2.4}$$

where the supremum is taken over bounded functions. In the methods presented below, it will be convenient to request that (for $\rho_0 \in (0,1)$)

$$H(\mu^N|\nu^N_{\rho_0(\cdot)}) = o(N). \tag{2.5}$$

Note that if $\tilde{\rho}_0 : \mathbb{T} \to (0,1)$, $H(\nu^N_{\tilde{\rho}_0(\cdot)}|\nu^N_{\rho_0(\cdot)}) = o(N)$ implies $\tilde{\rho}_0 = \tilde{\rho}$. Also, if $H(\mu^N|\nu_{\rho(\cdot)}) = o(N)$, μ^N is associated with the density profile $\rho(\cdot)$, in the sense that (see [71, Chapter 6, Corollary 1.3]), for any continuous function $H : \mathbb{T} \to \mathbb{R}$,

$$\mu^N \left(\left| \frac{1}{N} \sum_{x \in \mathbb{T}_N} \phi(x/N)\eta(x) - \int_{\mathbb{T}} \phi\rho(t,\cdot) \right| \right) \underset{N\to\infty}{\longrightarrow} 0. \tag{2.6}$$

The question is then to describe the evolution of the density in the system. Intuitively, one would like to understand the variations of $\mathbb{P}_{\mu^N}(\eta_t(x) = 1)$ as time evolves. In practice, this quantity is difficult to access and we rather define the density at x using local averages of η_t as e.g. in (2.6). But the most important point is that, before the first order of this density becomes different from ρ_0, one needs to wait for a time that diverges with N. Typically the time scale is N^2 for symmetric systems, N for asymmetric systems. We will focus on the first case.

Question 12. *Can the distribution μ^N_t of η_{tN^2} under $\mathbb{P}_{\mu^N}$ be associated with a profile $\rho(t,\cdot)$? How can we describe $\rho(t,\cdot)$?*

In the compact setting we chose, it is standard that any limit point of the distribution μ^N_t has a density w.r.t. the Lebesgue measure. The question of identifying the density is more interesting. Let us sketch what happens for the simple case of the SSEP. It is well known that the following quantity is a martingale.

$$\eta_{tN^2}(x) - \eta_0(x) - N^2 \int_0^t [\mathcal{L}_{\mathrm{SSEP}}\eta(x)]_{sN^2} ds, \tag{2.7}$$

with predictable quadratic variation $[\eta(x-1) - \eta(x)]^2 + [\eta(x+1) - \eta(x)]^2$. The integrated quantity can be rewritten as $\eta_{sN^2}(x+1) + \eta_{sN^2}(x-1) - 2\eta_{sN^2}(x)$. The fact that this is the discrete Laplacian of a local function is crucial to the rest of the story, and is called the *gradient property*. $\mathcal{L}\eta(x)$ can always be written as the gradient of the current: $j_{x-1,x} - j_{x,x+1}(\eta)$, where $j_{x,x+1}$ is the signed current between x and $x+1$, i.e. the rate at which a particle jumps from x to $x+1$ minus the rate at which a particle jumps from $x+1$ to x. The gradient property says that $j_{x,x+1}$ itself can be written as the gradient of a local function. In the SSEP case,

$$j_{x,x+1}(\eta) = \eta(x)(1-\eta(x+1)) - \eta(x+1)(1-\eta(x)) = \eta(x) - \eta(x+1). \tag{2.8}$$

Let us take the average of (2.7), and assume we can replace $\mathbb{E}_{\mu^N}[\eta_{sN^2}(x)]$ by $\rho(s, x/N)$ at the cost of a negligible error. Choosing $x = Nu$ for some $u \in \mathbb{T}$, we get

$$\rho(t,u) - \rho_0(u) = \int_0^t \Delta_N \rho(s,u) ds + \text{error}, \tag{2.9}$$

with $\Delta_N h(u) = N^2(h(u+1/N) + h(u-1/N) - 2h(u))$. This gives a good guess for $\rho(t,\cdot)$: it should be the solution at time t to the heat equation on $\mathbb{T}$ with initial condition ρ_0[1]:

$$\partial_t \rho = \Delta\rho, \quad \rho(0,\cdot) = \rho_0. \tag{2.10}$$

Let us now discuss two standard methods to reach the conclusion that μ^N_t can be associated to the density profile $\rho(t,\cdot)$. The reader already familiar with these methods may skip to the next section.

[1] I have not yet proved anything. In reality, one would write an equation satisfied by the empirical measure against a test function and reach the conclusion that ρ is a weak solution of the heat equation – rather than a mild solution as appears here.

Entropy method Let us use (2.7) and integration by parts to write that, for a smooth function ϕ,

$$\frac{1}{N}\sum_{x\in\mathbb{T}_N}\phi(x/N)\eta_{tN^2}(x)-\frac{1}{N}\sum_{x\in\mathbb{T}_N}\phi(x/N)\eta_0(x)-\int_0^t\frac{1}{N}\sum_{x\in\mathbb{T}_N}\Delta_N\phi(x/N)\eta_{sN^2}(x)ds \tag{2.11}$$

is a martingale whose predictable quadratic variation

$$\sum_{x\in\mathbb{T}_N}[\eta(x+1)-\eta(x)]^2\left[\phi\left(\frac{x+1}{N}\right)-\phi(x/N)\right]^2$$

vanishes when $N\to\infty$. Note that, in the SSEP case, (2.11) is merely a function of the empirical measure, which allows to identify possible limit points for μ_t^N rather easily.

However, when one deals with more general gradient systems, the integral term involves $\tau_x h(\eta_{sN^2})$ for some local function h, rather than just $\eta_{sN^2}(x)$. In particular, this term cannot be written as a function of the empirical measure. The idea is then to replace $\tau_x h(\eta_{sN^2})$ by $\bar{h}\left(\tau_x\eta^{\varepsilon N}_{sN^2}\right)$, where $\bar{h}(\rho)=\nu_\rho^N(h)$, ν_ρ is "the" stationary measure at density ρ and $\eta^\ell(x)=\frac{1}{2\ell+1}\sum_{|y|\le\ell}\eta(x+y)$. This allows to obtain a close equation on any limit point of the empirical measures.

Define

$$V_\ell(\eta):=\left|\frac{1}{2\ell+1}\sum_{|y|\le\ell}\tau_y h(\eta)-\bar{h}(\eta^\ell(0))\right| \tag{2.12}$$

In order to show that the replacement does not modify the limit, one shows that for some $\alpha\in(0,1)$,

$$\limsup_{\varepsilon\to0}\limsup_{N\to\infty}\sup_f\int\frac{1}{N}\sum_{x\in\mathbb{T}_N}\tau_x V_{\varepsilon N}(\eta)f(\eta)\nu_\alpha^N(d\eta)\le0, \tag{2.13}$$

where the supremum is taken over densities whose Dirichlet forms satisfy the same general bound as (the time-average of) $\mu^N_{sN^2}$.

This is generally performed in two steps: one replacing εN by ℓ and taking the $\ell\to\infty$ limit after the N limit (the *one-block estimate*); the second by estimating the cost of replacing η^ℓ by $\eta^{\varepsilon N}$ (the *two-block estimate*). We focus on the first part, which will play the most important role in following discussions.

One-block estimate:
$$\limsup_{\ell\to\infty}\limsup_{N\to\infty}\sup_f\int\frac{1}{N}\sum_{x\in\mathbb{T}_N}\tau_x V_\ell(\eta)f(\eta)\nu_\alpha^N(d\eta)\le0, \tag{2.14}$$

where $D_N(f):=-\nu_\alpha^N(\sqrt{f}L\sqrt{f})\le C/N$. The idea is to take advantage of the fact that V_ℓ is a local function (for fixed ℓ), ν_α^N is translation invariant, and we can therefore replace the integrand by $V_\ell(\eta)\tilde{f}_\ell(\eta)$, where $\tilde{f}=\frac{1}{N}\sum_{x\in\mathbb{T}_N}\tau_x f$ and $\tilde{f}_\ell$ is the density obtained from $\tilde{f}$ by averaging the configuration outside $\{-\ell,\dots,\ell\}$. Taking the limit $N\to\infty$, it turns out that any limit density has to satisfy $D_\ell(f)=0$ and therefore we only have to show that

$$\limsup_\ell\sup_{f:D_\ell(f)=0}\int V_\ell f d\nu_\alpha^\ell\le0. \tag{2.15}$$

A density f such that $D_\ell(f)=0$ is invariant under particle jumps. This tells us that in fact, a limit density is a mixture of uniform measures on the sets of configurations with a fixed number of particles. These measure being well understood, it is now a standard problem to show the estimate above.

Relative entropy method I describe this method in the context of the SSEP. This gives a rather circonvoluted proof of a result that can be showed more efficiently (see the paragraph above). However, this does allow to keep the exposition simple.

The relative entropy method, as its name suggests, uses the relative entropy of μ_t^N w.r.t. $\nu^N_{\rho(t,\cdot)}$, where ρ solves the equation (2.10). The idea is to show that $H_N(t) := H(\mu_t^N|\nu^N_{\rho(t,\cdot)})$ satisfies a Gronwall lemma:

$$H_N(t) \leq o(N) + C\int_0^t H_N(s)ds. \tag{2.16}$$

This in turn implies that

$$H(\mu^N|\nu^N_{\rho_0(\cdot)}) = o(N), \quad \Longrightarrow \quad H(\mu_t^N|\nu^N_{\rho(t,\cdot)}) = o(N), \tag{2.17}$$

and therefore that μ_t^N can indeed be associated with the density profile $\rho(t,\cdot)$. In turn, the Gronwall lemma (2.16) is proved through a bound on the derivative of $H_N(t)$. Let us denote for fixed $\alpha \in (0,1)$ and $\rho_0 \in (0,1)$

$$\psi_t^N(\eta) = \frac{d\nu^N_{\rho(t,\cdot)}}{d\nu^N_\alpha}(\eta), \quad \lambda_t^N(u) = \log\frac{\rho(t,u)(1-\alpha)}{(1-\rho(t,u))\alpha}, \quad f_t^N(\eta) = \frac{d\mu_t^N}{d\nu^N_\alpha}(\eta). \tag{2.18}$$

Then it holds [104]:

$$H_N'(t) \leq \int\left[\frac{N^2L^*\psi_t^N - \partial_t\psi_t^N}{\psi_t^N}\right] f_t^N d\nu_\alpha^N, \tag{2.19}$$

and the game is to bound the term in the integral. Except f_t^N, everything in the right-hand side can be exactly computed. The computation is a bit long, but not difficult. A Taylor expansion, continuous approximations of discrete derivatives and the fact that $\rho(\cdot,\cdot)$ solves the heat equation show that

$$\begin{aligned}\frac{N^2L^*\psi_t^N - \partial_t\psi_t^N}{\psi_t^N} &= o(N) + \sum_{x\in\mathbb{T}_N}\Delta\lambda_t^N(x/N)\left[\eta(x) - \rho(t,x/N)\right] \\ &+ \sum_{x\in\mathbb{T}_N}\nabla\lambda_t^N(x/N)^2\left[\tau_x g(\eta) - \bar g(\rho(t,x/N)) - \bar g'(\rho(t,x/N))(\eta(x)-\rho(t,x/N))\right],\end{aligned} \tag{2.20}$$

where $g(\eta) = \eta(0)+\eta(1)-\eta(0)\eta(1)$, and $\bar g(\rho) = \nu_\rho^N(g)$. Discrete integration by parts allow easily to replace the $\eta(x)$ above by $\eta^\ell(x) := \frac{1}{2\ell+1}\sum_{y=-\ell}^\ell \eta(x+y)$, up to an error term that is again $o(N)$ (for fixed ℓ). It is now time to use again the *one-block estimate*:

$$\lim_{\ell\to\infty}\lim_{N\to\infty}\int\frac{1}{N}\sum_{x\in\mathbb{T}_N}\nabla\lambda_t^N(x/N)^2\left[g(\eta(x)) - \bar g(\eta^\ell(x))\right] f_t^N d\nu_\alpha^N = 0. \tag{2.21}$$

This means the only non-negligible terms from (2.20) are

$$\begin{aligned}&\sum_{x\in\mathbb{T}_N}\Delta\lambda_t^N(x/N)\left[\eta^\ell(x) - \rho(t,x/N)\right] \\ &\quad+ \sum_{x\in\mathbb{T}_N}\nabla\lambda_t^N(x/N)^2\left[\bar g(\eta^\ell(x)) - \bar g(\rho(t,x/N)) - \bar g'(\rho(t,x/N))(\eta^\ell(x)-\rho(t,x/N))\right].\end{aligned} \tag{2.22}$$

Notice the Taylor expansions in both lines. For ℓ large, under $\nu^N_{\rho(t,\cdot)}$, $\eta^\ell(x)$ should be close to $\rho(t,x/N)$, and the averages of these terms should therefore be small. Working under $\nu^N_{\rho(t,\cdot)}$ rather than μ_t^N is feasible, at the cost of adding a term $H(t)$ to the estimate, and working in the large deviations regime: for any $\gamma > 0$

$$\int (2.22) f_t^N d\nu_\alpha \leq \gamma^{-1}H_N(t) + \gamma^{-1}\log\int\exp\left[\gamma\times(2.22)\right]d\nu^N_{\rho(t,\cdot)}. \tag{2.23}$$

As hinted above, the last term, by large deviations methods, goes to 0 as $N\to\infty$, then $\ell\to\infty$. This concludes the (sketch of) proof.

2.1.2 Fluctuations

An answer to Question 12 can be understood as a law of large numbers for $(\eta_{tN^2})_{t\geq 0}$. In turn, the next natural question is to consider the process $(\eta_{tN^2} - \rho(t, \frac{\cdot}{N}))_{t\geq 0}$ and ask whether it can be scaled in a way that gives it a non-trivial limit (like a CLT).

Question 13. *Can the fluctuations of η_{tN^2} around its hydrodynamic limit $\rho(t,\cdot)$ be described by a random process?*

We will in fact focus on the equilibrium case where the initial density $\rho_0 \equiv \rho$ is constant. Let us define the *fluctuation field*, for functions H in the Schwartz space $\mathcal{S}(\mathbb{R})$,

$$\mathcal{Y}_t^N(H) = \frac{1}{\sqrt{N}} \sum_{x\in\mathbb{Z}} H(x/N)\left(\eta_{tN^2}(x) - \rho\right). \tag{2.24}$$

Using again (2.7), we get that

$$\mathcal{Y}_t^N(H) - \mathcal{Y}_0^N(H) - \int_0^t N^{-1/2} \sum_{x\in\mathbb{Z}} (\eta_{sN^2}(x) - \rho)\Delta_N H(x/N) ds \tag{2.25}$$

is a martingale whose predictable quadratic variation at time t converges to $2\rho(1-\rho)t\int_{\mathbb{R}}(\nabla H)^2$ when $N \to \infty$. This martingale converges to a Brownian motion with diffusion coefficient $2\rho(1-\rho)\int_{\mathbb{R}}(\nabla H)^2$. In turn, one can show that the process $(\mathcal{Y}_t^N)_t$ converges (as a càdlàg process in $\mathcal{S}'(\mathbb{R})$) to the infinite dimensional Ornstein-Uhlenbeck process, solution to

$$d\mathcal{Y}_t = \Delta\mathcal{Y}_t dt + \sqrt{2\rho(1-\rho)}\nabla d\mathcal{W}_t, \tag{2.26}$$

where $d\mathcal{W}_t$ is a standard white noise on $\mathcal{S}'(\mathbb{R})$. This makes the fluctuations of the SSEP fall in the Gaussian universality class.

In the eighties [68], physicists identified a different equation (the KPZ equation), describing the height fluctuations of growing interfaces:

$$dh = \sigma\Delta h dt + \lambda(\nabla h)^2 dt + \sqrt{D} d\mathcal{W}_t. \tag{2.27}$$

In particular, it is expected that for a large class of *weakly asymmetric* conservative particle systems, the fluctuations should be described by this equation. For instance, the weakly asymmetric simple exclusion process (WASEP) has generator

$$\begin{aligned}\mathcal{L}_{\text{WASEP}} f(\eta) = &\sum_{x\in\mathbb{Z}} (1+\frac{1}{N})\eta(x)(1-\eta(x+1))\left[f(\eta^{x,x+1}) - f(\eta)\right] \\ &+ \sum_{x\in\mathbb{Z}} (1-\frac{1}{N})\eta(x+1)(1-\eta(x))\left[f(\eta^{x,x+1}) - f(\eta)\right].\end{aligned} \tag{2.28}$$

A companion equation can be obtained from (2.27) by formally taking the gradient of the solution. This object naturally lives in the space of distributions and solves the *Stochastic Burgers Equation*:

$$d\mathcal{Y}_t = \sigma\Delta\mathcal{Y}_t dt + \lambda\nabla\left[\mathcal{Y}_t^2\right] dt + \sqrt{D}\nabla d\mathcal{W}_t. \tag{2.29}$$

The weak KPZ conjecture mentioned above can therefore be reformulated as asking to describe the limit of the fluctuation fields of weakly asymmetric particle systems as a solution of the Stochastic Burgers Equation.

One issue with this program is that (2.27) is ill-defined if one uses only standard stochastic analysis tools: the regularity of putative solutions is too low to define properly the square appearing in the equation. There is in fact no uniqueness of solutions to (2.27), (2.29). A standard way of going around this difficulty is to consider the Cole-Hopf transform, which turns (2.27) into a stochastic heat equation. In [12], the authors managed to exhibit a microscopic Cole-Hopf transform of the WASEP, which they were able to show converges to the Cole-Hopf transform of (2.27). This also allows to show that the limit fluctuation field satisfies a version of the stochastic Burgers equation

$$d\mathcal{Y}_t = \Delta\mathcal{Y}_t dt + \nabla\left[\mathcal{Y}_t^2\right] dt + \sqrt{D}\nabla d\mathcal{W}_t. \tag{2.30}$$

2.2 The facilitated exclusion process

2.2.1 Model definitions

In this section, we consider the Facilitated Exclusion Process (FEP). Each of the particles in this system jumps at rate one to a uniformly chosen nearest neighbor, at the condition that 1) the target site is empty, and 2) the particle has an occupied neighbor before the jump. The generator of the process is thus defined as

$$\mathcal{L}_{\mathrm{FEP}} f(\eta) = \sum_{x\in\mathbb{Z}} \eta(x)\eta(x-1)(1-\eta(x+1))[f(\eta^{x,x+1}) - f(\eta)] + \sum_{x\in\mathbb{Z}} \eta(x)\eta(x+1)(1-\eta(x-1))[f(\eta^{x,x-1}) - f(\eta)]. \quad (2.31)$$

This may look similar to the KCMs described in Chapter 1 (reversing the roles of 0s and 1s), but a fundamental difference is that there is no reversibility in the FEP – even in a weak sense. Indeed, it is perfectly possible to perform a jump that cannot be reversed immediately (e.g. $1100 \to 1010$).

The FEP has been introduced in the physics literature [89] to illustrate a universality class of absorbing phase transition that does not fall in that of directed percolation. Symmetric or (totally) asymmetric variations, have also been studied, both in the physics and mathematics literature [86, 10, 9, 91, 8, 52]. The FEP has multiple absorbing states, namely all configurations with isolated particles. It also has a conserved field – the density – which should explain the difference with directed percolation at criticality. One expects that at small densities, the system should eventually reach an absorbing state (at least locally), while at high densities it should remain active forever. These two cases should be sharply separated by a critical value of the density.

So far, all mathematical results on the FEP hold only in dimension 1. To exemplify the difficulty involved in going to higher dimensions, note that it is still not proved that in dimension 2, at small enough densities, the system eventually freezes (locally). However, in the one-dimensional setting, there is a special tool that simplifies the study a lot, and has no equivalent in higher dimension (that we know of). The one-dimensional FEP can indeed be mapped to a zero-range process, through the following procedure, which is shown in Figure 2.1 on a torus $\mathbb{Z}/N\mathbb{Z}$ but could be generalized to other settings. 1) Fix a configuration $\eta \in \{0,1\}^{\mathbb{Z}/N\mathbb{Z}}$. 2) Number the empty sites from 1 to k, starting from the first empty site to the right of zero and moving from left to right. 3) For each $i = 1, \dots, k$, let ω_i^η be the number of particles separating the empty site i from the empty site $i+1$ ($k+1$ being identified with 1). This yields $\omega^\eta \in \mathbb{N}^{\mathbb{Z}/k\mathbb{Z}}$. This mapping can be extended to follow the FEP dynamics: when a particle jumps in η, consider that it exchanges its position with a numbered empty site; the jump then results in a jump of a particle in ω^η from a site with at least two particles to a neighboring site. The resulting dynamics is that of a zero-range process with rates $g(k) = \mathbf{1}_{k\geq 2}$.

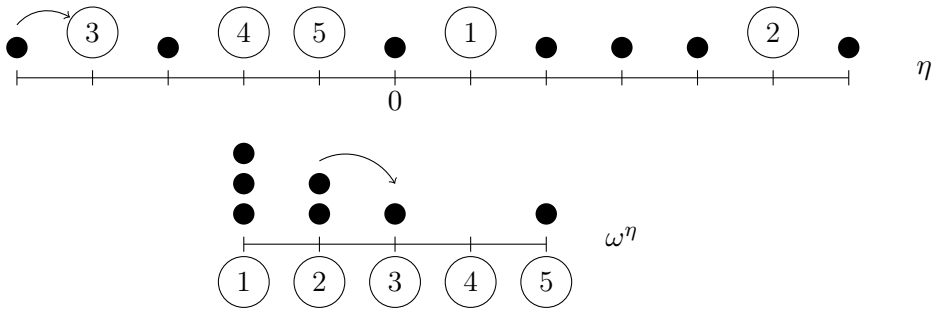

Figure 2.1: The mapping to a zero-range process on a segment with periodic boundary condition with $N = 12$ sites and $k = 5$ empty sites. When the jump indicated by an arrow occurs in the exclusion configuration, it results in the jump indicated by an arrow in the zero-range configuration.

With this mapping at hand, it becomes easy to prove that the absorbing phase transition mentioned above does occur in the one-dimensional FEP, at density 1/2.

Proposition 5. *If $\eta \in \{0,1\}^{\mathbb{Z}/N\mathbb{Z}}$ is a configuration with $N-k$ particles, the FEP started from η*

- *freezes in finite time if $k \geq N-k$;*
- *remains active forever if $k < N-k$.*

Moreover, if $k < N-k$, the FEP is an irreducible Markov chain on the set

$$E_k^N = \{\eta : \sum_{x\in\mathbb{Z}/N\mathbb{Z}} \eta(x) = N-k, \text{ and } \forall x \in \mathbb{Z}/N\mathbb{Z}, \eta(x)+\eta(x+1) \geq 1\}, \tag{2.32}$$

and in the symmetric case the uniform measure π_k^N on E_k^N is reversible.

In finite volume, this result identifies without ambiguity the active/frozen phases: density $\leq 1/2$ leads to a configuration with isolated particles (frozen), while density $> 1/2$ leads to the set of configurations with isolated holes (active).

Taking the limit of π_k^N as $N, k \to \infty$ with $N/k \to 1-\rho \in [0, \frac{1}{2})$ gives a limiting distribution π_ρ on $\{0,1\}^{\mathbb{Z}}$, which has an explicit expression and is reversible for the symmetric FEP dynamics on $\mathbb{Z}$. The measures π_ρ, $\rho > 1/2$ describe the supercritical (active) phase of the infinite volume FEP, while the frozen phase is just the set

$$F = \{\eta \in \{0,1\}^{\mathbb{Z}} : \forall x \in \mathbb{Z}\ \eta(x)\eta(x+1) = 0\}, \tag{2.33}$$

in which the density is $\leq 1/2$. More precisely, we proved the following result, which is essential to describe the scaling limit of the FEP.

Theorem 13 ([B6])**.** *Let ν be a translation invariant stationary measure for the symmetric FEP on $\mathbb{Z}$. Then there exist $\lambda \in [0,1]$, β a probability measure on $(1/2,1]$ and μ_F a probability measure on F such that*

$$\nu = \lambda\mu_F + (1-\lambda)\int_{(\frac{1}{2},1]} \pi_\rho d\beta(\rho). \tag{2.34}$$

One can think of this theorem as saying that "the" invariant measure at density $\rho > 1/2$ is given by π_ρ.

2.2.2 A Stefan problem

We now have all the tools to guess the hydrodynamic limit of the symmetric FEP. Let us follow the same steps as for the exclusion process. Computing $\mathcal{L}\eta(x)$, we see that

$$\eta_{tN^2}(x) - \eta_0(x) - N^2\int_0^t \Delta_{\mathrm{d}} h(\tau_x\eta_{sN^2})ds \tag{2.35}$$

is a martingale, with Δ_{d} the discrete Laplacian and $h(\eta) = \eta(0)\eta(1)+\eta(0)\eta(-1)-\eta(-1)\eta(0)\eta(1)$. As for the SSEP, the fact that we can directly write $\mathcal{L}\eta(x)$ as a discrete Laplacian (gradient property; recall the discussion below (2.7)) is very important to us. If we want to take an average, we see that we need to understand not only the local densities, but also the densities of pairs/triples of neighboring particles. It is natural to assume that, if the local density around x is ρ, the configuration around x should have a distribution close to π_ρ. With that guess, we obtain that in the hydrodynamic limit, the density should satisfy

$$\begin{cases} \partial_t\rho = \Delta(H(\rho)), \\ \rho(0,\cdot) = \rho_0, \end{cases} \tag{2.36}$$

where $H(\rho) = 0$ if $\rho \leq 1/2$, $H(\rho) = \pi_\rho(h) = \frac{2\rho-1}{\rho}$ if $\rho \in (1/2,1]$.

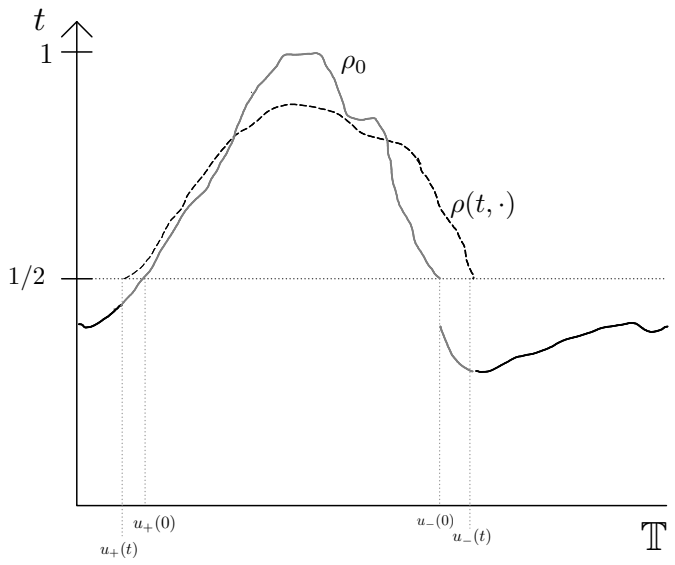

Figure 2.2: A sketch of the evolution described by (2.36). The full line represents the initial profile. Whereever the density is above 1/2, the dynamics is described by an elliptic diffusion equation. At the boundary of the $\{\rho \geq 1/2\}$ region (here labeled $u_\pm$), this creates a spill of mass into the $\rho < 1/2$ region which results in the displacement of the boundary. The density at a point in the $\rho < 1/2$ region remains unchanged until it is reached by the free boundary (full black line).

This requires some explaining: there is a priori no reason why (2.36) makes sense. Indeed, the degeneracy in H means that, even starting from a smooth initial condition ρ_0, the density profile will generically create discontinuities (see what happens around the u_+ in Figure 2.2). (2.36) is in fact a *free boundary* problem. Equations of this kind are used in the literature [96, 85, 4] to describe the spreading of a liquid phase into a frozen one. In the particular instance of (2.36), density would stand for enthalpie in the standard interpretation, the liquid phase corresponding to the zone where $\rho > 1/2$ and the profile is a solution to a standard elliptic equation, while the frozen phase corresponds to $\rho < 1/2$ where the profile does not evolve. It should be intuitively clear that the liquid phase increases as matter pours into the frozen phase.

A standard way of going around the inherent lack of regularity of a putative solution ρ is to use the notion of *weak solution*. Let us work on the torus, since this is the setting of [B6].

Definition 4. *A measurable function $\rho : [0,T] \times \mathbb{T} \to [0,1]$ is a solution to (2.36) up to time $T \in [0,\infty]$ if, for any test function $\phi \in \mathcal{C}^{1,2}([0,T] \times \mathbb{T})$,*

$$\int_{\mathbb{T}} \rho(T,\cdot)\phi(T,\cdot) - \int_{\mathbb{T}} \rho_0\phi(0,\cdot) = \int_0^T \int_{\mathbb{T}} \rho(t,\cdot)\partial_t\phi(t,\cdot)dt + \int_0^T \int_{\mathbb{T}} \rho(t,\cdot)\Delta\phi(t,\cdot)dt. \tag{2.37}$$

It is a nice feature of (2.36) (which is a consequence of H being a non-decreasing function) that it has a unique weak solution. However, one would like to describe the solution in terms of sub- and super-critical phases, at least for simple initial profiles as in Figure 2.2. This requires a stronger notion of solution. For simplicity, let us restrict the definition to the case where the supercritical phase in ρ_0 is an interval. Then it is convenient to consider that a strong solution to (2.36) is given by a triple (ρ, u_+, u_-), where ρ is the density, and $u_\pm$ describe the evolution of the two interfaces between liquid/frozen phases.

Definition 5. *Assume that $\{x \in \mathbb{T} : \rho_0(x) \geq 1/2\}$ is an interval. A triple (ρ, u_+, u_-), where $\rho : \mathbb{R}_+ \times \mathbb{T} \to [0,1]$ and $u_\pm : \mathbb{R}_+ \to \mathbb{T}$, is a strong solution to (2.36) if*

1. *u_- (resp. u_+) is non-decreasing (resp. non-increasing), with*

$$\{x \in \mathbb{T} : \ \rho_0(x) \geq 1/2\} = [u_+(0), u_-(0)];$$

2. *there exists $\tau \in \mathbb{R}_+ \cup \{\infty\}$ such that $u_-(t) = u_+(t)$ iff $t \geq \tau$, and $u_\pm$ are constant on $[\tau, \infty)$; the time τ is called the* merging time*;*

3. *$\mathcal{H} \circ \rho$ is Lipschitz;*

4. *for any $t \in \mathbb{R}_+$, if $u \in (u_-(t), u_+(t))$, then $\rho_t(u) = \tilde{\rho}_0(u)$;*

5. *if $\big\{t \leq \tau$ and $u \in (u_+(t), u_-(t))\big\}$ or if $t > \tau$, then*

$$\rho_t(u) > \tfrac{1}{2}, \qquad \text{and} \qquad \partial_t \rho_t(u) = \partial_u^2 \mathcal{H}(\rho_t(u));$$

6. *if $t \in (0, \tau)$, then*

$$u'_\pm(t) = -\frac{4\partial_u \rho_t(u_\pm(t)^\pm)}{\frac{1}{2} - \rho_0(u_\pm(t))}.$$

For sufficiently nice initial profiles, such a solution will indeed exist (and by uniqueness correspond to the weak solution).

Proposition 6. *Assume $\rho_0 : \mathbb{T} \to [0,1]$ is in $\mathcal{C}^2(\mathbb{T})$, $|\rho_0^{-1}(\{1/2\})| = 2$ and $\rho'(x) \neq 0$ whenever $\rho(x) = 1/2$.*

Then there exists a strong solution (ρ, u_-, u_+) to the Stefan problem (2.36)*.*

Note that the hypotheses above are by no means minimal. We could certainly allow for more interfaces (but this would make the notations even more cumbersome). Also, the regularity of ρ_0 matters mostly at the interface points and can be relaxed elsewhere. Finally, the situation is in fact simpler if instead of a non-zero derivative at the interface points we have a discontinuity (without the discontinuity, $u'_\pm$ diverges as $t \to 0$, and thus the definition of solution requires an extra argument).

A proof of the above fact is given in [B6], but essentially follows from arguments present in [85, 4]. The case of initial profiles with a discontinuity at the boundary of the supercritical region is treated via a fixed point argument in the space of plausible candidates for the interface trajectories $u_\pm$. The continuous case is deducted from this by sandwiching the initial configuration between two profiles with a jump.

2.2.3 Hydrodynamic limit and local convergence to the active/frozen phase

Now that we understand better the natural candidate for the hydrodynamic limit of the FEP, it remains to give a proper statement of convergence.

Theorem 14. *[B5, B6] Let $\rho_0 : \mathbb{T} \to [0,1]$ be a measurable initial profile. For any $t > 0$, $\delta > 0$ and $\phi : \mathbb{T} \to \mathbb{R}$ smooth, we have (recall* (2.2)*)*

$$\mathbb{P}_{\mu^N}\left(\left|\frac{1}{N}\sum_{x \in \mathbb{T}_N} \eta_{tN^2}(x)\phi(x/N) - \int_{\mathbb{T}} \rho_t \phi\right| > \delta\right) \underset{N\to\infty}{\longrightarrow} 0. \tag{2.38}$$

In [B5], we prove this result in the case $\rho_0 > 1/2$. In this regime, the non-ellipticity of the equation (2.36) is in fact not relevant: on the macroscopic side, the equation we deal with behaves like a nice elliptic equation. The difficulty appears in the microscopic regime. We use the standard entropy method as described in Section 2.1.1. This requires technical work to show decorrelation properties of the stationary measures π_ρ and adapt the method to non-product grand-canonical measures (π_ρ replacing ν_α). But the most important obstacle is the fact that our initial distribution is a priori not absolutely continuous w.r.t. any grand-canonical measure:

the supremum taken over densities f in e.g. (2.14) has no relevance here. Therefore, we need as a preliminary result to establish that in subdiffusive time (in fact, poly-log), the system reaches the active set of configurations $E^N := \cup_{k<N/2} E_k^N$ (recall (2.32)); this is Theorem 15 below. The subdiffusivity of this time is important because it means that during this preliminary phase, the initial density profile does not have time to evolve. It is then possible to follow the steps sketched in Section 2.1.1. In particular, we show that any limit law is invariant under allowed jumps; given the irreducibility properties of the sets E_k^N, this tells us that understanding the π_k^N is enough to conclude.

Theorem 15. *[B5] Assume $\rho_0 > 1/2$. Then*

$$\mathbb{P}_{\mu_N}\left(\eta_{(\log N)^8} \in E^N\right) \underset{N\to\infty}{\longrightarrow} 1. \tag{2.39}$$

To prove this result, we work with the zero-range process associated with the FEP, because we rely a lot on monotonicity properties. Recall that in the zero-range world, the critical density becomes 1.

On the one hand we show that with high probability, the initial configuration is regular, in the sense that in mesoscopic boxes the density is at least slightly above 1, and the particles are not too concentrated toward the right boundary. This is a rather technical proof, but since we work with product initial measures it is tractable. For more general initial measures, this would need to be proved anew.

Then we start from a given configuration η_0 whose zero-range image satisfies the above requirements, and show that (2.39) holds with μ_N replaced by η_0. The principle of the proof is the following: fix x. By waiting a long enough time T, we make sure that with high probability each particle initially present in $\{x+1, ..., x+\ell\}$ will either have encountered an empty site in the box $\{x+1, ..., x+\ell\}$, and gotten stuck there forever; or have left the box $\{x+1, ..., x+\ell\}$ at some point. At most ℓ particles fall into the first case, therefore, if the box initially contains at least $(1+\delta)\ell$ particles, $\delta\ell$ particles or more will have left the box at least once before T. In order for the site x to remain empty up until T, all of these particles must have left through the other boundary $x+\ell$. This means that particles in $\{x+1, ..., x+\ell\}$ have performed an abnormally large number of steps to the right.

In order to make the above sketch rigorous, we use a chain of couplings between, successively, our original zero-range process, a zero-range with the same rates but in finite volume, a rate one zero-range process, and finally independent simple random walks. In the end, we reduce the proof to standard estimates on simple random walks.

In our second work [B6], we really wanted to address the formation of interfaces. This meant giving up on the entropy method and relying instead on the study of Young measures, as used by Funaki in [48]. The entropy method is not suited in our context, because of the inherent reducibility of the inactive phase. The chosen strategy bypasses the need to work with Dirichlet forms, and relies instead on being able to prove that any stationary, translation invariant measure for the infinite volume dynamics satisfies a replacement lemma, in the sense that for any such ν,

$$\nu\left(\left|\frac{1}{2\ell+1}\sum_{x=-\ell}^{\ell} h\circ\tau_x(\eta) - H(\eta^\ell(x))\right|\right) \underset{\ell\to\infty}{\longrightarrow} 0, \tag{2.40}$$

where $\eta^\ell(x) := \frac{1}{2\ell+1}\sum_{y=-\ell}^{\ell}\eta(x+y)$. This is fortunately implied by Theorem 13. In fact, under μ_F, all terms inside the expectation are essentially 0. On the other hand, the decorrelation properties of the measures π_ρ imply (2.40) with ν replaced by ν_ρ. That allows us to extend the statement of Theorem 14 to general initial profiles. It is unclear to me whether it is feasible to show (2.40) without first identifying all possible translation invariant stationary measures.

One drawback of the result in Theorem 14 is that we do not say much about the microscopic behavior of the system in terms of active/frozen phase, as it approaches its limit (where the two phases are well identified e.g. under the conditions of Proposition 6). In fact, it turns out that, on a time scale much shorter than the diffusive one, and for ρ_0 such as in Proposition 6,

the microscopic system first separates into an active and frozen phase (and remains thus split forever). When this happens, the local densities have not had time to evolve, and only on the longer diffusive time scale do they start to follow the Stefan problem (2.36). In particular, it makes sense to speak of microscopic interfaces and we can then complement Theorem 14 with the statement that micro– and macroscopic interfaces match.

Let us define the set of configurations which are (microscopically) split into an active and frozen regions.

$$\mathcal{P}_N := \left\{ \eta \in \{0,1\}^{\mathbb{T}_N} : \exists a, b \in \mathbb{T}_N, \begin{cases} \forall x \in [a,b] & \eta(x) + \eta(x+1) \geq 1 \text{ and} \\ \forall x \in (b,a) & \eta(x)\eta(x+1) = 0 \end{cases} \right\}. \tag{2.41}$$

Note that this is a closed set for the dynamics: $\eta_t \in \mathcal{P}_N \Rightarrow \eta_{t+s} \in \mathcal{P}_N$ for any $s > 0$.

Theorem 16. *[B6] Assume that $\rho_0 : \mathbb{T} \to [0,1]$ is in $\mathcal{C}^2(\mathbb{T})$, $|\rho_0^{-1}(\{1/2\})| = 2$ and $\rho'(x) \neq 0$ whenever $\rho(x) = 1/2$.*

1. $\mathbb{P}_{\mu_N}\left(\eta_{N^{-1/4}} \in \mathcal{P}_N\right) \underset{N\to\infty}{\longrightarrow} 1$.

   On this event, let us define for $t > 0$, $u_-^N(t)$ (resp. $u_+^N(t)$) as the largest b (resp. smallest a) for which η_{tN^2} satisfies the conditions in (2.41). On the complement of this event, $u_\pm(t)$ may be defined arbitrarily.

2. *For any $t \in (0,\tau] \cap \mathbb{R}_+$ (recall that τ is the merging time defined in Definition 5), for any $\varepsilon > 0$*

$$\mathbb{P}_{\mu_N}\left(|u_\pm^N(t) - u_\pm(t)| \geq \varepsilon\right) \underset{N\to\infty}{\longrightarrow} 0. \tag{2.42}$$

The proof of the first point is where the core work lies. It relies on the same type of arguments as Theorem 15, but the estimates are more delicate around the interfaces. We did not aim for optimal bounds, and in particular the $(\log N)^8$ and $N^{-1/4}$ appearing above have no significant meaning.

The second point of Theorem 16 is in fact an easy consequence of the first, and Theorem 14.

2.2.4 Open problems

All the results above rely crucially on the mapping with the zero-range process, especially through Theorems 13, 15 and 16 (first point). This is a significant problem when one tries to study the system in higher dimension.

There are several ways one could define a multidimensional version of the FEP. Perhaps the most natural is to consider that particles jump at rate one to empty neighbors, provided they have a neighbor. Already the question of guessing the limit equation is not trivial, since it requires knowing the infinite volume stationary measures. So far we do not have a guess for a multidimensional analog of the measures π_ρ. In some sense, they should be uniformly distributed on configurations such that no particle is isolated, and empty sites have either 0 or more than 2 neighbors.

In dimension 1, the function H appearing in (2.36) vanishes for $\rho < 1/2$, which leads to the free boundary problem we discussed. This is ultimately a consequence of the fact that configurations with at most half sites occupied by particles eventually freeze. In higher dimension, it is intuitively clear that for densities small enough, this should also be true. However, even this simple statement currently does not have a proof in infinite volume. In finite volume, it is clear that configurations with density $< 1/2$ eventually freeze, but simulations[2] indicate that this takes an exponentially long time for densities between $1/3$ and $1/2$ (compared with polynomial for densities less than $1/3$). This is usually indicative that in infinite volume the system remains

[2]Kindly communicated by Clément Erignoux, Marielle Simon and Alexandre Roget.

active at those densities. Our current conjecture is therefore that $H(\rho)$ should be 0 (resp. > 0) for $\rho < 1/3$ (resp. $> 1/3$).

A different kind of question concerns the fluctuations in the FEP in the weakly asymmetric regime. We expect that in the bulk of the active region, the fluctuations should be similar to those of the exclusion process, i.e. described by the KPZ equation on the real line. On the other hand, we could also study the fluctuations around the active/inactive interface, where the situation should be different. In an ongoing project with Guillaume Barraquand and Marielle Simon, we aim to show that, in the case of a step initial condition ($\eta(x) = 1$ iff $x \leq 0$), the fluctuations can be described by the KPZ equation on the half-line with Dirichlet boundary condition and δ_0' initial condition.

2.3 The porous medium model

The second model we consider in this chapter is an example of KCLG, already mentioned in Section 1.1.2: the porous medium model (PMM) in dimension 1.

2.3.1 Definition and hydrodynamic limit

In the simplest version, x and $x+1$ exchange their contents at rate $\eta(x-1)+\eta(x+2)$. We will also consider a more general version, which requires at least $m-1$ neighboring particles for a jump to occur. Fix an integer $m \geq 2$. The jump rates of the m–PMM are defined as

$$c^m_{x,x+1}(\eta) = \sum_{k=1}^{m} \prod_{j \in \{k-m,\dots,k\} \setminus \{0,1\}} \eta(x+j) \tag{2.43}$$

We recover the simple version in the case $m = 2$.

The generator of the PMM is then simply

$$\mathcal{L}^m_{\mathrm{PMM}} f(\eta) = \sum_{x \in \mathbb{T}_N} c^m_{x,x+1}(\eta) \left[f(\eta^{x,x+1}) - f(\eta) \right]. \tag{2.44}$$

(2.43) may seem like a rather specific and convoluted way of requiring "at least $m-1$ particles" in the neighborhood of $\{x, x+1\}$. The reason for this particular choice is that it gives a gradient model: $\mathcal{L}^m_{\mathrm{PMM}}\eta(x)$ can be rewritten as a discrete Laplacian of (a translation of)

$$h^m(\eta) = \sum_{k=1}^{m} \prod_{j=k-m}^{k-1} \eta(j) - \sum_{k=1}^{m-1} \prod_{j \in \{k-m,\dots,k\} \setminus \{0\}} \eta(j). \tag{2.45}$$

In the simple $m = 2$ case (dropping the superscript m), this just gives $h(\eta) = \eta(0)\eta(1) + \eta(0)\eta(-1) - \eta(-1)\eta(1)$.

Now to guess the hydrodynamic limit of the PMM, we just have to understand its invariant measures. It is easy to check that the Bernoulli product measures μ_p are reversible w.r.t. the PMM dynamics (recall that it is a general property of KCMs – see Section 1.1.3). Since $\mu_\rho(h^m) = \rho^m$, a natural guess is therefore that the hydrodynamic limit is described by

$$\partial_t \rho = \Delta(\rho^m). \tag{2.46}$$

This is known as the Porous Medium Equation (PME), which in turn gives its name to the PMM. As the name indicates, it was introduced to model the flow of an ideal gas through a porous medium [102]. Mathematically, it provides another example of a free boundary problem: the positivity region spreads with finite speed, contrary to what happens for the heat equation. In fact, the boundary between the $\{\rho > 0\}$ and $\{\rho = 0\}$ regions may even remain static for a non-zero period of time before starting to move. This depends on the pressure applied on the boundary by the mass inside the $\{\rho > 0\}$ region. When such a boundary exists, generically

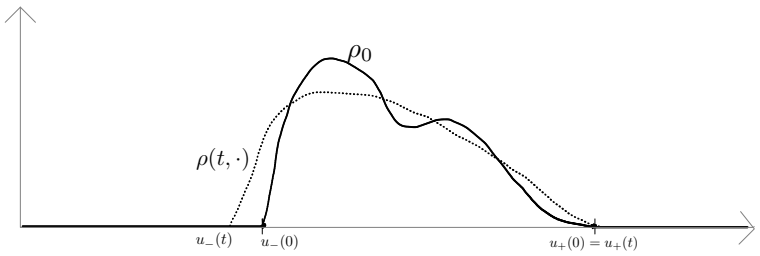

Figure 2.3: A sketch of the evolution described by the PME (2.46). u_-, u_+ mark the boundaries of the $\{\rho > 0\}$ region.

the solution will not be differentiable: the derivative is not continuous at the interface (see Figure 2.3).

The convergence of the PMM to the PME in the diffusive limit was proved in [53]. Recall that μ_t^N denotes the law of the process started from μ^N at time tN^2.

Theorem 17. *[53] Assume that there exists $\delta \in (0, 1/2)$ s.t. $\rho_0 : \mathbb{T} \to [\delta, 1-\delta]$, and $\rho_0 \in \mathcal{C}^{2+\delta}(\mathbb{T})$. Then,*

$$H(\mu^N | \nu^N_{\rho_0(\cdot)}) = o(N), \quad \Longrightarrow \quad H(\mu_t^N | \nu^N_{\rho(t,\cdot)}) = o(N), \tag{2.47}$$

where ρ is the solution to (2.46) with initial condition $\rho(0,\cdot) = \rho_0$.

The hypotheses on ρ_0 ensure that (2.46) has a unique classical solution. Requiring that ρ_0 be bounded away from 0 is a necessity in the context of the relative entropy technique: if ρ_0 vanishes on an interval, the finite speed propagation property of the PME means $\rho(t,\cdot)$ also vanishes on an interval for t small enough. The theorem in [53] is only stated and proved in the case $m = 2$, but the proof extends without difficulty to the more general case.

2.3.2 Fluctuations around the equilibrium

The results of this section appear in [B7]. The goal is to understand the fluctuation field (2.24) for weakly asymmetric versions of the PMM, in which particles (when allowed) jump right (resp. left) with rate $1 + N^{-\gamma}$ (resp. $1 - N^{-\gamma}$) for some positive $\gamma \geq 1/2$. We denote by $L^S = \mathcal{L}^m_{\text{PMM}}$ the symmetric part of the generator (with rates 1) and L^A the antisymmetric part:

$$L^A f(\eta) = N^{-\gamma} \sum_{x \in \mathbb{Z}} c^m_{x,x+1} \left[\eta(x)(1-\eta(x+1)) - \eta(x+1)(1-\eta(x))\right] \left[f(\eta^{x,x+1}) - f(\eta)\right]. \tag{2.48}$$

The starting point of the study is again the fact that

$$\mathcal{Y}_t^N(H) - \mathcal{Y}_0^N(H) - I_t^N(H) - B_t^N(H) \tag{2.49}$$

is a martingale, with $I_t^N = \int_0^t N^2 \mathcal{L}^S \mathcal{Y}_s^N(H) ds$ and $B_t^N(H) = \int_0^t N^2 \mathcal{L}^A \mathcal{Y}_s^N(H) ds$. The integral terms can be computed, and it turns out that for $\gamma \leq 3/2$ and $\rho \neq \frac{m}{m+1}$, the antisymmetric term is of order $N^{3/2-\gamma}$ while the symmetric term remains of order 1, as does the martingale. In fact, in this situation, the leading behavior of the fluctuation field is a transport phenomenon: if one accelerates the process by $N^{2-\gamma}$ rather than N^2, the limit solves a transport equation

$$d\mathcal{Y}_t = F'(\rho) \nabla \mathcal{Y}_t dt, \quad \text{with } F(\rho) = 2m\rho^{m-1}\rho(1-\rho). \tag{2.50}$$

Note that $D(\rho) = m\rho^{m-1}$ is the diffusion coefficient appearing in the hydrodynamic limit equation (the PME) and $\rho(1-\rho) = \nu_\rho((\eta(0) - \rho)^2) : \chi(\rho)$, so that $F(\rho) = 2D(\rho)\chi(\rho)$. In order to retain the fluctuating behavior, we can either impose $\rho = \frac{m}{m+1}$, or modify the definition of the fluctuation field to follow the transport wave. From now on, our definition of the fluctuation field will be

$$\mathcal{Y}_t^N(H) = \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{Z}} H\left(\frac{x - N^{2-\gamma} F'(\rho)}{N}\right) (\eta_{sN^2}(x) - \rho). \tag{2.51}$$

Theorem 18. *[B7]*

1. *If $\gamma > 1/2$, the fluctuation field converges to the Ornstein-Uhlenbeck (OU) process solution of*
$$d\mathcal{Y}_t = D(\rho)\Delta\mathcal{Y}_t dt + \sqrt{D(\rho)\chi(\rho)}\nabla d\mathcal{W}_t. \tag{2.52}$$
2. *If $\gamma = 1/2$, the fluctuation field converges to the (unique) energy solution of the Stochastic Burgers Equation (SBE)*
$$d\mathcal{Y}_t = D(\rho)\Delta\mathcal{Y}_t + F''(\rho)\nabla(\mathcal{Y}_t^2)dt + \sqrt{D(\rho)\chi(\rho)}\nabla d\mathcal{W}_t. \tag{2.53}$$

In order to understand this result, we need 1) to give a precise definition of the notions of solutions to the SPDEs appearing in the theorem; 2) to dive further into (2.49) to understand where the different terms of the SPDEs arise at the microscopic level. I leave the issue of tightness out of the discussion.

Notions of solutions Let us start with the Ornstein-Uhlenbeck process. A solution to (2.52) is a process such that for all t, $\mathcal{Y}_t$ is a space white noise (in $\mathcal{S}'(\mathbb{R})$) with covariances $\mathbb{E}[\mathcal{Y}_t(H)Y_t(G)] = \frac{1}{2}\chi(\rho)\int_{\mathbb{R}} HG$, and
$$\mathcal{Y}_t(H) - \mathcal{Y}_0(H) - D(\rho)\int_0^t \mathcal{Y}_s(\Delta H)ds \tag{2.54}$$
is a continuous martingale with quadratic variation $tD(\rho)\chi(\rho)\int_{\mathbb{R}}(\nabla H)^2$. It is well known [71] that there exists a unique such solution.

For the Stochastic Burgers Equation (SBE), we need to be careful: the naive extension of the above notion leads to substracting the quadratic term in the martingale, but it is not clear what $\mathcal{Y}_t^2$ should be. We will use the notion of *energy solutions*, which involves an extra assumption used to control the quadratic term (point 2 in Definition 6). This notion has the double advantage that it provides uniqueness [57] and that the extra assumption has a microscopic counterpart which can be checked for a number of microscopic systems [54] (including the PMM, as we show in [B7]).

Definition 6. *A process $\{\mathcal{Y}_t\,;\, t \in [0,T]\}$ in $\mathcal{C}([0,T],\mathcal{S}'(\mathbb{R}))$ is a stationary energy solution of (2.29) if*

1. *for each $t \in [0,T]$, $\mathcal{Y}_t$ is a $\mathcal{S}'(\mathbb{R})$-valued white noise with variance $\chi(\rho)$;*
2. *there exists a positive constant κ such that for any $H \in \mathcal{S}(\mathbb{R})$ and $0 < \delta < \epsilon < 1$*
$$\mathbb{E}\Big[\big(\mathcal{B}^{\epsilon}_{s,t}(H) - \mathcal{B}^{\delta}_{s,t}(H)\big)^2\Big] \le \kappa\epsilon(t-s)\|\nabla H\|_2^2, \tag{2.55}$$
where
$$\mathcal{B}^{\epsilon}_{s,t}(H) := \int_s^t \int_{\mathbb{R}} \big(\mathcal{Y}_r * \iota_\epsilon(x)\big)^2\, \nabla H(x)\, dx\, dr$$
and $\iota_\epsilon(x)$ is an approximation of the identity;
3. *for $H \in \mathcal{S}(\mathbb{R})$,*
$$\mathcal{Y}_t(H) - \mathcal{Y}_0(H) - \tfrac{1}{2}\int_0^t \mathcal{Y}_s(D(\rho)\Delta H)\, ds + \tfrac{1}{2}F''(\rho)\mathcal{B}_t(H)$$
is a Brownian motion of variance $D(\rho)\chi(\rho)\int_{\mathbb{R}}(\nabla H)^2$, where $\mathcal{B}_t(H) := \lim_{\epsilon\to 0}\mathcal{B}^{\epsilon}_{0,t}(H)$, the limit being in L^2;
4. *the reversed process $\{\mathcal{Y}_{T-t}\,;\, t \in [0,T]\}$ satisfies (3).*

Theorem 19 ([57])**.** *There exists a unique stationary energy solution of* (2.29)*.*

Analysis of (2.49) First of all, it is not hard (though tedious to compute) to show that the quadratic variation of the martingale converges to $tD(\rho)\chi(\rho)\int_{\mathbb{R}}(\nabla H)^2$. It remains to analyse the integral terms.

For simplicity, let us fix $m = 2$ and $\rho = 2/3$ (so that the corrective term $N^{2-\gamma}F'(\rho)$ in (2.51) vanishes). We get the following expressions for the integral terms:

$$I_t^N(H) = \int_0^t \frac{1}{\sqrt{N}} \sum_{x\in\mathbb{Z}} (\tau_x h(\eta_{sN^2}) - \rho^2)\Delta_N H(x/N)ds, \tag{2.56}$$

$$B_t^N(H) = -\int_0^t \frac{N^{1-\gamma}}{\sqrt{N}} \sum_{x\in\mathbb{Z}} \tau_x f(\eta_{sN^2})\nabla_N H(x/N)ds, \tag{2.57}$$

where f is a finite sum of local functions, each taking one of the forms

$$C(\rho)\bar\eta(y), \quad C(\rho)\bar\eta(x)\bar\eta(x+y) \quad \text{or} \quad C(\rho)\bar\eta(x)\eta(x+y)\eta(x+z), \tag{2.58}$$

where $\bar\eta(x) = \eta(x) - \rho$.

In fact, one can rewrite

$$\tau_x h(\eta) - \rho^2 = 2\rho\bar\eta(x) + \bar\eta(x)\bar\eta(x+1) + \bar\eta(x)\bar\eta(x-1) - \bar\eta(x-1)\bar\eta(x+1). \tag{2.59}$$

The contribution to (2.56) resulting from the first term is $2\rho\int_0^t \mathcal{Y}_t^N(\Delta_N H)ds$ – this will become the Laplacian term in the SBE. The contributions to (2.57) coming from the degree one terms in (2.58) vanish in the limit because[3] of our choice of $\rho = 2/3$ (alternatively because we used the corrected expression (2.51) for the fluctuation field). Three types of terms remain:

- terms of the form $\int_0^t \frac{1}{\sqrt{N}} \sum_{x\in\mathbb{Z}} \Delta_N H(x/N)\bar\eta(x+y)\bar\eta(x+z)$ with $y \neq z$,
- terms of the form $\int_0^t \frac{N^{1-\gamma}}{\sqrt{N}} \sum_{x\in\mathbb{Z}} \nabla_N H(x/N)\bar\eta(x+y)\bar\eta(x+z)$ with $y \neq z$,
- terms of the form $\int_0^t \frac{N^{1-\gamma}}{\sqrt{N}} \sum_{x\in\mathbb{Z}} \nabla_N H(x/N)\bar\eta(x)\bar\eta(x+y)\bar\eta(x+z)$ with $0, y, z$ pairwise distinct.

It is not too hard (see Section 6 in [B7]) to show that the first and third kinds of terms do not contribute in the limit. On the other hand, the second kind will give rise to the non-linearity in the SBE when $\gamma = 1/2$ and has to be handled with special care. This is where the main result of [B7] comes in.

Theorem 20 ([B7] Second order Boltzmann-Gibbs principle for degree two polynomial functions)**.** *Fix $v : \mathbb{Z} \to \mathbb{R}$ (possibly depending on N) such that $\|v\|_{2,N}^2 := N^{-1}\sum_{x\in\mathbb{Z}} v^2(x)$ is bounded. Then for all $y \in \mathbb{Z}$, ℓ sufficiently large,*

$$\mathbb{E}_\rho^N \left[\left(\int_0^t \sum_{x\in\mathbb{Z}} v(x) \left\{ \bar\eta_{sN^2}(x)\bar\eta_{sN^2}(x+y) - \left(\vec\eta_{sN^2}^{\ell}(x)\right)^2 + \frac{\chi(\rho)}{\ell} \right\} \right)^2 \right] \leq Ct \left(\frac{\ell}{N} + \frac{tN}{\ell^2} \right) \|v\|_{2,N}^2, \tag{2.60}$$

where $\vec\eta^{\ell}(x) = \frac{1}{\ell}\sum_{y=x+1}^{x+\ell} \bar\eta(x)$. In particular, choosing $\ell = \varepsilon N$, the left-hand side vanishes as $N \to \infty$ then $\varepsilon \to 0$.

This type of result was first proved in [54] for models with a sharp bound on the spectral gap and rates bounded above and below by a positive constant. It was later extended to other settings [55, 44, 56], though never with degenerate rates. The proof is similar in spirit to the one-block/two-block estimates used in the entropy method, though more quantitative. Let us explain what the estimate relies on.

For instance, let us try replacing $\bar\eta(x)\bar\eta(x+y)$ by $\bar\eta(x)\left(\bar\eta(x+y) - \vec\eta^{\ell_0}(x)\right)$. In the original proof, the error is estimated by $O(\ell_0^3/N^2)$. ℓ_0^2/N^2 is the inverse of the spectral gap of the

[3]The proof is merely an integration by parts and using Cauchy-Schwarz inequality.

dynamics restricted to a box of size ℓ_0; ℓ_0 the number of terms to which a spectral inequality is applied. Making a replacement from scale ℓ to scale 2ℓ is less costly, because the upper bound will be divided by a variance of order $1/\ell^2$, thus leading to an estimate $O(\ell/N^2)$.

The issue with the PMM is of course that we cannot estimate the spectral gap in this way. This is mainly due to the existence of *blocked configurations*, in which no two particles are at distance less than 2. Fortunately, such configurations are very unlikely under the equilibrium measures μ_ρ: in an interval of size ℓ, $\mu_\rho(\text{blocked}) \leq (1-\rho^2)^{\lfloor \ell/2 \rfloor}$. Our strategy relies on this observation, which we use to condition η_{sN^2} to be non-blocked in the box where we want to perform a replacement.

The next crucial observation [53] is that, if a configuration η is non-blocked in an interval of size ℓ, for x, y in the interval, the PMM dynamics allow to transition from η to $\eta^{x,y}$ using at most $C\ell$ jumps. This *path argument* allows us te replace the spectral gap bound and get the one-block estimate

Proposition 7 ([B7] One-block estimate)**.** *Let $\ell_0, \ell \in \mathbb{N}$ and let $\varphi : \mathcal{E} \to \mathbb{R}$ be a mean zero local function whose support does not intersect the set of points $\{1, \dots, \ell\}$. Let $\mathcal{G}(x)$ be the event that the configuration in $\{x+\ell+1, \dots, x+\ell+\ell_0\}$ is non-blocked. Then for any $t > 0$, $N \in \mathbb{N}$ and $y \in \{1, \cdots, \ell\}$:*

$$\mathbb{E}_\rho^N \Big[\Big(\int_0^t \sum_{x \in \mathbb{Z}} v(x)\, \tau_x \varphi(\eta_{sn^2}) \big(\bar{\eta}_{sN^2}(x+y) - \vec{\eta}^{\ell}_{sN^2}(x) \big)\, \mathbf{1}_{\mathcal{G}(x)}(\eta_{sn^2})\, ds \Big)^2 \Big] \leq Ct \frac{(\ell+\ell_0)^2}{n} \mathrm{Var}(\varphi) \|v\|_{2,n}^2. \quad (2.61)$$

The interested reader will note that one needs to be careful with the supports of the functions involved. This is due to the fact that, in our path argument, we do not want the exchange to perturb either $\mathbf{1}_{\mathcal{G}}$ or $\tau_x \phi$. In turn, this is the main reason why, in Theorem 20, we need to consider $\vec{\eta}^{\ell}$ rather than $\bar{\eta}^{\ell}$.

The rest of the proof consists in performing replacements on increasing length scales; during this process, we need to take special care of taking replacements on appropriate intervals (in particular, we need some non-intersection properties of the successive intervals that make the argument more delicate).

2.3.3 Open problems

The statement of the hydrodynamic limit in Theorem 17 is somewhat frustratingly incomplete. Indeed, the condition that ρ_0 takes values in $[\delta, 1-\delta]$ means that this result leaves out initial conditions where the free boundary problem appears. Another result of [53], relying on the entropy method, is that the hydrodynamic limit holds for any initial profile ρ_0, but for a slightly modified dynamics with generator

$$\mathcal{L}_\theta = \mathcal{L}_{\mathrm{PMM}} + N^{\theta-2} \mathcal{L}_{\mathrm{SSEP}}, \quad (2.62)$$

for $\theta \in (0,2)$. This allows to compensate the lack of good ergodicity properties of the PMM (recall that there are frozen configurations as soon as there are e.g. less than $N/3$ particles for $m = 2$).

With Clément Cancès, Makiko Sasada and Marielle Simon, we tried to make the relative entropy work by considering as the reference density an approximation ρ_0^N of ρ_0, taking values in $[\delta_N, 1-\delta_N]$ for a well-chosen δ_N. We did manage to get a one-block estimate of the form

$$\int_0^T \int \frac{1}{N} \sum_{x \in \mathbb{T}_N} V_\ell f_t^N d\nu_\alpha^N \, dt \leq \frac{1}{\gamma N} \int_0^T H_N(t) dt + \varepsilon, \quad (2.63)$$

for ℓ, N large enough. It is also true that we can write (2.20), despite the fact that the approximations of continuous gradients and Laplacians now involve diverging terms. However, the remaining divergences make the large deviation argument fail.

On the other hand, the method we used in [B6] gives us hope to prove a result with arbitrary initial conditions. Using the technique detailed in Appendix A, it is possible to classify all invariant measures for the PMM. Then the picture is similar as what happens for the FEP: the frozen as well as the active components all satisfy the replacement lemma. Extending this to higher dimension is on the other hand a completely open problem.

Chapter 3

Random walks in dynamic random environments

3.1 Overview

The study of random walks in random environments (RWRE) constitutes a very wide and active field of study. Let me approach it through a non-standard route. We have seen our first example of RWRE in Section 1.2.2, when we studied the motion of a tracer in a KCM. There, we were mostly interested in the physically meaningful quantity that is the diffusion constant, and we only skimmed briefly through the means of obtaining limit theorems for the tracer. This is however a mathematical manuscript and it is worth taking a closer look at the features that allow us to state a law of large numbers (LLN) and an invariance principle for the tracer. This will also lead us naturally to more general problems of random walks in (dynamic) random environment.

We are working in the following setting. We have a Markov process $(\eta_t, X_t)_{t\geq 0}$, where $(\eta_t)_{t\geq 0}$ is itself a Markov process (the environment) on $\{0,1\}^{\mathbb{Z}^d}$ with generator L_{env}. The environment is stationary, i.e. $\eta_t \overset{(d)}{=} \eta_0$ for all $t \geq 0$. Let us call μ the stationary measure. $(X_t)_{t\geq 0}$ on the other hand is a (a priori non-Markovian) process on $\mathbb{Z}^d$, which jumps to $x+y$ at rate $r(y, \tau_x \eta_t)$ at time t when it is in position x. The collection of rates r depend on the second coordinate only through a fixed neighborhood of the origin[1]. We could define the process $(\eta_t, X_t)_{t\geq 0}$ through its generator

$$\mathcal{L}f(\eta, X) = L_{\text{env}} f(\cdot, X)(\eta) + \sum_{y\in\mathbb{Z}^d} r(y, \tau_X \eta) \left[f(\eta, X+y) - f(\eta, x) \right]. \tag{3.1}$$

A more convenient way of approaching the problem is to look at the *process seen from the particle*, namely $(\omega_t)_{t\geq 0} := (\tau_{X_t}\eta_t)_{t\geq 0}$. This has the nice feature that it is again a Markov process with generator

$$Lf(\omega) = L_{\text{env}} f(\omega) + \sum_{y\in\mathbb{Z}^d} r(y,\omega)[f(\tau_y\omega) - f(\omega)]. \tag{3.2}$$

Moreover, this process generically contains all the information on $(\eta_t, X_t)_{t\geq 0}$. In particular, one extremely useful fact is that the following quantity is a martingale

$$X_t - \int_0^t j(\omega_s) ds, \tag{3.3}$$

where $j(\omega) := \sum_{y\in\mathbb{Z}^d} y r(y,\omega)$ is the instantaneous drift of a particle sitting at the origin in environment ω. In order to obtain limit theorems for $(X_t)_{t\geq 0}$, it is therefore (almost) enough to establish them for the additive functional $\int_0^t j(\omega_s) ds$.

[1] In Section 1.2.2, we had $r(y,\cdot) \equiv 0$ for $y \notin \{\pm e_i, i = 1, \dots, d\}$ and $r(\pm e_i, \omega) = (1-\omega(0))(1-\omega(\pm e_i))$.

3.1.1 Reversible case

In the setting of Section 1.2.2, the process η is reversible with respect to some probability measure μ on $\{0,1\}^{\mathbb{Z}^d}$, and the rates r are chosen in such a way that the process seen from the particle $(\omega_t)_{t\geq 0}$ shares that property. In particular, the integral term in (3.3) is an additive functional of a reversible process. Additionally, it can be checked easily that this process is ergodic.

Random walks in random conductances [18] also fit in this framework – with some caveats. In that case, the environment is given by $(\eta_t(e))_{e\in\mathbb{E}^d}$, a Markov process over edges (rather than vertices), and the random walk jumps over an edge e at time t with rate $\eta_t(e)$. Since the total jump rate can depend on the current vertex, this is usually called the variable speed random walk. In order to avoid explosion, assume that e.g. η is bounded. Let us also assume that the conductances are reversible w.r.t. some probability measure μ, that μ is invariant under space shifts and ergodic. Then the process seen from the (variable speed) particle has the same properties. Constant speed random walks, in which the jump rate from x to y at time t is given by $\eta_t(\{x,y\})/\sum_{z\sim x}\eta_t(\{x,z\})$, can also be considered, but the reversible measure for the process seen from the particle is no longer equal to μ in general.

Law of Large Numbers

In both cases above, $(\omega_t)_{t\geq 0}$ is both *stationary* and *ergodic*. With these two features, the ergodic theorem applied to the additive functional $\int_0^t j(\omega_s)ds$ immediately implies the law of large numbers.

Annealed invariance principle

Once again, we would like to show a CLT or invariance principle for $\int_0^t j(\omega_s)ds$. The reversibility of ω allows to use powerful spectral tools [70, 37, 95, 73, 80].

As a warm-up, imagine for a second that we can find a function g, say in $L^2(\mu)$, s.t. $j = Lg$. Then we can write

$$\int_0^t j(\omega_s)ds = \underbrace{\left[\int_0^t Lg(\omega_s)ds - g(\omega_t) + g(\omega_0)\right]}_{M_t \text{ martingale}} + \underbrace{[g(\omega_t) - g(\omega_0)]}_{R_t}. \tag{3.4}$$

R_t being bounded in L^2 will go to 0 when divided by $\sqrt{t}$; it remains to apply one of the numerous invariance principle results for martingales [26] and we are essentially done.

In general this is too strong a condition. It can however be efficiently relaxed by considering g_λ a solution to

$$j = (-L + \lambda)g_\lambda$$

instead of the Poisson equation. Then one can hope that the corresponding martingale and rest M_t^λ, R_t^λ converge as $\lambda \to 0$ to quantities that satisfy the same conditions as above.

A celebrated result of Kipnis and Varadhan [70] provides an optimal condition for this convergence and $\int_0^t j(\omega_s)ds$ to satisfy an invariance principle: that

$$\int_0^\infty \mu(je^{tL}j)dt < \infty. \tag{3.5}$$

In [37], De Masi, Ferrari, Goldstein and Wick noticed that in our context, $(X_t)_{t\geq 0}$ is antisymmetric under time reversal, while $\int_0^t j(\omega_s)ds$ is symmetric, and that allows to satisfy automatically the condition (see [94, Section II.6.2] for a short proof). [37, 95] also derive a variational formula for the limiting diffusion matrix. Altogether, this leads us in particular to Proposition 2.

Quenched invariance principle

A lot of effort has been recently dedicated to establishing *quenched invariance principles* for random walks in random conductances (see e.g. [3, 2, 19]). The techniques used to build and

control the corrector are quite different from what is presented above, and exposing them in more details would take us too far for the purposes of this manuscript.

3.1.2 Stationary case

When $(\omega_s)_s$ is only stationary, the same arguments that prove the LLN in the reversible case apply. For the CLT or invariance principle however, it is no longer obvious that condition (3.5) is sufficient. In general, one needs additional conditions to control the convergence of g_λ and Lg_λ, e.g. a (graded) sector condition. This has been used for instance to establish ([75], after a long list of contributions) that in doubly stochastic random walks, it is enough to have $j \in \mathcal{H}_{-1}$, i.e.

$$\sup\{2 < j, \varphi > -\mu(\varphi(-L)\varphi)\} < \infty.$$

Another sufficient condition is that L has positive spectral gap or is a perturbation of a normal operator [73, Sections 2.7.2, 2.7.5]. In [B1], we show that a LLN and CLT hold also for cases in which L is a perturbation of a process with positive spectral gap (also see [74]). This result is discussed in greater length in Section 3.2.

3.1.3 Mixing environments

In fact, most RWRE are irremediably far from stationarity. This deserves a comment: for bounded environments, the existence of invariant measures for the process seen from the particle is guaranteed by compactness, so it may seem we can circle back to the stationary case even in a very general setting. However, in most cases, uniqueness of the invariant measure and absolute continuity are either hard to establish or plainly do not hold. This means that we cannot easily transfer results that would hold starting from an invariant measure to the setting in which the random walk is dropped at an arbitrary position in a stationary environment. In this context, it is hard to identify under which criterion LLN and invariance principles should hold.

Let us start by noting that it is simply not true in general that RWRE obey LLN and CLT. A prominent counter-example is the one-dimensional nearest-neighbor random walk in static i.i.d. environment. Depending on the distribution of the environment, the random walk can be recurrent with fluctuations of order $(\log n)^2$ [93], transient with zero speed and/or converge to a stable law... [69, 39]. These exotic behaviors are highly related to the presence of traps, i.e. regions in which the environment is organized in such a way that the random walk needs an unusually long time to exit them. The setting best understood for random walks in static random environments is the one where the environment is assumed to be i.i.d., and such that the random walk has *a priori* a ballistic behavior in a given direction [97] (see (3.18)). This ensures that the RW exits traps efficiently and keeps discovering fresh environment.

It can be expected that in a dynamic environment, local traps should dissolve and their effect on the long time behavior of the random walk should be less radical than changing its order of magnitude (see e.g. [64]). A statement to that effect would of course require identifying the mixing properties of the environment (on which the dissolution of traps depends), and even more to the point: the mixing properties of the environment seen from the particle [7, 87].

Let me make a simple comment here. The dynamic case may look more complex and more general than the static one. However, it is always possible to understand time as an additional space direction, and to view a d–dimensional random walk in dynamic random environment as a $(d+1)$–dimensional random walk in static environment whose last coordinates evolves deterministically (and in particular this RW naturally satisfies a ballisticity condition). The interest of the dynamic point of view therefore is not to generalize the static case. Rather, it offers a setting in which it is natural to formulate conditions on the mixing properties of the environment that are weaker than the i.i.d. hypothesis, but still strong enough that the (time–)ballisticity ensures e.g. the dissolution of traps. For instance, a popular choice is to define the environment as an interacting particle system.

Prominent works in the field have focused on environments in which correlations decay uniformly, e.g. (η is always assumed to be stationary):

$$\sup_{A,B} |\mathbb{P}((\eta_u)_{u\geq t+s} \in B | (\eta_u)_{u\leq t} \in A) - \mathbb{P}(\eta \in B)| \underset{s\to\infty}{\longrightarrow} 0,$$

where the supremum is taken over local events B and (almost) all possible events A ([36, 7, 15]). While such a condition may cover a wide range of applications, it also fails in many cases, such as conservative interacting particle systems or KCMs. With my coauthors, we tried to improve available results in both these directions.

For conservative systems, we work with a special choice of environment that offers a lot of possibilities for hands-on analysis. In [B8, B10], $(\eta_t)_{t\geq 0}$ is given by a Poisson cloud of independent particles performing simple random walks on $\mathbb{Z}$. We work in the low- and high-density regimes. In our settings, the random walker has an a-priori drift, say to the right, which implies that as it moves forward, it meets new particles. Intuitively, this drift is important because the mixing properties of the environment along a tilted line are stronger than on a straight line. In fact in this case, the knowledge we have of the environment is precise enough that we can turn this information into the construction of regeneration times for the trajectory of the random walker. See Section 3.3 for more details.

In [B9] on the other hand, we looked for a statement in the spirit of "if the environment mixes sufficiently fast, the random walker satisfies a LLN". The two goalposts we reached were 1) the minimal mixing that we require is only polynomial (to compare with [B1] where it is exponential), 2) our mixing condition only involves a decay of covariances and in particular requires no uniformity. This result is detailed in Section 3.4.

3.2 Close-to-stationary case

This section describes the contents of [B1, B2], written in collaboration with Luca Avena and Alessandra Faggionato.

3.2.1 Motivating example

In Section 1.2.2, we studied the motion of a tracer through Kinetically Constrained Models, to which the results mentioned in Section 3.1.1 apply. A natural variation of the experience consists in pulling the tracer in a given direction, e.g. through a magnetic or chemical field. This setting has been considered in [65], along with a related problem, in which the tracer takes up the place of a particle in the environment. The authors of [65] show numerically that the resulting speed of the tracer is monotone w.r.t. the force applied in the first case, non-monotone in the second. The latter phenomenon is interpreted as a signature of heterogeneity in the surrounding system: non-monotonicity is a well-documented feature in the presence of traps [47, 11]. It is interesting however that it seems to occur only when the tracer participates in the trap.

Mathematically however, it is hard to say anything at first, since it is not even obvious that the speed of the tracer is a well-defined quantity (i.e. a LLN holds). Let us take a closer look at the setting (recall the notations of Section 3.1). L_{env} is the generator of a KCSM with positive spectral gap (e.g. the East or FA-1f model, see Chapter 1).

$$r(y,\omega) = (1-\omega(0))(1-\omega(y))\tilde{r}_\varepsilon(y),$$

with $\tilde{r}_\varepsilon(y) = 0$ if $y \notin \{\pm e_1, \dots, \pm e_d\}$, $\tilde{r}_\varepsilon(e_1) = \frac{2}{1+e^{-\varepsilon}}$, $\tilde{r}_\varepsilon(-e_1) = e^{-\varepsilon}r_\varepsilon(e_1)$, $\tilde{r}_\varepsilon(y) = 1$ if $y \in \{\pm e_2, \dots, \pm e_d\}$. In words, when it is allowed to move, the tracer has a drift in direction e_1, which becomes stronger as the parameter ε increases.

It is very natural to believe that the tracer should satisfy a law of large numbers for any ε, and that the limit speed should be proportional to ε for small ε (Einstein relation). The results of [B1] allow to show LLN, CLT and an approximate Einstein relation for small enough ε[2], in the regime of densities where the spectral gap is positive (see Section 1.2.1.

[2]Note that the later result [B9] allows to prove LLN for any ε, but only in dimension $d = 1$.

3.2.2 Description of the hypotheses

Let us describe more precisely the main hypotheses we make. As usual, they are twofold: hypotheses on the environment, and on the random walker.

Environment The environment (with generator L_{env}) should be a Feller process that is stationary w.r.t. a translation invariant measure μ. More crucially, it should satisfy a Poincaré inequality: there exists $\gamma > 0$ such that for all $f \in D(L_{\text{env}})$ with $\mu(f) = 0$,

$$\gamma\mu(f^2) \leq -\mu(fL_{\text{env}}f). \tag{3.6}$$

Note that this implies in particular exponential decay of correlations: for bounded test functions f_1, f_2,

$$\mathbb{E}_\mu\left[f_1\left((\eta_t)_{t\in[0,T]}\right) f_2\left((\eta_t)_{t\geq T+T'}\right)\right] \leq Ce^{-cT'}. \tag{3.7}$$

Random walker We assume that the rates describing the random walker jumps $r(y,\eta)$ admit the following decomposition

$$r = r_0 + \hat{r}. \tag{3.8}$$

In addition to technical conditions, we ask the two following crucial properties

1. the process seen from the particle *with jump rates given by* r_0 (a.k.a. *unperturbed process*) is stationary w.r.t. μ (it has generator $L_{\text{env}} + L_0$, where $L_0 f(\eta) = \sum_y r_0(y,\eta)[f(\tau_y\eta) - f(\eta)]$);
2. the operator norm of $\hat{L}f(\eta) := \sum_y \hat{r}(y,\eta)[f(\tau_y\eta) - f(\eta)]$ satisfies $\hat{\varepsilon} := \|\hat{L}\| < \gamma$.

In particular, the generator L of the environment seen from the particle is a perturbation of an operator with positive spectral gap.

The example of Section 3.2.1 fits in this framework, provided ε is small enough w.r.t. the spectral gap of the environment KCM. Another example that works well is the so-called ε–random walk, which jumps to the right (resp. left) with rate $1+\varepsilon$ (resp. $1-\varepsilon$) when it is in a position occupied by a particle, and vice-versa when it is on an empty site. In this case, the rates r_0 are just those of a simple random walk that does not feel the environment.

3.2.3 Main results

Some of the results are rather technical in nature, so let me provide a summary. Mainly, we expand the semi-group associated with L into a series whose terms are bounded by successive powers of $\hat{\varepsilon}$. This allows us to derive consequences on the environment seen from the particle, and the trajectory of the random walker itself. Results below may be subject to technical additional assumptions on the rates.

The expansion involves a series of operators defined recursively. We denote by $S_\varepsilon(\cdot)$ the semi-group associated with the law of the environment seen from the random walker (with generator $L_{\text{env}} + L_0 + \hat{L}$), $S(\cdot)$ the semigroup of the environment seen from the *unperturbed* random walker (with generator $L_{\text{env}} + L_0$). We define recursively

$$S_\varepsilon^{(0)}(t) = S(t); \quad S_\varepsilon^{(n+1)}(t) = \int_0^t S(t-s)\hat{L}S_\varepsilon^{(n)}(s)ds. \tag{3.9}$$

Then it is known that the $S_\varepsilon^{(n)}$ give a series expansion of S_ε in $L^2(\mu)$, and we show that for $f \in L^2\mu$,

$$\|S_\varepsilon(t)f - \sum_{k=0}^{n} S_\varepsilon^{(n)}(t)f\|_{L^2(\mu)} \leq (\varepsilon/\gamma)^{n+1}\frac{2\gamma}{\gamma-\varepsilon}\|f - \mu(f)\|_{L^2(\mu)}. \tag{3.10}$$

Environment seen from the particle It has a unique stationary measure $\hat{\mu}$ and $\hat{\mu}, \mu$ are mutually absolutely continuous. For $f \in L^2(\mu)$, we have the following expansion[3] for $\hat{\mu}(f)$:

$$\hat{\mu}(f) = \mu(f) + \sum_{k=0}^{\infty} \int_0^{\infty} \mu\big(\hat{L} S_\varepsilon^{(n)}(s) f\big) ds. \tag{3.11}$$

Additionally, for finite range jump rates and environments with finite speed of propagation, $\hat{\mu}$ becomes exponentially close to μ far from the origin (see Theorem 3 in [B1] for a proper statement).

Law of large numbers There exists $v \in \mathbb{R}^d$ such that

$$\frac{X_t}{t} \underset{t\to\infty}{\longrightarrow} v \quad \text{a.s.}, \tag{3.12}$$

and v can be written as a series expansion that converges exponentially fast.

Central limit theorem There exists a covariance matrix D such that, under the annealed law starting from $\hat{\mu}$,

$$\frac{X_t - vt}{\sqrt{t}} \tag{3.13}$$

converges weakly to the Brownian motion with covariance matrix D. Moreover, D is non degenerate if e.g. the unperturbed process is reversible and the rates $\hat{r}$ small enough.

3.2.4 A curious antisymmetry feature

In [B2], we noted an intriguing property of the speed v when the environment and jump rates satisfy some additional assumptions. Let me just state it for the ε–random walk described at the end of Section 3.2.2 – more general conditions can be found in [B2, Assumption 3]. Assume that both the environment and the unperturbed process are reversible w.r.t. μ. Then, for ε small enough that we can apply the results of Section 3.2.3, we have the relation

$$v(-\varepsilon) = -v(\varepsilon). \tag{3.14}$$

Numerics suggest that this holds for any[4] ε, but our proof exploits the series expansion of v which is only valid for small enough ε. Frustratingly, we do not have a very good explanation for this phenomenon. The proof consists in showing that terms of even degree in the series expansion vanish thanks to symmetries induced by reversibility. It is unclear how we could use these symmetries in the regime where the series expansion blows up.

Let me just conclude this section by recalling that a reformulation of this antisymmetry is that the asymptotic speed of the ε–random walk is the same in a given reversible environment and in its image by (spatial) mirror symmetry. When the environment is not invariant under such mirror symmetry (which is the case e.g. with the East model), it should be clear that the full trajectories are quite different in the two environments.

3.3 Ballistic walks on random walks

This section describes the contents of [B8, B10], written in collaboration with Marcelo Hilàrio, Renato dos Santos, Vladas Sidoravicius and Augusto Teixeira and in direct continuation of their work [62]. The environment we consider is a Poisson cloud of independent simple random walks. This has two features of interest for us. On the one hand, it is conservative and consequently has

[3] Note that the spaces matter here: the fact that this expansion is valid only for $f \in L^2(\mu)$ (not $L^2(\hat{\mu})$) complicates everything. An illustration of this issue is that we did not manage to get a Poincaré inequality for S_ε in $L^2(\mu_\varepsilon)$: we could only show $\|S_\varepsilon(t) - \hat{\mu}(f)\|_{L^2(\hat{\mu})} \le Ce^{-ct}\|f - \mu(f)\|_\infty$.

[4] Recall that we now know $v(\varepsilon)$ exists for all ε in dimension 1 through the results of [B9].

poor mixing properties, in the sense *e.g.* that covariance between occupation variables at a given site decays only as $1/\sqrt{t}$ in time. On the other hand, it is a very well understood system, in which it is possible to follow very precisely the propagation of information. We work in perturbative cases (either high or low densities of the Poisson cloud) under assumptions such that in the reference case (0 or infinite density), the random walker has a non-zero drift. This will allow us to boost the mixing properties of the environment, by looking to forget information along a tilted space-time line rather than a vertical one.

3.3.1 Settings and assumptions

Let me describe the setting more precisely. In both [B8, B10], at time 0 we place (independently on each site of $\mathbb{Z}^d$) a number of particles chosen with distribution $\mathcal{P}(\rho)$. At later times, each of these particles performs independently a discrete time simple random walk on $\mathbb{Z}^d$. Note that $\otimes_{x\in\mathbb{Z}^d}\mathcal{P}(\rho)$ is reversible for this dynamics. The above definition is in fact imprecise, and we will need to distinguish the *lazy* case from the non-lazy. In the non-lazy version of the environment, at each time step every particle in the environment performs a jump chosen uniformly from $\{\pm e_i, i = 1,\dots,d\}$. In the lazy version, each particle has a positive probability to remain in place during each time step.

Now for the walker. Each jump is sampled from a distribution that depends on the current number of particles on the site it occupies:

$$\mathbb{P}^\eta(X_{n+1} = x + y | X_n = x) = p(\eta(x), y), \tag{3.15}$$

where $\mathbb{P}^\eta$ denotes the quenched law of the random walker $(X_n)_{n\in\mathbb{N}}$, for each $k \in \mathbb{N}$, $y \in \mathbb{Z}^d$, $p(k,y) \geq 0$ and $\sum_{y\in\mathbb{Z}^d} p(k,y) \leq 1$. We always require that the possible jumps are finite range, as well as a drift assumption that depends on the density regime we are after.

High density [62, B8]

A simplified version of the main assumption is that for k large enough,

$$\sum_{y\in\mathbb{Z}^d} y\cdot e_1 \; p(k,y) \geq v_\bullet > 0. \tag{3.16}$$

This means that on sites occupied by a large number of particles, our random walker has a positive drift in the first coordinate. For high densities ρ, we expect that this will be the most frequent situation.

In [62], the setting was further restricted to dimension 1, with nearest-neighbor jumps for the random walker, and $p(k,\cdot) = p(1,\cdot)$ for $k \geq 1$.

Low density [B10]

Here we take a perturbative approach around $\rho = 0$ rather than $\rho = \infty$. In this setting, we are only able to work in one dimension. For simplicity, we consider $p(k,\cdot) = p(1,\cdot)$ for all $k \geq 1$, and nearest-neighbor jumps. We denote $p_\bullet = p(1,1) = 1 - p(1,-1)$ and $p_\circ = p(0,1) = 1 - p(0,-1)$. Our drift assumption is modified accordingly, to

$$v_\circ := 2p_\circ - 1 > 0. \tag{3.17}$$

This means that in the enpty environment, the random walker is a biased random walk with drift to the right. When the density ρ is small, we expect that the random walker will spend most of its time on empty sites, which should give it a drift to the right.

In the non-lazy environment, this effect may however be countered by the strength of the drift $v_\bullet := 2p_\bullet - 1 < 0$, if it is close to -1. Indeed, in the limit case $p_\bullet = 0$, the walker can never move to the right of a particle it has met. As a result, it remains to the left of an infection cluster, which propagates to the left with non-zero speed.

3.3.2 Results

Our results essentially state that in a small region of parameters close to the limit cases, the leading behavior of the walker is similar to the one it has in the limit case. Notice however the change in behavior in the neighborhood of $\rho = 0, p_\bullet = 0$ depending on the (non)-laziness of the environment particles (see Figure 3.1).

Theorem 21. *The random walker satisfies a LLN* (3.12) *and CLT* (3.13) *in the following cases:*

1. *under Assumption* (3.16)*, for ρ high enough, with $v \cdot e_1 > 0$;*
2. *under Assumption* (3.17)*, if $p_\bullet > 0$ (*permeable environment*), for ρ small enough, with $v > 0$;*
3. *in a non-lazy environment, if $p_\circ > 0$, for $p_\bullet$ small enough, with $v < 0$.*

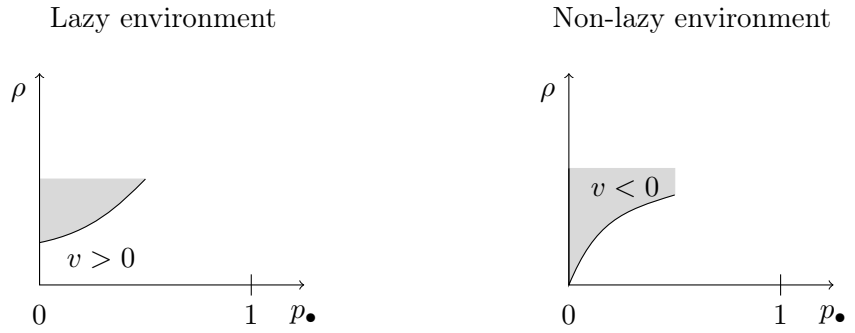

Figure 3.1: Phase diagrams around the $\rho = 0$, $p_\bullet = 0$ parameters, corresponding to lazy and non-lazy particles ($p_\circ > 1/2$).

In each case, a key step is to establish that the random walker (or its opposite when the dominating drift is to the left) satisfies a *ballisticity condition.*

Theorem 22. *Assume one of the following conditions:*

1. *Assumption* (3.16) *and ρ high enough;*
2. *Assumption* (3.17)*, the environment is lazy or $p_\bullet > 0$ (*permeable environment*), and ρ small enough;*
3. *in a non-lazy environment, $p_\bullet$ small enough.*

Then a ballisticity condition is satisfied, i.e.

$$\mathbb{P}\left(\mathrm{sgn}(\tilde{v})X_n \cdot e_1 < n\tilde{v} - L\right) \leq Ce^{-c(\log L)^\gamma}, \tag{3.18}$$

with $\tilde{v} > 0$ in the first two cases, $\tilde{v} < 0$ in the third case; $\gamma \in (1, 3/2)$.

The ballisticity condition (3.18) is reminiscent of Sznitman's (T') condition [97], widely used in the study of random walks in static random environments. In [62, B8, B10], it is established through a renormalization procedure which I detail further in the upcoming Section 3.3.3. Note that in general, the $\tilde{v}$ we obtain need not be the actual speed of the random walker, but merely a lower bound.

This kind of condition turns out to be a crucial first step to establish LLN and CLT for the random walker. Indeed, with no clear martingale property, the best way to get CLT is to show that the increments of the random walk have good mixing properties. In turn, these mixing properties should be inherited from those of the environment. But in our environment of symmetric random walks, vertical (i.e. in time) decorrelations decay rather slowly. The ballisticity condition allows in principle to use the decorrelations of the environment between space-time regions that

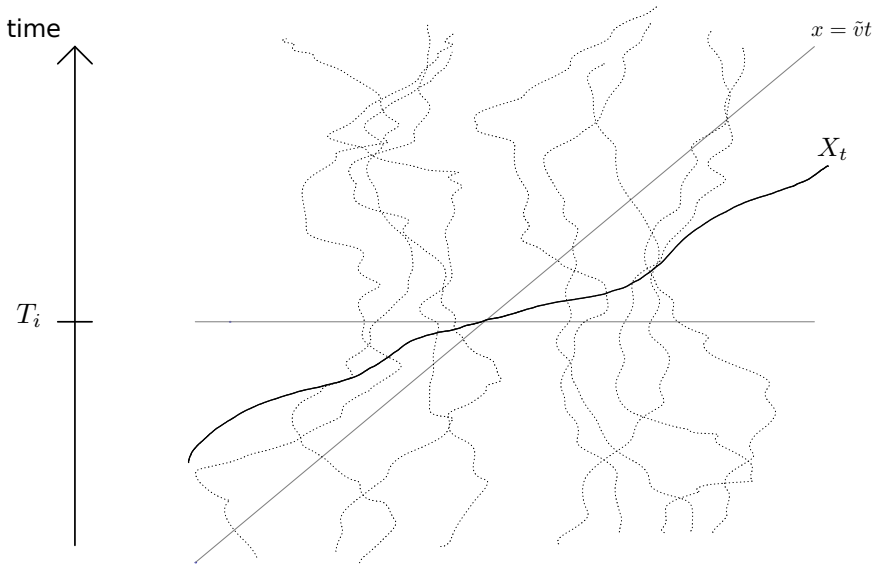

Figure 3.2: The regeneration times T_i are built in such a way that the trajectory of the random walker remains inside the double-sided cone with slope $\tilde{v}$. Moreover, the clouds of environment particles (represented by dashed lines) that intersect the upper and lower parts of the cone respectively never intersect.

are well separated both in time and space. This means that with high probability the particle trajectories intersecting them are disjoint, and dramatically reduces their interdependence.

In practice however, we are not well-equipped to transfer these decorrelation properties directly to the environment seen from the random walk. Instead we rely on the delicate procedure of building regeneration times for the random walker. This requires constructing (random) times T_i such that the particle trajectories crossed by the random walker before and after T_i are *disjoint*, and controlling the tail of the their distribution. This construction is very similar to the work my coauthors carried in [62] so I do not detail it further. The additional ellipticity requirements on $p_\bullet, p_\circ$ (that appear in Theorem 21 but not Theorem 22) come from the needs of this construction.

3.3.3 Renormalization strategies

The general strategy used to establish the ballisticity condition (3.18) is similar in [62, B8, B10], but each setting leads to technical issues that have to be solved. Let me briefly describe the strategy in the setting of [62] (one dimension, high density), and then explain what adjustments we need to make in the other cases.

The basis of the proof is a multiscale argument. Assume we have fixed a sequence of scales $(L_k)_{k\in\mathbb{N}}$, diverging to infinity. In practice, we will also choose this sequence so that L_{k+1}/L_k diverges, so the increase is very fast. The idea is to look at

$$p_k := \mathbb{P}(X_{L_k} > L_k\tilde{v}_k), \tag{3.19}$$

for an appropriate sequence $v_k \underset{k\to\infty}{\longrightarrow} \tilde{v}$, and show a recursive inequality of the kind

$$p_{k+1} \leq N_k p_k^2 + \varepsilon_k, \tag{3.20}$$

where N_k is an entropic factor and ε_k an error term. In spirit, this follows from the fact that, in order to 'run fast" between times 0 and L_{k+1}, the random walker has to "run fast" during at least two time steps of length L_k, separated by a time at least L_k. The meaning of "run fast" is

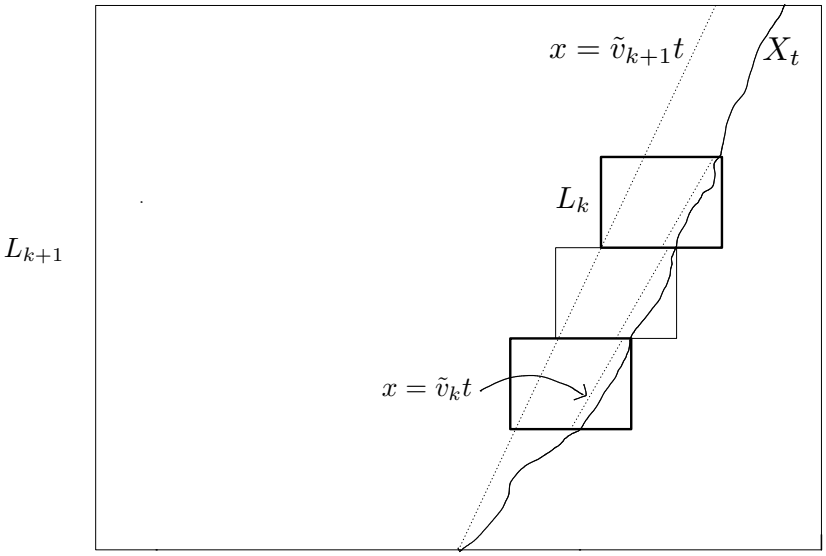

Figure 3.3: Schematic of the renormalization procedure: on the event $X_{L_{k+1}} \geq \tilde{v}_{k+1}L_{k+1}$, we can find three sub-boxes (and therefore two that are separated by a time length at least L_k) of side length L_k in which the trajectory has a slope larger than $\tilde{v}_k$.

given by the choice of the sequence $(\tilde{v}_k)_k$. This part of the argument is deterministic and requires tuning $(\tilde{v}_k)_k$ in terms of $(L_k)_k$. We need that $(L_k)_k$ diverges fast enough so that $\tilde{v}_k \to \tilde{v} > 0$.

As such though, this observation is not quite enough to obtain (3.20); we actually need to use a modified version of p_k, which I do not detail for simplicity. Suffice to say, it does allow to get (3.20), with N_k of order $(L_{k+1}/L_k)^{d+1}$ (counting essentially the number of space-time boxes of side L_k needed to pave a box of side L_{k+1}), and the error ε_k is N_k multiplied by an upper bound on the covariance between two events separated by time L_k.

Things get a bit subtle at this point, because as mentioned above, our environment does not have good mixing properties, and if we just use the $1/\sqrt{L_k}$ decay of correlations, it seems hopeless to use (3.20). The trick is to allow ourselves to consider that the two events involved in the recursion actually look at versions of the system *with slightly different densities*. This trick, called *sprinkling*, effectively replaces the $1/\sqrt{L_k}$ by a stretched exponential in L_k and dramatically reduces the error ε_k. In particular, it becomes true that for any $\gamma < 3/2$, and k large enough

$$p_k \leq e^{-(\log L_k)^\gamma} \quad \Rightarrow \quad p_{k+1} \leq e^{-(\log L_{k+1})^\gamma}. \tag{3.21}$$

That ensures a very quick decay of $(p_k)_k$, provided this bound is satisfied for some k_0 large enough. The cost of using sprinkling is that it becomes essential that the events we consider in the definition of p_k be *increasing* (seen as measurable functions on the environment state space $\mathbb{N}^{\mathbb{Z}^d \times \mathbb{N}}$).

The last ingredient needed is then a *triggering* lemma, stating that we can find such a k_0. In the setting of [62] this is simply done by tuning ρ so that on the scale k_0 at which (3.21) becomes true, with overwhelming probability a box of side L_{k_0} is completely filled.

The three sets of Assumptions in Theorem 22 require different adjustments. Let us look at each of them.

1. In the multidimensional, non-nearest-neighbor, high-density case, the issue is that the event defining p_k is no longer monotone. Indeed, monotonicity in the one-dimensional nearest-neighbor case comes from the fact that two trajectories have to meet in order to cross, so that if $X_0 \leq X_0'$ and X' sees more particles than X, the trajectories X, X' can be built in

such a way that $X \leq X'$. This is intrinsically no longer true in higher dimension, or for random walkers taking non-nearest-neighbor jumps.

What we do in [B8] is give a general construction of events whose probability satisfy (3.20) if they happen to be monotone. Then we show that for the RWRE, we can choose the complement of these events as requiring that *any eligible trajectory* for the random walker has to spend most of its time on sites occupied by many particles. This condition is increasing, so (3.20) is satisfied. The triggering lemma easily follows by taking the density high enough, so we have a very fast decay of the probability that *there exists* a bad trajectory, that does not spend enough time on crowded sites. In particular, with overwhelming probability the actual trajectory has a drift in the direction e_1 most of the time, and this implies ballisticity.

2. In the one-dimensional, nearest-neighbor, lazy, low-density case, the difficulty lies in the triggering lemma. In fact, in order to use sprinkling, we need some room to adjust the density. This gives a lower bound on ρ_k (the density seen at scale L_k) of order $L_k^{-1/16}$, below which we are not able to establish (3.20). If we wanted to do the triggering as in the high density case, we would need to take the density small enough that a box of side L_k is empty, meaning $\rho_k \ll L_k^{-1}$ for k large, which is incompatible with our first condition. A big part of [B10] is therefore dedicated to showing that for any L, at density $L^{-1/16}$,

$$\mathbb{P}(X_L < L^{15/16}) \leq Ce^{-c(\log L)^2}. \tag{3.22}$$

This triggering lemma relies heavily on the monotonicity mentioned above, which is only valid in one dimension and for nearest-neighbor random walks. It uses the fact that with positive probability, every time the random walker meets a particle, it may jump as if it had not seen it (this is what we call *permeability*).

3. In the non-lazy one-dimensional environment with small $p_\bullet$, the argument is perturbative around the $p_\bullet = 0$ case, so we first need to understand how the latter behaves. The main property we use is: if at some time the random walker is left of a particle[5], this remains true at any later time. So consider the set of particles initially sitting on $\mathbb{Z}_+$. The infection cluster we mentioned above grows by incorporating every particle it meets along the way. The technology developed in [B8] allows us to show that the infection front (the leftmost particle) will spend a positive portion of its time in a situation where it has a drift to the left. In turn, this allows us to establish the ballisticity property (3.18) with negative speed for this front (and therefore for the random walker with $p_\bullet = 0$). This improves slightly the previously available bounds on the speed of propagation of an infection cluster.

   Turning back to the problem for small $p_\bullet$, the recursive inequality is not difficult to establish. The triggering lemma is proved by noticing that for fixed L, the left hand-side of (3.18) is a continuous function of $p_\bullet$.

3.4 Fast-mixing environments

This section describes the contents of [B9], written in collaboration with Marcelo Hilário and Augusto Teixeira. We work with a dynamical environment with polynomial mixing (with high enough exponent), in dimension 1. The goal is to obtain a law of large numbers for a random walker in this type of environment, when the notion of polynomial mixing is taken in the weak sense of correlations. Applications include Markovian environments with positive spectral gap (e.g. KCMs) or the (backward) contact process[6]. Though we are restricted to dimension 1 and nearest-neighbor jumps for the random walker, our setting covers situations where (contrary to the other sections) the random walker does not need to evolve "on top of" the environment (see e.g. Section 3.4.2).

[5] In fact, everything in the rest of this paragraph should be restricted to the sub-lattice of sites which have the same parity as the position of the random walker.

[6] A nice feature of our assumptions is that they are not sensitive to time-reversal.

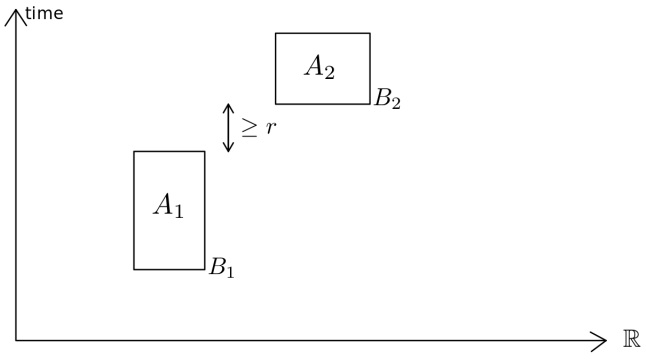

Figure 3.4: The mixing hypothesis on the environment implies that the covariance between events A_1, A_2 (measurable w.r.t. the environment restricted to B_1, B_2 respectively) decays as a power of r, when B_1, B_2 are vertically separated as shown in the picture.

3.4.1 Hypotheses and main result

We consider a dynamical environment in dimension $1+1$ $(\eta_t(x))_{x\in\mathbb{Z},t\in\mathbb{R}_+}$, with values in a countable state space S (mostly, $S=\{0,1\}$ or $\mathbb{N}$). We assume that $(\eta_t)_t$ is a càdlàg process that is also stationary and translation invariant. Contrary to the other works presented in this manuscript, we do not need $(\eta_t)_t$ to be a Markov process. Let P be the law of $(\eta_t)_t$.

The main assumption we make on the environment is a decorrelation property.

Definition 7 (Decoupling inequality)**.** *P satisfies a decoupling inequality with exponent α if the following holds: for all $r\geq 1$, all boxes B_1, B_2 with side lengths less than $5r$ and time-distance larger than r, for any functions f_1, f_2 measurable w.r.t. the environment restricted to B_1, B_2 respectively,*

$$\mathrm{Cov}(f_1,f_2)\leq Cr^{-\alpha}. \tag{3.23}$$

The constant 5 in the definition is a bit arbitrary; it is chosen so that it covers without difficulty the examples we have in mind. This assumption is rather weak, though we will need to take α larger enough in order to use it.

The strongest assumptions we need concern the random walker. We define it by setting a collection of possible jumps times $\mathcal{T}=(T_i^x)_{i\in\mathbb{N},x\in\mathbb{Z}}$. The main example we have in mind is choosing the $(T_i^x)_i$ as independent Poisson point processes, also independent from the environment, in which case we recover the standard setting described in the other sections. But we may also allow for instance the T_i^x to be times at which the environment itself updates.

The jumps of the random walker are determined by the environment, the collection $\mathcal{T}$, a collection $\mathcal{U}=(U_i^x)_{i\in\mathbb{N},x\in\mathbb{Z}}$ of iid $\mathcal{U}[0,1]$ variables independent from everything else, through a function

$$g: S^{\{-\ell,\dots,\ell\}}\times[0,1]\to\{-1,0,1\}. \tag{3.24}$$

In words, each possible jump time T_i^x carries extra randomness encoded in U_i^x. When the random walker attempts to jump, g looks at the environment in a final interval around its current position and the extra randomness and outputs a jump.

Fix a starting point $(x_0,t_0)\in\mathbb{Z}\times\mathbb{R}_+$. The random walker trajectory $(X_t^{(x_0,t_0)})_{t\geq t_0}$ started from (x_0,t_0) is constructed as follows:

- $X_{t_0}^{(x_0,t_0)}=x_0$.
- For $T\geq t_0$, if $X_T^{(x_0,t_0)}=X$, let $I=\min\{i\,:\,T_i^X\geq T\}$. For $t\in[T,T_I^X)$, set $X_t^{(x_0,t_0)}=X$, and
$$X_{T_I^X}^{(x_0,t_0)}=g\left(\eta_{T'}(X-\ell),\dots,\eta_{T'}(X+\ell);U_I^X\right). \tag{3.25}$$

When $(x_0, t_0) = (0, 0)$, the superscript will be omitted from the notation.

When the T_i^x are independent Poisson point processes independent from the environment, this framework allows to recover the standard setting described in the previous sections.

Let us now list the assumptions we require on $\mathcal{T}$. The main one is that the environment enriched with $\mathcal{T}$ (and $\mathcal{U}$) should still be stationary, translation invariant, and satisfy a decoupling inequality. We also need that $\mathcal{T}$ has finite speed of propagation properties.

Definition 8. *1. An* allowed *path is a possible trajectory for the random walker, i.e. with nearest-neighbor jumps such that jumps from x occur only at times in $\{T_i^x, i \in \mathbb{N}\}$.*

2. The speed of an allowed path $(\gamma_t)_{t\in[0,T]}$ is $(\gamma_T - \gamma_0)/T$. Define

$$B_T(v) := \{\exists \text{ an allowed path of length } T \text{ starting from } [0,T) \times \{0\}] \text{ with speed } \geq v\} \tag{3.26}$$

$$\tilde{B}_T(v) := \{\exists \text{ an allowed path of length } T \text{ starting from } [0,T) \times \{0\}] \text{ with speed } \leq v\}. \tag{3.27}$$

3. T has the finite speed of propagation property (FSP) *if*[7]

(a) For all $v > 1$, $\liminf_{T\to\infty} \mathbb{P}(B_T(v)) = 0$ and $\liminf_{T\to\infty} \mathbb{P}(\tilde{B}_T(-v)) = 0$.
(b) $\mathbb{P}(B_T(2) \cup \tilde{B}_T(-2)) \leq Ce^{-cT}$.

4. For all $x \in \mathbb{Z}$, almost surely

$$\{T_i^x, i \in \mathbb{N}\} \cap \{T_i^x, i \in \mathbb{N}\} = \emptyset. \tag{3.28}$$

It should be clear that the constants 1 and 2 appearing in the FSP property do not play a special role; they are chosen so that, when $\mathcal{T}$ is a collection of iid PPP(1), FSP is satisfied. (3.28) may look anecdotical and is certainly satisfied for a wide range of natural choices for $\mathcal{T}$; it is however crucial to our strategy (see Section 3.4.3).

We are now ready to state the main result of [B9].

Theorem 23. *Assume that $\mathbb{P}$ satisfies the decoupling inequality with exponent $\alpha > 8$, and $\mathcal{T}$ has the FSP property and* (3.28)*. Then there exists $v \in [-1, 1]$ such that*

$$\frac{X_t}{t} \underset{t\to\infty}{\longrightarrow} v \quad \mathbb{P}\text{–a.s.} \tag{3.29}$$

Moreover, for every $\varepsilon > 0$, for t large enough,

$$\mathbb{P}\left(\left|\frac{X_t}{t} - v\right| \geq \varepsilon\right) \leq t^{-\alpha/4}. \tag{3.30}$$

It is a priori difficult to extract information on the value of v from our proofs (see Section 3.4.3 to understand why). In particular, in general we cannot give its sign. However, in the special case where g takes its values in $\{0, 1\}$ (so that clearly $v \geq 0$), it is possible to give a criterion implying the positivity of v.

Theorem 24. *Assume that $\mathbb{P}$ satisfies the decoupling inequality with exponent $\alpha > 8.5$, $\mathcal{T}$ has the FSP property,* (3.28) *and $\mathbb{P}(X_1 \geq 1) > 0$. Then $v > 0$.*

It is natural to ask whether the threshold on the exponents present in Theorems 23 and 24 are optimal. The answer is no: they are chosen to satisfy a number of conditions appearing in the several multiscale arguments involved in the proof, and the additional aesthetic requirement of being easy to remember. It is true however that our proofs cannot generalize to decoupling exponents much below 8 (see [63] for a result with the same flavor in a well-known environment with

[7] The second condition should in fact be a bit different for technical reasons.

Figure 3.5: A simulation of the ergodic environment defined in [B9] in which a random walker may not satisfy a LLN. Black and white rectangles are deposited like a superposition of Poisson processes at increasing scales. Points in the plane untouched by this deposition process are left gray. The random walker has a drift to the right in black regions, to the left in white regions. Inside a big black/white region, it will have a positive/negative speed; since both happen with positive probability, this rules out the possibility of a LLN.

smaller α). It is still unclear just how mixing an environment should be for the random walker to satisfy a LLN; would the conclusion in Theorem 23 still hold if we just knew $\mathrm{Cov}(f_1, f_2) \underset{r\to\infty}{\longrightarrow} 0$ for f_1, f_2 as in Definition 7?

In [B9], we provide an example of a random walker that does not satisfy a LLN. The environment is chosen stationary, translation invariant, ergodic with respect to space-time shifts, and in fact satisfies $\mathrm{Cov}(f_1, f_2) \underset{r\to\infty}{\longrightarrow} 0$ for f_1, f_2 with support in boxes vertically separated by r but of side length less than r^β for some $\beta \in (0, 1)$. I do not attempt to describe it here, just see Figure 3.5 for an inspirational picture.

3.4.2 Applications

Let us give some settings in which Theorem 23 applies.

The (backward) supercritical contact process The contact process is a standard interacting particle systems decribing the spread of an infection (which we already met in Section 1.3.2.2). Each infected site recovers at rate 1 and infects its neighbors at rate λ. In dimension 1, the process has generator

$$\mathcal{L}_{\mathrm{contact}} f(\eta) = \sum_{x\in\mathbb{Z}} \left[\eta(x) + (1-\eta(x))(\eta(x-1)+\eta(x+1))\right]\left[f(\eta^x) - f(\eta)\right]. \tag{3.31}$$

It is well known that this process exhibits a phase transition: there exists λ_c such that, for $\lambda \le \lambda_c$ the process started from a unique infection converges to the empty state almost surely, while for $\lambda > \lambda_c$, with positive probability the infection survives. In the latter case, the process started from the all-infected state converges to the so-called upper-invariant measure ν_λ, which by construction is stationary and translation-invariant.

The backward contact process is merely the time-reversal of the contact process. It appears naturally as a model for genealogies of populations.

Due to monotonicity properties (the process started with more initial infections has more infections at time t), it is not hard to show that the supercritical contact process satisfies our decoupling inequality. In particular, Theorem 23 applies in this context.

Lemma 2. *For $\lambda > \lambda_c$, the contact process started from ν_λ satisfies a decoupling inequality with any exponent $\alpha > 0$.*

While there are several strong results for RWRE on a supercritical contact process [14, 38, 82], the backward contact process is generally much harder to handle [16, 17]. For our hypotheses however, time-reversal is transparent, and Theorem 23 applies immediately to the backward process as well.

Positive spectral gap An important class of environments also covered by our results is the family of Markov processes with positive spectral gap[8]. The spectral gap is the inverse of the relaxation time appearing in Definition 3.

Lemma 3. *If η is a càdlàg strongly continuous Markov process stationary w.r.t. ν, with positive spectral gap in $L^2(\nu)$, it satisfies a decoupling inequality with any exponent $\alpha > 0$.*

It is not hard to deduce this property from (1.8). This allows to apply our results in particular to the following processes:

- independent spin flip dynamics on $\mathbb{Z}$;
- Glauber dynamics for the Ising model on $\mathbb{Z}$;
- one-dimensional KCMs in the supercritical regime.

The latter case provides an example where it is useful to consider more general families of update times $\mathcal{T}$ than merely independent PPP. In the beginning of Section 1.3.1.2, I mentioned a notion very useful in the study of the East model: the *distinguished zero*. Let us define it within the framework of this section. The allowed jump times $\mathcal{T}$ are given by the Poisson clocks used in the graphical construction of the East process described below Definition 1. The function g only looks at the configuration to the right of the current position: $g(\eta_{|\{-1,0,1\}}; u) = 1-\eta(1)$, meaning the distinguished zero jumps to the right at the time of an update at its current position iff the site on its right is empty. By construction, if $\eta_0(0) = 0$, $\eta_t(X_t) = 0$ and moreover $\eta_{|\{0,\dots,X_t-1\}} \sim \mu_p$ regardless of the initial distribution. This property makes the distinguished zero a very useful tool to study the non-equilibrium dynamics of the East model.

In order to apply our results, we need the environment to be stationary, which seems to contradict the purpuse described above. However, the East model has the additional property that it is *oriented*, implying for instance that the trajectory of the distinguished zero does not depend on the initial distribution on $\mathbb{Z}_- = \{\dots, -2, -1, 0\}$. A consequence of our results is therefore the following.

Proposition 8. *Consider the East process started from any distribution such that $\eta_0(0) = 0$ and $(\eta_0)_{|\mathbb{N}^*} \sim \mu_p$. The distinguished zero trajectory satisfies a LLN with positive speed.*

A similar argument along with a simple modification of our proofs also allows to recover the LLN for the front in the East process (recall Section 1.3.2) in the special case where $(\eta_0)_{|\mathbb{N}^*} \sim \mu_p$ (or a distribution absolutely continuous w.r.t. μ_p).

Independent renewal chains This is an example in which the covariance decay is sub-exponential (contrary to the examples above).

Let us fix a sequence $(a_n)_{n\in\mathbb{N}}$ of positive real numbers satisfying

$$\forall n \; 0 < a_n \leq Ce^{-\log^2 n}, \text{ and } \sum_n a_n = 1. \tag{3.32}$$

The environment we consider is given by independent copies of the stationary renewal Markov chain with transition probabilities given by the a_n, i.e. the process with generator

$$Lf(n) = \begin{cases} f(n-1) - f(n) & \text{if } n > 0, \\ \sum_{k>0} a_k \left(f(k) - f(0)\right) & \text{if } n = 0. \end{cases} \tag{3.33}$$

[8] As far as I know, it is still an open question whether the contact process has positive spectral gap.

Lemma 4 ([62]). *The environment defined above satisfies a decoupling inequality with any exponent $\alpha > 0$.*

3.4.3 Strategy

Let me now describe the strategy of proof behind Theorem 23.

The first step is to identify upper and lower speeds v_+, v_-. We want v_+ to be such that the random walker has very small probability to exceed speed v_+, but at the same time has a non-vanishing probability of attaining speeds close to v_+ (and similarly for v_-). In the final argument we want to use these two properties to show that in fact v_+ and v_- coincide.

The idea is that v_+ and v_- are chosen so that the random walker has a good chance of attaining speeds close to both of these values over sufficiently long time scales. On the other hand, the probability that it runs faster than $v_+ + \varepsilon$ (and also slower than $v_- - \varepsilon$) vanishes fast. Assume by contradiction that $v_+ > v_-$. By the fact that the random walker cannot run faster than v_+, the moments when its speed stays close to v_- should delay it sufficiently to prevent it from attaining a speed close to v_+ over long time scales. This would give rise to a contradiction implying that $v_- = v_+$.

The proof is not so straightforward however, since our basic properties on v_+, v_- do not rule out that, by time H, the random walker moves faster than $v_+ + \varepsilon_H$ (or slower than $v_- - \varepsilon_H$) for some well-chosen ε_H. The main part of the proof will use multiscale arguments to make our naive guess work.

Definition of v_+, v_- Let us define the events[9] that the random walker has speed at least (resp. at most) v on scale H.

$$A_H(v) := \{\exists x \in [0, H) \text{ s.t. } X_H^{(x,0)} - x \geq vH\}, \tag{3.34}$$

$$\tilde{A}_H := \{\exists x \in [0, H) \text{ s.t. } X_H^{(x,0)} - x \leq vH\}. \tag{3.35}$$

The corresponding probabilities are denoted $p_H(v) = \mathbb{P}(A_H(v))$, $\tilde{p}_H(v) = \mathbb{P}(\tilde{A}_H(v))$. We can now define the upper and lower speeds v_+, v_-.

$$v_+ := \inf\{v \in \mathbb{R} : \liminf_{H\to\infty} p_H(v) = 0\}, \tag{3.36}$$

$$v_- := \sup\{v \in \mathbb{R} : \liminf_{H\to\infty} \tilde{p}_H(v) = 0\}. \tag{3.37}$$

The finite speed propagation property clearly implies $v_+ \leq 1$, $v_- \geq -1$. This definition gives us for free the second property we wanted our upper/lower speeds to have (with non-vanishing probability, the random walker attains speeds close to those values). The first one remains to be shown: the definition only says that the upper/lower spees cannot be exceeded along a subsequence. The desired property is the first substantial result in [B9].

Proposition 9. *Assume that $\mathbb{P}$ satisfies the decoupling inequality with exponent $\alpha > 5$, and $\mathcal{T}$ has the FSP property and* (3.28)*. Then for any $\varepsilon > 0$, for all $H > 0$*

$$p_H(v_+ + \varepsilon) \leq C(\varepsilon)H^{-\alpha/4} \quad \textit{and} \quad \tilde{p}_H(v_- - \varepsilon) \leq C(\varepsilon)H^{-\alpha/4}. \tag{3.38}$$

The argument is close to the one described in Section 3.3.3, with a little twist. We fix a sequence $(L_k)_k$ such that $L_{k+1} \asymp L_k^{5/4}$ and a zooming parameter $h \in \mathbb{R}_+^*$ to be chosen later. We consider here the sequence $p_{k,h} = p_{hL_k}(v_k)$, where v_k is chosen so that

$$p_{k+1,h} \leq C(L_{k+1}/L_k)^4 \left(p_{k,h}^2 + c(hL_k)^{-\alpha}\right), \tag{3.39}$$

where the second term in parenthesis comes from the decoupling inequality. This implies that *uniformly in $h \geq 1$*, if $\alpha > 5$, $p_{k,h} \leq L_k^{-\alpha/2}$ implies $p_{k+1,h} \leq L_{k+1}^{-\alpha/2}$. The triggering step consists

[9] In fact, we use a slighlty more involved definition to handle discretization effects.

in finding h large enough so that $p_{k_0,h} \leq L_{k_0}^{-\alpha/2}$, using the definition of v_+. This gives us our first intermediate result: there exists $h \geq 1$ such that for all $k \geq k_0$,

$$p_{k,h} \leq L_k^{-\alpha/2}. \tag{3.40}$$

To conclude the proof of Proposition 9, it remains to use an interpolation argument that I do not detail here.

Setting traps We want to argue that $v_+ = v_-$ by contradiction. Assuming $\delta := (v_+ - v_-)/4 > 0$, we attempt to prove that the random walk spends too much time having speeds in $[v_-, v_- + \delta]$ to be able to reach speeds in $[v_+ - \delta, v_+]$. To this end, we identify traps in which the random walker may fall, resulting in a delay.

The construction of these traps relies heavily on our hypothesis that the random walker only performs nearest-neighbor jumps, and that we are in dimension 1. Indeed, with our construction, that and (3.28) imply

$$x \leq x' \Longrightarrow X_s^{(x,t)} \leq X_s^{(x',t)} \text{ for all } s \geq 0. \tag{3.41}$$

This would no longer be true with longer range jumps, or in higher dimensions (not even coordinate-wise). This implies that if a trajectory started from (x,t) has speed close to v_-, it will slow down trajectories going through points slightly to the left of (x,t).

Definition 9. *The point (x,t) is H–trapped if there exists $y \in [x + \delta H, x + 2\delta H]$ such that*

$$X_H^{(y,t)} - y \leq (v_- + \delta)H. \tag{3.42}$$

(3.41) implies that, if (x,t) is H–trapped, then for all $y \in [x, x + \delta H]$,

$$X_H^{(y,t)} - y \leq (v_+ - \delta)H, \tag{3.43}$$

so that a trap at (x,t) causes a delay in any trajectory crossing $[x, x + \delta H]$ at time t.

A simple consequence of our definition of v_- (along with invariance under space-time translations) is that traps occur with positive probability, uniformly over large scales.

Lemma 5. *There exists $c > 0$ such that, for H large enough, for all $(x,t) \in \mathbb{Z} \times \mathbb{R}_+$,*

$$\mathbb{P}((x,t) \text{ is } H\text{–trapped}) \geq c. \tag{3.44}$$

One could think of the set of H–trapped points as a percolation-type environment of obstacles. Every time the random walker passes next to such an obstacle it will be delayed for a time H. Furthermore, if the probability that a point is trapped could be made very high, then every allowed path would have to approach these obstacles at time scales smaller than H and we would be done. However, we only know that this probability is positive and it could, in principle, be very small. Therefore, the random walker could always avoid these trapped points, or it could spend only a negligible fraction of the time next to them.

Our next step is to boost the probability of meeting a trap by extending the vision range of the random walker.

Definition 10. *The point (x,t) is (H,r)–threatened if*[10] *there exists $j \in \{0, \dots, r-1\}$ such that $(x + jHv_+, t + jH)$ is trapped.*

Threatened points have both properties we hoped for when we set traps: they induce a delay in trajectories that meet them, and they occur with high probability, making their encounter inevitable.

Lemma 6. *1. If (x,t) is (H,r)–threatened, either*

[10] Technically, we use a slightly different definition, involving a sublattice that I do not describe here.

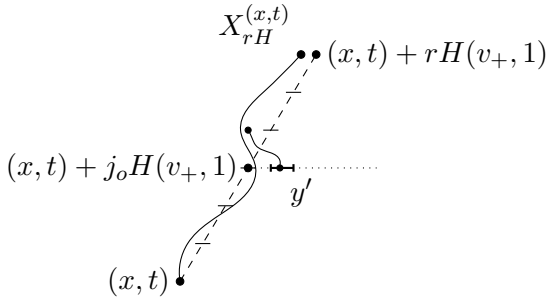

Figure 3.6: The point (x,t) is (H,r)-threatened, since $(x,t)+j_oH(v_+,1)$ is H-trapped (by the trajectory started at y'). This results in a delay for $X_{rH}^{(x,t)}$.

- *$X^{(x,t)}$ has a speed larger than $v_+ + \delta/(2r)$ during one of the $[jH,(j+1)H]$ time intervals;*
- *or the trajectory $X^{(x,t)}$ is delayed:*

$$X^{(x,t)} - x \leq \big(v_+ - \delta/(2r)\big)rH. \tag{3.45}$$

2. *If $\mathbb{P}$ satisfies a decoupling inequality with exponent $\alpha > 1$, $\mathcal{T}$ has the FSP property and* (3.28)*, there exists $c > 0$ such that for H large enough, for any $r \geq 1$,*

$$\mathbb{P}((x,t) \text{ is } (H,r)\text{–threatened}) \geq 1 - cr^{-\alpha}. \tag{3.46}$$

By Proposition 9, having speed larger than $v_+ + \delta/(2r)$ is very unlikely, so that with high probability it is the delay scenario that occurs. The first part of Lemma 6 is a deterministic proof that follows simply from our definitions. The second part on the other hand is proved through a renormalization scheme in two steps. We first show that the probability to not be (H,r)–threatened decays to 0 (at this stage, the rate of decay is not important). For this part, the triggering is merely done by choosing the decay appropriately. We then use this intermediate result as a triggering argument in a renormalization scheme providing the right rate of decay.

Note that at this point we still only need $\alpha > 5$. The next step (perhaps the most involved) consists in showing that with high probability, every allowed path spends a positive proportion of its time (in fact, larger than $1/2$) next to threatened points. It is at this stage that we need to restrict our conditions to $\alpha \geq 8$. Let me not detail this step which gets rather technical.

In order to conclude the proof, it remains to argue that the accumulation of delays induced by the density of threatened points implies that there exists $\eta > 0$ and a sequence $H_k \to \infty$ such that

$$p_{H_k}(v_+ - \eta) \underset{k\to\infty}{\longrightarrow} 0. \tag{3.47}$$

This contradicts the definition of v_+ and concludes the proof.

3.4.4 Open problems

As I hinted above, the conditions $\alpha > 8$ or 8.5 in Theorems 23 and 24 are not meaningful thresholds for the behavior of the random walker. The same proofs would in fact work a bit below these values, and we could get a deviation estimate (see (3.30)) with another exponent than $\alpha/4$. However, additional arguments would be required to reach lower values of α, e.g. $\alpha = 2$, which is the relevant value for many conservative particle systems. For instance, in [63], the authors perform a study similar to that in [B9], but with the environment given by a Simple Symmetric Exclusion Process. The extra arguments involve the use of lateral decoupling (i.e., decoupling at large spatial distances), monotonicity and sprinkling to get the law of large numbers for all values of the density (except maybe two). It is unclear whether there exist

environments satisfying the decoupling inequality with some $\alpha > 0$ and random walks on them that do not satisfy a law of large numbers.

A major restriction in [B9] is that we can only deal with one-dimensional nearest-neighbor random walks. These assumptions (even the second one alone) do not seem easy to lift (but keep an eye on Julien Allasia's thesis).

Finally, a natural follow-up question, even in the simple setting of [B9], is whether a CLT holds as well as a LLN. As we have seen in the previous sections, when the environment is sufficiently well-known, it is sometimes possible to design regeneration arguments. This is what the authors do in [63] to get a CLT when the speed in the law of large numbers is non-zero. It would be nice however to find a way to use our multiscale arguments to prove something at the level of fluctuations.

Appendix A

Invariant measures for the one-dimensional FA-1f model

This note is a reformulation (with extra detail) of [84], transposed to the case of the FA-1f model in dimension 1. In the original version, $c_x(\eta) = 1 - \eta(x-1) + 1 - \eta(x+1)$, instead of $c_x(\eta) = 1 - \eta(x-1)\eta(x+1)$ for FA-1f, but the same proof is valid. Since this paper does not seem well-known in the KCM community, I revisit it here. The notations are those of Chapter 1. Let us abbreviate

$$r_x(\eta) := p(1 - \eta(x) + q\eta(x). \tag{A.1}$$

Theorem A1. *[84] For the FA-1f model in dimension 1 with parameter p, every invariant measure is a linear combination of μ_p and $\delta_{\mathbf{1}}$.*

Proof. Let $\Lambda \Subset \mathbb{Z}$, ν a probability measure on Ω, ν_t the distribution of the FA-1f process started from distribution ν. For $\sigma \in \Omega_\Lambda$, $\nu_t(\sigma)$ denotes $\nu_t(\{\eta \in \Omega : \ \eta_{|\Lambda} = \sigma\})$ and $\mathbf{1}_\sigma(\eta) = \mathbf{1}_{\{\eta_{|\Lambda}=\sigma\}}$. The idea is to consider the finite volume entropy

$$H_\Lambda(\nu_t) := \sum_{\sigma\in\Omega_\Lambda} \nu_t(\sigma) \log\left[\frac{\nu_t(\sigma)}{\mu_p(\sigma)}\right]. \tag{A.2}$$

This is only possible if $\nu_t(\sigma) > 0$ for any $\sigma \in \Omega_\Lambda$. Therefore, we need a first lemma guaranteeing that this holds if ν is invariant. Let $\Lambda_n := [-n, n] \cap \mathbb{Z}$ and $\delta_{\mathbf{1},n}$ the configuration identically equal to 1 on Λ_n.

Lemma A1. *If ν is invariant, for any $n \in \mathbb{N}^*$, if $\nu(\delta_{\mathbf{1},n}) < 1$, for all $\sigma \in \Omega_{\Lambda_n}$, $\nu(\sigma) > 0$.*

Note that the second condition is necessary, since $\delta_{\mathbf{1}}$ is invariant.

Proof of Lemma A1. This follows by standard arguments from the irreducibility of the dynamics on $\Omega_{\Lambda_n} \setminus \{\delta_{\mathbf{1},n}\}$. Indeed, by invariance of ν,

$$\sum_{x\in\Lambda_n} [\nu(c_x r_x \mathbf{1}_{\sigma^x}) - \nu(c_x r_x \mathbf{1}_\sigma)] = 0. \tag{A.3}$$

Assume that there exists $\sigma \in \Omega_{\Lambda_n} \setminus \{\delta_{\mathbf{1},n}\}$ such that $\nu(\sigma) = 0$. Then the terms coming with a minus sign are zero for this choice of σ. Therefore, for all $x \in \Lambda_n$, $0 = \nu(c_x r_x \mathbf{1}_{\sigma^x}) \geq c^{\mathbf{1}}_{\Lambda_n,x}(\sigma)(p \wedge q)\nu(\sigma^x)$. Therefore, $\nu(\sigma^x) = 0$ for all $x \in \Lambda_n$ neighboring a zero in σ. In the same way, we reach the conclusion that for all $\sigma' \in \Omega_{\Lambda_n} \setminus \{\delta_{\mathbf{1},n}\}$ accessible by spin flips allowed by the constraints, $\nu(\sigma') = 0$. But every configuration with at least a zero is accessible, and therefore this contradicts $\nu(\delta_{\mathbf{1},n}) < 1$.

It remains to eliminate the case $\nu(\delta_{\mathbf{1},n}) = 0$. Let us apply the previous result to Λ_{n+1}. If $\nu(\delta_{\mathbf{1},n+1}) < 1$ (which certainly holds if $\nu(\delta_{\mathbf{1},n}) = 0$), the configuration with one zero at $n+1$ and ones on $\Lambda_n \cup \{-n\}$ has positive probability. This forbids $\nu(\delta_{\mathbf{1},n}) = 0$. □

Let us go back to the proof of Theorem A1. If ν is invariant and $H_\Lambda(\nu)$ well-defined, clearly $\frac{d}{dt}H_\Lambda(\nu_t) \equiv 0$. On the other hand,

$$\frac{d}{dt}\left[\sum_{\sigma\in\Omega_\Lambda}\nu_t(\sigma)\log\nu_t(\sigma)\right]_{|t=0} = \sum_{\sigma\in\Omega_\Lambda}(1+\log\nu(\sigma))\nu(L\mathbf{1}_\sigma) = \sum_{\sigma\in\Omega_\Lambda}\log\left(\nu(\sigma)\right)\nu(L\mathbf{1}_\sigma), \tag{A.4}$$

$$\frac{d}{dt}\left[\sum_{\sigma\in\Omega_\Lambda}\nu_t(\sigma)\log\mu_p(\sigma)\right]_{|t=0} = \sum_{\sigma\in\Omega_\Lambda}\log(\mu_p(\sigma))\nu(L\mathbf{1}_\sigma), \tag{A.5}$$

where the second equality in the first line follows from $\sum_{\sigma\in\Omega_\Lambda}\mathbf{1}_\sigma = 1$. Let us now choose $\Lambda = \Lambda_n$ and define, for $x \in \Lambda_n$, $\sigma \in \Omega_{\Lambda_n}$,

$$\Gamma_n(x,\sigma) = \int_\Omega d\nu(\eta)c(x,\eta)r_x(\eta)\mathbf{1}_\sigma(\eta). \tag{A.6}$$

Notice that

$$\Gamma_n(x,\sigma) = \sum_{\zeta\in\Omega_{n+1},\zeta_{|\Lambda_n}=\sigma}\Gamma_{n+1}(x,\zeta). \tag{A.7}$$

From (A.4), (A.5), we conclude that for any $n \in \mathbb{N}^*$,

$$0 = \frac{d}{dt}H_{\Lambda_n}(\nu_t)_{|t=0} = \sum_{\sigma\in\Omega_{\Lambda_n}}\log\left(\frac{\nu(\sigma)}{\mu_p(\sigma)}\right)\sum_{x\in\Lambda_n}\left[\Gamma_n(x,\sigma^x)-\Gamma_n(x,\sigma)\right]. \tag{A.8}$$

The key lemma that we will show is the following.

Lemma A2. *If ν is invariant, for all $n \in \mathbb{N}^*$, $\sigma \in \Omega_{\Lambda_n}$, $x \in \Lambda_n$,*

$$\Gamma_n(x,\sigma) = \Gamma_n(x,\sigma^x). \tag{A.9}$$

Proof of Lemma A2. If $\nu(\delta_{\mathbf{1},n+1}) = 1$, the result is clear. Otherwise, we can apply Lemma A1 and (A.8) holds. Notice that the right-hand side of (A.8) can be symmetrized into

$$\frac{1}{2}\sum_{x\in\Lambda_n}\sum_{\sigma\in\Omega_{\Lambda_n}}\left[\log\left(\frac{\mu_p(\sigma^x)}{\mu_p(\sigma)}\right)-\log\left(\frac{\nu(\sigma^x)}{\nu(\sigma)}\right)\right]\left[\Gamma_n(x,\sigma^x)-\Gamma_n(x,\sigma)\right]. \tag{A.10}$$

Let us split this sum into $I_n + B_n$, with

$$I_n := \frac{1}{2}\sum_{x=-n+1}^{n-1}\sum_{\sigma\in\Omega_{\Lambda_n}}\left[\log\left(\frac{\mu_p(\sigma^x)}{\mu_p(\sigma)}\right)-\log\left(\frac{\nu(\sigma^x)}{\nu(\sigma)}\right)\right]\left[\Gamma_n(x,\sigma^x)-\Gamma_n(x,\sigma)\right], \tag{A.11}$$

and B_n containing the boundary terms. Since the rates are reversible w.r.t. μ_p, and I_n contains only terms in the bulk, it can be rewritten as

$$I_n = -\frac{1}{2}\sum_{x=-n+1}^{n-1}\sum_{\sigma\in\Omega_{\Lambda_n}}\log\left(\frac{\nu(\sigma^x)r_x(\sigma^x)}{\nu(\sigma)r_x(\sigma)}\right)\left[\Gamma_n(x,\sigma^x)-\Gamma_n(x,\sigma)\right] \tag{A.12}$$

$$= -\frac{1}{2}\sum_{x=-n+1}^{n-1}\sum_{\sigma\in\Omega_{\Lambda_n}}\log\left(\frac{\Gamma_n(x,\sigma^x)}{\Gamma_n(x,\sigma)}\right)\left[\Gamma_n(x,\sigma^x)-\Gamma_n(x,\sigma)\right]. \tag{A.13}$$

Let us define $\Phi : (u,v) \mapsto \log(u/v)(v-u)$, and notice that it is convex, subadditive and non-negative. In particular, $I_n \le 0$. Moreover, letting

$$\alpha_n(x) := \sum_{\sigma\in\Omega_{\Lambda_n}}\log\left(\frac{\Gamma_n(x,\sigma^x)}{\Gamma_n(x,\sigma)}\right)\left[\Gamma_n(x,\sigma^x)-\Gamma_n(x,\sigma)\right], \tag{A.14}$$

we check easily that $(\alpha_n(x))_{n\geq x+1}$ is non-decreasing (by subadditivity of Φ and (A.7)). Consequently, if $I_n \to 0$ as $n \to \infty$, $\alpha_n(x) = 0$ for all $n \geq |x| + 1$.

Let us now turn to $B_n = -I_n$ and show that it converges to 0. Using(A.7), we can rewrite

$$2B_n = \sum_{\sigma\in\Omega_{\Lambda_n}} \sum_{\zeta\in\Omega_{\Lambda_{n+1}},\zeta_{|\Lambda_n}=\sigma} \left[\log\left(\frac{\mu_p(\sigma^{-n})}{\mu_p(\sigma)}\right) - \log\left(\frac{\nu(\sigma^{-n})}{\nu(\sigma)}\right)\right] \left[\Gamma_{n+1}(-n,\zeta^{-n}) - \Gamma_{n+1}(-n,\zeta)\right] + \sum_{\sigma\in\Omega_{\Lambda_n}} \sum_{\zeta\in\Omega_{\Lambda_{n+1}}} \left[\log\left(\frac{\mu_p(\sigma^{n})}{\mu_p(\sigma)}\right) - \log\left(\frac{\nu(\sigma^{n})}{\nu(\sigma)}\right)\right] \left[\Gamma_{n+1}(n,\zeta^{n}) - \Gamma_{n+1}(n,\zeta)\right]. \tag{A.15}$$

First, we can bound the terms involving μ_p by

$$\sum_{\sigma\in\Omega_{\Lambda_{n+1}}} \log\left(\frac{\mu_p(\sigma^{\pm n})}{\mu_p(\sigma)}\right) \left[\Gamma_{n+1}(\pm n,\zeta^{\pm n}) - \Gamma_{n+1}(\pm n,\zeta)\right] \leq 2\log\left(\frac{p\vee q}{p\wedge q}\right) \sum_{\sigma\in\Omega_{\Lambda_{n+1}}} \left|\Gamma_{n+1}(\pm n,\zeta^{\pm n}) - \Gamma_{n+1}(\pm n,\zeta)\right|. \tag{A.16}$$

We will then split the sum in (A.15) according to the value of $\frac{\nu(\sigma^{\pm n})}{\nu(\sigma)}$. If $\frac{\nu(\sigma^{\pm n})}{\nu(\sigma)} \in [M, M']$ with $M' \geq M \geq 1$, we can bound $-\log\left(\frac{\nu(\sigma^{\pm n})}{\nu(\sigma)}\right)\left[\Gamma_{n+1}(\pm n,\zeta^{\pm n}) - \Gamma_{n+1}(\pm n,\zeta)\right]$ by 0 if $\Gamma_{n+1}(\pm n,\zeta^{\pm n}) \geq \Gamma_{n+1}(\pm n,\zeta)$, and $\log(M')\Gamma_{n+1}(\pm n,\zeta) \leq \log(M')(p\vee q)\nu(\zeta)$ else. Summing over ζ such that $\zeta_{|\Lambda_n} = \sigma$, we get in that case

$$\sum_{\zeta_{|\Lambda_n}=\sigma} -\log\left(\frac{\nu(\sigma^{\pm n})}{\nu(\sigma)}\right) \left[\Gamma_{n+1}(\pm n,\zeta^{\pm n}) - \Gamma_{n+1}(\pm n,\zeta)\right] \leq \log(M')(p\vee q)\nu(\sigma) \leq \frac{\log(M')(p\vee q)}{M}\nu(\sigma^{\pm n}).$$

Similarly, if $\frac{\nu(\sigma^{\pm n})}{\nu(\sigma)} \in [\delta', \delta]$ with $0 < \delta' < \delta < 1$,

$$-\log\left(\frac{\nu(\sigma^{\pm n})}{\nu(\sigma)}\right) \left[\Gamma_{n+1}(\pm n,\zeta^{\pm n}) - \Gamma_{n+1}(\pm n,\zeta)\right] \leq -\log(\delta')(p\vee q)\delta\nu(\sigma). \tag{A.17}$$

Let us now fix $M > 1$ and define $M_k = Me^k$, $\delta_k = M^{-1}e^{-k}$. Using the above bound, and

$$\sum_{\zeta\in\Omega_{\Lambda_{n+1}}} \Gamma_{n+1}(\cdot,\zeta) \leq 1,$$

we can bound

$$\begin{aligned} 2B_n &\leq \left(\log M + \log\left(\frac{p\vee q}{p\wedge q}\right)\right) \sum_{\zeta\in\Omega_{\Lambda_{n+1}}} |\Gamma_{n+1}(n,\zeta^n) - \Gamma_{n+1}(n,\zeta)| \\ &\quad + \left(\log M + \log\left(\frac{p\vee q}{p\wedge q}\right)\right) \sum_{\zeta\in\Omega_{\Lambda_{n+1}}} \left|\Gamma_{n+1}(-n,\zeta^{-n}) - \Gamma_{n+1}(-n,\zeta)\right| \\ &\quad + 2\sum_{k\geq 0}(\log M + k + 1)e^{-k}/M \qquad (A.18) \\ &< \infty. \qquad (A.19) \end{aligned}$$

By monotonicity of the $(\alpha_n(x))_n$, I_n is non-increasing and B_n is therefore non-decreasing. We conclude that $\lim_{n\to\infty} B_n$ exists, and also $\lim_{n\to\infty} I_n = -\frac{1}{2}\lim_{n\to\infty}\sum_{x=-n+1}^{n-1}\alpha_n(x)$. Since $0 \leq \alpha_{n+1}(\pm n) \leq I_n - I_{n+1}$, this implies $\alpha_{n+1}(\pm n)$ vanishes as $n \to \infty$.

Let us now show that [76, Lemma IV.5.8]

$$\sum_{\sigma\in\Omega_{\Lambda_{n+1}}} \left|\Gamma_{n+1}(\pm n,\sigma^{\pm n}) - \Gamma_{n+1}(\pm n,\sigma)\right| \leq \sqrt{2\alpha_{n+1}(\pm n)}. \tag{A.20}$$

By (A.18), this will conclude the proof. Let

$$M = \sum_{\zeta \in \Omega_{\Lambda_{n+1}}} \Gamma_{n+1}(\pm n, \zeta^{\pm n}) \vee \Gamma_{n+1}(\pm n, \zeta), \tag{A.21}$$

$$m = \sum_{\zeta \in \Omega_{\Lambda_{n+1}}} \Gamma_{n+1}(\pm n, \zeta^{\pm n}) \wedge \Gamma_{n+1}(\pm n, \zeta). \tag{A.22}$$

The square of the left-hand side of (A.20) can be reformulated and estimated

$$\begin{aligned} &\Big[\sum_{\zeta \in \Omega_{\Lambda_{n+1}}} \big| \Gamma_{n+1}(\pm n, \zeta^{\pm n}) - \Gamma_{n+1}(\pm n, \zeta) \big| \Big]^2 = (M-m)^2 \leq (M-m) M \log \frac{M}{m} \quad \leq M \Phi(M,m) \\ &\leq 2 \sum_{\zeta \in \Omega_{\Lambda_{n+1}}} \Phi\Big(\Gamma_{n+1}(\pm n, \zeta^{\pm n}) \vee \Gamma_{n+1}(\pm n, \zeta), \Gamma_{n+1}(\pm n, \zeta^{\pm n}) \wedge \Gamma_{n+1}(\pm n, \zeta) \Big) = 2\alpha_{n+1}(\pm n). \end{aligned} \tag{A.23}$$

□

Let us now conclude the proof of Theorem A1, by showing that an invariant measure ν satisfying (A.9) is of the form $\lambda \delta_{\mathbf{1}} + (1-\lambda)\mu_p$ for some $\lambda \in [0,1]$. $\nu(\delta_{\mathbf{1},n})$ is non-increasing in n, so it has a limit in $[0,1]$; let us call it λ. A consequence of (A.9) is that, for any $\sigma \in \Lambda_n$, $x \in \Lambda_{n-1}$, if $c_x(\sigma) > 0$, then

$$\nu(\sigma^x) = \nu(\sigma) \frac{r_x(\sigma)}{r_x(\sigma^x)}.$$

In particular, since it is possible to connect any $\sigma \in \Omega_{\Lambda_n} \setminus \{\delta_{\mathbf{1},n}\}$ to (say) $\delta_{\mathbf{0},n}$ through allowed spin flips, for any such σ,

$$\nu(\sigma) = \nu(\delta_{\mathbf{0},n}) \left(\frac{p}{q} \right)^{|\sigma|},$$

where $|\sigma| = \sum_{x \in \Lambda_n} \sigma(x)$. $\nu(\sigma)$ can therefore be expressed in terms of $\nu(\delta_{\mathbf{1},n})$:

$$\nu(\sigma) = \left(\frac{p}{q} \right)^{|\sigma|} \frac{q^{2n+1}}{1-p^{2n+1}} (1 - \nu(\delta_{\mathbf{1},n})). \tag{A.24}$$

Now fix $m \in \mathbb{N}^*$, $\zeta \in \Omega_{\Lambda_m}$. Denote by $0 \cdot \zeta \cdot 0$ the configuration equal to ζ on Λ_m and 0 on $-m-1, m+1$ (and similarly $1 \cdot \zeta \cdot 0, 0 \cdot \zeta \cdot 1, 1 \cdot \zeta \cdot 1$).

$$\nu(\zeta) = \nu(0 \cdot \zeta \cdot 0) + \nu(1 \cdot \zeta \cdot 0) + \nu(0 \cdot \zeta \cdot 1) + \nu(1 \cdot \zeta \cdot 1) \tag{A.25}$$

$$= \left(\frac{p}{q} \right)^{|\zeta|} \left[1 + 2\frac{p}{q} + \frac{p^2}{q^2} \right] \frac{q^{2m+3}}{1-p^{2m+3}} (1 - \nu(\delta_{\mathbf{1},m+1})) \tag{A.26}$$

$$= \left(\frac{p}{q} \right)^{|\zeta|} \frac{q^{2m+1}}{1-p^{2m+3}} (1 - \nu(\delta_{\mathbf{1},m+1})). \tag{A.27}$$

Iterating this, we get that for all $n \geq m$,

$$\nu(\zeta) = \left(\frac{p}{q} \right)^{|\zeta|} \frac{q^{2m+1}}{1-p^{2n+1}} (1 - \nu(\delta_{\mathbf{1},n})), \tag{A.28}$$

which fixes the value of $\nu(\zeta)$ at $\mu_p(\zeta)(1-\lambda)$ since $\nu(\delta_{\mathbf{1},n})$ converges to Λ as $n \to \infty$.

□

Contributions

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Stochastic dynamics with degenerate rates

Kinetically constrained models; hydrodynamic limits; random walks in random environment

Abstract: The works presented here concern several examples of stochastic models on discrete lattices. Their common feature is that some transitions are forbidden, which makes them lose properties like monotonicity or ellipticity. This makes their study more challenging.
In the first chapter we consider the equilibrium/non-equilibrium relaxation of Kinetically Constrained Models (KCM). At equilibrium, we use Poincaré inequalities or variational formulas to study quantities that characterize the speed of the dynamics: relaxation time or a tracer diffusion coefficient. For non-equilibrium dynamics, results are sparser. In this chapter we study the propagation of a one-dimensional front in the FA-1f model.
In the second chapter, we show hydrodynamic limits for conservative interacting particle systems with constraints. We study the facilitated exclusion process and the porous medium model. In the first case, we show that the hydrodynamic limit is given by a Stefan problem; in the second we describe the equilibrium fluctuations of the weakly asymmetric version of the model.
The last chapter concerns random walks in dynamic random environment. The results presented there aim to study environments which – like the KCM – have rather weak mixing properties, and obtain law of large numbers or CLT. The elected tools range from perturbation theory of operators to renormalisation techniques.

Keywords: Kinetically constrained models; hydrodynamic limits; random walks in random environment

Dynamiques aléatoires avec taux dégénérés

Systèmes de particules avec contraintes cinétiques, limites hydrodynamiques, marches aléatoires en milieu aléatoire

Résumé: Les travaux présentés dans ce manuscrit s'intéressent à différents exemples de modèles stochastiques sur réseau discret, dont le point commun est que l'interdiction de certaines transitions rend leur étude plus difficile en faisant perdre des propriétés de monotonie ou d'ellipticité.
La première classe de modèles considérée est celle des systèmes de particules avec contraintes cinétiques (KCM), dont on étudie la relaxation à l'équilibre et hors équilibre. Dans le premier cas, des outils du type inégalités de Poincaré ou formules variationnelles permettent d'étudier des quantités caractérisant la vitesse de la dynamique, comme le temps de relaxation ou le coefficient de diffusion d'un traceur. Dans le cas hors équilibre, les résultats sont plus rares. On étudie dans ce chapitre la propagation d'un front unidimensionnel dans le modèle FA-1f.
Le deuxième chapitre s'intéresse à des résultats de type limite hydrodynamique pour des systèmes de particules conservatifs avec contraintes. On étudie le modèle d'exclusion facilité ainsi que le modèle des milieux poreux. Pour le premier, on montre que la limite hydrodynamique est donnée par un problème de Stefan, et on décrit les fluctuations à l'équilibre du deuxième dans sa version faiblement asymétrique.
Le dernier chapitre porte sur des marches aléatoires en milieu aléatoire dynamique. Les travaux présentés visent à étudier des environnements ayant – comme les KCM – de faibles propriétés de mélanges, pour obtenir loi des grands nombres ou TCL. Les outils proposés vont de la perturbation d'opérateurs à des techniques de renormalisation.

Mots clés: Systèmes de particules avec contraintes cinétiques, limites hydrodynamiques, marches aléatoires en milieu aléatoire

Image en couverture : Simulation du processus d'exclusion facilité.

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