Random Markov property for random walks in random environments

Julien Allasia * Rangel Baldasso † Oriane Blondel ‡ Augusto Teixeira §

September 20, 2024

Abstract

We consider random walks in dynamic random environments and propose a criterion which, if satisfied, allows to decompose the random walk trajectory into i.i.d. increments, and ultimately to prove limit theorems. The criterion involves the construction of a random field built from the environment, that has to satisfy a certain random Markov property along with some mixing estimates. We apply this criterion to correlated environments such as Boolean percolation and renewal chains featuring polynomial decay of correlations.

Keywords and phrases. Random walk, dynamic random environment, limit laws
MSC 2010: *Primary.* 60K37, 60F15, 60F05; *Secondary.* 82B41, 60K10.

1 Introduction

Random walks in random environments have been a subject of intensive study since the 60s-70s, when the first models motivated by applications in biophysics or chemistry were introduced [10, 25]. One of the main challenges in their study is that memory of the past trajectory of the random walk can be conserved through the environment currently explored. A large part of the literature on the subject focuses on the identification of conditions (on the random walk jumps and the law of the environment) that imply memory loss in some sense.

One of the most prominent example in this direction is [24]. The environment is assumed to be i.i.d., and the random walk to satisfy some ballisticity condition. This allows to build regeneration times for the trajectory of the random walk. Let us note that it is natural to request some kind of directional transience, since otherwise the random walk keeps revisiting previously explored environment and there is no reason why it should forget what it saw (indeed

*Email: allasia@math.univ-lyon1.fr; Universite Claude Bernard Lyon 1, CNRS, Ecole Centrale de Lyon, INSA Lyon, Université Jean Monnet, ICJ UMR5208, 69622 Villeurbanne, France.

†Email: rangel@mat.puc-rio.br; Department of Mathematics, PUC-Rio, Rua Marquês de São Vicente 225, Gávea, 22451-900 Rio de Janeiro, RJ - Brazil.

‡Email: blondel@math.univ-lyon1.fr; Universite Claude Bernard Lyon 1, CNRS, Ecole Centrale de Lyon, INSA Lyon, Université Jean Monnet, ICJ UMR5208, 69622 Villeurbanne, France.

§Email: augusto@impa.br; IMPA, Estrada Dona Castorina 110, 22460-320 Rio de Janeiro, RJ - Brazil and IST, University of Lisbon, Portugal.

in general it does not [23]). The i.i.d. assumption on the other hand is clearly non-optimal, and later works have focused on relaxing it. A landmark paper in that direction is [11], where the authors introduce a *cone-mixing* condition that the environment has to satisfy. It allows to build a sequence of approximate regeneration times (which yield increments that are close to being i.i.d.).

This has later been refined in various ways [6, 7], especially in the context of random walks in dynamic random environments (RWDRE). In this case, one may consider the time direction as the direction of transience of a random walk in dimension $d+1$ whose last coordinate deterministically increases by 1 at each step. Ballisticity is then automatic, and the setting allows to consider weaker decoupling properties on the environment. The main drawback of the cone-mixing property is that is requires a *uniform* decoupling property, which many natural dynamic environments do not satisfy.

In recent years, there has been a growing interest in studying RWDRE when the environment has a non-uniform decoupling property [4], especially using renormalization techniques [8, 17, 5, 3, 2, 1]. What we propose in the present paper is to give a framework in which one can build a sequence of regeneration times for the RWDRE, and use renormalization to control the tails of these regeneration times in terms of decorrelation properties of the environment. We can then apply standard limit results for sums of i.i.d. random variables to the trajectory of the RWDRE.

It is worth mentioning that the same type of problems have been studied by means of analyzing the environment seen from the particle [12, 13, 4].

Our strategy relies on the construction of a field $\eta \in \{0,1\}^{\mathbb{Z}^{d+1}}$, which is model-dependent (see examples in Section 5). η has to be built from the environment (and possibly extra independent randomness) in such a way that it has the *Random Markov Property* which we define below (Definition 1.1). Intuitively, the Random Markov Property says that, conditional on $\eta_x = 1$, the future of a random walk trajectory going through x is independent of its past; this will allow us to decompose the trajectory of the RWDRE into i.i.d. increments (Lemma 3.2), conditioning on the event that η is 1 at the origin. Note that this step is less straightforward than it might appear from the intuitive interpretation of the Random Markov Property. For instance, it necessitates resampling the field η (Lemma 3.1) in order to erase irrelevant information. The tail of the regeneration times is controlled by the decorrelation properties of η (see Definition 1.2 and Lemma 4.10), which in turn are inherited from those of the environment in our examples. The tail estimates are proved through a renormalization procedure.

The paper is organised as follows. In Section 1.2, we give a short definition of the random walk in dynamic random environment. In Section 1.3, we explain what we need from the auxiliary field η, and in particular the Random Markov Property (RMP). We conclude Section 1 with the statement of our main result Theorem 1.3. In Section 2 we give a formal construction of our random walk and the surrounding objects. Section 3 is devoted to the construction of regeneration times for the trajectory of the random walk, using the RMP and a resampling procedure. Section 4 presents the renormalization strategy that allows to control the tails of the regeneration times, and the proof of our main results is given in Section 4.5. We conclude in Section 5 with examples of dynamic random environments for which we construct an appropriate field η: Boolean percolation with polynomial tails for the radii, and independent renewal chains with polynomial tails for the interarrival distribution.

1.1 Notation

For $x \in \mathbb{Z}^d$ and $t \in \mathbb{Z}$, we see $z = (x,t)$ as an element of $\mathbb{Z}^{d+1}$. We denote by $|\cdot|$ the L^∞ norm on $\mathbb{Z}^{d+1}$. Mind that depending on the context, 0 can mean the origin of $\mathbb{Z}^d$ or $\mathbb{Z}^{d+1}$. For instance, when we write $(0,t)$ with $t \in \mathbb{Z}$ and a point in $\mathbb{Z}^{d+1}$ was expected, what we mean is $0 \in \mathbb{Z}^d$.

$\mathbb{N}$ denotes the set of natural integers starting from 0. $\mathbb{N}^*$ is $\mathbb{N} \setminus \{0\}$. For any two integers $a < b$, denote by $[\![a,b]\!]^d = [a,b]^d \cap \mathbb{Z}^d$.

1.2 Random walk in dynamic random environment

We start by defining somewhat informally the environment and random walk that we consider. In Section 2 we propose a formal construction of those objects.

Fix S a countable set and R a positive integer. We work with an environment $\omega = (\omega_z)_{z\in\mathbb{Z}^{d+1}} \in S^{\mathbb{Z}^{d+1}}$ that will be chosen with some fixed probability distribution P. We assume throughout the text that S is endowed with a distance that makes it a Polish metric space.

To define the random walk in random environment, for each state $s \in S$ we choose a probability kernel $p(s,\cdot)$ on $[\![-R,R]\!]^d$. For $\omega \in \Omega$, the random walk in the environment ω is the process $X = (X_t)_{t\in\mathbb{N}}$ such that $X_0 = 0$ and, for $t \in \mathbb{N}$, $x \in \mathbb{Z}^d$ and $y \in [\![-R,R]\!]^d$,

$$\mathrm{P}^\omega(X_{t+1} = x+y | X_t = x) = p(\omega_{(x,t)}, y). \tag{1.1}$$

We say that R is the *range* of X, and that X is *finite-range.*

We assume that X is *uniformly elliptic*, meaning that

$$\min_{y\in[\![-1,1]\!]^d} \inf_{s\in S} p(s,y) = c_0 > 0. \tag{1.2}$$

Mind that although X has range R, we only ask for uniform ellipticipty for the jumps in $[\![-1,1]\!]^d$.

We see the trajectory of our random walk as a path in $\mathbb{Z}^{d+1}$ by considering $Z = (Z_t)_{t\in\mathbb{N}}$, where

$$Z_t = (X_t, t). \tag{1.3}$$

1.3 The field η: Random Markov Property and decoupling

Our main assumption is the existence of a random field $\eta = (\eta_z)_{z\in\mathbb{Z}^{d+1}} : \mathbb{Z}^{d+1} \to \{0,1\}$ on the same probability space as ω, which will help us construct a renewal structure based on times where the random walk hits a point where $\eta_z = 1$. Let us start by some technical assumptions.

Assumption 1. *We ask ω and η to satisfy the following conditions.*

1. *(ω,η) is translation-invariant: for any $z \in \mathbb{Z}^{d+1}$, the law of $(\omega_{z+\cdot}, \eta_{z+\cdot})$ under* P *is the same as that of (ω,η).*

2. *There exists $c_1 > 0$ such that*

$$\mathrm{P}(\eta_0 = 1) \geqslant c_1. \tag{1.4}$$

3. There exists $s_\star \in S$ such that

$$\forall z \in \mathbb{Z}^{d+1},\ \eta_z = 1 \Rightarrow \omega_z = s_\star. \tag{1.5}$$

The translation invariance is a natural requirement in our setting. The second assumption will be used to ensure that the random walk often meets points where η is 1. The third assumption is a technicality that helps us remove an extra condition in the construction of our renewal times. Bear in mind that in our examples, η will be a *non-local* function of ω (and sometimes extra randomness).

The crucial assumptions on η are given by two definitions that we present now. For $z = (x,t) \in \mathbb{Z}^{d+1}$, let us define the future and past cones rooted at z.

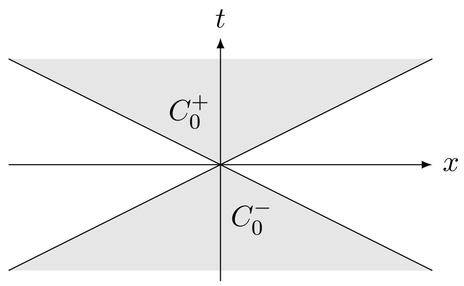

Figure 1: Future and past cones rooted at the origin when $d = 1$ and $R = 2$.

$$\begin{aligned} C_0^+ &= \{(x,t) \in \mathbb{Z}^{d+1} \ :\ |x| \leqslant Rt\}, \\ C_0^- &= \{(x,t) \in \mathbb{Z}^{d+1} \ :\ t \leqslant 0, |x| \leqslant R|t|\}, \\ C_z^\pm &= z + C_0^\pm. \end{aligned} \tag{1.6}$$

Note that, if $X_t = x$, since X has range R, we have

$$\begin{aligned} &Z_{t+s} \in C^+_{(x,t)} \text{ for all } s \in \mathbb{N}, \\ &Z_{t-s} \in C^-_{(x,t)} \text{ for all } s \in [\![0,t]\!]. \end{aligned} \tag{1.7}$$

Let us denote, for $z \in \mathbb{Z}^{d+1}$,

$$\begin{aligned} \mathcal{F}_z^- &= \sigma\left((\omega_z, \eta_z), z \in C_z^-\right), \\ \mathcal{F}_z^+ &= \sigma\left(\omega_z, z \in C_z^+\right). \end{aligned} \tag{1.8}$$

Mind the subtlety in the definition of $\mathcal{F}_z^-$ and $\mathcal{F}_z^+$: for the former, we look at both processes ω and η in the past cone; for the latter, we only look at ω in the future cone.

Definition 1.1 (Random Markov property). *We say the field (ω, η) satisfies the Random Markov property if, for any $z \in \mathbb{Z}^{d+1}$ and for any bounded random variable W measurable with respect to $\mathcal{F}_z^+$,*

$$\mathrm{E}[W|\mathcal{F}_z^-]\,\eta_z = \mathrm{E}[W|\eta_z]\,\eta_z. \tag{1.9}$$

Remark 1. It is important to emphasize that $\mathcal{F}^-$ depends on both ω and η in the past cone C^-, while $\mathcal{F}^+$ depends only on ω in C^+. This is a crucial difference that makes the proof more complicated in many places. In other words, it would have been much easier to prove a CLT under the hypothesis (1.9) if $\mathcal{F}^+$ was allowed to depend on η as well. However, this would rule out several possible applications of the model, as in the case of independent renewal chains that we present below.

Additionally, we need a decoupling property for the field η. Let $A \subseteq \mathbb{Z}^{d+1}$. We say that A is a box if it is of the form $\prod_{i=1}^{d+1} [\![a_i, b_i]\!]$ with $a_i \leqslant b_i$ relative integers. If A is a box, we define its spatial diameter $r(A)$ and height $h(A)$ by

$$\begin{aligned} r(A) &:= \max_{i=1,\dots,d} (b_i - a_i), \\ h(A) &:= b_{d+1} - a_{d+1}. \end{aligned} \tag{1.10}$$

For $A = \prod_{i=1}^{d+1} [\![a_i, b_i]\!]$, $A' = \prod_{i=1}^{d+1} [\![a'_i, b'_i]\!]$ two boxes, we call $\mathrm{sep}(A, A')$ their vertical separation defined by

$$\mathrm{sep}(A, A') = \begin{cases} a_{d+1} - b'_{d+1} & \text{if } b'_{d+1} < a_{d+1}, \\ a'_{d+1} - b_{d+1} & \text{if } b_{d+1} < a'_{d+1}, \\ 0 & \text{else.} \end{cases} \tag{1.11}$$

Definition 1.2 (Decoupling property)**.** *Let $\alpha \geqslant 0$. We say that η satisfies the α–decoupling property if there exists $c_2 = c_2(\alpha) > 0$ such that for any $r \in \mathbb{N}$, for any boxes $A, B \subseteq \mathbb{Z}^{d+1}$ satisfying*

$$r(A) \vee r(B) \leqslant (2R+1)r, \quad h(A) \vee h(B) \leqslant r, \quad \mathrm{sep}(A, B) \geqslant r,$$

and for any $f_1 : \{0,1\}^A \to \{0,1\}$, $f_2 : \{0,1\}^B \to \{0,1\}$, we have

$$\mathrm{E}[f_1(\eta_A) f_2(\eta_B)] \leqslant \mathrm{E}[f_1(\eta_A)]\, \mathrm{E}[f_2(\eta_B)] + c_2 r^{-\alpha}.$$

Remark 2. In most cases, such as those presented in Section 5, η will satisfy a stronger decoupling property that works for larger sets of boxes. We use the following vocabulary in that context. Let $\varepsilon : \mathbb{R}_+^3 \to \mathbb{R}_+$. We say that ε is a decoupling function for η if for any $r, h, s \in \mathbb{N}$, for any boxes $A, B \subseteq \mathbb{Z}^{d+1}$ such that

$$r(A) \vee r(B) \leqslant r, \quad h(A) \vee h(B) \leqslant h, \quad \mathrm{sep}(A, B) \geqslant s, \tag{1.12}$$

and for any $f_1 : \{0,1\}^A \to \{0,1\}$, $f_2 : \{0,1\}^B \to \{0,1\}$, we have

$$\mathrm{E}[f_1(\eta_A) f_2(\eta_B)] \leqslant \mathrm{E}[f_1(\eta_A)]\, \mathrm{E}[f_2(\eta_B)] + \varepsilon(r, h, s).$$

The α–decoupling property then holds if there exists $c_2 > 0$ such that $\varepsilon((2R+1)r, r, r) \leq c_2 r^{-\alpha}$ for any $r \in \mathbb{N}$.

1.4 Main results

Our main theorem is a law of large numbers and an annealed central limit theorem, meaning that it holds for the annealed law defined by $\mathbb{P} = \int \mathrm{P}^{\omega}(\cdot)\,\mathrm{d}\,\mathrm{P}(\omega)$ (see Section 2).

Theorem 1.3. *Assume that X is finite-range and uniformly elliptic and that Assumption 1 as well as the Random Markov property and the α–decoupling property are satisfied.*

1. *Law of large numbers. If $\alpha > 1$, then, as t goes to infinity, $\frac{X_t}{t}$ converges $\mathbb{P}$-almost surely to a certain limit $a \in \mathbb{R}^d$.*
2. *Central limit theorem. If $\alpha > 2$, then, as t goes to infinity, $\frac{X_t - ta}{\sqrt{t}}$ converges in $\mathbb{P}$-distribution to a Gaussian random variable in $\mathbb{R}^d$ with zero mean value.*

The proof of Theorem 1.3 is divided into two parts, following the road map of the renewal method from [24]. The first one, namely Section 3, consists in defining a sequence of renewal times $(T_k)_{k\in\mathbb{N}}$ and show that conditioned on $T_0 = 0$, the increments of the random walk along this sequence of times is i.i.d. This is a consequence of the Random Markov property. In the second part of the proof, namely Section 4, we need to control the moments of T_1 conditioned on $T_0 = 0$. Here, the mixing behavior that we demand for η will be instrumental. The final proof of Theorem 1.3 can be found in Section 4.5. A construction will be provided for a and the variance of the limiting Gaussian distribution. In Section 5, we give examples of environments that satisfy our assumptions.

Acknowledgements. The research of JA was supported by a doctoral contract and travel funds provided by CNRS Mathématiques. RB has counted on the support of "Conselho Nacional de Desenvolvimento Científico e Tecnológico – CNPq" grants "Produtividade em Pesquisa" (308018/2022-2) and "Projeto Universal" (402952/2023-5). The research of OB was partially supported by the ANR grant MICMOV (ANR- 19-CE40-0012) of the French National Research Agency (ANR). During this period, AT has also been supported by grants "Projeto Universal" (406250/2016-2) and "Produtividade em Pesquisa" (304437/2018-2) from CNPq and "Jovem Cientista do Nosso Estado" (202.716/2018) from FAPERJ.

2 Formal construction

Let us endow $S^{\mathbb{Z}^{d+1}}$ with the product topology and the subsequent Borel σ-algebra. Let $(\Omega, \mathcal{T}, \mathrm{P})$ be a probability space and ω, η two random variables on $(\Omega, \mathcal{T})$ taking values in $S^{\mathbb{Z}^{d+1}}$ and $\{0,1\}^{\mathbb{Z}^{d+1}}$ respectively. Mind that we will also use notation ω to denote elements of Ω. We denote by $\hat{\mathrm{P}}$ the probability distribution P conditioned on $\eta_0 = 1$ (recall (1.4)), that is, $\hat{\mathrm{P}}(A) = \mathrm{P}(A|\eta_0 = 1)$.

We now assume that environment ω is fixed and provide a formal construction of the random walk in ω. What we want to do, besides having equality (1.1) satisfied, is to couple random walks starting from different points together. In order to do so, we use uniform random variables to encapsulate the randomness needed for the jumps of the random walks and allocate them to points in $\mathbb{Z}^{d+1}$ instead of allocating them to the random walks themselves, depending on their starting points.

For each $s \in S$, partition $[0,1]$ into $(2R+1)^d$ intervals $(I_s^y)_{y\in[\![-R,R]\!]^d}$ so that the length of I_s^y is $p(s,y)$. Set, for $u \in [0,1]$,

$$g(s,u) = \sum_{y\in[\![-R,R]\!]^d} y\mathbf{1}_{I_s^y}(u). \tag{2.1}$$

This is a measurable function from $S \times [0,1]$ to $[\![-R,R]\!]^d$ that will determine the jump of a random walk if the state of the environment at its location is s.

Now, let $(U_z)_{z\in\mathbb{Z}^{d+1}}$ be a collection of i.i.d. uniform random variables in $[0,1]$ defined on a probability space $(\Omega', \mathcal{T}', \mathrm{P}')$. Let

$$\Omega_0 = \Omega \times \Omega', \quad \mathcal{T}_0 = \mathcal{T} \otimes \mathcal{T}', \text{ and } \mathbb{P} = \mathrm{P} \otimes \mathrm{P}'. \tag{2.2}$$

We also set, for $\omega \in \Omega$,

$$\mathrm{P}^\omega = \delta_{\{\omega\}} \otimes \mathrm{P}'. \tag{2.3}$$

We have $\mathbb{P}(\cdot) = \int_\Omega \mathrm{P}^\omega(\cdot)\,\mathrm{d}\,\mathrm{P}(\omega)$. We also denote by $\hat{\mathbb{P}}(\cdot) = \int_\Omega \mathrm{P}^\omega(\cdot)\,\mathrm{d}\hat{\mathrm{P}}(\omega)$.

Define the random walk Z^z started at $z \in \mathbb{Z}^{d+1}$ driven by the environment ω as a random variable on $(\Omega_0, \mathcal{T}_0)$ as follows:

$$\begin{cases} Z_0^z = z; \\ Z_{t+1}^z = Z_t^z + (g(\omega_{Z_t^z}, U_{Z_t^z}), 1), t \in \mathbb{N}, . \end{cases} \tag{2.4}$$

We simply write Z for Z^0, where 0 is the origin in $\mathbb{Z}^{d+1}$. Note that, with this construction, we do have equality (1.1).

Finally, define a shifting operator by setting, for any $\omega \in \Omega$ and $z, z_0 \in \mathbb{Z}^{d+1}$,

$$(\omega \circ \theta_{z_0})_z = \omega_{z_0+z}. \tag{2.5}$$

3 Independent decomposition

Recall that $\hat{\mathrm{P}}$ denotes the distribution P conditioned on $\eta_0 = 1$. Let us now define the resampling operator $\mathcal{R}$. This operator uses the information about ω in the cone of the future and some additional randomness to resample the value of the field η. More precisely, if $\tilde{U}$ is sampled uniformly in $[0,1]$ and independently from $\omega \sim \hat{\mathrm{P}}$, then $\mathcal{R}(\omega, \tilde{U}) = \big(\mathcal{R}(\omega, \tilde{U})_z\big)_{z\in\mathbb{Z}^{d+1}} \in \{0,1\}^{\mathbb{Z}^{d+1}}$ is a field such that

1. $\mathcal{R}(\omega, \tilde{U}) \sim \eta$ under $\hat{\mathrm{P}}$,
2. There exists an $\mathcal{F}_0^+$-measurable function $\tilde{\mathcal{R}}$ such that $\mathcal{R}(\omega, \tilde{U}) = \tilde{\mathcal{R}}(\omega, \tilde{U})$ for $\hat{\mathrm{P}}$-almost every $(\omega, \tilde{U})$

Recall the shift operator introduced in (2.5). We write $\mathcal{R}_z(\omega, u) = \mathcal{R}(\omega \circ \theta_z, u) \circ \theta_{-z}$ for the resampling operator that uses the cone fixed at $z \in \mathbb{Z}^{d+1}$.

Let us note that, although it is not immediately clear that the operator above is always well-defined, this is a consequence of the existence of regular conditional probabilities, as stated in the next lemma.

Lemma 3.1. *Assume that the space Ω is enriched to allow for the definition of an independent random variable $\tilde{U}$ uniformly distributed in $[0,1]$. In this case the operator $(\omega, \tilde{U}) \mapsto \mathcal{R}(\omega, \tilde{U})$ is well defined for $\hat{\mathrm{P}}$-almost every ω and $\tilde{U}$.*

We postpone the proof of this lemma to the end of this section and define now a sequence of random times for the walk starting at z. In the following we enlarge our probability space to allow for an additional collection of independent uniform random variables $(\tilde{U}_z)_{z\in\mathbb{Z}^{d+1}}$ that will be used in combination with the resampling operator introduced above. Define now the sequence $(T_k^z)_{k\geqslant 0}$ as

$$\begin{aligned} &T_0^z = T_0^z(\omega) = \inf\left\{t \geqslant 0,\, \eta_{Z_t^z} = 1\right\}, \\ &T_{k+1}^z = T_{k+1}^z(\omega) = \inf\left\{t > T_k^z,\, \mathcal{R}_{Z^z_{T_k^z}}\big(\omega, \tilde{U}_{Z^z_{T_k^z}}\big)_{Z_t^z} = 1\right\}, \text{ for all } k \in \mathbb{N}, \end{aligned} \tag{3.1}$$

where the infimum of the empty set is taken to be $+\infty$. We will often write simply T_k without the upper script z to refer to the times T_k^0 associated with the random walk starting at the origin. Observe that when $\eta_0 = 1$, we have $T_0 = 0$.

We also set, for $k \in \mathbb{N}$,

$$Y_k = \begin{cases} Z_{T_k} & \text{if } T_k < \infty; \\ \infty & \text{otherwise.} \end{cases} \tag{3.2}$$

We set $Y_{k+1} - Y_k = +\infty$ if $T_{k+1} = +\infty$.

The next lemma is quite general and does not require any decoupling assumption on η.

Lemma 3.2. *Let $n \geqslant 0$ and let $f_0, \ldots, f_n$ be bounded measurable functions on $\mathbb{Z}^{d+1}$. We have*

$$\begin{aligned} &\mathbb{E}\left[\mathbf{1}_{T_{n+1}<\infty}\, f_0(Y_1 - Y_0)\cdots f_n(Y_{n+1} - Y_n)\ \middle|\ \eta_0 = 1\right] \\ &\quad = \prod_{k=0}^{n} \mathbb{E}[\mathbf{1}_{T_1<\infty}\, f_k(Y_1) \,|\, \eta_0 = 1]. \end{aligned} \tag{3.3}$$

In particular, under $\mathbb{P}\big(\cdot\,|\eta_0 = 1\big)$, if $T_1 < \infty$ almost surely, then the $(Y_{k+1} - Y_k)_{k\geqslant 0}$ are i.i.d. and have the same distribution as $Y_1 = (X_{T_1}, T_1)$.

We will see in Lemma 4.10 that whenever $\alpha > 0$, the α-decoupling property implies that $T_1 < \infty$ almost surely, so the second part of the statement will hold.

Proof. We first need to strengthen the Random Markov Property. We let

$$\mathcal{G}_z = \mathcal{F}_z^- \vee (U_y)_{y\in\mathcal{C}_z^-\setminus\{z\}}. \tag{3.4}$$

Note that for any bounded measurable function f on $(S\times[0,1]^2)^{\mathcal{C}_z^+}$, we have

$$\begin{aligned} &\mathbb{E}\left[f\left((\omega_y, U_y, \tilde{U}_y)_{y\in\mathcal{C}_z^+}\right)\middle|\mathcal{G}_z\right]\eta_z \\ &\quad = \mathbb{E}\left[f\left((\omega_y, U_y, \tilde{U}_y)_{y\in\mathcal{C}_z^+}\right)\middle|\mathcal{F}_z^-\right]\eta_z \\ &\quad = \int_{([0,1]^2)^{\mathcal{C}_z^+}} \mathrm{E}\left[f\left((\omega_y, u_y, \tilde{u}_y)_{y\in\mathcal{C}_z^+}\right)\middle|\mathcal{F}_z^-\right]\eta_z \prod_{y\in\mathcal{C}_z^+} \mathrm{d}u_y \mathrm{d}\tilde{u}_y \end{aligned}$$

$$
\begin{aligned}
&= \int_{([0,1]^2)^{\mathcal{C}_z^+}} \mathrm{E}\left[f\left((\omega_y, u_y, \tilde{u}_y)_{y\in\mathcal{C}_z^+}\right)\Big|\eta_z\right]\eta_z \prod_{y\in\mathcal{C}_z^+} \mathrm{d}u_y \mathrm{d}\tilde{u}_y \\
&= \mathbb{E}\left[f\left((\omega_y, U_y, \tilde{U}_y)_{y\in\mathcal{C}_z^+}\right)\Big|\eta_z\right]\eta_z \\
&= \mathbb{E}\left[f\left((\omega_y, U_y, \tilde{U}_y)_{y\in\mathcal{C}_z^+}\right)\Big|\eta_z = 1\right]\eta_z.
\end{aligned}
$$

The first equality uses that $(U_y)_{y\in\mathcal{C}_z^-\setminus\{z\}}$ is independent of $\mathcal{F}_z^-$ and $(\omega_y, U_y, \tilde{U}_y)_{y\in\mathcal{C}_z^+}$. In the second and last equalities, we used that $(U_y, \tilde{U}_y)_{y\in\mathcal{C}_z^+}$ is independent of ω and $\mathcal{F}_z^-$. In the third equality, the Random Markov property was used.

Let us now show an auxiliary identity. Let f be a bounded measurable function on $\mathbb{Z}^{d+1}$ and g a bounded measurable function on $(S\times[0,1]^2)^{\mathcal{C}_0^+}$. Under $\eta_0 = 1$, let us define $\tilde{T} = \inf\{t > 0,\ \eta_{Z_t}(\omega) = 1\}$ and $\tilde{Y} = Z_{\tilde{T}}$. Mind that Y and Y_1 may differ, since the latter is defined using a resampled environment. We have

$$
\begin{aligned}
&\mathbb{E}\left[\mathbf{1}_{T_1<\infty} f(Y_1)\, g\left((\omega, U, \tilde{U})\circ\theta_{Y_1}|_{\mathcal{C}_0^+}\right)\Big|\eta_0 = 1\right] \\
&\quad= \mathbb{E}\left[\mathbf{1}_{\tilde{T}<\infty} f(\tilde{Y})\, g\left((\omega, U, \tilde{U})\circ\theta_{\tilde{Y}}|_{\mathcal{C}_0^+}\right)\Big|\eta_0 = 1\right] \\
&\quad= \sum_{z\in\mathbb{Z}^{d+1}} f(z)\,\mathbb{E}\left[\mathbf{1}_{\tilde{T}<\infty}\mathbf{1}_{\tilde{Y}=z}\, g\left((\omega_y, U_y, \tilde{U}_y)_{y\in\mathcal{C}_z^+}\right)\Big|\eta_0 = 1\right] \\
&\quad= \sum_{z\in\mathbb{Z}^{d+1}} f(z)\,\mathbb{E}\left[\mathbf{1}_{\tilde{T}<\infty}\mathbf{1}_{\tilde{Y}=z}\,\mathbb{E}\left[g\left((\omega_y, U_y, \tilde{U}_y)_{y\in\mathcal{C}_z^+}\right)\Big|\mathcal{G}_z\right]\Big|\eta_0 = 1\right] \\
&\quad= \sum_{z\in\mathbb{Z}^{d+1}} f(z)\,\mathbb{E}\left[\mathbf{1}_{\tilde{T}<\infty}\mathbf{1}_{\tilde{Y}=z}\,\mathbb{E}\left[g\left((\omega_y, U_y, \tilde{U}_y)_{y\in\mathcal{C}_z^+}\right)\Big|\eta_z = 1\right]\Big|\eta_0 = 1\right] \\
&\quad= \mathbb{E}\left[\mathbf{1}_{\tilde{T}<\infty} f(\tilde{Y})\Big|\eta_0 = 1\right]\,\mathbb{E}\left[g\left((\omega_y, U_y, \tilde{U}_y)_{y\in\mathcal{C}_0^+}\right)\Big|\eta_0 = 1\right] \\
&\quad= \mathbb{E}\left[\mathbf{1}_{T_1<\infty} f(Y_1)|\eta_0 = 1\right]\,\mathbb{E}\left[g\left((\omega, U, \tilde{U})|_{\mathcal{C}_0^+}\right)\Big|\eta_0 = 1\right].
\end{aligned}
$$

In the first and last equalities, condition 3 in the definition of the resampled environment was used. The third equality relies on the fact that $\{\eta_0 = 1\}$, $\tilde{T}$ and $\tilde{Y}$ are $\mathcal{G}_z$-measurable. In the fourth equality, we used the strengthened Random Markov property, while, the fifth equality follows from translation invariance.

Now, let us prove (3.3). It is obvious when $n = 0$. Suppose that (3.3) is satisfied for $n-1$ with some $n \geqslant 1$. Let $f_0, \dots, f_n$ be bounded measurable functions on $\mathbb{Z}^{d+1}$. First note that there exists a bounded measurable function g as in the previous identity such that

$$
\mathbf{1}_{T_{n+1}<\infty} f_1(Y_2 - Y_1)\cdots f_n(Y_{n+1} - Y_n) = g\left((\omega, U, \tilde{U})\circ\theta_{Y_1}|_{\mathcal{C}_0^+}\right).
$$

Then, applying the previous identity with $f = f_0$ and g, we get the result after noticing that under $\eta_0 = 1$, we have

$$
g\left((\omega, U, \tilde{U})|_{\mathcal{C}_0^+}\right) = \mathbf{1}_{T_n<\infty} f_1(Y_1)\cdots f_n(Y_n - Y_{n-1})
$$

and applying the induction assumption.

Now, assume that $\mathbb{P}(T_1 < \infty \,|\, \eta_0 = 1) = 1$. Applying (3.3) with $n \in \mathbb{N}$ and $f_0 = \ldots = f_n = 1$, we get $\mathbb{P}(T_{n+1} < \infty | \, \eta_0 = 1) = 1$. Therefore, (3.3) becomes

$$\mathbb{E}\left[f_0(Y_1 - Y_0) \cdots f_n(Y_{n+1} - Y_n) \,|\, \eta_0 = 1\right] = \prod_{k=0}^{n} \mathbb{E}[f_k(Y_1) \,|\, \eta_0 = 1], \tag{3.5}$$

for any bounded measurable functions $f_0, \ldots, f_n$. This concludes the proof of lemma. □

We end this section with the proof of Lemma 3.1.

Proof of Lemma 3.1. Recall that $\mathcal{F}_0^+ = \sigma\big(\omega_z, z \in C_0^+\big)$. Since ω takes values in a Polish space, the regular conditional probability $K(\omega, \cdot) = \hat{\mathrm{P}}(\,\cdot\,|\mathcal{F}_0^+)(\omega)$ is well defined for $\hat{\mathrm{P}}$-almost every ω[1] (see for example [14, Theorem 5.1.9]).

Although we kernel K allows us to sample pairs $(\tilde{\omega}, \tilde{\eta})$, we only need the latter and use the variable $\tilde{U}$ to sample a field $\tilde{\eta} \in \{0,1\}^{\mathbb{Z}^{d+1}}$ from $K(\omega, \cdot)$, thus defining $\mathcal{R}(\omega, \tilde{U}) = \tilde{\eta}$. For the first property of the resampling operator, simply notice that

$$\hat{\mathrm{P}}\big(\tilde{\eta} \in A\big) = \hat{\mathrm{E}}\big[K(\omega, \{\eta \in A\})\big] = \hat{\mathrm{E}}\big[\hat{\mathrm{P}}\big(\eta \in A \big| \mathcal{F}_0^+\big)\big] = \hat{\mathrm{P}}\big(\eta \in A\big), \tag{3.6}$$

for any measurable set $A \subset \{0,1\}^{\mathbb{Z}^{d+1}}$.

Let us now verify the second property. Notice that, from the definition of the kernel $K(\cdot, \cdot)$, for every finite-dimensional set $A \subset \{0,1\}^{\mathbb{Z}^{d+1}}$, there exists a set $G_A \subset \Omega$ with $\hat{\mathrm{P}}(G_A) = 1$ such that $K(\omega, \{\eta \in A\}) = \hat{\mathrm{P}}(\{\eta \in A\} | \mathcal{F}_0^+)$ in G_A. Consider now $G = \cap_A G_A$ (the intersection runs over all finite-dimensional sets) and notice that an application of Dynkin's $\pi - \lambda$ Theorem yields that, for any measurable set $B \subset \{0,1\}^{\mathbb{Z}^{d+1}}$, $K(\omega, \{\eta \in B\}) = \hat{\mathrm{P}}(\eta \in B | \mathcal{F}_0^+)$ in G. By further restricting the set G, we may assume that, for all $\omega \in G$, $B \mapsto K(\omega, B)$ is a probability measure. In particular, in G, the function $\omega \mapsto K(\omega, \cdot)$, that maps configurations to probability measures, is measurable with respect to $\mathcal{F}_0^+$. The last ingredient missing is to note that, given $\tilde{U}$, the sampling of $\mathcal{R}$ is measurable with respect to the regular conditional probability kernel $K(\omega, \cdot)$. In particular, in the set G, $\mathcal{R}(\omega, \tilde{U})$ coincides with a $\mathcal{F}_0^+$ measurable function $\hat{\mathrm{P}}$-almost surely, concluding the verification of the second property. □

4 Renewal tails

4.1 Threatened points

Throughout this section we consider the random walk Z^z on top of the random environment ω, governed by the law P^ω defined in (1.1).

[1] This means that $K(\omega, A)$ is a transition kernel that satisfies

1. $A \mapsto K(\omega, A)$ is a probability measure on $\mathcal{T}$ for $\hat{\mathrm{P}}$-almost every ω.
2. For every A, $\omega \mapsto K(\omega, A)$ is a $\hat{P}$-version of $\hat{\mathrm{P}}(A|\mathcal{F}_0^+)$.

We fix a collection of *traps* $\Sigma \subseteq \mathbb{Z}^{d+1}$ and estimate the time it takes for the random walk to hit one of these traps. More precisely, consider the stopping time

$$T^z = \inf\{t > 0; Z^z_t \in \Sigma\}. \tag{4.1}$$

The dependence of quantities such as T^z on the set Σ will be omitted through the section, since Σ is considered fixed.

Later, when we are going to apply the results of this section to our random walk, the set Σ will be chosen as $\{z \in \mathbb{Z}^{d+1}; \eta_z = 1\}$, so that conditioned on $\eta_0 = 1$, the stopping time T^z will coincide with T^z_1 introduced in (3.1). However, the results of this section work for any set $\Sigma \subseteq \mathbb{Z}^{d+1}$ and they are not constrained to being given by the times where we can construct a decoupling of ω's past and future (as in the definition of η).

Our goal is to give conditions under which the random walk falls on some trap sufficiently quickly. In the definition below, we denote by $\pi_{d+1} : \mathbb{Z}^{d+1} \to \mathbb{Z}$ the projection in the $(d+1)$-coordinate and by $\bar{\pi}_d : \mathbb{Z}^{d+1} \to \mathbb{Z}^d$ the projection in the first d coordinates, so that a point in $\mathbb{Z}^{d+1}$ can be written as $x = (\bar{\pi}_d(x), \pi_{d+1}(x)) \in \mathbb{Z}^d \times \mathbb{Z}$.

Definition 4.1. *Let $a < b \in \mathbb{N}$. We say that a path $\sigma : [\![a, b]\!] \to \mathbb{Z}^{d+1}$ is an R-allowed path if*

- *$\pi_{d+1}(\sigma(s)) = s$, for every $s \in [\![a, b]\!]$: "$d + 1$ is time";*
- *$\bar{\pi}_d \circ \sigma$ is R-Lipschitz in the $|\cdot|_\infty$ norm.*

We call $b - a$ the length of the allowed path σ. By construction, the sample paths of the random walks Z^z on any time interval are R-allowed paths.

Definition 4.2. *For $H \geqslant 1$, we say that a point $z = (x, t) \in \mathbb{Z}^{d+1}$ is H-threatened (with respect to Σ) if there exists a 1-allowed path $\sigma : [\![t, t + H]\!] \to \mathbb{Z}^{d+1}$ such that $\sigma(t) = z$ and σ passes through some point $z' \in \Sigma$.*

Observe that we require the points z and z' to be connected through a 1-allowed path instead of an R-allowed path. The reason behind this is that we intend to use the uniform ellipticity provided by (1.2), which guarantees that our random walk has a positive probability to follow the exact steps of any 1-allowed path. Therefore, whenever $z = (x, t)$ is H-threatened with respect to Σ, we have, for every ω,

$$\mathrm{P}^\omega \left(\text{there exists } s \in [\![t, t + H]\!] \text{ such that } Z^z_s \in \Sigma\right) \geqslant c_0^H, \tag{4.2}$$

which can be deduced from (1.2) by instructing the random walk to follow step-by-step the 1-allowed path from Definition 4.2.

We now define the *number of threats* along a given path.

Definition 4.3. *Given an R-allowed path $\sigma : [\![a, b]\!] \to \mathbb{Z}^{d+1}$ and $H \geqslant 1$, let*

$$M(\sigma, H) := \left|\left\{t \in [\![a, b]\!] : \sigma(t) \text{ is } H\text{-threatened and } \pi_{d+1}(\sigma(t)) \in H\mathbb{Z}\right\}\right|. \tag{4.3}$$

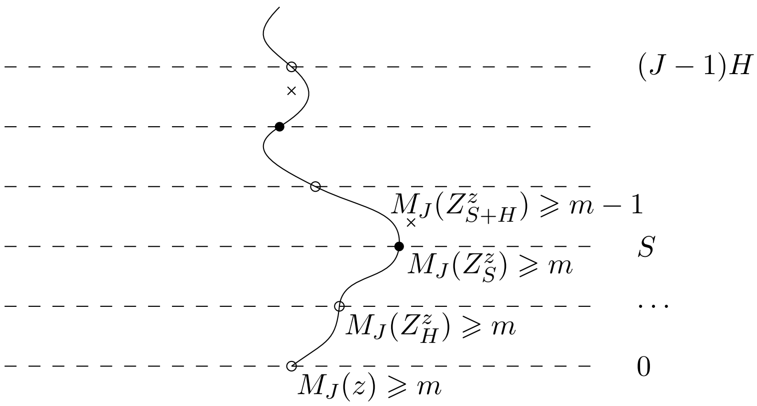

Figure 2: An illustration of the proof of Lemma 4.4. The horizontal dashed lines represent times that are multiple of H. The filled dots along the random walk trajectory represent threatened points, while the crosses represent traps in Σ. Note how the lower bound on M_J drops from m to $m-1$ after crossing the first threat.

Observe that we only count a threat if it is encountered at a time which is a multiple of H, this is done in order to keep the attempts to fall into different traps independent.

The connection between threats and the desired estimates on the tail of T^z are made evident in the following result.

Lemma 4.4. *For every $\omega \in \Omega$ and $H, J \in \mathbb{N}^*$,*

$$\mathrm{P}^\omega\left(T^0 > JH\right) \leqslant (1-c_0^H)^{M_J}, \tag{4.4}$$

where $M_J = M_J(\Sigma)$ is given by

$$M_J := \inf\Big\{M(\sigma, H); \sigma : [\![0, JH]\!] \to \mathbb{Z}^{d+1} \text{ R-allowed path starting at } 0\Big\}.$$

Before we provide the proof of the lemma, let us explain the simple intuition behind it. Note that the path σ' given by the random walk $(Z^0_t)_{t\in[\![0,JH]\!]}$ itself is an R-allowed path starting at zero, therefore $M(\sigma', H) \geqslant M_J$. In other words, independently of what the random walk does, it will be threatened M_J times and in well separated locations. Also, in each of these threatened times, by (4.2), there is a chance of at least c_0^H that the walk falls into the corresponding trap. The proof below makes the above intuition rigorous.

Proof. We follow an induction argument on M_J, but in order to set this up, we need to introduce a more general version of M_J that includes alternative starting points for the path. Given $z = (x, jH) \in \mathbb{Z}^d \times (H\mathbb{Z})$ with $j \in [\![0, J-1]\!]$, define

$$M_J(z) = \inf\Big\{M(\sigma, H); \sigma : [\![jH, JH]\!] \to \mathbb{Z}^{d+1} \text{ starts at } z \text{ and is } R\text{-allowed}\Big\}, \tag{4.5}$$

which depends solely on Σ. Recall the definition of T^z from (4.1). Our objective is to prove that, for all Σ,

$$\begin{aligned}&\text{for every } z = (x, jH) \text{ (with } j = 0, \dots, J-1\text{), for every } \omega \text{ and } m \geqslant 1\\ &\quad\text{if } M_J(z) \geqslant m, \text{ then } \mathrm{P}^\omega(T^z > (J-j)H) \leqslant (1-c_0^H)^m.\end{aligned} \tag{4.6}$$

Note that (4.6) is enough to establish Lemma 4.4 if we apply it to $z = (0,0)$ and $m = M_J$. We now turn to the proof of (4.6) by using induction on m.

It is clear that the claim is valid for $m = 0$ and from now on we suppose that $m > 0$ and (4.6) is valid for $m - 1$. Take then an arbitrary point $z = (x, jH)$ with $j \in \{0, \dots, J-1\}$ such that $M_J(z) \geqslant m$ and we turn to the proof of (4.6). Define first the stopping time (with respect to the filtration $\sigma(Z^z_s, s \leqslant t)$ on the space of random walk trajectories)

$$S = \inf\big\{s \in \{0, H, 2H, \dots\}; Z^z_s \text{ is } H\text{-threatened}\big\}, \tag{4.7}$$

which is smaller than or equal to $(J - j - 1)H$ almost surely because $M_J(z) \geqslant m \geqslant 1$. We then estimate, using the Strong Markov Property twice (at times S and $S + H$, see Figure 2),

$$\begin{aligned} \mathrm{P}^\omega(T^z > (J-j)H) &= \mathrm{E}^\omega\Big[\mathrm{P}^\omega\big(T^z > (J-j)H \big| Z^z_0, \dots, Z^z_S\big)\Big] \\ &\leqslant \mathrm{E}^\omega\Big[\mathrm{P}^\omega\big(T^{Z^z_S} > (J-j)H - S\big)\Big] \\ &\leqslant \mathrm{E}^\omega\Big[\mathrm{P}^\omega\Big(T^{Z^z_S} > H, T^{Z^z_{S+H}} > (J-j-1)H - S\Big)\Big] \\ &\leqslant \mathrm{E}^\omega\Big[\Big(1 - c_0^H\Big) \sup_{z'\,;\, M_J(z') \geqslant m-1} \mathrm{P}^\omega\Big(T^{z'} > (J-j-1)H - S\Big)\Big] \\ &\leqslant \Big(1 - c_0^H\Big)^m. \end{aligned} \tag{4.8}$$

Notice that, in the estimate above, the supremum is taken over all points $z' = (x, (j+1)H + S)$ such that $M_J(z') \geqslant m - 1$, which allows for the direct application of the induction hypothesis. This concludes the proof of (4.6) and consequently the lemma. □

4.2 Finding traps on intervals

$$L_k := 4^k, \qquad \ell_k = \Big\lfloor \frac{L_k}{2dk^2} \Big\rfloor. \tag{4.9}$$

Given integers $a < b$, we introduce the event

$$F(a,b) := \big\{\eta_{(0,a)} = \eta_{(0,a+1)} = \dots = \eta_{(0,b-1)} = 0\big\}. \tag{4.10}$$

Observe that

$$F\big(0, L_{k+1}\big) \subseteq F\big(0, L_k\big) \cap F\big(3 \cdot L_k, L_{k+1}\big). \tag{4.11}$$

From (4.11), applying the α–decoupling property (recall Definition 1.2) and translation invariance,

$$\mathrm{P}\left(F(0, L_{k+1})\right) \leqslant \mathrm{P}\left(F(0, L_k)\right)^2 + c_2 L_k^{-\alpha}. \tag{4.12}$$

We now write

$$q_k := \mathrm{P}\left(F(0, L_k)\right), \tag{4.13}$$

which allows for (4.12) to be rewritten as

$$q_{k+1} \leqslant q_k^2 + c_2 L_k^{-\alpha}. \tag{4.14}$$

Recall the constant c_1 from (1.4).

Lemma 4.5. *Under the α–decoupling property, there exists $c_3 = c_3(\alpha, c_1, c_2) \in (0,1)$ such that*

$$q_k \leqslant c_3^k, \text{ for every } k \geqslant 0. \tag{4.15}$$

Note that the decay could be very slow, this will be improved in the next lemma.

Proof. We first fix a constant $c_4 = c_4(\alpha, c_1) \in (0,1)$ such that

$$c_4 > 4^{-\alpha/2} \qquad \text{and} \qquad c_4^2 \geqslant \mathrm{P}\left(\eta_{(0,0)} = 0\right). \tag{4.16}$$

Note that this is possible using (1.4). Choose also an integer $k_0 = k_0(\alpha, c_1, c_2) \geqslant 1$ such that

$$c_2 4^{-\alpha\left(k_0 - \frac{1}{2}\right)} \leqslant 1 - c_4. \tag{4.17}$$

We want to prove by induction in $k \geqslant 2$ that

$$q_{k_0+k} \leqslant c_4^k, \tag{4.18}$$

which is sufficient to establish (4.15).

Taking $k = 2$, observe that

$$q_{k_0+2} = \mathrm{P}\left(F(0, L_{k_0+2})\right) \leqslant \mathrm{P}\left(\eta(0,0) = 0\right) \overset{(4.16)}{\leqslant} c_4^2, \tag{4.19}$$

so that (4.18) holds for $k = 2$.

Assuming now the validity of (4.18) for a certain value of $k \geqslant 2$, we get

$$\begin{aligned} \frac{q_{k_0+k+1}}{c_4^{k+1}} &\overset{(4.14)}{\leq} \frac{q_{k_0+k}^2 + c_2 L_{k_0+k}^{-\alpha}}{c_4^{k+1}} \overset{(4.18)}{\leq} c_4^{k-1} + c_2 \frac{L_{k_0+k}^{-\alpha}}{c_4^{k+1}} \\ &\overset{(4.16)}{\leqslant} c_4^{k-1} + c_2 4^{-\alpha(k_0+k)+\alpha(k+1)/2} \overset{(4.17)}{\leqslant} 1. \end{aligned} \tag{4.20}$$

This finishes the proof of (4.18) and we can bootstrap it into (4.15) by choosing c_3 sufficiently close to one. □

Lemma 4.6. *Under the α–decoupling property, there exists $c_5 = c_5(\alpha, c_2, c_1)$ such that*

$$q_k \leqslant c_5 L_k^{-\alpha}, \tag{4.21}$$

for every $k \geqslant 0$.

Proof. Using Lemma 4.5, we can obtain $k_0 = k_0(\alpha, c_2, c_1) \geqslant 1$ such that

$$q_{k_0+1} \leqslant \frac{1}{2} L_1^{-\alpha} \qquad \text{and} \qquad 2c_2 4^{-\alpha(k_0-1)} \leqslant \frac{1}{2}. \tag{4.22}$$

We are going to prove by induction that,

$$q_{k_0+k} \leqslant \frac{1}{2} L_k^{-\alpha}, \text{ for every } k \geqslant 1. \tag{4.23}$$

Note that the case $k = 1$ is covered in the left hand side of (4.22). Suppose that (4.23) holds for a certain $k \geqslant 1$ and estimate

$$\begin{aligned} \frac{q_{k_0+k+1}}{(1/2)L_{k+1}^{-\alpha}} &\overset{(4.14)}{\leqslant} 2\frac{q_{k_0+k}^2 + c_2 L_{k_0+k}^{-\alpha}}{L_{k+1}^{-\alpha}} \overset{(4.23)}{\leqslant} 2 \cdot \frac{1}{4} \cdot L_k^{-2\alpha} L_{k+1}^{\alpha} + 2c_2 L_{k+k_0}^{-\alpha} L_{k+1}^{\alpha} \\ &\leqslant \frac{1}{2} 4^{-\alpha(k-1)} + 2c_2 4^{-\alpha(k_0-1)} \overset{(4.22)}{\leqslant} \frac{1}{2} + \frac{1}{2} = 1. \end{aligned} \tag{4.24}$$

This finishes the proof of (4.23) by induction. Estimate (4.21) now follows from (4.23) by properly choosing $c_5 = c_5(k_0) = c_5(\alpha, c_2, c_1)$. □

Corollary 4.7. *There exists $c_6 = c_6(\alpha, c_2, c_1)$ such that, for any positive integer L*

$$\mathrm{P}(F(0, L)) \leqslant c_6 L^{-\alpha}. \tag{4.25}$$

Proof. The proof is a simple interpolation: fix L and k such that $L_k \leqslant L < L_{k+1}$.

$$\mathrm{P}(F(0, L)) \leqslant q_k \leqslant c_5 L_k^{-\alpha} \leqslant c_5 4^{\alpha} L_{k+1}^{-\alpha} \leqslant c_5 4^{\alpha} L^{-\alpha}. \tag{4.26}$$

□

With the uniform ellipticity (1.2), this implies that the probability that *every* point in a large box of $\mathbb{Z}^d$ be threatened is large. This will be useful in the next section where we show that every allowed path meets many threatened points with high probability.

Corollary 4.8. *There exists $c_7 = c_7(\alpha, c_2, c_1)$ such that for any $w \in \mathbb{Z}^{d+1}$, for any integer L,*

$$\mathrm{P}(\exists z \in w + [\![0, L-1]\!]^d \times \{0\} \textit{ s.t. } z \textit{ is not } L\textit{–threatened}) \leqslant c_7 L^{-\alpha}. \tag{4.27}$$

Proof. If L is uneven, let us call $x_0 = (\lfloor L/2 \rfloor, \dots, \lfloor L/2 \rfloor) \in \mathbb{Z}^d$ the middle point of $[\![0, L-1]\!]^d$. If there is a trap on $w + \{x_0\} \times [\![\lfloor L/2 \rfloor, L]\!]$, by construction, every point in $w + [\![0, L-1]\!]^d \times \{0\}$ is L–threatened. We then use translation invariance and apply Corollary 4.7. If L is even, we have, using translation invariance and a union bound,

$$\begin{aligned} &\mathrm{P}(\exists z \in w + [\![0, L-1]\!]^d \times \{0\} \text{ s.t. } z \text{ is not } L\text{–threatened}) \\ &\quad \leqslant \mathrm{P}(\exists z \in w + [\![0, L-1]\!]^d \times \{0\} \text{ s.t. } z \text{ is not } (L-1)\text{–threatened}) \\ &\quad \leqslant 2^d\, \mathrm{P}(\exists z \in w + [\![0, L-2]\!]^d \times \{0\} \text{ s.t. } z \text{ is not } (L-1)\text{–threatened}) \\ &\quad \leqslant 2^d c (L-1)^{-\alpha}. \end{aligned}$$

The result follows by choosing c_7 accordingly. □

4.3 Number of threats on allowed paths

For $k \in \mathbb{N}$, $H \geqslant 1$ and $w \in \mathbb{Z}^{d+1}$, we let

$$\begin{aligned} B_k &= [\![-RL_k, (R+1)L_k - 1]\!]^d \times [\![0, L_k - 1]\!] \subseteq \mathbb{Z}^{d+1}; \\ R_k &= [\![0, L_k - 1]\!]^d \times \{0\}. \end{aligned}$$

We also define

$$A_{k,H}(w) = \left\{ \begin{array}{c} \text{there exists an allowed path } \sigma \text{ of length } L_k - 1 \\ \text{starting in } w + R_k \text{ such that } M(\sigma, H) < k^2 \end{array} \right\}, \tag{4.28}$$

and $A_{k,H} = A_{k,H}(0)$.

The goal of this section is to show the following result.

Lemma 4.9. *For every $\beta < \alpha$, there exist $c_9 = c_9(\alpha, \beta, c_2, c_1) > 0$ and two integers $k_0 = k_0(\alpha, \beta, c_2, c_1)$ and $H = H(\alpha, \beta, c_2, c_1)$ satisfying, for every $k \geqslant k_0$,*

$$\mathrm{P}(A_{k,H}) \leqslant c_9 L_k^{-\beta}.$$

4.3.1 Renormalization argument

For now, H is just a fixed parameter; it will be chosen in Section 4.3.2. The final goal is to prove that the probability of $A_{k,H}$ goes to zero quickly enough when k goes to infinity, for a good choice of H. In this section, we use renormalization methods to show an inequality, (4.30), which will be instrumental in the induction step in the proof of Lemma 4.9.

The core of the argument is the following. Fix $k \geqslant 3$, $H \in \mathbb{N}^*$ and suppose that $A_{k+1,H}$ occurs. We are therefore given an allowed path $\sigma : [\![0, L_{k+1} - 1]\!] \to \mathbb{Z}^{d+1}$ starting in R_{k+1} such that $M(\sigma, H) < (k+1)^2$. Recalling (4.3), this means that there are at most $(k+1)^2$ times $t \in \left\{0, H, \dots, \left\lfloor \frac{L_{k+1}-1}{H} \right\rfloor H \right\}$ such that $\sigma(t)$ is H-threatened.

Since σ is an allowed path starting in R_{k+1}, by construction it takes values in B_{k+1}. Let C_k be the subset of B_{k+1} of cardinality 4^{d+1} satisfying

$$\bigcup_{w \in C_k} (w + R_k) = B_k \cap (\mathbb{Z}^d \times L_k \mathbb{Z}). \tag{4.29}$$

Our allowed path σ can be divided into four disjoint allowed paths $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ of length $L_k - 1$, starting respectively in $w_1 + R_k$, $w_2 + R_k$, $w_3 + R_k$ and $w_4 + R_k$, where $w_1, w_2, w_3, w_4 \in C_k$.

Let us assume by contradiction that we cannot find three among these four allowed paths σ_i satisfying $M(\sigma_i, H) < k^2$. This implies that we can find at least two i's such that $M(\sigma_i, H) \geqslant k^2$. Therefore,

$$M(\sigma, H) = \sum_{i=1}^{4} M(\sigma_i, H) \geqslant 2k^2 \geqslant (k+1)^2,$$

since $k \geqslant 3$. This contradicts the assumption we made on σ.

Therefore, for three i's the event $A_{k,H}(w_i)$ occurs. Now, $A_{k,H}(w_i)$ is measurable with respect to η restricted to the box $w_i + B_k$. In particular two events $A_{k,H}(w_i)$ supported by boxes that are L_k-separated in time occur.

At the end of the day, noting that the invariance of η implies that $\mathbb{P}(A_{k,H}(w))$ does not depend on w, we have

$$\begin{aligned}\mathrm{P}(A_{k+1,H}) &\leqslant 16^{d+1}(\mathrm{P}(A_{k,H})^2 + c_2 L_k^{-\alpha}) \\ &= 16^{d+1}\,\mathrm{P}(A_{k,H})^2 + c_8 L_k^{-\alpha},\end{aligned} \tag{4.30}$$

where $c_8 > 0$.

4.3.2 Triggering

Let $k \in \mathbb{N}$ and $\beta < \alpha$. Let us set $H_k = \lfloor L_k/k^2 \rfloor$ and define C'_k to be a minimal subset of B_k satisfying

$$\bigcup_{w \in C'_k} (w + [\![0, H_k - 1]\!]^d \times \{0\}) = B_k \cap (\mathbb{Z}^d \times H_k\mathbb{Z}).$$

The cardinality of C'_k satisfies $|C'_k| \leqslant c\, k^{2(d+1)}$. Then, by Corollary 4.8,

$$\begin{aligned}\mathrm{P}(A_{k,H_k}) &\leqslant \mathrm{P}\left(\exists w \in C'_k,\ \exists z \in w + [\![0, H_k - 1]\!]^d \times \{0\},\ z \text{ is not } H_k\text{–threatened}\right) \\ &\leqslant c k^{2(d+1)} c_7 H_k^{-\alpha} \\ &\leqslant c c_7 k^{2(d+1-\alpha)} L_k^{-\alpha}.\end{aligned}$$

At the end of the day, there exists $c_9 = c_9(\alpha, \beta, c_2, c_1) > 0$ such that

$$\mathrm{P}(A_{k,H_k}) \leqslant c_9 L_k^{-\beta}. \tag{4.31}$$

4.3.3 Induction

We fix $k_0 = k_0(\alpha, \beta, c_2, c_1) \in \mathbb{N}$ satisfying

$$16^{d+1} c_9 4^{-\beta(k_0-1)} + 4^{\beta} c_8 c_9^{-1} 4^{-(\alpha-\beta)k_0} \leqslant 1. \tag{4.32}$$

We fix $H = H_{k_0} = \lfloor L_{k_0}/k_0^2 \rfloor$. By (4.31) we have

$$\mathrm{P}(A_{k_0,H}) \leqslant c_9 L_{k_0}^{-\beta}.$$

Suppose that $\mathrm{P}(A_{k,H}) \leqslant c_9 L_k^{-\beta}$ for some $k \geqslant k_0$. Then, using (4.30), we have

$$\begin{aligned}\frac{\mathrm{P}(A_{k+1,H})}{c_9 L_{k+1}^{-\beta}} &\leqslant 16^{d+1}\,\mathrm{P}(A_{k,H})^2 c_9^{-1} L_{k+1}^{\beta} + c_8 L_k^{-\alpha} c_9^{-1} L_{k+1}^{\beta} \\ &\leqslant 16^{d+1} c_9 4^{-\beta(k-1)} + 4^{\beta} c_8 c_9^{-1} 4^{-(\alpha-\beta)k} \\ &\leqslant 16^{d+1} c_9 4^{-\beta(k_0-1)} + 4^{\beta} c_8 c_9^{-1} 4^{-(\alpha-\beta)k_0} \\ &\leqslant 1,\end{aligned}$$

using (4.32). This concludes the proof of Lemma 4.9 via induction.

4.4 Moments of T_1

We can now combine Lemmas 4.4 and 4.9 to obtain bounds on the moments of T_1.

Recall that $\hat{\mathbb{P}}$ denotes the law of the random walk together with the environment conditioned on $\eta_0 = 1$.

Lemma 4.10. *If $\alpha > 0$, and $p \in (0, \alpha)$, then*

$$\hat{\mathbb{E}}[T_1^p] < \infty. \tag{4.33}$$

In particular, T_1 is $\hat{\mathbb{P}}$-almost surely finite.

Proof. The first observation is that

$$L_k \leqslant JH \Longrightarrow A_{k,H}^c \subseteq \{M_J \geqslant k^2\}. \tag{4.34}$$

Set $\beta \in (p, \alpha)$, fix $J \in \mathbb{N}$, and H as chosen in Lemma 4.9. Let us choose $k \in \mathbb{N}$ such that $L_k \leqslant JH < L_{k+1}$; in particular $k \geqslant \frac{\ln J + \ln H}{\ln 4} - 1 \geqslant c \ln J$ for some universal constant $c > 0$. Now, note that conditioned on $\eta_0 = 1$, T_1 is equal to T^0 associated to $\Sigma = \{z \in \mathbb{Z}^{d+1}, \eta_z = 1\}$. Therefore, using Lemma 4.4, we have

$$\begin{aligned}
\hat{\mathbb{P}}(T_1 > JH) &\leqslant \frac{1}{\mathrm{P}(\eta_0 = 1)} \mathrm{E}\left[(1 - c_0^H)^{M_J}\right] && \text{by Lemma 4.4} \\
&\leqslant c_1^{-1} \left(\mathrm{E}\left[(1 - c_0^H)^{M_J} \mathbf{1}_{A_{k,H}^c}\right] + \mathrm{E}\left[(1 - c_0^H)^{M_J} \mathbf{1}_{A_{k,H}}\right] \right) && \text{by (1.4)} \\
&\leqslant c_1^{-1}(1 - c_0^H)^{k^2} + c_1^{-1}\,\mathrm{P}(A_{k,H}) && \text{by (4.34)} \\
&\leqslant c_1^{-1}(1 - c_0^H)^{(c \ln J)^2} + c_1^{-1} c_9 (H/4)^{-\beta} J^{-\beta} && \text{by Lemma 4.9} \\
&\leqslant c J^{-\beta}.
\end{aligned}$$

The result follows from the fact that $\beta > p$ and

$$\hat{\mathbb{E}}[T_1^p] = \int_0^\infty \hat{\mathbb{P}}\big(T_1 \geqslant x^{1/p}\big)\, \mathrm{d}x. \tag{4.35}$$

□

4.5 Proof of Theorem 1.3

4.5.1 Law of large numbers

We now assume $\alpha > 1$ and we study the almost sure convergence of Z_t/t. For $t \geqslant 0$, set

$$K_t = \min\big\{k \geqslant 0, T_k \geqslant t\big\}. \tag{4.36}$$

We then write, for $t > 0$,

$$\frac{Z_t}{t} = \frac{Y_{K_t}}{K_t} \frac{K_t}{T_{K_t}} \frac{T_{K_t}}{t} + \frac{Z_t - Y_{K_t}}{t}. \tag{4.37}$$

Note that, since $\alpha > 1$, Lemma 4.10 yields

$$\hat{\mathbb{E}}[|Z_{T_1}|] \leqslant R\,\hat{\mathbb{E}}[T_1] < \infty. \tag{4.38}$$

Therefore, by Lemma 3.2, under $\hat{\mathbb{P}}$, $(Y_{k+1} - Y_k)_{k\geqslant 0}$ is an i.i.d. sequence of integrable random variables with distribution Z_{T_1}.

Notice also that Lemma 3.2 implies that $\hat{\mathbb{P}}$-almost surely, $T_k \underset{k\to\infty}{\longrightarrow} \infty$. Therefore $K_t \underset{t\to\infty}{\longrightarrow} \infty$ as well. At the end of the day, the strong law of large numbers implies that, $\hat{\mathbb{P}}$-almost surely,

$$\frac{Y_{K_t}}{K_t} \underset{t\to\infty}{\longrightarrow} \hat{\mathbb{E}}[Z_{T_1}]. \tag{4.39}$$

In particular, since $Z_t = (X_t, t)$, it follows from the above that', $\hat{\mathbb{P}}$-almost surely,

$$\frac{T_{K_t}}{K_t} \underset{t\to\infty}{\longrightarrow} \hat{\mathbb{E}}[T_1]. \tag{4.40}$$

Note now that $\frac{T_{K_t}}{t} = 1 + \frac{T_{K_t}-t}{t}$, where $\frac{T_{K_t}-t}{t} \leqslant \frac{|Z_t - Y_{K_t}|}{t}$. Therefore, the only thing left to do is study the convergence of the last term in (4.37), that is, $\frac{Z_t - Y_{K_t}}{t}$. Now, if we fix $\varepsilon > 0$ abd $\beta = \frac{1+\alpha}{2}$, Markov's inequality yields

$$\begin{aligned}\hat{\mathbb{P}}\left(\frac{|Z_t - Y_{K_t}|}{t} \geqslant \varepsilon\right) &\leqslant \hat{\mathbb{P}}(T_{K_t} - t \geqslant \varepsilon t/R)\\ &\leqslant \hat{\mathbb{P}}(T_1 \geqslant \varepsilon t/R) \leqslant c\,\hat{\mathbb{E}}\big[T_1^{\beta}\big]t^{-\beta},\end{aligned}$$

which is summable in t, since $\beta \in (1,\alpha)$ and $\hat{\mathbb{E}}\big[T_1^{\beta}\big] < \infty$, by Lemma 4.10. Therefore, using Borel-Cantelli's lemma, $\hat{\mathbb{P}}$-almost surely,

$$\frac{Z_t - Y_{K_t}}{t} \underset{t\to\infty}{\longrightarrow} 0. \tag{4.41}$$

Putting all the convergences together in (4.37), we obtain that $\hat{\mathbb{P}}$-almost surely,

$$\frac{Z_t}{t} \underset{t\to\infty}{\longrightarrow} \frac{\hat{\mathbb{E}}[Z_{T_1}]}{\hat{\mathbb{E}}[T_1]}. \tag{4.42}$$

From now on, we let

$$v = \frac{\hat{\mathbb{E}}[Z_{T_1}]}{\hat{\mathbb{E}}[T_1]} \in \mathbb{R}^{d+1}. \tag{4.43}$$

It remains to show that $\mathbb{P}$-almost surely, $\frac{Z_t}{t} \underset{t\to\infty}{\longrightarrow} v$. First note that if $T_0 < \infty$, for any $t > T_0$, we have

$$\frac{Z_t}{t} = \frac{Z_t - Z_{T_0}}{t - T_0}\,\frac{t - T_0}{t} + \frac{Z_{T_0}}{t},$$

where $\mathbb{P}$-almost surely, $\mathbf{1}_{T_0<\infty}\frac{t-T_0}{t} \underset{t\to\infty}{\longrightarrow} 1$ and $\mathbf{1}_{T_0<\infty}\frac{Z_{T_0}}{t} \underset{t\to\infty}{\longrightarrow} 0$.

Now, $\mathbf{1}_{T_0<\infty}(Z_{T_0+t} - Z_{T_0})_{t\geqslant 0}$ under $\mathbb{P}$ has the same distribution as $(Z_t)_{t\geqslant 0}$ under $\hat{\mathbb{P}}$. Therefore, $\mathbb{P}$-almost surely,

$$\mathbf{1}_{T_0<\infty}\frac{Z_t - Z_{T_0}}{t - T_0} \underset{t\to\infty}{\longrightarrow} v. \tag{4.44}$$

Finally, since $T_0 \leqslant T_1 < \infty$ almost surely, we obtain the result.

4.5.2 Central limit theorem

For the central limit theorem, we strongly rely on [20], following ideas of [22]. Mind that Theorem 1.4 in [20] is stated for real-valued processes, but it can easily be generalized to higher dimensions; when the proof refers to Anscombe's theorem, we use Theorem 3.1 in [16], that can easily be stated and proven in dimension d.

We assume that $\alpha > 2$. All assumptions of Theorem 1.4 in [20] (considering multidimensional processes) are satisfied (note that we need to apply the theorem with the probability measure $\hat{\mathbb{P}}$), namely:

- $(Z_t)_{t\in\mathbb{N}}$ has regenerative increments over $(T_n)_{n\in\mathbb{N}}$, which means that the
$$(Z_t - Z_{T_k},\ 0 \leqslant t \leqslant T_{k+1} - T_k)_{k\in\mathbb{N}}$$
are i.i.d. Note that this requirement is slightly stronger than Lemma 3.2, but it is clear that it is also satisfied, with the same arguments as in the proof of Lemma 3.2.

- The three quantities $\hat{\mathbb{E}}[T_1]$, $\hat{\mathbb{E}}[Z_{T_1}]$ and $\mathrm{Cov}(Z_{T_1} - vT_1|\eta_0 = 1)$ (where v is defined in (4.43)) are finite, since $\alpha > 2$ (recall Lemma 4.10).

- $\hat{\mathbb{P}}\Big(\sup_{0\leqslant t\leqslant T_1} |Z_t| < \infty\Big) = 1$, which is clear since $|Z_t| \leqslant Rt$ and $T_1 < \infty$ almost surely.

Therefore, we have the following convergence in distribution, under $\hat{\mathbb{P}}$:

$$\frac{Z_t - tv}{\sqrt{t}} \underset{t\to\infty}{\rightharpoonup} \mathcal{N}(0,\Sigma), \tag{4.45}$$

with $\Sigma = \dfrac{\mathrm{Cov}(Z_{T_1} - T_1 v|\eta_0 = 1)}{\hat{\mathbb{E}}[T_1]}$. In order to conclude, we use the same strategy as for the law of large numbers, by writing, when $T_0 < \infty$ and $t > T_0$,

$$\frac{Z_t - tv}{\sqrt{t}} = \frac{Z_t - Z_{T_0} - (t - T_0)v}{\sqrt{t - T_0}}\sqrt{\frac{t - T_0}{t}} + \frac{Z_{T_0} - T_0 v}{\sqrt{t}}. \tag{4.46}$$

Under $\mathbb{P}$, we have $\mathbf{1}_{T_0<\infty}\frac{Z_t - Z_{T_0} - (t-T_0)v}{\sqrt{t-T_0}} \rightharpoonup \mathcal{N}(0,\Sigma)$, $\mathbf{1}_{T_0<\infty}\sqrt{\frac{t-T_0}{t}} \xrightarrow{a.s.} 1$, and $\mathbf{1}_{T_0<\infty}\frac{Z_{T_0} - T_0 v}{\sqrt{t}} \xrightarrow{a.s} 0$. It follows from Slutsky's theorem that the convergence in (4.45) also holds under $\mathbb{P}$, which proves the result.

5 Examples

Let us now collect examples of environments where our main result can be applied. This amounts to verifying which environments satisfy the Random Markov Property (1.9) and the decoupling condition from Definition 1.2.

5.1 Boolean percolation

Start with a Poisson point process Φ in $\mathbb{R}^{d+1} \times (0,+\infty)$ with intensity $\mathrm{Leb}(\mathrm{d}x) \otimes \nu(\mathrm{d}\rho)$, where ν is a fixed distribution in $(0,+\infty)$ that satisfies

$$\nu[\rho,\infty) \leqslant c_{10}\rho^{-\beta}, \tag{5.1}$$

for some $\beta > 0$, and define the occupied set as the collection of open balls (in the Euclidean norm)

$$\mathcal{O} = \bigcup_{(x,\rho)\in\Phi} B(x,\rho) \subseteq \mathbb{R}^{d+1}. \tag{5.2}$$

The environment $\omega = (\omega_z)_{z\in\mathbb{Z}^{d+1}}$ is now given as

$$\omega_z = \begin{cases} 1, & \text{if } z \in \mathcal{O}, \\ 0, & \text{if } z \notin \mathcal{O}. \end{cases} \tag{5.3}$$

In order to define the field $\eta = (\eta_z)_{z\in\mathbb{Z}^{d+1}}$, consider, for $z \in \mathbb{Z}^{d+1}$, the event

$$\mathcal{A}_z = \Big\{\text{No ball } B(x,\rho) \text{ with } (x,\rho)\in\Phi \text{ intersects both } C_z^+ \text{ and } C_z^-\Big\}, \tag{5.4}$$

and set $\eta_z = \mathbf{1}_{\mathcal{A}_z}$. Notice that $\eta_z = 1$ immediately implies $\omega_z = 0$, which verifies (1.5). Furthermore, the collection (ω_z, η_z) inherits the shift invariance property from Boolean percolation.

Let us now verify that this construction yields a field that satisfies (1.9).

Lemma 5.1. *The field $(\omega,\eta) = \big((\omega,\eta)_z\big)_{z\in\mathbb{Z}^{d+1}}$ defined as above satisfies the Random Markov Property* (1.9)*.*

Proof. Let $z \in \mathbb{Z}^{d+1}$ and let W be an $\mathcal{F}_z^+$-measurable random variable. We need to prove that, for every $E \in \mathcal{F}_z^-$,

$$\mathrm{E}\big[W\eta_z\mathbf{1}_E\big] = \mathrm{E}\big[\,\mathrm{E}[W|\eta_z]\eta_z\mathbf{1}_E\big]. \tag{5.5}$$

Define now the disjoint sets

$$\begin{aligned} A^+ &= \{(x,t)\in\mathbb{R}^{d+1}\times[0,+\infty) : B(x,t)\cap C_z^+ \neq \emptyset \text{ and } B(x,t)\cap C_z^- = \emptyset\}, \\ A^- &= \{(x,t)\in\mathbb{R}^{d+1}\times[0,+\infty) : B(x,t)\cap C_z^- \neq \emptyset \text{ and } B(x,t)\cap C_z^+ = \emptyset\}, \\ A^\pm &= \{(x,t)\in\mathbb{R}^{d+1}\times[0,+\infty) : B(x,t)\cap C_z^+ \neq \emptyset \text{ and } B(x,t)\cap C_z^- \neq \emptyset\}, \end{aligned} \tag{5.6}$$

and notice that W is determined by $\Phi \cap (A^+ \cup A^\pm)$, E is determined by $\Phi\cap(A^-\cup A^\pm)$ and $\eta_z = \mathbf{1}_{\Phi\cap A^\pm=\emptyset}$. Furthermore, since the sets above are disjoint, $\Phi\cap A^+$, $\Phi\cap A^-$, and $\Phi\cap A^\pm$ are all independent. Note also that on the event that $\Phi\cap A^\pm = \emptyset$ (that is, $\eta_z = 1$), W can be replaced by $f(\Phi \cap A^+)$ where f is a measurable function. This then implies

$$\begin{aligned}
\mathrm{E}\Big[W\eta_z \mathbf{1}_E\Big] &= \mathrm{E}\Big[W\mathbf{1}_{\Phi\cap A^\pm=\emptyset}\mathbf{1}_E\Big] \\
&= \mathrm{E}\Big[f(\Phi\cap A^+)\mathbf{1}_{\Phi\cap A^\pm=\emptyset}\mathbf{1}_E\Big] \\
&= \mathrm{E}\Big[\mathrm{E}\ \big[f(\Phi\cap A^+)\big]\mathbf{1}_{\Phi\cap A^\pm=\emptyset}\mathbf{1}_E\Big] \\
&= \mathrm{E}\Big[\mathrm{E}\ \big[f(\Phi\cap A^+)\,|\,\eta_z=1\big]\mathbf{1}_{\Phi\cap A^\pm=\emptyset}\mathbf{1}_E\Big] \\
&= \mathrm{E}\Big[\mathrm{E}\ \big[W|\eta_z=1\big]\eta_z\mathbf{1}_E\Big] \\
&= \mathrm{E}\Big[\mathrm{E}\ \big[W|\eta_z\big]\eta_z\mathbf{1}_E\Big].
\end{aligned} \tag{5.7}$$

In the third equality, we used that $\Phi \cap A^+$ is independent of $\Phi \cap (A^- \cup A^\pm)$. In the fourth equality, we used that $\Phi \cap A^+$ is independent of η. This concludes the proof. □

Finally, it still remains to verify that the field η satisfies the decoupling property.

Proposition 5.2. *There exists $c_{11} = c_{11}(\beta, d) > 0$ such that the function defined by*

$$\varepsilon(r, h, s) = c_{11} r^d h s^{-\beta+d+1} \tag{5.8}$$

is a decoupling function for η (recall Remark 2).

In order to prove the proposition above, we need a preliminary lemma which requires some additional notation. Given $s > 0$, define

$$\mathcal{A}_z^s = \left\{ \begin{array}{c} \text{No disk } B(x,\rho) \text{ with } (x,\rho) \in \Phi \text{ and } \mathrm{d}(x,z) < \frac{s}{2} \\ \text{intersects both } C_z^+ \text{ and } C_z^- \end{array} \right\}, \tag{5.9}$$

and set $\eta_z^s = \mathbf{1}_{\mathcal{A}_z^s}$. Notice that η_z^s is completely determined by the Φ restricted to the set $\big(z + \big(-\frac{s}{2}, \frac{s}{2}\big)^{d+1}\big) \times [0, +\infty)$. We now prove the following lemma

Lemma 5.3. *Assume $\beta > d$. There exists a constant $c_{12} = c_{12}(\beta, d) > 0$ such that, for any $s > 0$ and $z \in \mathbb{Z}^{d+1}$,*

$$\mathrm{P}\left(\eta_z \neq \eta_z^s\right) \leqslant c_{12} s^{-\beta+d+1}. \tag{5.10}$$

Proof. Assume without loss of generality that $z = 0$ and notice that

$$\mathrm{P}\left(\eta_0 \neq \eta_0^s\right) \leqslant \mathrm{P}\left(\mathcal{B}_s\right), \tag{5.11}$$

where

$$\mathcal{B}_s = \left\{ \begin{array}{c} \text{There exists a disk } B(x,\rho) \text{ with } (x,r) \in \Phi \text{ and } |x| > \frac{s}{2} \\ \text{that intersects both } C_0^+ \text{ and } C_0^- \end{array} \right\}. \tag{5.12}$$

In order to bound the probability of $\mathcal{B}_s$, we further introduce the event

$$\tilde{\mathcal{B}}_s = \left\{ \begin{array}{c} \text{There exists a disk } B(x,\rho) \text{ with } (x,\rho) \in \Phi, \\ x \in [0,+\infty)^{d+1} \text{ and } |x| > \frac{s}{2} \text{ that intersects } C_0^- \end{array} \right\}, \tag{5.13}$$

and notice that rotation invariance implies

$$\mathrm{P}\left(\mathcal{B}_s\right) \leqslant 2^{d+1}\, \mathrm{P}\left(\tilde{\mathcal{B}}_s\right). \tag{5.14}$$

We now notice that the distance between a point $x \in [0,+\infty)^{d+1}$ and C_0^- is lower bounded by the distance from x to the set $\{(x,t) \in [0,+\infty)^d \times (-\infty, 0] : |x| = -Rt\}$, which yields the bound

$$\mathrm{d}(x, C_0^-) \geqslant c_{13}|x|, \tag{5.15}$$

for some $c_{13} > 0$. From this is follows immediately that

$$\begin{aligned} \mathrm{P}\left(\tilde{\mathcal{B}}_s\right) &\leqslant \int_{[0,+\infty)^{d+1}\setminus B\left(0,\frac{s}{2}\right)} \nu\big[\,\mathrm{d}(x, C_0^-), +\infty)\,\mathrm{d}\,x \\ &\leqslant \int_{[0,+\infty)^{d+1}\setminus B\left(0,\frac{s}{2}\right)} \nu\big[c_{13}|x|, +\infty)\,\mathrm{d}\,x \\ &\leqslant \int_{[0,+\infty)^{d+1}\setminus B\left(0,\frac{s}{2}\right)} c_{10}c_{13}^{-\beta}|x|^{-\beta}\,\mathrm{d}\,x \\ &\leqslant c_{10}c_{13}^{-\beta}c\int_{c'\frac{s}{2}}^{+\infty} u^{-\beta+d}\,\mathrm{d}\,u \\ &\leqslant \tilde{c}s^{-\beta+d+1}, \end{aligned} \tag{5.16}$$

concluding the proof. □

We are now ready to prove Proposition 5.2.

Proof of Proposition 5.2. Let $A = \prod_{i=1}^{d+1}[\![a_i, b_i]\!]$ and $B = \prod_{i=1}^{d+1}[\![a_i', b_i']\!]$ be two boxes satisfying (1.12) and $f_1 : \{0,1\}^A \to \{0,1\}$, $f_2 : \{0,1\}^B \to \{0,1\}$. Consider now the events

$$\begin{aligned} G_A &= \Big\{\eta_z = \eta_z^s,\ \text{for all } z \in A\Big\}, \\ G_B &= \Big\{\eta_z = \eta_z^s,\ \text{for all } z \in B\Big\}, \end{aligned} \tag{5.17}$$

and notice that G_A and G_B are determined by Φ restricted to the sets $\prod_{i=1}^{d+1}\left(a_i - \frac{s}{2}, b_i + \frac{s}{2}\right) \times \left[0,+\infty\right)$ and $\prod_{i=1}^{d+1}\left(a_i' - \frac{s}{2}, b_i' + \frac{s}{2}\right) \times \left[0,+\infty\right)$, respectively. Due to the assumption that $\mathrm{sep}(A,B) \geqslant s$, these two sets are disjoint and thus the configurations η_z^s in the boxes A and B are independent. This then yields

$$\begin{aligned} \mathrm{E}[f_1(\eta_A)f_2(\eta_B)] &\leqslant \mathrm{E}\left[f_1\left(\eta_A^s\right)f_2(\eta_B^s)\right] + \mathrm{P}(G_A) + \mathrm{P}(G_B) \\ &= \mathrm{E}\left[f_1\left(\eta_A^s\right)\right]\mathrm{E}\left[f_2\left(\eta_B^s\right)\right] + \mathrm{P}(G_A) + \mathrm{P}(G_B) \\ &\leqslant \mathrm{E}[f_1(\eta_A)]\,\mathrm{E}[f_2(\eta_B)] + 2\,\mathrm{P}(G_A) + 2\,\mathrm{P}(G_B). \end{aligned} \tag{5.18}$$

To conclude, observe that union bound together with Lemma 5.3 implies

$$\mathrm{P}(G_A) \leqslant c_{12}r^d h s^{-\beta+d+1}. \tag{5.19}$$

An analogous bound is satisfied by $\mathrm{P}(G_B)$, concluding the proof. □

5.2 Independent renewal chains

We now assume that the environment ω is composed of independent copies of a renewal chain than we introduce now. Fix an aperiodic[2] reference distribution μ on the non-negative integers with finite expectation. For each $x \in \mathbb{Z}^d$, the environment $\omega(x, \cdot\,)$ will be given by an independent copy of a renewal process, defined as the Markov chain with transition probabilities $p(k, k-1) = 1$, if $k \geqslant 1$, and $p(0,k) = \mu(k)$. We call μ the *interarrival distribution* of the renewal chain and assume that the initial state $\omega(x, 0)$ is such that each of these chains is stationary. This amounts to taking $\omega(x,0)$ with distribution $\hat{\mu}$ defined as

$$\hat{\mu}(k) = \frac{1}{\mathrm{E}[\xi]+1}\,\mathrm{P}(\xi \geqslant k), \text{ with } \xi \sim \mu. \tag{5.20}$$

Let us note for future reference that, if $\xi \sim \mu, \hat{\xi} \sim \hat{\mu}$ and $\mathrm{E}(\xi^{1+\beta}) < \infty$ for some $\beta > 0$, then

$$\mathrm{E}[\hat{\xi}^{\beta}] < \infty. \tag{5.21}$$

Let us also recall the following theorem from [21].

Theorem 5.4 ([21])**.** *Let $\mathrm{P}_\mu^{\nu,\nu'}$ denote the distribution of two independent renewal chains X and $\tilde{X}$ with independent initial distributions ν and ν' and same interarrival distribution μ. Denote the coupling time of these chains by*

$$T = \inf\{n \geqslant 0 : X_n = \tilde{X}_n = 0\}. \tag{5.22}$$

1. *If μ is aperiodic and has finite $1+\beta$ moment for some $\beta > 0$, and both ν and ν' have finite β moment, then $\mathrm{E}_\mu^{\nu,\nu'}\left[T^\beta\right] < \infty$.*
2. *If μ is aperiodic and has finite $1+\beta$ moment for some $\beta \geqslant 0$, and both ν and ν' have finite $1+\beta$ moment, then $\mathrm{E}_\mu^{\nu,\nu'}\left[T^{1+\beta}\right] < \infty$.*

5.2.1 Graphical construction

Let

$$\gamma_\mu := \inf_{k:\ \hat{\mu}(k)>0} \frac{\mu(k)}{\hat{\mu}(k)}. \tag{5.23}$$

We assume[3] $\gamma_\mu > 0$. Notice that any non-trivial μ for which $\gamma_\mu > 0$ is necessarily aperiodic. Consider three i.i.d. independent collections $(\hat{W}_t^x)_{x\in\mathbb{Z}^d,t\in\mathbb{Z}}$, with distribution $\hat{\mu}$, $(Z_t^x)_{x\in\mathbb{Z}^d,t\in\mathbb{Z}}$, with distribution $\mathrm{Ber}(\gamma_\mu)$, and $(Y_t^x)_{x\in\mathbb{Z}^d,t\in\mathbb{Z}}$, with distribution[4] $\frac{1}{1-\gamma_\mu}(\mu - \gamma_\mu\hat{\mu})$. Note that the latter is indeed a probability measure on the non-negative integers, thanks to the definition of γ_μ. We now define

$$W_t^x := Z_t^x \hat{W}_t^x + (1 - Z_t^x) Y_t^x. \tag{5.24}$$

The W_t^x are i.i.d. with distribution μ.

We aim to construct $(\omega(x,t))_{x\in\mathbb{Z}^d,t\in\mathbb{Z}}$ such that

[2] We say that μ is aperiodic if $\gcd\{k \in \mathbb{Z}_+ : \mu(k+1) > 0\} = 1$.
[3] There are non-trivial distributions for which $\gamma_\mu = 0$ (take $\mu(0) = 0$ or $\mu(2k) \propto 2^{-k}$, $\mu(2k+1) \propto k^{-2}$).
[4] $\gamma_\mu < 1$ unless $\mu = \delta_0$; in the latter case the distribution of Y is irrelevant and can be chosen arbitrarily.

- the processes $(\omega(x,\cdot))_{x\in\mathbb{Z}^d}$ are independent;
- for all $x \in \mathbb{Z}^d$, $(\omega(x,t))_{t\in\mathbb{Z}}$ is a stationary renewal chain that uses the collection W^x to determine its jumps, i.e. $\omega(x,t) = W^x_t$ if $\omega(x,t-1) = 0$.

By independence, it suffices to fix $x \in \mathbb{Z}^d$ and construct $(\omega(x,t))_{t\in\mathbb{Z}}$ satisfying the second property. For $K \in \mathbb{N}$, define $X^{(K)}$ as the renewal chain started from 0 at time $-K$ whose jumps are determined by W^x: for $t \geqslant -K$,

$$\begin{aligned} &X^{(K)}_{-K} = 0, \\ &X^{(K)}_{t+1} = \begin{cases} X^{(K)}_t - 1 & \text{if } X^{(K)}_t \geqslant 1, \\ W^x_{t+1} & \text{if } X^{(K)}_t = 0. \end{cases} \end{aligned} \tag{5.25}$$

Lemma 5.5. *If μ is aperiodic and has finite moment of order $1+\delta$, for some $\delta > 0$,*

1. *for any $t \in \mathbb{Z}$, $\omega(x,t) := \lim_{K\to\infty} X^{(K)}_t$ exists a.s.;*
2. *the process $\omega(x,\cdot)$ thus obtained is a stationary renewal chain whose jumps are determined by W^x.*

Proof. Since our construction is invariant under time translations, it is enough to prove the first item for $t = 0$. We claim that there exists $c \in \mathbb{R}_+$ such that

$$\mathrm{P}(X^{(K)}_0 \neq X^{(K+1)}_0) \leqslant cK^{-(1+\delta)}. \tag{5.26}$$

Indeed, let $T := \inf\{k \geqslant 0\colon X^{(K)}_{-K+k} = X^{(K+1)}_{-K+k}\}$ denote the coupling time of the chains started at time $-K$ from 0 and from $X^{(K+1)}_{-K} = W^x_{-K}$. Note that, up to time T, the chains $(X^{(K)}_{-K+t})_{t\geqslant 0}$ and $(X^{(K+1)}_{-K+t})_{t\geqslant 0}$ are independent (they are constructed using disjoint sets of independent variables). By Theorem 5.4, T has finite $1+\delta$ moment. Also note that the distribution of T does not depend on K.

It remains to notice

$$\mathrm{P}\left(X^{(K)}_0 \neq X^{(K+1)}_0\right) \leqslant cK^{-(1+\delta)}. \tag{5.27}$$

Indeed, let $T := \inf\{k \geqslant 0\colon X^{(K)}_{-K+k} = X^{(K+1)}_{-K+k}\}$ denote the coupling time of the chains started at time $-K$ from 0 and from $X^{(K+1)}_{-K} = W^x_{-K}$. Note that, up to time T, the chains $(X^{(K)}_{-K+t})_{t\geqslant 0}$ and $(X^{(K+1)}_{-K+t})_{t\geqslant 0}$ are independent (they are constructed using disjoint sets of independent variables). By Theorem 5.4, T has finite $1+\delta$ moment. Furthermore, distribution of T does not depend on K.

It remains to notice that

$$\mathrm{P}\left(X^{(K)}_0 \neq X^{(K+1)}_0\right) \leqslant \mathrm{P}(T > K) \tag{5.28}$$

and to apply Markov's inequality.

To show that $(X_0^{(K)})_K$ converges almost surely, let us now estimate

$$\begin{aligned}\mathrm{P}(\{\exists K_0 \text{ s.t. } \forall K \geqslant K_0,\ X_0^{(K)} = X_0^{(K_0)}\}^c) &\leqslant \mathrm{P}(\forall K_0,\ \exists K \geqslant K_0 \text{ s.t. } X_0^{(K)} \neq X_0^{(K+1)}) \\ &\leqslant \inf_{K_0} \sum_{K \geqslant K_0} \mathrm{P}(X_0^{(K)} \neq X_0^{(K+1)}) \\ &\leqslant c \inf_{K_0} \sum_{K \geqslant K_0} \frac{1}{K^{1+\delta}} = 0,\end{aligned}$$

since $\delta > 0$. The first item is proved.

It is now immediate to check that $\omega(x, \cdot)$ is a renewal chain whose jumps are determined by W^x: by our construction, if $(X_{t_0}^{(K)})_{K \geqslant K_0}$ is constant, so is $(X_t^{(K)})_{K \geqslant K_0}$ for any $t \geqslant t_0$. Therefore, for all $t_0 \in \mathbb{Z}$ there exists a random K s.t. $(\omega(x,t))_{t \geqslant t_0} = (X_t^{(K)})_{t \geqslant t_0}$.

To show that $\omega(x, 0) \sim \hat{\mu}$, consider $\hat{X}^{(K)}$ the (stationary) chain started at time $-K$ from an independent variable $\hat{X}_{-K}^{(K)} \sim \hat{\mu}$ and using the W^x variables as jumps. Then

$$\begin{aligned}\mathrm{P}\left(\omega(x,0) \neq \hat{X}_0^{(K)}\right) &\leqslant \mathrm{P}\left(\omega(x,0) \neq X_0^{(K)}\right) + \mathrm{P}\left(X_0^{(K)} \neq \hat{X}_0^{(K)}\right) \\ &\leqslant \mathrm{P}\left(\omega(x,0) \neq X_0^{(K)}\right) + \mathrm{P}(\hat{T} > K),\end{aligned} \tag{5.29}$$

where $\hat{T}$ denotes the coupling time between $(X_t^{(K)})_{t \geqslant -K}$ and $(\hat{X}_t^{(K)})_{t \geqslant -K}$. By Theorem 5.4, $\hat{T}$ has finite moment of order δ, implying that the right-hand side above tends to 0 as K grows. $\square$

5.2.2 Construction and properties of η

We now build η. Let us fix $(x_0, t_0) \in \mathbb{Z}^d \times \mathbb{Z}$ and construct $\eta(x_0, t_0)$. Define the outer boundary of $C^-_{(x_0,t_0)}$ as

$$\partial C_0^- = \{(x, -\lceil |x|/R \rceil + 1), x \in \mathbb{Z}^d\}, \quad \partial C^-_{(x_0,t_0)} = \partial C_0^- + (x_0, t_0), \tag{5.30}$$

and $t^x := -\lceil |x|/R \rceil + 1$.

Consider the random times

$$T^x = T^x(x_0, t_0) := \inf\{t \geqslant t_0 + t^x : \omega(x,t) = 0 \text{ and } Z_{t+1}^x = 1\}. \tag{5.31}$$

We can now define η.

Definition 5.6. *Let $\eta(x_0, t_0)$ be the indicator function of the event*

$$\{\forall x \in \mathbb{Z}^d \setminus \{x_0\},\ T^x < t_0 + |x - x_0|/R, \text{ and } T^{x_0} = t_0\}. \tag{5.32}$$

We now verify that, with this construction, (ω, η) satisfies the hypotheses of our theorem. Note that η is not exactly measurable w.r.t. ω itself, but rather w.r.t. the random variables which we use to sample ω. In other words, to apply Theorem 1.3, we need to consider that the environment is given by $(\omega(x,t), \hat{W}_t^x, Z_t^x, Y_t^x)_{(x,t) \in \mathbb{Z}^d \times \mathbb{Z}}$.

Let us first estimate the probability that $\eta(x_0, t_0) = 1$.

Lemma 5.7. *If μ is a non-trivial distribution with $\gamma_\mu > 0$ (see (5.23)) and finite moment of order $1+\beta$, for some $\beta > d$, then*

$$\mathrm{P}(\eta(x_0, t_0) = 1) > 0. \tag{5.33}$$

Proof. By translation invariance we can assume $x_0 = t_0 = 0$. By independence of the renewal chains at different sites,

$$\begin{aligned} \mathrm{P}(\eta(0,0) = 1) &= \mathrm{P}(T^0 = 0) \prod_{x \neq 0} \mathrm{P}(T^x < |x|/R) \\ &\geqslant \mathrm{P}(T^0 = 0) \prod_{\substack{|x| \leqslant R/2 \\ x \neq 0}} \mathrm{P}\Big(T^x < \frac{|x|}{R}\Big) \prod_{|x| > R/2} \mathrm{P}\Big(T^x - t^x < 2\frac{|x|}{R} - 1\Big). \end{aligned} \tag{5.34}$$

Let us reformulate the distribution of $T^x - t^x$. Let $(Z_i)_{i\in\mathbb{N}}, (Y_i)_{i\in\mathbb{N}}, \hat{W}$ be independent random variables, with $Z_i \sim \mathrm{Ber}(\gamma_\mu)$, $Y_i \sim \frac{1}{1-\gamma_\mu}(\mu - \gamma_\mu \hat{\mu})$, $\hat{W} \sim \hat{\mu}$. Let $S = \inf\{i \geqslant 1 \colon Z_i = 1\}$. Then

$$T^x - t^x \sim \hat{W} + \sum_{i=1}^{S-1} (Y_i + 1). \tag{5.35}$$

In particular, $T^x - t^x$ has finite moment of order β.

Since $\beta > d$, by Markov inequality, the last product converges, and it is not difficult to check that every term in the (finite) remaining product is positive, concluding the proof. □

We now need to check that η satisfies the decoupling property in 1.2. To that end, we proceed as in the case of Boolean percolation, by introducing finite range approximations of η. For any integer $s \geqslant 1$, let $\eta^s(x_0, t_0)$ be the indicator function of the event

$$\{\forall x \in \mathbb{Z}^d \setminus \{x_0\} \text{ s.t. } |x - x_0| \leqslant s, T^x < t_0 + |x - x_0|/R, \text{ and } T^{x_0} = t_0\}. \tag{5.36}$$

Lemma 5.8. *Assume that μ is a non-trivial distribution with $\gamma_\mu > 0$ and finite moment of order $1+\beta$, for some $\beta > d+1$. There exists $c_{14} > 0$ such that, for any integer $s \geqslant 1$ and $(x,t) \in \mathbb{Z}^d \times \mathbb{Z}_+$,*

$$\mathbb{P}\big(\eta(x,t) \neq \eta^s(x,t)\big) \leqslant c_{14} s^{d-\beta+1}. \tag{5.37}$$

Proof. By translation invariance we can assume $x_0 = t_0 = 0$. By union bound, for $s > R/2$,

$$\mathbb{P}\big(\eta(x,t) \neq \eta^s(x,t)\big) \leqslant \sum_{|x|>s} \mathrm{P}(T^x \geqslant |x|/R) \tag{5.38}$$

$$\leqslant c \sum_{|x|>s} \frac{1}{|x|^\gamma} \leqslant c' \sum_{k>s} \frac{1}{k^{\beta-d}} \leqslant c'' s^{d-\beta+1}, \tag{5.39}$$

where we used Markov inequality and (5.35) in the second inequality. It then remains to adjust the value of c_{14} to accommodate $s \leqslant R/2$.

□

In order to verify the decoupling property, we an analogous result that holds for one-dimensional renewal chains. The proof is basically the same as that of [18, Lemma 2.1].

Lemma 5.9. *Assume μ is a non-trivial distribution with $\gamma_\mu > 0$ and finite moment of order $1+\beta$, for some $\beta > 0$. Consider a stationary renewal process $(X_n)_{n\in\mathbb{Z}}$ with interarrival distribution μ. There exists a positive constant c_{15} such that, for all $m \in \mathbb{Z}, n \in \mathbb{Z}_+$ and any pair of events A and B with*

$$A \in \sigma(X_k : k \leqslant m) \quad \text{and} \quad B \in \sigma(X_k : k \geqslant m+n), \tag{5.40}$$

it holds that

$$\mathrm{P}(A \cap B) \leqslant \mathrm{P}(A)\,\mathrm{P}(B) + c_{15} n^{-\beta}. \tag{5.41}$$

We are now in position to prove the decoupling property for η.

Proposition 5.10. *Assume that μ is a non-trivial distribution with $\gamma_\mu > 0$ and has finite moment of order $1+\beta$, for some $\beta > d+1$. There exists a positive constant c_{16} such that the function*

$$\varepsilon(r,h,s) = c_{16}(r+s)^d h s^{d-\beta+1}, \tag{5.42}$$

is a decoupling function for the field η (see Remark 2).

Proof. Let $A = \prod_{i=1}^{d+1} [\![a_i, b_i]\!]$ and $B = \prod_{i=1}^{d+1} [\![a'_i, b'_i]\!]$ be two boxes satisfying (1.12) and consider two measurable functions $f_1 : \{0,1\}^A \to \{0,1\}$, $f_2 : \{0,1\}^B \to \{0,1\}$. Assume without loss of generality that $b_{d+1} \leqslant a'_{d+1}$.

Lemma 5.8 implies

$$\begin{aligned} \mathrm{P}\left(f_1(\eta_{|A}) \neq f_1(\eta^{s/3}_{|A})\right) &\leqslant \mathrm{P}\left(\eta_{x,t} \neq \eta^{s/3}_{x,t}, \text{ for some } (x,t) \in A\right) \\ &\leqslant c_{14} 3^{-d+\beta-1} r^d h s^{d-\beta+1}. \end{aligned} \tag{5.43}$$

Observe now that $f_1(\eta^{s/3}_{|A})$ depends only on the fields ω and W restricted to the set $A' = \prod_{i=1}^{d+1} [\![a_i - \frac{s}{3}, b_i + \frac{s}{3}]\!]$. Similarly,

$$\mathrm{P}\left(f_2(\eta_{|B}) \neq f_2(\eta^{s/3}_{|B})\right) \leqslant c_{14} 3^{-d+\beta-1} r^d h s^{d-\beta+1}, \tag{5.44}$$

and $f_2(\eta^{s/3}_{|B})$ depends only on the fields ω and W restricted to $B' = \prod_{i=1}^{d+1} [\![a'_i - \frac{s}{3}, b'_i + \frac{s}{3}]\!]$. In particular, by successive conditioning, Lemma 5.9 yields

$$\mathrm{E}\left[f_1(\eta^{(s/3)}_{|A}) f_2(\eta^{(s/3)}_{|B})\right] \leqslant \mathrm{E}\left[f_1(\eta^{(s/3)}_{|A})\right] \mathrm{E}\left[f_2(\eta^{(s/3)}_{|B})\right] + c_{15} 3^\beta (r+s)^d s^{-\beta}. \tag{5.45}$$

This, when combined with (5.43), implies

$$\begin{aligned} \mathrm{E}[f_1(\eta_{|A}) f_2(\eta_{|B})] &\leqslant \mathrm{E}[f_1(\eta_{|A})]\,\mathrm{E}[f_2(\eta_{|B})] \\ &\qquad + c_{15} 3^\beta (r+s)^d s^{-\beta} + 4 c_{14} 3^{-d+\beta-1} r^d h s^{d-\beta+1}, \end{aligned} \tag{5.46}$$

concluding the proof. □

It remains to prove that the field satisfies the Random Markov Property (1.9).

Proposition 5.11. *If μ is a non-trivial distribution with $\gamma_\mu > 0$ and finite moment of order $1+\beta$, for some $\beta > d$, the field (ω, η) satisfies the Random Markov Property* (1.9)*.*

Proof. Fix $z \in \mathbb{Z}^d \times \mathbb{Z}_+$, a $\mathcal{F}_z^+$-measurable function $f(\omega_{|C_z^+})$ and an event $E \in \mathcal{F}_z^-$. Our goal is to prove that

$$\mathrm{E}\left[f(\omega_{|C_z^+})\eta_z \mathbf{1}_E\right] = \mathrm{E}\left[\,\mathrm{E}[f(\omega_{|C_z^+})|\eta_z]\eta_z \mathbf{1}_E\right] \tag{5.47}$$

By space-time translation invariance, we may and do assume $z = 0$. We also assume that f has finite space support: there exists $\Lambda \subset \mathbb{Z}^d$ such that f depends only on ω restricted to $C_0^+ \cap (\Lambda \times \mathbb{Z})$. The general claim follows from this particular case through standard arguments.

Let us introduce some notation. For $x \in \mathbb{Z}^d$, $t \in \mathbb{Z}$, let

$$\mathcal{G}_t^x = \sigma\left(\hat{W}_s^x, Z_{s+1}^x, Y_s^x; s \leqslant t\right), \tag{5.48}$$

and notice that $T^x (= T^x(0,0))$ defined in (5.31) is a finite stopping time for the filtration $(\mathcal{G}_t^x)_t$. Let $\mathcal{G}_{T^x}^x$ be the associated stopped σ-algebra and $\mathcal{G} = \otimes_{x\in\mathbb{Z}^d}\mathcal{G}_{T^x}^x$. Finally, we define

$$\mathcal{L} = \{(x, T^x), x \in \Lambda\}, \tag{5.49}$$

on the event $\eta_0 = 1$, and $\mathcal{L} = \emptyset$ else. Let $\mathfrak{L}$ be the set of possible non-empty values for $\mathcal{L}$.

For $L = \{(x, t_x), x \in \Lambda\} \in \mathfrak{L}$, we define $\mathcal{G}_L := (\otimes_{x\in\Lambda}\mathcal{G}_{t_x}) \otimes (\otimes_{x\in\Lambda^c} G_{T^x}^x)$, and $\mathcal{H}_L = \otimes_{x\in\Lambda}\sigma\left(\hat{W}_s^x, Z_{s+1}^x, Y_s^x; s > t_x\right)$.

The following claims imply (5.47):

1. $\eta_0 = \sum_{L\in\mathfrak{L}} \mathbf{1}_{\mathcal{L}=L}$;
2. for $L \in \mathfrak{L}$, $\{\mathcal{L} = L\}$ is $\mathcal{G}_L$-measurable;
3. for $L \in \mathfrak{L}$, on $\{\mathcal{L} = L\}$, $\mathbf{1}_E$ coincides with a $\mathcal{G}_L$-measurable variable G^L;
4. for $L \in \mathfrak{L}$, on $\mathcal{L} = L$, $\omega_{|C_0^+\cap(\Lambda\times\mathbb{Z})}$ coincides with a $\mathcal{H}_L$-measurable random variable ω^L whose distribution does not depend on L.

Let us check that we reach the desired conclusion before proving the claims.

$$\begin{aligned}
\mathrm{E}[f(\omega_{|C_0^+})\eta_0 \mathbf{1}_E] &= \sum_{L\in\mathfrak{L}} \mathrm{E}[f(\omega_{|C_0^+})\mathbf{1}_{\mathcal{L}=L}\mathbf{1}_E] && (5.50)\\
&= \sum_{L\in\mathfrak{L}} \mathrm{E}[f(\omega^L)\mathbf{1}_{\mathcal{L}=L}G^L] && (5.51)\\
&= \sum_{L\in\mathfrak{L}} \mathrm{E}[f(\omega_{|C_0^+})]\,\mathrm{E}[\mathbf{1}_{\mathcal{L}=L}\mathbf{1}_E] && (5.52)\\
&= \mathrm{E}[f(\omega_{|C_0^+})]\,\mathrm{E}[\eta_0\mathbf{1}_E], && (5.53)
\end{aligned}$$

where the first claim was used in the first and last lines, the other claims in the second line. In the third line, we made use of the independence of $\mathcal{G}_L$ and $\mathcal{H}_L$ as well as the third and fourth claims. By taking E as the whole sample space above we get

$$\begin{aligned}\mathrm{E}[f(\omega_{|C_0^+})|\eta_0 = 1] &= \mathrm{P}(\eta_0 = 1)^{-1}\,\mathrm{E}[f(\omega_{|C_0^+})\mathbf{1}_{\eta_0=1}] \\ &= \mathrm{P}(\eta_0 = 1)^{-1}\,\mathrm{E}[f(\omega_{|C_0^+})\eta_0] \\ &= \mathrm{P}(\eta_0 = 1)^{-1}\,\mathrm{E}[f(\omega_{|C_0^+})]\,\mathrm{E}[\eta_0] \\ &= \mathrm{E}[f(\omega_{|C_0^+})],\end{aligned} \tag{5.54}$$

so that, since $\mathrm{E}[f(\omega_{|C_0^+})|\eta_0]\eta_0 = \mathrm{E}[f(\omega_{|C_0^+})|\eta_0 = 1]\eta_0$,

$$\mathrm{E}[\mathrm{E}[f(\omega_{|C_0^+})|\eta_0]\eta_0\mathbf{1}_E] = \mathrm{E}[f(\omega_{|C_0^+})]\,\mathrm{E}[\eta_0\mathbf{1}_E]. \tag{5.55}$$

Let us now prove the claims. The first and second ones are clear from the definition of $\mathcal{L}$ and $\mathcal{G}_L$, let us focus on the third one. Let $L = \{(x, t_x), x \in \Lambda\} \in \mathfrak{L}$. $\mathbf{1}_E$ is a function of $(\omega_z, \eta_z)_{z\in C_0^-}$. By definition of $\mathfrak{L}$, ω_z is $\mathcal{G}_L$-measurable for $z \in C_0^-$. Let us check that η_z coincides with a $\mathcal{G}_L$-measurable random variable for $z = (x_0, t_0) \in C_0^-$. Let $\tilde{T}^x = T^x(x_0, t_0)$ as in (5.31). Since $\tilde{T}^x \leqslant T^x$ for any $x \in \Lambda^c$, $\{\tilde{T}^x < t_0 + |x - x_0|/R\}$ is $\mathcal{G}^x_{T^x}$-measurable. For $x \in \Lambda$, either $t_x \geqslant t_0+|x-x_0|/R$, in which case $\{\tilde{T}^x < t_0+|x-x_0|/R\}$ is $\mathcal{G}_{t_x}$-measurable; or $t_x < t_0+|x-x_0|/R$ and on $\mathcal{L} = L$ the events $\{\tilde{T}^x < t_0 + |x - x_0|/R\}$ and $\{\tilde{T}^x < t_x\}$ coincide since $\tilde{T}^x \leqslant T^x$.

Finally, it remains to verify the fourth claim. On $\{\mathcal{L} = L\}$, $\omega_{|C_0^+}$ coincides with the process started from the $\hat{W}^x_{T^x+1}$ on $\{(x, T^x + 1), x \in \Lambda\}$ which uses (W^x_t) with $(x, t) \in \{(x, t) \colon x \in \Lambda, t > T^x + 1\}$ as jumps. This process is stationary and, in particular, its distribution in C_0^+ does not depend on L. It is also clearly $\mathcal{H}_L$-measurable.

This concludes the proof of the claims, and thus of the Random Markov Property. □

Corollary 5.12. *If μ is a non-trivial distribution with $\gamma_\mu > 0$ and finite moment of order $1+\delta$, for some $\delta > d + 1$, Theorem 1.3 applies to the environment given by independent stationary renewal chains with interarrival distribution μ, with $\alpha = \delta - d - 1$.*

6 Open problems

We finish the paper with two simple examples of dynamics for which the techniques presented have not proved sufficient to establish a CLT. Namely, we explain in detail Finitary Factors of i.i.d. fields and Kinetically Constrained Models. Other interesting open problems are: the Renewal Contact Process [15] and Gaussian Fields with fast decaying covariances [5].

Finitary factors of i.i.d. A random environment $(\omega_z)_{z\in\mathbb{Z}^{d+1}}$ is said to be a *factor of i.i.d.* if there exists a measurable function $f : [0, 1]^{\mathbb{Z}^{d+1}} \to S$ such that $\omega_z = f(V \circ \theta_z)$, for all $z \in \mathbb{Z}^{d+1}$ where $V = (V_{z'})_{z'\in\mathbb{Z}^{d+1}}$ is an i.i.d. field of uniform $[0, 1]$ variables and θ_z stands for the shift by z.

One way to further improve its mixing conditions is to impose that it be a *finitary factor of i.i.d.* (ffiid). Intuitively speaking, we say ω is a ffiid if for every $v \in [0, 1]^{\mathbb{Z}^{d+1}}$, there exists a finite (and random) coding radius $\rho < \infty$, such that knowing the values of the uniform variables inside the ball $B(0, \rho)$, fully determines the value of f. See [19] for a very interesting application of this definition on the subject of proper colorings of $\mathbb{Z}^d$.

Assuming that the coding radius ρ has light tails, can we prove a CLT for the random walk on top of a ffiid field? Our attempts to find an appropriate definition of η that would satisfy Definition 1.1 have failed.

Kinetically constrained models Kinetically constrained models (KCM) are a class of interacting particle systems whose non-equilibrium study poses many challenges. They evolve under (reversible) Glauber dynamics, subject to a constraint that forbids any update unless a certain neighborhood is free of particles. We refer to [9] for a general definition. The most emblematic example of such systems is the East model, which in dimension 1 can be defined through its generator

$$\mathcal{L}f(\sigma) = \sum_{x \in \mathbb{Z}} (1 - \sigma(x+1))(p(1 - \sigma(x)) + (1-p)\sigma(x)) \left[f(\sigma^x) - f(\sigma) \right], \tag{6.1}$$

where $p \in (0, 1)$ is a density parameter and σ^x denotes the configuration $\sigma \in \{0, 1\}^{\mathbb{Z}}$ flipped at x.

KCM provide a class of environments that display fast, but non-uniform mixing. For instance, the East model is exponentially mixing in $L^2(\mu_p)$ at any density (μ_p being the product Bernoulli measure with density p), but the constraint makes it non-uniformly mixing. They are also generically non-attractive. As a result, few criteria implying limit theorems for RWRE apply to them, although it is expected that at least LLN and CLT should follow from the fast mixing.

In [4], law of large numbers and CLT were proved for random walks on KCM with exponential mixing (e.g. the East model at any density), but only in a perturbative regime where the jumps rates of the random walker are close to rates that would make the environment seen from the walker stationary w.r.t. μ_p. The results in [8, 2] show LLN for random walks on the one-dimensional East model assuming only finite-range jumps. However, this is still limited to dimension 1, and does not yield a CLT.

Can one prove a CLT for random walks on top of the East Model over all range of parameters?

References

[1] J. Allasia. "Asymptotic direction of a ballistic random walk in a two-dimensional random environment with nonuniform mixing". In: *arXiv preprint arXiv:2307.01603* (2023).

[2] J. Allasia. "Law of large numbers for a finite-range random walk in a dynamic random environment with nonuniform mixing". In: *arXiv preprint arXiv:2304.03143* (2023).

[3] W. S. Arcanjo, R. Baldasso, M. R. Hilário, and R. S. dos Santos. "Law of large numbers for ballistic random walks in dynamic random environments under lateral decoupling". In: *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques* (2024+).

[4] L. Avena, O. Blondel, and A. Faggionato. “Analysis of random walks in dynamic random environments via L^2-perturbations”. In: *Stochastic Processes and their Applications* 128.10 (2018), pp. 3490–3530.

[5] R. Baldasso, M. R. Hilario, D. Kious, and A. Teixeira. “Fluctuation bounds for symmetric random walks on dynamic environments via Russo-Seymour-Welsh”. In: *arXiv preprint arXiv:2304.05771* (2023).

[6] S. A. Bethuelsen and F. Völlering. “Absolute continuity and weak uniform mixing of random walk in dynamic random environment”. In: *Electronic Journal of Probability* 21 (2016).

[7] S. A. Bethuelsen and F. Völlering. “Limit laws for random walks in a dynamic path-cone mixing random environment”. In: *arXiv preprint arXiv:2303.06756* (2023).

[8] O. Blondel, M. R. Hilario, and A. Teixeira. “Random Walks on Dynamical Random Environments with Non-Uniform Mixing”. In: *Annals of Probability* 48.4 (2020).

[9] N. Cancrini, F. Martinelli, C. Roberto, and C. Toninelli. “Kinetically constrained spin models”. In: *Probability theory and related fields* 140.3 (2008), pp. 459–504.

[10] A. A. Chernov. “Replication of a multicomponent chain by the “lightning” mechanism”. In: *Biophysics* 12.2 (1967), pp. 336–341.

[11] F. Comets and O. Zeitouni. “A law of large numbers for random walks in random mixing environments”. In: *The Annals of Probability* 32.1B (2004), pp. 880–914.

[12] D. Dolgopyat, G. Keller, and C. Liverani. “Random Walk in Markovian environment”. In: *Annals of probability* 36.5 (2008), pp. 1676–1710.

[13] D. Dolgopyat and C. Liverani. “Non-perturbative approach to random walk in Markovian environment”. In: *Electronic Communications in Probability* 14 (2009), pp. 245–251.

[14] R. Durrett. *Probability: theory and examples*. Vol. 49. Cambridge university press, 2019.

[15] L. R. Fontes, T. S. Mountford, D. Ungaretti, and M. E. Vares. “Renewal Contact Processes: phase transition and survival”. In: *Stochastic Processes and their Applications* 161 (2023), pp. 102–136.

[16] A. Gut. *Stopped random walks*. Springer, 2009.

[17] M. R. Hilário, D. Kious, and A. Teixeira. “Random walk on the simple symmetric exclusion process”. In: *Communications in Mathematical Physics* 379.1 (2020), pp. 61–101.

[18] M. R. Hilario, M. Sá, R. Sanchis, and A. Teixeira. “Phase transition for percolation on a randomly stretched lattice”. In: *arXiv preprint arXiv:1912.03320* (2019).

[19] A. E. Holroyd, O. Schramm, and D. B. Wilson. “Finitary coloring”. In: *The Annals of Probability* (2017), pp. 2867–2898.

[20] S. Janson. “On a central limit theorem in renewal theory”. In: *Statistics & Probability Letters* 204 (2024), p. 109948.

[21] T. Lindvall. “On coupling of discrete renewal processes”. In: *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* 48.1 (1979), pp. 57–70.

[22] R. Serfozo. *Basics of applied stochastic processes*. Springer Science & Business Media, 2009.

[23] F. Solomon. "Random walks in a random environment". In: *The annals of probability* 3.1 (1975), pp. 1–31.

[24] A.-S. Sznitman and M. Zerner. "A law of large numbers for random walks in random environment". In: *The Annals of Probability* 27.4 (1999), pp. 1851–1869.

[25] D. E. Temkin. "One-dimensional random walks in a two-component chain". In: *Soviet Mathematics Doklady* 13 (1972), pp. 1172–1176.