ANALYSIS OF RANDOM WALKS IN DYNAMIC RANDOM ENVIRONMENTS VIA L^2–PERTURBATIONS

L. AVENA[1], O. BLONDEL[2], AND A. FAGGIONATO[3]

Abstract. We consider random walks in dynamic random environments given by Markovian dynamics on $\mathbb{Z}^d$. We assume that the environment has a stationary distribution μ and satisfies the Poincaré inequality w.r.t. μ. The random walk is a perturbation of another random walk (called "unperturbed"). We assume that also the environment viewed from the unperturbed random walk has stationary distribution μ. Both perturbed and unperturbed random walks can depend heavily on the environment and are not assumed to be finite–range. We derive a law of large numbers, an averaged invariance principle for the position of the walker and a series expansion for the asymptotic speed. We also provide a condition for non-degeneracy of the diffusion, and describe in some details equilibrium and convergence properties of the environment seen by the walker. All these results are based on a more general perturbative analysis of operators that we derive in the context of L^2–bounded perturbations of Markov processes by means of the so–called Dyson–Phillips expansion.

Keywords: perturbations of Markov processes, Poincaré inequality, Dyson–Phillips expansion, random walk in dynamic random environment, asymptotic velocity, invariance principle.

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1. Introduction

Random motion in random media has been the subject of intensive studies in the physics and mathematics literature over the last decades. The main motivation to our work is the analysis of rather general continuous–time Random Walks (RWs) on $\mathbb{Z}^d$, whose transition rates are given as a function of an underlying (autonomous) Markov process playing the role of a dynamic random environment.

A number of results (as LLN, CLT, large deviation estimates) have been obtained in the past under various conditions that allow some control on the strong dependence between the trajectories of the random walk and the environment. We mention space and/or time independence assumptions on the environment (see e.g. [10] for quenched CLT of perturbation of simple random walks using cluster expansion, [11] for diffusive bounds by using renormalization techniques, [37] for quenched invariance principles by analyzing the environment as seen by the walk, [5] for a law of

[1]Mathematisch instituut Universiteit Leiden. Postbus 9512 2300 RA Leiden, The Netherlands. Supported by NWO Gravitation Grant 024.002.003-NETWORKS

[2]CNRS, Univ Lyon, Université Claude Bernard Lyon 1, ICJ, CNRS UMR 5208; 43 blvd. du 11 novembre 1918 F-69622 Villeurbanne cedex, France

[3]Dipartimento di Matematica, Università di Roma La Sapienza. P.le Aldo Moro 2, 00185 Roma, Italy

E-mail addresses: `l.avena@math.leidenuniv.nl, blondel@math.univ-lyon1.fr, faggiona@mat.uniroma1.it`.

large numbers and a high-dimensional quenched invariance principle by constructing regeneration times) and balanced conditions (cf. [15] for averaged invariance principles under reversibility of the environment as seen by the walker, [14] and [30], respectively, for quenched diffusive statements for balanced walks and for zero-mean doubly-stochastic walks). When allowing non–trivial space-time correlation structures, in [1] for some uniformly elliptic walks and in [20] for non–elliptic ones, laws of large numbers via regeneration times have been established by assuming mixing conditions on the environment that are uniform on the initial configuration (i.e. adaptation of *cone-mixing* conditions borrowed from [13] for static random environments). In a similar setting, a quenched CLT has been established in [17], and a quite general asymptotic analysis has been pursued in the recent [39], again by using a uniform mixing condition expressed in terms of a coupling. When dealing with poorly–mixing environments, some progress has been recently achieved by using highly model dependent techniques [2, 19, 21].

In this work, we require that the environment satisfies an *exponential L^2–mixing* hypothesis (namely, the Poincaré inequality w.r.t. an invariant distribution μ) and that the random walk is "close to nice", in the sense that it is a perturbation of a random walk such that μ is an invariant distribution for the environment viewed by the walker. We stress that even though we are in a perturbative setting, the reference unperturbed random walk is allowed to depend strongly on the environment. Moreover, unlike most of the references above, we do not require finite range for the jumps of the walk. As discussed in Section 2, we establish several results for the RW and for the environment seen from it. For the latter, we show that there exists a unique invariant distribution absolutely continuous w.r.t. μ, we analyze convergence to this invariant measure and ergodicity, we derive an expansion of its density w.r.t. μ and show that the effect of the perturbation on the density is sharply localized around the origin, and we derive an exponential L^2–mixing property similar to the Poincaré inequality (see Theorems 1, 2, 3). For the random walk itself, we prove a LLN and an averaged invariance principle, as well as the non-degeneracy of the diffusion matrix under suitable conditions (see Theorems 1, 4). We point out that we do not derive quenched CLTs. Even by quenching only on the initial configuration, the problem becomes much harder and further uniform and/or balanced assumptions might be necessary. Also for our unperturbed walk (when present) in equation (5), it is not immediate to obtain quenched statements in this generality, e.g. the reference in [30] might be used for the latter but still under their further technical assumptions.

One of the basic tools for the above results is the so–called Dyson–Phillips expansion, which we use to derive a series expansion for the semigroup of the environment seen from the walker. This perturbative analysis is very general, and indeed in Section 3 it is carried on for a generic Markov process stationary w.r.t. some invariant and ergodic distribution μ and satisfying the Poincaré inequality. We assume that the generator of the perturbed Markov process is (roughly speaking) obtained by a $L^2(\mu)$–bounded perturbation of the generator of the original, unperturbed, Markov process. In Theorem 5 we prove that the perturbed process admits a unique invariant distribution absolutely continuous w.r.t. μ (which is also ergodic), write a series expansion for its density w.r.t. μ and for the perturbed Markov semigroup, and estimate the convergence to equilibrium for the latter. In addition, in Corollary

1 and Proposition 3.6, we state a law of large numbers and an invariance principle for additive functionals of the perturbed Markov process, respectively.

Let us further comment on some closely related works with perturbative techniques. In [1] the Dyson–Phillips expansion has also been used in a similar fashion in one of the main results therein, but the authors only focus on the law of large numbers for the walk and work under the more restrictive sup–norm instead of the L^2–norm. In [29] the authors work with hypotheses very similar to our own for Theorem 5 (even allowing more general perturbations), but the obtained results present some differences. In particular, in [29] the uniqueness of the invariant distribution for the perturbed process is proved inside the smaller class of distributions whose density w.r.t. μ is bounded in $L^2(\mu)$. In addition, in Theorem 5 we derive information on the exponential convergence of the perturbed semigroup (which is relevant to get the invariance principle in Proposition 3.6), while in [29] the exponential convergence of the perturbed densities is derived. For more detailed comments on the relation between [29] and our Theorem 5 we refer to Remark 3.4. We point out that the main goal in [29] is to establish the Einstein relation for the speed of the walker, hence we have not focused on this issue since already treated there. Finally, we mention the recent work [35], where the author considers perturbations of infinite dimensional diffusions with known invariant measure (not necessarily reversible), satisfying the log-Sobolev inequality (which is stronger than the Poincaré inequality). The invariant measure for the perturbed process is analyzed and its density is expressed in terms of a series expansion similar to (28), (33) below.

Finally, we mention that the results we present herein can be pushed to obtain more detailed information when dealing with explicit examples of random walks in dynamic random environments. This path has been pursued in [3], where we consider one-dimensional examples in which the dynamic environments are given by kinetically constrained models.

Outline of the paper. In Section 2 we present our main results concerning random walks in dynamic random environments, i.e. Theorems 1, 2, 3 and 4. The main results concerning perturbations of more general Markov processes, i.e. Theorem 5, Corollary 1 and Proposition 3.6, are stated in Section 3. The other sections, from 4 to 12, are devoted to the proofs of the above statements. In particular, in Section 8 we present a coupling construction allowing to compare perturbed and unperturbed walkers which is independent of the small perturbation assumption. Finally, in Appendix A we derive some simple but useful analytic results.

2. Random walks in dynamic random environment

In this section we start with a stochastic process $(\sigma_t)_{t\geq 0}$, called *dynamic random environment*, with state space $\Omega := S^{\mathbb{Z}^d}$, S being a compact Polish space and Ω being endowed with the product topology. We assume it has càdlàg paths in the Skohorod space $D[\mathbb{R}_+;\Omega)$. We will then introduce two random walks $(X_t)_{t\geq 0}$ and $(X_t^{(\varepsilon)})_{t\geq 0}$, on $\mathbb{Z}^d$, whose jump rates depend on the dynamic environment. The random walk $(X_t^{(\varepsilon)})_{t\geq 0}$ will be thought of as a perturbation of $(X_t)_{t\geq 0}$ and the parameter ε will quantify the perturbation. More precisely, we give conditions in terms of Markov generators ensuring that the process "environment viewed from the walker $X_t^{(\varepsilon)}$" (i.e. $\tau_{X_t^{(\varepsilon)}}\sigma_t$) is a perturbation of the process "environment viewed from the walker

X_t" (i.e. $\tau_{X_t}\sigma_t$). In the above notation, τ_x denotes the translation operator on Ω such that $\tau_x\eta(y) = \eta(x+y)$ for $x, y \in \mathbb{Z}^d, \eta \in \Omega$.

In Subsection 2.1 we introduce the main mathematical objects under investigation and our assumption. In Subsection 2.3 we present our main results concerning random walks in dynamic random environments, while in Subsection 2.2 we discuss examples and collect some comments.

2.1. **Processes and assumptions.**

Assumption 1. *The dynamic random environment is a Feller process and is stationary w.r.t. a probability measure μ on Ω. Moreover, μ is translation invariant.*

We denote by $\big(S_{\mathrm{env}}(t)\big)_{t\geq 0}$ the Markov semigroup in $L^2(\mu)$ associated with the dynamic random environment, and by $L_{\mathrm{env}} : \mathcal{D}(L_{\mathrm{env}}) \subset L^2(\mu) \to L^2(\mu)$ the corresponding generator. In particular, given $f \in L^2(\mu)$, it holds $(S_{\mathrm{env}}(t)f)(\sigma) := \mathbb{E}^{\mathrm{env}}_\sigma\big[f(\sigma_t)\big]$ μ–a.s., where $\mathbb{E}^{\mathrm{env}}_\sigma$ is the expectation for the dynamic random environment starting at σ.

Assumption 2. *The dynamic random environment commutes with translations, i.e.*

$$S_{\mathrm{env}}(t)(f\circ\tau_x) = (S_{\mathrm{env}}(t)f)\circ\tau_x, \qquad \forall f\in L^2(\mu)\,,\ t\geq 0\,. \tag{1}$$

Moreover, the generator L_{env} satisfies the Poincaré inequality, i.e. there exists $\gamma > 0$ such that

$$\gamma\|f\|^2 \leq -\mu(fL_{\mathrm{env}}f) \qquad \forall f\in\mathcal{D}(L) \text{ with } \mu(f) = 0\,. \tag{2}$$

We point out that (2) is equivalent to the bound $\|S_{\mathrm{env}}(t)f - \mu(f)\| \leq e^{-\gamma t}\|f - \mu(f)\|$ for all $t \geq 0$ and $f \in L^2(\mu)$, $\|\cdot\|$ being the norm in $L^2(\mu)$ (see Lemma A.3 in Appendix A).

We now want to introduce two random walks on $\mathbb{Z}^d$, whose jump rates depend on the dynamic random environment. To this aim, we require the following:

Assumption 3. *There are given continuous functions $r_\varepsilon(y,\cdot)$, $r(y,\cdot)$ and $\hat r_\varepsilon(y,\cdot)$ on Ω, parametrized by $y\in\mathbb{Z}^d$. These functions are zero for $y = 0$, $r_\varepsilon(y,\cdot)$ and $r(y,\cdot)$ are nonnegative and $r_\varepsilon(y,\cdot)$ can be decomposed as*

$$r_\varepsilon(y,\cdot) := r(y,\cdot) + \hat r_\varepsilon(y,\cdot)\,. \tag{3}$$

We also require that, for some $n\geq 1$, the above functions have finite n-th moment:

$$\sum_{y\in\mathbb{Z}^d}|y|^n\sup_{\eta\in\Omega} r(y,\eta) < \infty\,, \qquad \sum_{y\in\mathbb{Z}^d}|y|^n\sup_{\eta\in\Omega}|\hat r_\varepsilon(y,\eta)| < \infty\,. \tag{4}$$

Let now $(X_t)_{t\geq 0}$ be the continuous time random walk on $\mathbb{Z}^d$ jumping from site $x \in \mathbb{Z}^d$ to site $x + y \in \mathbb{Z}^d$ at rate $r(y, \tau_x\eta)$, given that the dynamic random environment is in state $\eta\in\Omega$. Due to dependence on the environment, such a random walk is not Markovian itself, but the joint process $(\sigma_t, X_t)_{t\geq 0}$ on state space $\Omega\times\mathbb{Z}^d$ is a Markov process with formal generator[1]

$$L_{\mathrm{rwre}}f(\eta,x) := L_{\mathrm{env}}f(.,x)(\eta) + \sum_{y\in\mathbb{Z}^d} r(y,\tau_x\eta)\big[f(\eta,x+y) - f(\eta,x)\big],\ \ (\eta,x)\in\Omega\times\mathbb{Z}^d\,. \tag{5}$$

[1]The notation L_{rwre} is thought to stress that we are referring to the joint process describing both the random walk and the random environment.

We do not insist here with a precise description of the generator, since it will not be used in the sequel. On the other hand, below we will discuss carefully the generator of the process "environment viewed from the walker". Due to (4), no explosion takes place and therefore the random walk $(X_t)_{t\geq 0}$ is well defined (a universal construction is given in Section 8). In what follows we write $P_{\eta,x}$ for the law on the càdlàg space $D(\mathbb{R}_+;\Omega\times\mathbb{Z}^d)$ of this joint process starting at (η,x).

As in the construction of the joint Markov process in (5), we define a new joint Markov process $(\sigma_t, X_t^{(\varepsilon)})_{t\geq 0}$ on state space $\Omega\times\mathbb{Z}^d$ with formal generator:

$$L^{(\varepsilon)}_{\rm rwre} f(\eta,x) := L_{\rm env} f(.,x)(\eta) + \sum_{y\in\mathbb{Z}^d} r_\varepsilon(y,\tau_x\eta)\big[f(\eta,x+y)-f(\eta,x)\big], \quad (\eta,x)\in\Omega\times\mathbb{Z}^d\,. \tag{6}$$

In what follows we write $P^{(\varepsilon)}_{\eta,x}$ for the law on the càdlàg space $D(\mathbb{R}_+;\Omega\times\mathbb{Z}^d)$ of this joint process starting at (η,x). We refer to this new walker $(X_t^{(\varepsilon)})_{t\geq 0}$ as the *perturbed walker*.

One of the most common approaches to study random motion in random media is to analyze the so called *environment seen by the walker*. In our case, we are interested in the Markov processes on Ω given by $\tau_{X_t}\sigma_t$ and $\tau_{X_t^{(\varepsilon)}}\sigma_t$, where $(\sigma_t,X_t)_{t\geq 0}$ and $(\sigma_t,X_t^{(\varepsilon)})_{t\geq 0}$ are the joint Markov processes defined above.

We write $C(\Omega)$ for the space of real continuous functions on Ω endowed with the uniform norm. Since, by assumption, the dynamic random environment is a Feller process, it has a well defined Markov semigroup on $C(\Omega)$, and we denote by[2] $\mathcal{L}_{\rm env}:\mathcal{D}(\mathcal{L}_{\rm env})\subset C(\Omega)\to C(\Omega)$ the associated Markov generator. We define $\mathcal{L}_{\rm jump}f(\eta)=\sum_{y\in\mathbb{Z}^d} r(y,\eta)\big[f(\tau_y\eta)-f(\eta)\big]$ for $f\in C(\Omega)$ and $\hat{\mathcal{L}}_\varepsilon f(\eta)=\sum_{y\in\mathbb{Z}^d}\hat{r}_\varepsilon(y,\eta)\big[f(\tau_y\eta)-f(\eta)\big]$ for $f\in C(\Omega)$. Then, by Assumption 3, the operators $\mathcal{L}_{\rm jump},\hat{\mathcal{L}}_\varepsilon: C(\Omega)\to C(\Omega)$ are well posed and bounded.

Assumption 4. *The environment seen from the unperturbed walker $\big(\tau_{X_t}\sigma_t\big)_{t\geq 0}$ and the one seen from the perturbed walker $\big(\tau_{X_t^{(\varepsilon)}}\sigma_t\big)_{t\geq 0}$ are Feller processes on Ω with generators on $C(\Omega)$ given respectively by $\mathcal{L}_{\rm env}+\mathcal{L}_{\rm jump}$ and $\mathcal{L}_{\rm env}+\mathcal{L}_{\rm jump}+\hat{\mathcal{L}}^{(\varepsilon)}$, both having domain $\mathcal{D}(\mathcal{L}_{\rm env})$.*

The above assumption is typically satisfied in all common applications:

Proposition 2.1. *Suppose that $\mathcal{L}_{\rm env}$ is the closure of a Markov pregenerator $\mathbb{L}$ as in* [32, Def.2.1, Chp.I]*, satisfying the criterion in* [32, Prop.2.2, Chp.I]*. Then $\mathbb{L}+\mathcal{L}_{\rm jump}$ and $\mathbb{L}+\mathcal{L}_{\rm jump}+\hat{\mathcal{L}}^{(\varepsilon)}$ are Markov pregenerators, whose closures are Markov generators of Feller processes (cf.* [32, Def.2.7, Chp.I]*). The resulting Markov generators are given respectively by the operators $\mathcal{L}_{\rm env}+\mathcal{L}_{\rm jump}$ and $\mathcal{L}_{\rm env}+\mathcal{L}_{\rm jump}+\hat{\mathcal{L}}^{(\varepsilon)}$, both having domain $\mathcal{D}(\mathcal{L}_{\rm env})$.*

The proof of the above proposition is similar to the proof of [16, Lemma 2.1]. The interested reader can find the proof of Prop. 2.1 in [4, Appendix A].

Assumption 5. *The environment seen by the unperturbed walker $\big(\tau_{X_t}\sigma_t\big)_{t\geq 0}$ has invariant distribution μ.*

[2]We denote consistently with curved $\mathcal{L}$ generators on $C(\Omega)$ and with straight L their version living in $L^2(\mu)$.

Remark 2.2. *Due to Assumption 1, Assumption 5 is equivalent to the fact that* $\mu(\mathcal{L}_{\rm jump} f) = 0$ *for any* $f \in C(\Omega)$ *(or for any* f *in a dense subset of* $C(\Omega)$*, since* $\mathcal{L}_{\rm jump}$ *is a bounded operator due to* (4)*).*

We can state our last main assumption, which is indeed related to the perturbative approach. Consider the operator $\hat{L}_\varepsilon : L^2(\mu) \to L^2(\mu)$ defined as

$$\hat{L}_\varepsilon f(\eta) := \sum_{y\in\mathbb{Z}^d} \hat{r}_\varepsilon(y,\eta)\,[f(\tau_y\eta) - f(\eta)]\,, \qquad f \in L^2(\mu)\,. \tag{7}$$

It is indeed a bounded operator in $L^2(\mu)$. For example, by Schwarz inequality and by Assumption 3, given $f \in L^2(\mu)$ we can write

$$\mu\left(\big[\hat{L}_\varepsilon f\big]^2\right) \leq \Big[\sum_{y\in\mathbb{Z}^d} \sup_\eta |\hat{r}_\varepsilon(y,\eta)|\Big] \sum_{y\in\mathbb{Z}^d} \sup_\eta |\hat{r}_\varepsilon(y,\eta)| \mu([f(\tau_y\cdot) - f]^2)\,,$$

and by the translation invariance of μ we conclude that

$$\|\hat{L}_\varepsilon\| \leq 2 \sum_y \sup_\eta |\hat{r}_\varepsilon(y,\eta)|\,. \tag{8}$$

Assumption 6. *The operator* $\hat{L}_\varepsilon$ *has norm* $\varepsilon := \|\hat{L}_\varepsilon\|$ *satisfying* $\varepsilon < \gamma$*, where* γ *has been introduced in Assumption 2.*

2.2. **Examples.** We give here some explicit examples of environments, unperturbed walks and perturbed walks for which our results apply.

Dynamic environments: Natural examples of environments satisfying our assumptions are given by Interacting Particle Systems (IPSs) with state space $\Omega = \{0,1\}^{\mathbb{Z}^d}$. A first class of such IPSs is that of translation invariant *stochastic Ising models* in a "high–temperature" regime (see [33, Thm.4.1] and [32, Thm.4.1, Chp.I]), among which, the simplest case is the independent spin-flip dynamics. The latter is the Markov process with generator $\mathcal{L}_{\rm env} f(\sigma) = \gamma \sum_{x\in\mathbb{Z}} f(\sigma^x) - f(\sigma)$, where $\gamma > 0$, and σ^x is the configuration obtained by $\sigma \in \Omega$ by flipping the spin at x. As a variant of these processes, one could consider some Kawasaki dynamics superposed to a high–noise spin–flip dynamics. When the exponential convergence of the Markov semigroup holds in the stronger L^∞–norm one could also apply [1, Sec. 3] to derive some of the results presented here (as the existence of the limiting velocity). On the other hand, several of our results have not been derived in the existing literature, even under the assumption of L^∞–convergence; moreover, there are several models where the Poincaré inequality holds while the log-Sobolev inequality is violated or has not been proved. One of the motivations which prompted the present study is to consider the class of so–called Kinetically Constrained Spin Models (KCSMs), for which (2) was proved in great generality (in the ergodic regime) in [12]. Their generator is given by $\mathcal{L}_{\rm env} f(\sigma) = \sum_{x\in\mathbb{Z}} c_x(\sigma)(\rho(1-\sigma(x)) + (1-\rho)\sigma(x))\,[f(\sigma^x) - f(\sigma)]$ with $\rho \in (0,1)$ and c_x encodes a kinetic constraint which should be of the type "there are enough empty sites in a neighbourhood of x". We refer to [12] for precise conditions that the constraints need to satisfy and identification of the regime where (2) is satisfied. Examples of constraints include the FA-jf model, where $c_x(\sigma) = \mathbf{1}_{\sum_{y\sim x}(1-\sigma(y))\geq j}$ with $j \leq d$, or generalized East processes $c_x(\sigma) = 1 - \prod_{i=1}^d \sigma(x+e_i)$ with $(e_i)_{i=1,\cdots,d}$ the canonical basis of $\mathbb{R}^d$. The presence of the constraint gives rise to a number of difficulties as for instance the lack of attractivity. Consequently, most of the general existing results, as e.g. [1, 5, 10, 11, 17, 37, 39], do not apply to this class.

Unperturbed walks: We give here three simple cases of different nature. Let us emphasize that the unperturbed walk is in general an auxiliary process subject to Assumptions 3, 4 and 5. In particular, it does not need to be effectively present, and this is our first simplest case: that is, $r(y,\cdot) \equiv 0$ for all $y \in \mathbb{Z}^d$, and hence environment and environment from the unperturbed walker coincide and the above mentioned assumptions are trivially satisfied. As a second case, we can consider unperturbed walks whose rates do not dependent on the environment, as for example a simple symmetric random walks, that is, $r(y,\cdot) = 1/2$ for $y = \pm 1$ and 0 else, for which it is immediate to check the above assumptions. The third and most delicate case is when the unperturbed walk is present and depends effectively on the environment, for which, in order to check the crucial Assumption 5, the specific choice of the environment is essential. To give an example of recent interest in the physics literature, we mention environments given by a KCSM as in [22] and [7], for which, it is straight-forward to check that e.g. in dimension one the unperturbed walk with nearest–neighbor jumps and defined by $r(\pm 1,\eta) = (1-\eta(0))(1-\eta(\pm 1))$ satisfies the required assumptions.

Perturbed walks: By considering any of the environments and of the unperturbed walks just discussed, our results apply to arbitrary perturbed walks essentially provided the small-perturbation Assumption 6 is in force. We mention that in case the unperturbed walk is a simple symmetric random walk, an interesting example is for $r_\varepsilon(y,\eta) = \pm\varepsilon(2\eta(0)-1)\mathbb{1}_{\{y=\pm 1\}}$ for which the resulting walk has the tendency to stick to the space-time interfaces between empty and occupied regions in the environment. A more detailed analysis of this walk on the East model, mainly based on the results in this work, can be found in [3]. In case of an unperturbed walk effectively dependent on the environment, as mentioned above, it is necessary to specify the environment. For an interesting example of perturbed walk in this setup, we mention again environments given by KCSM, as in [22], one could consider a probe particle driven by a constant external field (as perturbed walk) in the KCSM started from a stationary distribution μ left invariant by the non–driven probe (as unperturbed walk). In the one-dimensional case one possibility is $r(\pm 1,\eta) = (1-\eta(0))(1-\eta(\pm 1))$, $r_\varepsilon(\pm 1,\eta) = (1-\eta(0))(1-\eta(\pm 1))\tilde{r}_\varepsilon(\pm 1)$, $\tilde{r}_\varepsilon(1) = 2/(1+e^{-\varepsilon}) = e^\varepsilon \tilde{r}_\varepsilon(-1)$, the other rates are zero and ε is small enough.

2.3. **Main results.** In the rest of this section, we suppose Assumptions 1,...,6 to be satisfied without further mention.

Concerning the environment seen by the walker $(\tau_{X_t}\sigma_t)_{t\geq 0}$, we denote by $\mathbb{P}_\nu$ its law on $D(\mathbb{R}_+;\Omega)$, and by $\mathbb{E}_\nu$ the associated expectation, when the initial distribution is ν (if $\nu = \delta_\eta$, we simply write $\mathbb{P}_\eta$ and $\mathbb{E}_\eta$). We denote by $S(t)$ its Markov semigroup on $L^2(\mu)$, i.e. $(S(t)f)(\eta) := \mathbb{E}_\eta\big(f(\eta_t)\big)$ μ–a.s., and we write L_{ew} for its infinitesimal generator. For the perturbed version $(\tau_{X_t^{(\varepsilon)}}\sigma_t)_{t\geq 0}$ we use analogously the notation $\mathbb{P}_\eta^{(\varepsilon)}$, $\mathbb{E}_\eta^{(\varepsilon)}$ for the law and the expectation. Moreover, we define $(S_\varepsilon(t))_{t\geq 0}$ as the semigroup in $L^2(\mu)$ with infinitesimal generator $L_{\text{ew}}^{(\varepsilon)} = L_{\text{ew}} + \hat{L}_\varepsilon$ (see Section 9.1 for a detailed discussion). As proved in Section 9.1, $(S_\varepsilon(t)f)(\eta) = \mathbb{E}_\eta^{(\varepsilon)}(f(\eta_t))$ μ–a.s. at least for bounded continuous functions f.

Given $t \geq 0$ we define iteratively the operators $S_\varepsilon^{(n)}(t)$ as $S_\varepsilon^{(0)}(t) := S(t)$, $S_\varepsilon^{(n+1)}(t) := \int_0^t S(t-s)\hat{L}_\varepsilon S_\varepsilon^{(n)}(s)ds$. These operators enter in the Dyson expansion $S_\varepsilon(t) = \sum_{n=0}^\infty S_\varepsilon^{(n)}(t)$ discussed in detail in Section 3.

Theorem 1 (Asymptotic perturbed stationary state and velocity)**.**

(i) *The environment seen by the perturbed walker admits a unique distribution μ_ε on Ω which is invariant and absolutely continuous w.r.t. μ. Whenever the environment seen by the perturbed walker has initial distribution absolutely continuous w.r.t. μ, its distribution at time t weakly converges to μ_ε as $t \to \infty$. Moreover, μ_ε is ergodic w.r.t. time–translations and*

$$\mu_\varepsilon(f) = \mu(f) + \sum_{n=0}^\infty \int_0^\infty \mu\left(\hat{L}_\varepsilon S_\varepsilon^{(n)}(s) f\right) ds\,, \qquad f \in L^2(\mu)\,, \tag{9}$$

where $\int_0^\infty \left|\mu\left(\hat{L}_\varepsilon S_\varepsilon^{(n)}(s) f\right)\right| ds \leq (\varepsilon/\gamma)^{n+1}\|f - \mu(f)\|$.

(ii) *If the additional condition*

$$r(y,\eta) > 0 \implies r_\varepsilon(y,\eta) > 0 \tag{10}$$

is satisfied, then μ and μ_ε are mutually absolutely continuous.

Alternatively, if there exist subsets $V, V_\varepsilon \subset \mathbb{Z}^d$ such that

(a) *$r(y,\eta) > 0$ iff $y \in V$,*

(b) *$\hat{r}_\varepsilon(y,\eta) > 0$ iff $y \in V_\varepsilon$,*

(c) *each vector in V can be written as sum of vectors in V_ε,*

then μ and μ_ε are mutually absolutely continuous.

(iii) *If* (4) *holds with $n = 2$, then defining $v(\varepsilon) := \mu_\varepsilon(j^{(\varepsilon)})$ with $j^{(\varepsilon)}(\eta) := \sum_{y \in \mathbb{Z}^d} y r_\varepsilon(y,\eta), \eta \in \Omega$, it holds*

$$P_{\eta,0}^{(\varepsilon)}\Big(\lim_{t\to\infty} \frac{X_t^{(\varepsilon)}}{t} = v(\varepsilon)\Big) = 1 \tag{11}$$

for μ_ε–a.e. η and for η varying in a set of μ–probability larger than $1 - \varepsilon^2/(\gamma-\varepsilon)^2$. If μ and μ_ε are mutually absolutely continuous as in Item (ii), then (11) *holds for μ–a.e. η.*

(iv) *The asymptotic velocity $v(\varepsilon)$ can be expressed by a series expansion in ε as*

$$v(\varepsilon) = \mu(j^{(\varepsilon)}) + \sum_{n=0}^\infty \int_0^\infty \mu(\hat{L}_\varepsilon S_\varepsilon^{(n)}(s) j^{(\varepsilon)}) ds. \tag{12}$$

Moreover, $\left|\mu(\hat{L}_\varepsilon S_\varepsilon^{(n)}(s) j^{(\varepsilon)})\right| \leq \varepsilon^{n+1} e^{-\gamma s} s^n \|j^{(\varepsilon)}\|_\infty / n!$ for all $n \geq 0$.

Remark 2.3. *Further properties on the distribution μ_ε and on the semigroup $S_\varepsilon(t)$ are stated, in a more general context, in Section 3 (see in particular Proposition 3.3 and formulas* (30), (31), (32), (33) *and* (36) *in Theorem 5).*

The proof of Theorem 1 is given in Section 9.

Theorem 2. *Suppose that μ has the following decorrelation property: given functions f, g with bounded support, we have*

$$\lim_{|x|\to\infty} \mathrm{Cov}_\mu(f, \tau_x g) = 0\,. \tag{13}$$

Then, for any local function f, it holds

$$\lim_{|x|\to\infty} \mu_\varepsilon(\tau_x f) = \mu(f)\,. \tag{14}$$

The proof of Theorem 2 is given in Section 10.

Under stronger conditions, we can estimate the decay of $|\mu_\varepsilon(\tau_x f) - \mu(f)|$. To this aim we fix some notation and terminology. Given $x \in \mathbb{Z}^d$ and $\ell > 0$, we introduce the uniform box $B(x,\ell) = \{y \in \mathbb{Z}^d : |x-y|_\infty \le \ell\}$. If $x = 0$, we simply write $B(\ell)$.

Definition 2.4. *The stationary process* dynamic random environment *with generator $L_{\rm env}$ and initial distribution μ has* finite speed of propagation *if there exists a function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ vanishing at infinity (i.e. $\lim_{u\to\infty}\alpha(u) = 0$) and a constant $C > 0$ such that*

$$\big|\mathbb{E}^{\rm env}_\mu[XY] - \mathbb{E}^{\rm env}_\mu[X]\mathbb{E}^{\rm env}_\mu[Y]\big| \le \alpha(d(\Lambda,\Lambda')) \tag{15}$$

for any pair of random variables X, Y bounded in modulus by one and for any pair of sets $\Lambda, \Lambda' \subset \mathbb{Z}^d$, such that (for some $t \ge 0$) X is determined by $\big(\eta_s(x) : 0 \le s \le t,\ x \in \Lambda\big)$, Y is determined by $\big(\eta_s(x) : 0 \le s \le t,\ x \in \Lambda'\big)$, and $d(\Lambda,\Lambda') = \min\{|x-x'|_\infty : x \in \Lambda,\, x' \in \Lambda'\} \ge Ct$.

The above property is satisfied for example by many interacting particle systems on $\mathbb{Z}^d$, in particular it is fulfilled if the transition rates are bounded and have finite range, as can be easily checked from the graphical construction (see e.g. [32, Chap. III, Sec. 6], [33, Sec. 3.3]).

Theorem 3 (Quantitative approximation of μ_ε by μ at infinity)**.** *In addition to our main assumptions, assume the following properties:*

- *(i) translation invariance of the unperturbed dynamics, i.e. $S(t)\big(f\circ\tau_x\big) = \big(S(t)f\big)\circ\tau_x$, for any local function f, $x \in \mathbb{Z}^d$ and $t \ge 0$,*
- *(ii) the stationary process with generator $L_{\rm env}$ has finite speed of propagation with $\alpha(u) \le e^{-\theta u}$ for some $\theta > 0$,*
- *(iii) $r, \hat r_\varepsilon$ have finite range, i.e. $\exists R > 0$ such that $r(z,\cdot) \equiv 0$ and $\hat r_\varepsilon(z,\cdot) \equiv 0$ if $z \notin B(R)$ and such the support of $r(z,\cdot)$ and $\hat r_\varepsilon(z,\cdot)$ is included in $B(R)$.*

Then there exists $\theta' > 0$ (depending on ε and γ) such that, for any local function $f : \Omega \to \mathbb{R}$, it holds

$$|\mu_\varepsilon(\tau_x f) - \mu(f)| \le C(f,\varepsilon,\gamma)e^{-\theta'|x|_\infty}\,, \tag{16}$$

where $C(f,\varepsilon,\gamma)$ is a finite constant depending only on f, ε, γ.

Remark 2.5. *One could prove Theorem 3 without Assumption (i), and also its analogue for different decays in the finite speed propagation property, but the treatment would become very technical. Hence we have preferred to restrict to the above simpler case.*

The next lemma gives a sufficient condition for Assumption (i) in Theorem 3:

Lemma 2.6. *Assume* (1) *and that the unperturbed random walk is decoupled from the environment, i.e. $r(y,\eta)$ does not depend on η for any $y \in \mathbb{Z}^d$. Then the assumption in Item (i) of Theorem 3 is satisfied.*

The proofs of Theorem 3 and Lemma 2.6 are given in Section 10.

Our next result focuses on gaussian fluctuations of the random walk:

Theorem 4 (Invariance principle for the perturbed walker)**.** (i) *Suppose that* (4) *holds with* $n = 2$*. Then there exists a symmetric non–negative* $d \times d$ *matrix* D_ε *such that, under* $\int \mu_\varepsilon(d\eta) P^{(\varepsilon)}_{\eta,0}$*, as* $n \to \infty$ *the rescaled process*

$$\frac{X^{(\varepsilon)}_{nt} - v(\varepsilon)nt}{\sqrt{n}} \tag{17}$$

converges weakly to a Brownian motion with covariance matrix D_ε*.*
(ii) *Suppose in addition that* L_{env} *and* L_{ew} *are self–adjoint in* $L_2(\mu)$*, equivalently that* L_{env} *is self–adjoint and* r *satisfies*

$$r(y, \eta) = r(-y, \tau_y \eta). \tag{18}$$

Moreover, assume that (4) *holds with* $n = 4$*. Then the limiting Brownian motion has non-degenerate covariance matrix for* $\beta(\varepsilon)$ *small enough, where*[3]

$$\beta(\varepsilon) := \sum_{y \in \mathbb{Z}^d} |y| \sup_\eta |\hat{r}_\varepsilon(y, \eta)| \,. \tag{19}$$

The proof of Theorem 4 is given in Sections 11 and 12.

3. L^2–perturbation of stationary Markov processes

As already mentioned, the derivation of the results presented in Section 2 is based - between others - on a perturbative approach. In this section, starting from the Dyson–Phillips expansion of the Markov semigroup, we derive some results on perturbations of stationary Markov processes satisfying the Poincaré inequality. We will focus on the perturbed invariant distribution, the perturbed Markov semigroup, the LLN and invariance principle for additive functionals of the perturbed process. We have stated these results in full generality, while at the beginning of Section 9 we explain how the random walks in dynamic random environments analyzed in Section 2 fit into this general scheme.

We fix a metric space Ω, which is thought of as a measurable space endowed with the σ–algebra of its Borel sets. We consider a Markov process with state space Ω and with càdlàg paths in the Skorokhod space $D(\mathbb{R}_+; \Omega)$. We write $(\eta_t)_{t \in \mathbb{R}_+}$ for a generic path, denote by $\mathbb{P}_\nu$ the law on $D(\mathbb{R}_+; \Omega)$ of the process with initial distribution ν, and by $\mathbb{E}_\nu$ the associated expectation. If $\nu = \delta_\eta$, $\eta \in \Omega$, we simply write $\mathbb{P}_\eta$, $\mathbb{E}_\eta$. We suppose the process to have an *invariant distribution* μ on Ω. Then the family of operators $S(t)f(\eta) := \mathbb{E}_\eta\big[f(\eta_t)\big]$, $t \in \mathbb{R}_+$, gives a contraction semigroup in $L^2(\mu)$, which is indeed strongly continuous[4] in $L^2(\mu)$ (see Lemma A.2 in Appendix). We write L for its infinitesimal generator (in $L^2(\mu)$) and $\mathcal{D}(L)$ for the corresponding domain. In what follows we denote by $\| \cdot \|$ the norm in $L^2(\mu)$ and by $\mu(f)$ the μ–expectation of an arbitrary function f. We assume that L satisfies the *Poincaré inequality*, i.e. for some $\gamma > 0$

$$\gamma \|f\|^2 \le -\mu(fLf) \qquad \forall f \in \mathcal{D}(L) \text{ with } \mu(f) = 0 \,. \tag{20}$$

Note that the above Poincaré inequality is equivalent to the bound (cf. Lemma A.3 in Appendix)

$$\|S(t)f - \mu(f)\| \le e^{-\gamma t} \|f - \mu(f)\| \qquad \forall t \ge 0 \,,\ f \in L^2(\mu) \,. \tag{21}$$

[3]Note that by (8), a small $\beta(\varepsilon)$ implies that ε is small.

[4]Strongly continuous semigroup are often called C_0*–semigroups*

If μ is reversible w.r.t. L, then (20) corresponds to requiring that L has spectral gap bounded by γ from below.

Next, for a given fixed parameter $\varepsilon > 0$, we consider a new Markov process on Ω and call $\mathbb{P}^{(\varepsilon)}_\nu$ its law on $D(\mathbb{R}_+;\Omega)$ when starting with distribution ν, and $\mathbb{E}^{(\varepsilon)}_\nu$ the associated expectation. In the sequel we refer to this new Markov process as the *perturbed process.* Since typically μ is not an invariant distribution for the perturbed process, the map $\eta \mapsto \mathbb{E}^{(\varepsilon)}_\eta\big(f(\eta_t)\big)$ can be not well defined μ–a.s. for $f \in L^2(\mu)$. In particular, there is no naturally defined Markov semigroup in $L^2(\mu)$ associated with the perturbed process and therefore the same holds for the associated infinitesimal generator in $L^2(\mu)$. To overcome this difficulty and also to benefit of the existing theory of perturbations of strongly continuous semigroups, we will introduce a semigroup in $L^2(\mu)$ obtained by perturbing $(S(t))_{t\geq 0}$ in a purely analytical way. Afterwards we will assume (cf. Assumption 7 below) that for nice functions f the action of the semigroup at time t on f leads to the new function $\mathbb{E}^{(\varepsilon)}_{\cdot}\big(f(\eta_t)\big)$ which belongs to $L^2(\mu)$. Thanks to Lemma 3.2 below, in many applications this assumption will be automatically satisfied.

We introduce a bounded operator $\hat{L}_\varepsilon : L^2(\mu) \to L^2(\mu)$, with $\varepsilon := \|\hat{L}_\varepsilon\|$, and set

$$L_\varepsilon := L + \hat{L}_\varepsilon\,, \qquad \mathcal{D}(L_\varepsilon) := \mathcal{D}(L)\,. \tag{22}$$

It is known (cf. [18, Thm. 1.3, Chp. III]) that the operator $L_\varepsilon = L+\hat{L}_\varepsilon$ with domain $D(L_\varepsilon) = D(L)$ is the generator of a strongly continuous semigroup $(S_\varepsilon(t))_{t\geq 0}$ on $L^2(\mu)$. Moreover, it holds $S_\varepsilon(t) = e^{tL_\varepsilon}$, where the exponential of the operator L_ε is defined in [23, Ch. IX, Sec. 4] (cf. Problem 49 in [38][Ch. X]).

We fix our basic assumptions:

Assumption 7. *The unperturbed Markov process has invariant and ergodic distribution μ. The generator L of the $L^2(\mu)$–semigroup $S(t)$, $t \in \mathbb{R}_+$, satisfies the Poincaré inequality* (20). *Moreover, considering the semigroup $S_\varepsilon(\cdot)$ with generator $L_\varepsilon = L + \hat{L}_\varepsilon$ and the perturbed Markov process, it holds*

$$S_\varepsilon(t)f(\eta) = \mathbb{E}^{(\varepsilon)}_\eta\big(f(\eta_t)\big)\,, \qquad \mu\text{–a.s.}\,, \qquad \forall f \in C_b(\Omega)\,, \tag{23}$$

where we denote by $C_b(\Omega)$ the space of bounded continuous real functions on Ω.

Remark 3.1. *The above ergodicity of μ has to be thought w.r.t. time translations, i.e. any Borel set $A \subset D(\mathbb{R}_+,\Omega)$ which is left invariant by any time translation*[5] *θ_t has $\mathbb{P}_\mu$–probability equal to* 0 *or* 1. *Due to Theorem 6.9 in* [45] *(cf. also* [42, Chapter IV]*), this is equivalent to the following fact: $\mu(B) \in \{0,1\}$ if B is a Borel subset of Ω such that $\mathbb{1}_B(\eta_0) = \mathbb{1}_B(\eta_t)$ $\mathbb{P}_\mu$–a.s. for any $t \geq 0$. Note that for such a subset B it holds $S(t)\mathbb{1}_B = \mathbb{1}_B$ μ–a.s.. This observation allows to deduce the ergodicity of μ from the bound* (21)*, since we assume that $S(\cdot)$ satisfies the Poincaré inequality. Hence, the explicit hypothesis of μ ergodic could be removed from Assumption 7.*

In the following lemma we discuss a case, useful in applications, where the above property (23) is fulfilled (the proof is postponed to Section 4). The lemma covers numerous applications, e.g. interacting particle systems (cf. [32], in particular Chp. IV.4 there):

[5]Time translation $\theta_t : D(\mathbb{R}_+,\Omega) \to D(\mathbb{R}_+,\Omega)$ is defined as $(\theta_t\eta)_s := \eta_{t+s}$.

Lemma 3.2. *Suppose that Ω is compact and that the perturbed Markov process is Feller on $C(\Omega)$ endowed with the uniform norm. Consider the induced Markov semigroup $\tilde S_\varepsilon(t)$, $t \in \mathbb{R}_+$, on $C(\Omega)$: $\tilde S_\varepsilon(t)f(\eta) := \mathbb{E}_\eta^{(\varepsilon)}\big(f(\eta_t)\big)$ for $f \in C(\Omega)$. Call $\tilde L_\varepsilon : \mathcal{D}(\tilde L_\varepsilon) \subset C(\Omega) \to C(\Omega)$ its infinitesimal generator. Suppose that $\tilde L_\varepsilon$ has a core $\mathcal{C}_\varepsilon \subset \mathcal{D}(\tilde L_\varepsilon) \cap \mathcal{D}(L_\varepsilon)$ such that $\tilde L_\varepsilon f = L_\varepsilon f$ for all $f \in \mathcal{C}_\varepsilon$. Then identity (23) is satisfied.*

We recall, cf. [18, Cor. 1.7 and Eq. (IE*), Chp. III], the so called *variation of parameters formula*: for any $f \in L^2(\mu)$ it holds

$$
\begin{aligned}
S_\varepsilon(t)f &= S(t)f + \int_0^t S(t-s)\hat L_\varepsilon S_\varepsilon(s) f ds \\
&= S(t)f + \int_0^t S_\varepsilon(s)\hat L_\varepsilon S(t-s) f ds \,,
\end{aligned}
\tag{24}
$$

where the above integrals have to be understood in $L^2(\mu)$.

Given $t \geq 0$ we define iteratively the operators $S_\varepsilon^{(n)}(t)$ as

$$
S_\varepsilon^{(0)}(t) := S(t), \quad S_\varepsilon^{(n+1)}(t) := \int_0^t S(t-s)\hat L_\varepsilon S_\varepsilon^{(n)}(s) ds \;= \int_0^t S_\varepsilon^{(n)}(s)\hat L_\varepsilon S(t-s) ds \,.
\tag{25}
$$

The equivalence of the two forms of $S_\varepsilon^{(n+1)}$ in (25) can be checked by induction (see [4, App. A]). As explained in [18, Chp. III], $S_\varepsilon^{(n)}(\cdot)$ is a continuous function from $\mathbb{R}_+$ to the space $\mathcal{L}(L^2(\mu))$ of bounded operators in $L^2(\mu)$. Moreover, the *Dyson–Phillips expansion* holds:

$$
S_\varepsilon(t) = \sum_{n=0}^\infty S_\varepsilon^{(n)}(t) \,, \qquad t \geq 0 \,,
\tag{26}
$$

where the series converges in the operator norm of $\mathcal{L}(L^2(\mu))$, even uniformly as t varies in a bounded interval.

By means of the Poincaré inequality, we can derive more information on the Dyson–Phillips expansion and on the semigroup $(S_\varepsilon(t))_{t\geq 0}$:

Proposition 3.3 (Dyson–Phillips expansion)**.** *Let $\varepsilon < \gamma$, for any $f \in L^2(\mu)$ and $t \geq 0$ it holds*

$$
\|S_\varepsilon(t)f - \sum_{n=0}^{k-1} S_\varepsilon^{(n)}(t)f\| \leq (\varepsilon/\gamma)^k \left(\frac{2\gamma}{\gamma-\varepsilon}\right) \|f - \mu(f)\| \,, \qquad \forall k \geq 1.
\tag{27}
$$

The above proposition is proven in Section 5.

Theorem 5 (Invariant measure)**.** *Let Assumption 7 be satisfied and let $\varepsilon < \gamma$. Then there exists a probability measure μ_ε on Ω with the following properties:*

(i) Consider the perturbed Markov process with initial distribution ν absolutely continuous w.r.t. μ. Then its distribution at time t weakly converges to μ_ε as $t \to \infty$.

(ii) For each $f \in L^2(\mu)$ it holds

$$
\mu_\varepsilon(f) = \mu(f) + \sum_{n=0}^\infty \int_0^\infty \mu\left(\hat L_\varepsilon S_\varepsilon^{(n)}(s) f\right) ds \,,
\tag{28}
$$

where

$$\int_0^\infty \left|\mu\left(\hat{L}_\varepsilon S_\varepsilon^{(n)}(s)f\right)\right|ds \leq (\varepsilon/\gamma)^{n+1}\|f-\mu(f)\|\,. \tag{29}$$

Moreover, for $t \geq 0$*, the following estimates hold:*

$$\|S_\varepsilon(t)f - \mu(S_\varepsilon(t)f)\| \leq e^{-(\gamma-\varepsilon)t}\|f-\mu(f)\|\,, \tag{30}$$

$$\left|\mu(S_\varepsilon(t)f) - \mu_\varepsilon(f)\right| \leq \frac{\varepsilon}{\gamma-\varepsilon}e^{-(\gamma-\varepsilon)t}\|f-\mu(f)\|\,, \tag{31}$$

$$|\mu_\varepsilon(f) - \mu(f)| \leq \frac{\varepsilon}{\gamma-\varepsilon}\|f-\mu(f)\|\,. \tag{32}$$

(iii) μ_ε *is the unique distribution which is both absolutely continuous w.r.t.* μ *and invariant for the perturbed Markov process. The Radon–Nykodim derivative* $h_\varepsilon := d\mu_\varepsilon/d\mu$ *belongs to* $L^2(\mu)$ *and admits the expansion*[6]

$$h_\varepsilon = \mathbb{1} + \sum_{n=1}^\infty \int_0^\infty H_\varepsilon^{(n)}(t)\mathbb{1}dt, \tag{33}$$

where $H_\varepsilon^{(n)}(t) := [S_\varepsilon^{(n-1)}(t)]^*\hat{L}_\varepsilon^*$*,* $n \geq 1$*, are bounded operators on* $L^2(\mu)$ *satisfying the recursion:*

$$H_\varepsilon^{(n+1)}(t) = \int_0^t ds\, H_\varepsilon^{(n)}(s)S^*(t-s)\hat{L}_\varepsilon^* = \int_0^t ds\, S^*(t-s)\hat{L}_\varepsilon^* H_\varepsilon^{(n)}(s). \tag{34}$$

Moreover, it holds $\|h_\varepsilon - \mathbb{1}\| \leq \frac{\varepsilon}{\gamma-\varepsilon}$*.*

(iv) Suppose that for any $t > 0$ *and for any measurable* $B \subset \Omega$ *it holds*

$$\mu\big(\{\eta \in B^c \,:\, \mathbb{P}_\eta^{(\varepsilon)}(\eta_t \in B) = 0 \text{ and } \mathbb{P}_\eta(\eta_t \in B) > 0\}\big) = 0\,. \tag{35}$$

Then also μ *is absolutely continuous w.r.t.* μ_ε*.*

(v) For any $f \in L^\infty(\mu)$ *it holds*

$$\|S_\varepsilon(t)f - \mu_\varepsilon(f)\|_\varepsilon \leq \big(\frac{\gamma}{\gamma-\varepsilon}\big)^{3/2}e^{-\frac{\gamma-\varepsilon}{2}t}\|f-\mu(f)\|_\infty\,, \qquad t \geq 0\,, \tag{36}$$

where $\|\cdot\|_\varepsilon$*,* $\|\cdot\|_\infty$ *denote the norm in* $L^2(\mu_\varepsilon)$ *and* $L^\infty(\mu)$ *respectively.*

(vi) μ_ε *is ergodic w.r.t. time–translations, as in Remark 3.1.*

The proof of the above theorem is given in Section 6

Remark 3.4. *Theorem 5 presents some intersection with* [29, Thm. 2.2 and Thm. 4.1]*. There the authors consider also unbounded perturbations satisfying some sector condition and the analysis is not based on the Dyson–Phillips expansion. In particular, in* [29] *the content of Theorem 5–(i) is obtained only for* $\nu \ll \mu$ *with* $d\nu/d\mu \in L^2(\mu)$ *(while here the last condition is absent). The existence of a unique invariant distribution* $\mu_\varepsilon \ll \mu$ *for the perturbed process is obtained also in* [29] *and our expansion* (33) *is equivalent to the expansion (4.5) in* [29]*, see* [4, Appendix B] *for more details. In Theorem 5 we have collected information on the exponential convergence of semigroups (which is relevant to get the invariance principle in Proposition 3.6), while in* [29] *the exponential convergence of densities is derived.*

[6]We denote by A^* the adjoint of the operator A on $L^2(\mu)$

Remark 3.5. *Let h_ε be the Radon–Nykodim derivative of μ_ε w.r.t. μ. Let $A \subset \Omega$ be a Borel set such that $\mu_\varepsilon(A) = 0$. Since $0 = \mu_\varepsilon(A) = \mu(A) + \mu((h_\varepsilon - 1)\mathbb{1}_A)$, we have $\mu(A) = \mu((1 - h_\varepsilon)\mathbb{1}_A) \leq \|\mathbb{1} - h_\varepsilon\|\mu(A)^{1/2}$. Hence, by Theorem 5-*(iii)

$$\mu_\varepsilon(A) = 0 \;\Rightarrow\; \mu(A) \leq \varepsilon^2/(\gamma - \varepsilon)^2 \,. \tag{37}$$

This implies that any property that holds μ_ε–a.s. holds also μ–a.s. if $\mu \ll \mu_\varepsilon$ and anyway, in the general case, holds for all $\eta \in \Omega$ with exception of a set of μ–measure bounded by $\varepsilon^2/(\gamma - \varepsilon)^2$.

We now concentrate on additive functionals for the perturbed process. As an immediate consequence of Birkhoff ergodic theorem, Theorem 5 and (37) in Remark 3.5, we get:

Corollary 1 (Law of large numbers)**.** *Let Assumption 7 be satisfied, let $\varepsilon < \gamma$ and let $f : \Omega \to \mathbb{R}$ be a measurable function, nonnegative or in $L^1(\mu_\varepsilon)$ (e.g. bounded or in $L^2(\mu)$). Then*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(\eta_s) = \mu_\varepsilon(f)\,, \qquad \mathbb{P}^{(\varepsilon)}_\eta - \text{a.s.} \tag{38}$$

for μ_ε–a.e. η (recall Remark 3.5).

We conclude this general part with an invariance principle:

Proposition 3.6 (Invariance principle for additive functionals)**.** *Suppose that Ω is a Polish space and that the perturbed process on Ω is Feller. Let Assumption 7 be satisfied, let $\varepsilon < \gamma$ and let $f : \Omega \to \mathbb{R}$ be a function in $C_b(\Omega)$. Given $n \in \mathbb{N}$, define the process*

$$B^{(n)}_t(f) := \int_0^{nt} \frac{f(\eta_s) - \mu_\varepsilon(f)}{\sqrt{n}} ds\,, \qquad t \in \mathbb{R}_+\,.$$

Then there exists a constant $\sigma^2 \geq 0$ such that under $\mathbb{P}^{(\varepsilon)}_{\mu_\varepsilon}$ the process $\big(B^{(n)}_t\big)_{t\in\mathbb{R}_+}$ weakly converges to a Brownian motion with diffusion coefficient σ^2.

Proposition 3.6 is proved in Section 7, where a characterization of σ^2 is given.

4. Proof of Lemma 3.2

We first note that the semigroup $\tilde S_\varepsilon(t)$ is well defined since $C(\Omega) = C_b(\Omega)$ due to compactness. Let us prove the lemma. We claim that $\mathcal{D}(\tilde L_\varepsilon) \subset \mathcal{D}(L_\varepsilon)$ and that $\tilde L_\varepsilon f = L_\varepsilon f$ for all $f \in \mathcal{D}(\tilde L_\varepsilon)$. To prove our claim fix $f \in \mathcal{D}(\tilde L_\varepsilon)$. By definition of core, there exists $f_n \in \mathcal{C}_\varepsilon$ with $f_n \overset{\|\cdot\|_\infty}{\to} f$ and $\tilde L_\varepsilon f_n \overset{\|\cdot\|_\infty}{\to} \tilde L_\varepsilon f$. The convergence holds also in $L^2(\mu)$, while by assumption $f_n \in \mathcal{C}_\varepsilon \subset \mathcal{D}(L_\varepsilon)$ and $\tilde L_\varepsilon f_n = L_\varepsilon f_n$. Using that the operator L_ε is closed in $L^2(\mu)$ (being an infinitesimal generator), we get that necessarily $f \in \mathcal{D}(L_\varepsilon)$ and $\tilde L_\varepsilon f = L_\varepsilon f$, thus proving our claim. Let again $f \in \mathcal{D}(\tilde L_\varepsilon)$. Then (cf. [18, Lemma 1.3, Chapter 2]) $\tilde S_\varepsilon(t) f \in \mathcal{D}(\tilde L_\varepsilon)$. By the above claim we get that $\tilde S_\varepsilon(t) f \in \mathcal{D}(L_\varepsilon)$ and $L_\varepsilon \tilde S_\varepsilon(t) f = \tilde L_\varepsilon \tilde S_\varepsilon(t) f$. Since (cf. [18, Lemma 1.3, Chapter 2]) $\lim_{\delta\to 0} \frac{\tilde S_\varepsilon(t+\delta) - \tilde S_\varepsilon(t)}{\delta} f = \tilde L_\varepsilon \tilde S_\varepsilon(t) f = L_\varepsilon \tilde S_\varepsilon(t) f$ in uniform norm, the same must hold in $L^2(\mu)$ (if $t = 0$, the above limit has to be taken with $\delta \downarrow 0$). Collecting the above observations we get that the function $\varphi(t) : [0, +\infty) \ni t \mapsto \tilde S_\varepsilon(t) f \in L^2(\mu)$ has values in $\mathcal{D}(L_\varepsilon)$ and satisfies the Cauchy problem $\varphi'(t) = L_\varepsilon \varphi(t)$, $\varphi(0) = f$, where $\varphi'(0)$ has to be thought as right derivative. Since also the function $\bar\varphi(t) : [0, +\infty) \ni t \mapsto S_\varepsilon(t) f \in L^2(\mu)$ satisfies the same properties, by the uniqueness of the solution of the Cauchy problem (cf. [23, end of page 483]) we conclude that $\tilde{S}_\varepsilon(t)f = S_\varepsilon(t)f$, i.e. we get (23) for $f \in \mathcal{D}(\tilde{L}_\varepsilon)$. To extend (23) to any $f \in C(\Omega)$ its enough to take $f_n \in \mathcal{D}(\tilde{L}_\varepsilon)$ with $\|f - f_n\|_\infty \to 0$. Then also $\|f - f_n\| \to 0$. At this point it is enough to take the limit $n \to \infty$ in the identity $\tilde{S}_\varepsilon(t)f_n = S_\varepsilon(t)f_n$ and use that $\tilde{S}_\varepsilon(t)$ is a bounded operator in $C(\Omega)$, while $S_\varepsilon(t)$ is a bounded operator in $L^2(\mu)$.

5. Preliminary estimates on Dyson–Philipps expansion

In this section we prove Proposition 3.3 and the bound in (30). Let us first state a simple remark (whose proof is omitted since it is standard) that will be frequently used:

Remark 5.1. *Since μ is a stationary distribution for the unperturbed process and the Poincaré inequality (21) is satisfied, we have that (i) $S(t)f = f$ for all $t \geq 0$ iff f is a constant function, (ii) 0 is a simple eigenvalue of L, (iii) $\mu(S(t)f) = \mu(f)$ for any $f \in L^2(\mu)$. Moreover, since L_ε is a Markov generator, it must be $\hat{L}_\varepsilon f = 0$ for f constant.*

In the next proposition, by means of the Poincaré inequality, we improve known general bounds concerning the Dyson–Phillips expansion. In what follows, given $f \in L^2(\mu)$, we abbreviate (recall (25)) :

$$g_n(t) := S_\varepsilon^{(n-1)}(t)f, \text{ for any } n \geq 1, \tag{39}$$

so that the Dyson–Phillips expansion in equation 26 reads as

$$S_\varepsilon(t)f = \sum_{n=1}^{\infty} g_n(t)\,, \qquad f \in L^2(\mu)\,. \tag{40}$$

Proposition 5.2. *For each $f \in L^2(\mu)$ and $n \geq 1$ it holds*

$$\|g_n(t) - \mu(g_n(t))\| \leq e^{-\gamma t}\frac{(\varepsilon t)^{n-1}}{(n-1)!}\|f - \mu(f)\|\,, \tag{41}$$

$$|\mu\big(\hat{L}_\varepsilon g_n(t)\big)| \leq \varepsilon e^{-\gamma t}\frac{(\varepsilon t)^{n-1}}{(n-1)!}\|f - \mu(f)\|\,, \tag{42}$$

$$|\mu(g_{n+1}(t))| \leq (\varepsilon/\gamma)^n\|f - \mu(f)\|\,. \tag{43}$$

Moreover, $\mu(g_1(t)) = \mu(f)$ and, for each $n \geq 1$,

$$\lim_{t\to\infty} \mu(g_{n+1}(t)) = \int_0^\infty \mu\big(\hat{L}_\varepsilon g_n(s)\big)ds\,, \tag{44}$$

the integral being well posed due to (42). More precisely, it holds

$$\big|\mu(g_{n+1}(t)) - \int_0^\infty \mu\big(\hat{L}_\varepsilon g_n(s)\big)ds\big| \leq \|f - \mu(f)\| \int_t^\infty \varepsilon e^{-\gamma s}\frac{(\varepsilon s)^n}{n!}ds\,. \tag{45}$$

Proof. To prove (41) we bound

$$\|g_{n+1}(t) - \mu(g_{n+1}(t))\| = \Big\| \int_0^t S(t-s)\hat{L}_\varepsilon g_n(s)ds - \mu\Big(\int_0^t S(t-s)\hat{L}_\varepsilon g_n(s)ds\Big)\Big\|$$
$$\leq \int_0^t \big\|S(t-s)\hat{L}_\varepsilon g_n(s) - \mu\big(S(t-s)\hat{L}_\varepsilon g_n(s)\big)\big\|ds$$
$$\leq \int_0^t e^{-\gamma(t-s)}\big\|\hat{L}_\varepsilon g_n(s) - \mu\big(\hat{L}_\varepsilon g_n(s)\big)\big\|ds \leq \int_0^t e^{-\gamma(t-s)}\big\|\hat{L}_\varepsilon g_n(s)\big\|ds$$
$$= \int_0^t e^{-\gamma(t-s)}\big\|\hat{L}_\varepsilon\big(g_n(s) - \mu(g_n(s))\big)\big\|ds \leq \int_0^t e^{-\gamma(t-s)}\big\|\hat{L}_\varepsilon\big\|\big\|g_n(s) - \mu(g_n(s))\big\|ds,$$

where the second inequality follows from Item (iii) in Remark 5.1 and from the L^2–exponential decay (21), the third one uses that $\|f-\mu(f)\| \leq \|f\|$ for any $f \in L^2(\mu)$. With this established, we can check (41) inductively, noticing that for $n = 1$, the inequality is just a consequence of the L^2–exponential decay (21) and Item (iii) in Remark 5.1.

To prove (42), by Remark 5.1 we can bound $|\mu(\hat{L}_\varepsilon g_n(s))|$ by $|\mu(\hat{L}_\varepsilon(g_n(s)-\mu(g_n(s)))| \leq \|\hat{L}_\varepsilon\|\|g_n(s) - \mu(g_n(s))\|$. At this point the thesis follows from (41).

To prove (43) we write $\mu(g_{n+1}(t))$ as $\int_0^t \mu(\hat{L}_\varepsilon g_n(s))ds$. By (42) the last integral can be bounded by $\frac{\varepsilon^n}{(n-1)!}\|f-\mu(f)\| \int_0^\infty e^{-\gamma s}s^{n-1}ds$, thus leading to (43).

The identity $\mu(g_1(t)) = \mu(f)$ follows from Remark 5.1. As in the proof of (42), $\int_t^\infty \big|\mu\big(\hat{L}_\varepsilon g_n(s)\big)\big|ds \leq \int_t^\infty ds\varepsilon\, e^{-\gamma s}\frac{(\varepsilon s)^{n-1}}{(n-1)!}\|f-\mu(f)\|$, which goes to zero as $t \to \infty$. Hence, $\mu(g_{n+1}(t))$ has limit (44), which is finite, and also (45) holds. □

We have now the tools to prove some assertions of Section 3:

Proof of Prop. 3.3 and (30). Due to (41) and (43) we can bound the l.h.s. of (27) by

$$\|f-\mu(f)\|\Big\{\sum_{n=k}^\infty e^{-\gamma t}\frac{(\gamma t)^n}{n!}(\varepsilon/\gamma)^n + \sum_{n=k}^\infty (\varepsilon/\gamma)^n\Big\} \leq \|f-\mu(f)\| 2\sum_{n=k}^\infty (\varepsilon/\gamma)^n\,,$$

thus leading to (27).

Due to the Dyson–Phillips expansion, we can bound $\|S_\varepsilon(t)f - \mu(S_\varepsilon(t)f)\|$ by $\sum_{n\geq 1}\|g_n(t) - \mu(g_n(t))\|$, and (30) follows immediately from (41). □

6. Proof of Theorem 5

Let us denote by $\Gamma(f)$ the r.h.s. of (28). We first observe that by (42) the integral and series in the r.h.s. of (28) are absolutely convergent, hence $\Gamma(f)$ is well defined. Moreover, always by (42), we get $|\Gamma(f)| \leq \big(\gamma/(\gamma-\varepsilon)\big)\|f\|$.

Due to the Dyson–Phillips expansion, it holds $\mu(S_\varepsilon(t)f) = \sum_{n\geq 1}\mu(g_n(t))$. Hence, one easily derives (31) with $\mu_\varepsilon(f)$ replaced by $\Gamma(f)$ from (45). As a byproduct with (30) proved at the end of Section 5, we conclude that

$$\lim_{t\to\infty}\|S_\varepsilon(t)f - \Gamma(f)\| = 0\,, \qquad f \in L^2(\mu)\,. \tag{46}$$

6.1. **Proof of Item** (i)**.** Consider now the perturbed Markov process with initial distribution ν as in Item (i) and call $\nu_\varepsilon^{(t)}$ its distribution at time t. Take $f \in C_b(\Omega)$. We claim that

$$\nu_\varepsilon^{(t)}(f) = \mu\Big(\frac{d\nu}{d\mu}\mathbb{E}_\cdot^{(\varepsilon)}(f(\eta_t))\Big) = \mu\Big(\frac{d\nu}{d\mu}S_\varepsilon(t)f\Big) \underset{t\to\infty}{\longrightarrow} \Gamma(f)\,, \qquad f \in C_b(\Omega)\,. \tag{47}$$

(note that the first identity is trivial, while the second follows from (23)). To this aim it is enough to prove this equivalent claim: for any diverging sequence $t_n \nearrow \infty$ there exists a subsequence t_{n_k} such that $\mu\Big(\frac{d\nu}{d\mu}[S_\varepsilon(t_{n_k})f - \Gamma(f)]\Big) \to 0$ as $k \to \infty$. Since $S_\varepsilon(t_n)f - \Gamma(f) \to 0$ in $L^2(\mu)$, there exists a subsequence t_k such that $S_\varepsilon(t_{n_k})f - \Gamma(f) \to 0$ μ–a.s.. Hence $|S_\varepsilon(t_{n_k})f - \Gamma(f)|$ is a function bounded by $(1 + \gamma/(\gamma - \varepsilon))\|f\|_\infty$ (recall (23)) and converging to zero μ–a.s.. The equivalent claim then follows by the dominated convergence theorem.

We know that $\Gamma : L^2(\mu) \to L^2(\mu)$ is a bounded linear operator. By Riesz representation theorem, there exists $h_\varepsilon \in L^2(\mu)$ such that $\Gamma(f) = \mu(h_\varepsilon f)$ for each $f \in L^2(\mu)$. We observe that $h_\varepsilon \geq 0$ μ–a.s. since $\Gamma(f) \geq 0$ for any $f \in C_{b,+}(\Omega)$ (cf. Lemma A.1-(ii)). Let us define the nonnegative measure μ_ε as $d\mu_\varepsilon = h_\varepsilon d\mu$. By (47) we conclude that $\mu_\varepsilon(\mathbb{1}) = 1$, hence μ_ε is a probability measure. Using that $\Gamma(f) = \mu_\varepsilon(f)$, by (47) we get Item (i).

6.2. **Proof of Item** (ii)**.** Since $\mu_\varepsilon(f) = \Gamma(f)$, by the definition of $\Gamma(f)$ we get (28). We have already proved (30) at the end of Section 5, while at the beginning of this section we have shown that (31) holds with $\Gamma(f)$ instead of $\mu_\varepsilon(f)$. Since these two values are indeed equal, we get (31) and therefore Item (ii). (29) and (32) are a simple consequence of (28) and (42).

6.3. **Proof of Item** (iii)**.** By construction, $\mu_\varepsilon \ll \mu$ with Radon–Nikodym derivative h_ε. By (46) and since $\Gamma(f) = \mu(h_\varepsilon f) = \mu_\varepsilon(f)$ for any $f \in L^2(\mu)$, we have that $\mu_\varepsilon(f) = \lim_{t\to\infty} \mu_\varepsilon(S_\varepsilon(t)f)$ for any $f \in L^2(\Omega)$. For any $g \in C_b(\Omega)$, we use this once with $f := S_\varepsilon(s)g$ and another time with $f := g$. We conclude that $\mu_\varepsilon(S_\varepsilon(s)g) = \mu_\varepsilon(g)$ by using the semigroup property $S_\varepsilon(t+s)g = S_\varepsilon(t)S_\varepsilon(s)g$ to identify the two limits. By Assumption 7 this implies that $\mu_\varepsilon\big(\mathbb{E}_\cdot^{(\varepsilon)}[g(\eta_s)]\big) = \mu_\varepsilon(g)$ for any $g \in C_b(\Omega)$, hence the invariance of μ_ε for the perturbed Markov process. The uniqueness assertion follows from Item (i).

To derive the expansion (33), note first that for $n \geq 0$, and any $f \in L^2(\mu)$, we have

$$\mu\big(\hat{L}_\varepsilon S_\varepsilon^{(n)}(s)f\big) = \mu\big(\big([S_\varepsilon^{(n)}(s)]^*\hat{L}_\varepsilon^*\mathbb{1}\big)f\big) =: \mu\big(\big(H_\varepsilon^{(n+1)}(s)\mathbb{1}\big)f\big), \tag{48}$$

and the recursions in (34) easily follow. By (42) and (48), we then get $\|H_\varepsilon^{(n)}(s)\mathbb{1}\| \leq \varepsilon e^{-\gamma s}\frac{(\varepsilon s)^n}{n!}$. It then follows that the integrals and the series in the r.h.s. of (33) are absolutely convergent in $L^2(\mu)$, and therefore, by (28) and (48), the expansion (33) holds.

From (33) and the bound $\|H_\varepsilon^{(n)}(s)\mathbb{1}\| \leq \varepsilon e^{-\gamma s}\frac{(\varepsilon s)^n}{n!}$ we get $\|h_\varepsilon - \mathbb{1}\| \leq \frac{\varepsilon}{\gamma-\varepsilon}$.

6.4. **Proof of Item** (iv)**.** Some of the ideas are taken from [29][Sec. 3.1.2] although we show that some assumptions there can indeed be avoided.

We know that $\mu_\varepsilon \ll \mu$ (see Item (vi)). We call A_ε the μ–support of $h_\varepsilon = d\mu_\varepsilon/d\mu$. We only need to prove that $\mu(A_\varepsilon^c) = 0$. By stationarity of μ_ε w.r.t. the perturbed dynamics, we have

$$0 = \mu_\varepsilon(A_\varepsilon^c) = \mu_\varepsilon\left(S_\varepsilon(t)\mathbb{1}_{A_\varepsilon^c}\right) = \int \mu(d\eta)h_\varepsilon(\eta)\mathbb{P}_\eta^{(\varepsilon)}\left[\eta_t \in A_\varepsilon^c\right] . \tag{49}$$

Hence, $\mu\big(\{\eta \in A_\varepsilon : \mathbb{P}_\eta^{(\varepsilon)}\big[\eta_t \in A_\varepsilon^c\big] > 0\}\big) = 0$. By condition (35) we conclude that $\mu\big(\{\eta \in A_\varepsilon : \mathbb{P}_\eta\big[\eta_t \in A_\varepsilon^c\big] > 0\}\big) = 0$. This implies that the function $\eta \mapsto \mathbb{P}_\eta\big[\eta_t \in A_\varepsilon^c\big] = S(t)\mathbb{1}_{A_\varepsilon^c}(\eta) \in [0,1]$ is zero on A_ε μ–a.s., hence $0 \le S(t)\mathbb{1}_{A_\varepsilon^c} \le \mathbb{1}_{A_\varepsilon^c}$ μ–a.s.. Suppose by absurd that $\mu(\{\eta : S(t)\mathbb{1}_{A_\varepsilon^c}(\eta) < \mathbb{1}_{A_\varepsilon^c}(\eta)\}) > 0$. Then we would conclude that $\mu(S(t)\mathbb{1}_{A_\varepsilon^c}) < \mu(\mathbb{1}_{A_\varepsilon^c})$, in contradiction with the stationarity of μ w.r.t. $S(t)$. Hence, it must be $S(t)\mathbb{1}_{A_\varepsilon^c} = \mathbb{1}_{A_\varepsilon^c}$ μ–a.s.. Since this holds for each t, by the ergodicity of μ we conclude that $\mu(A_\varepsilon^c) \in \{0,1\}$. If $\mu(A_\varepsilon^c) = 1$, then $h_\varepsilon \equiv 0$ μ–a.s., while $\mu(h_\varepsilon) = 1$. It remains the case $\mu(A_\varepsilon^c) = 0$, which implies that $\mu \ll \mu_\varepsilon$.

6.5. **Proof of Item** (v)**.** Let $C_{b,+}(\Omega) := \{f \in C_b(\Omega) : f \ge 0\}$ and $L^2_+(\Omega) := \{f \in L^2(\mu) : f \ge 0 \ \mu\text{–a.s.}\}$. By Assumption 7 we have $S_\varepsilon(t)f \ge 0$ μ–a.s. for any $f \in C_{b,+}(\Omega)$. Since $C_{b,+}(\Omega)$ is $\|\cdot\|$–dense in $L^2_+(\Omega)$, as immediate consequence of Lemma A.1-(i), we conclude that $S_\varepsilon(t)f \ge 0$ μ–a.s. for any $f \in L^2_+(\mu)$. Since $\|f\|_\infty - f \in L^2_+(\mu)$ and $S_\varepsilon(t)\|f\|_\infty = \|f\|_\infty$, we then conclude that $S_\varepsilon(t)f \le \|f\|_\infty$ μ–a.s. for any $f \in L^\infty(\mu)$. By applying the last bound to $-f$, we get $|S_\varepsilon(t)f| \le \|f\|_\infty$ μ–a.s. for any $f \in L^\infty(\mu)$.

The above considerations and Schwarz inequality imply for any $f \in L^\infty(\mu)$ that

$$\begin{aligned}\|S_\varepsilon(t)f - \mu_\varepsilon(f)\|_\varepsilon^2 &= \mu_\varepsilon(|S_\varepsilon(t)f - \mu_\varepsilon(f)|^2) \le \|f - \mu_\varepsilon(f)\|_\infty \mu(h_\varepsilon|S_\varepsilon(t)f - \mu_\varepsilon(f)|) \\ &\le \|f - \mu_\varepsilon(f)\|_\infty \|h_\varepsilon\| \cdot \|S_\varepsilon(t)f - \mu_\varepsilon(f)\| .\end{aligned} \tag{50}$$

By (32) in Item (ii) we have $\|f - \mu_\varepsilon(f)\|_\infty \le \|f - \mu(f)\|_\infty + |\mu(f) - \mu_\varepsilon(f)| \le \frac{\gamma}{\gamma-\varepsilon}\|f - \mu(f)\|_\infty$. By Item (iii) we have $\|h_\varepsilon\| \le \gamma/(\gamma-\varepsilon)$ and by Item (ii) we have $\|S_\varepsilon(t)f - \mu_\varepsilon(f)\| \le [\gamma/(\gamma-\varepsilon)]e^{-(\gamma-\varepsilon)t}\|f - \mu(f)\|$. Hence the conclusion.

6.6. **Proof of Item** (vi)**.** Recall Remark 3.1. Let $B \subset \Omega$ be a Borel set satisfying $\mathbb{1}_B(\eta_t) = \mathbb{1}_B(\eta_0)$ $\mathbb{P}_{\mu_\varepsilon}^{(\varepsilon)}$–a.s. for all $t \ge 0$. By Lemma A.4[7] we then have $S_\varepsilon(t)\mathbb{1}_B = \mathbb{1}_B$ μ_ε–a.s.. Then, by (36), we conclude that $\mathbb{1}_B = \mu_\varepsilon(B)$ μ_ε–a.s., thus implying that $\mu_\varepsilon(B) \in \{0,1\}$.

7. Proof of the invariance principle in Proposition 3.6

Given $h \in L^2(\mu_\varepsilon)$, we introduce the functional $A_t(h) = \int_0^t h(\eta_s)ds$ defined on the path space $D(\mathbb{R}_+, \Omega)$. By Schwarz inequality and stationarity of μ_ε for the perturbed process, we can bound

$$\|A_t(h)\|_{L^2\left(\mathbb{P}_{\mu_\varepsilon}^{(\varepsilon)}\right)} = \mathbb{E}_{\mu_\varepsilon}^{(\varepsilon)}\left[A_t(h)^2\right]^{1/2} \le t\|h\|_\varepsilon . \tag{51}$$

The family of operators $\mathcal{S}_\varepsilon(t)h(\eta) := \mathbb{E}_\eta^{(\varepsilon)}\big[h(\eta_t)\big]$, $t \in \mathbb{R}_+$, is a well defined strongly continuous contraction semigroup in $L^2(\mu_\varepsilon)$ for $t \in \mathbb{R}_+$ (see Lemma A.2 and its proof). We write $\mathcal{L}_\varepsilon : \mathcal{D}(\mathcal{L}_\varepsilon) \subset L^2(\mu_\varepsilon) \to L^2(\mu_\varepsilon)$ for its infinitesimal generator. Do not confuse the above operators $\mathcal{S}_\varepsilon(t), \mathcal{L}_\varepsilon$ with the previously defined $S_\varepsilon(t), L_\varepsilon$ which live in $L^2(\mu)$. On the other hand, by (23), given $h \in C_b(\Omega)$ it holds $\mathcal{S}_\varepsilon(t)h = S_\varepsilon(t)h$ μ–a.s. and therefore μ_ε–a.s. (since $\mu_\varepsilon \ll \mu$).

[7]In the proof of Lemma A.4 we use Theorem 5 but not Item (vi).

Let $f \in C_b(\Omega)$, as in the theorem. Since along the proof ε is fixed, at cost of replacing f by $f - \mu_\varepsilon(f)$ we assume that $\mu_\varepsilon(f) = 0$. Due to (36) and the previous observations, we can bound

$$\int_0^\infty \|\mathcal{S}_\varepsilon(t) f dt\|_\varepsilon = \int_0^\infty \|S_\varepsilon(t) f dt\|_\varepsilon =: \kappa < \infty \,. \tag{52}$$

Hence, $g := \int_0^\infty \mathcal{S}_\varepsilon(t) f dt$ is a well defined element of $L^2(\mu_\varepsilon)$. Since $\mathcal{S}_\varepsilon(r)g - g = -\int_0^r \mathcal{S}_\varepsilon(t) f dt$ and since $\mathcal{S}_\varepsilon(t) f \to f$ in $L^2(\mu_\varepsilon)$ as $t \downarrow 0$, by definition of infinitesimal generator we get that $g \in \mathcal{D}(\mathcal{L}_\varepsilon)$ and $-\mathcal{L}_\varepsilon g = f$.

As a consequence we can write $A_t(f) = M_t + R_t$, where

$$M_t := g(\eta_t) - g(\eta_0) - \int_0^t \mathcal{L}_\varepsilon g(\eta_s) ds \,, \tag{53}$$

$$R_t := -g(\eta_t) + g(\eta_0) \,. \tag{54}$$

By (52), we get that

$$\|R_t\|_{L^2(\mathbb{P}^{(\varepsilon)}_{\mu_\varepsilon})} \le 2\kappa \,. \tag{55}$$

In what follows, we apply the invariance principle for martingales as stated in [28, Thm. 2.29, Chp. 2], which holds for càdlàg martingales w.r.t. filtrations satisfying the usual conditions. To this aim, we take the augmented filtration $(\bar{\mathcal{F}}_t)_{t\ge 0}$ w.r.t. $\mathbb{P}^{(\varepsilon)}_{\mu_\varepsilon}$ of the natural filtration $(\mathcal{F}_t)_{t\ge 0}$, where $\mathcal{F}_t := \sigma(\eta_s : 0 \le s \le t)$ [24, Chp. 2]. Since we have assumed that the perturbed process is Feller, then this filtration satisfies the usual condition w.r.t. $\mathbb{P}^{(\varepsilon)}_{\mu_\varepsilon}$ [24][Prop. 7.7, Chp. 2]. It is known (cf. [28, Chp. 2] that $(M_t)_{t\ge 0}$ is a martingale w.r.t. the augmented filtration $(\bar{\mathcal{F}}_t)_{t\ge 0}$. Below we work with the càdlàg modification of $(M_t)_{t\ge 0}$ (cf. [24, Thm. 3.13, Chp. 1]), that we still call M_t with some abuse of notation.

We split the rest of the proof in two parts. First we show an invariance principle for the martingale M_t, afterwards we prove that the rest R_t is negligible (cf. Lemma 7.1 and Lemma 7.2).

Lemma 7.1. *For any $t \ge 0$, define $M^{(n)}_t := \frac{M_{nt}}{\sqrt{n}}, n \in \mathbb{N}$. Then, under $\mathbb{P}^{(\varepsilon)}_{\mu_\varepsilon}$, the rescaled process $(M^{(n)}_t)_{t\in\mathbb{R}_+}$ weakly converges to a Brownian motion with diffusion constant $\sigma^2 := \mathbb{E}^{(\varepsilon)}_{\mu_\varepsilon}(M_1^2) \ge 0$.*

Proof. The martingale M_t is square integrable w.r.t. $\mathbb{P}^{(\varepsilon)}_{\mu_\varepsilon}$. According to [28, Thm. 2.29, Chp. 2] we only need to prove that $\langle M\rangle_k/k$ converges to $\mathbb{E}^{(\varepsilon)}_{\mu_\varepsilon}(M_1^2)$ both $\mathbb{P}^{(\varepsilon)}_{\mu_\varepsilon}$–a.s. and in $L^1(\mathbb{P}^{(\varepsilon)}_{\mu_\varepsilon})$. To this aim we write $\langle M\rangle_k = \sum_{j=0}^{k-1}(\langle M\rangle_{j+1} - \langle M\rangle_j)$ and we point out (cf. [28, chp.2]) that

$$\langle M\rangle_{j+1} - \langle M\rangle_j = \langle M\rangle_1 \circ \theta_j \quad \mathbb{P}^{(\varepsilon)}_{\mu_\varepsilon}\text{–a.s..}$$

Moreover, we have $\mathbb{E}^{(\varepsilon)}_{\mu_\varepsilon}\big(\langle M\rangle_1\big) = \mathbb{E}^{(\varepsilon)}_{\mu_\varepsilon}\big(M_1^2\big) < \infty$.

At this point the thesis follows from the a.s and the L^1–Birkhoff ergodic theorem and the ergodicity of $\mathbb{P}^{(\varepsilon)}_{\mu_\varepsilon}$ w.r.t. time translations. □

In the following lemma we give a bound to control the trajectories of the error R_t (cf. (54)). The proof of the lemma is based on a simple block-decomposition argument in the spirit of [36].

Lemma 7.2. *We have*

$$\lim_{T\to\infty} \frac{\sup_{t\le T}|R_t|}{\sqrt{T}} = 0 \qquad \text{in } \mathbb{P}^{(\varepsilon)}_{\mu_\varepsilon}\text{–probability} . \tag{56}$$

Since, as shown in (53), we can write $B^{(n)}_t(f) = M^{(n)}_t + \frac{R_{tn}}{\sqrt{n}}$, Proposition 3.6 follows from Lemma 7.1 and Lemma 7.2.

Proof of Lemma 7.2. Given a positive integer j, let $m_j := \lceil j^{4/5}\rceil$ and $\ell_j = \lceil j^{1/5}\rceil$. Consider the partition of the time interval $[0,j]$ in m_j sub–intervals $I^j_k := [kl_j,(k+1)l_j)$, $k = 0,\dots,m_j-1$, of measure ℓ_j (for simplicity of notation we assume that $m_j\ell_j = j$). From the decomposition $A_t(f) = M_t + R_t$ we can bound

$$\begin{aligned}\sup_{t\le j}|R_t| \le \max_{k=0,1,\dots,m_j}|R_{k\ell_j}| + \max_{k=0,1,\dots,m_j-1}\sup_{u\in I^j_k}|A_u(f)-A_{k\ell_j}(f)|\\ + \max_{k=0,1,\dots,m_j-1}\sup_{u\in I^j_k}|M_u - M_{k\ell_j}| =: C_{1,j}+C_{2,j}+C_{3,j}\end{aligned} \tag{57}$$

- **Step 1**. We first control the term

$$C_{3,j} := \max_{k=0,1,\dots,m_j-1}\sup_{u\in I^j_k}|M_u - M_{k\ell_j}| .$$

Due to Lemma 7.1 and the fact that for standard Brownian motion W_t, and any $C,T>0$, it holds $P(\sup_{t\le T}|W_t|\ge C)\le \exp\{-C^2/2T\}$, we have $\lim_{j\to\infty}\mathbb{P}^{(\varepsilon)}_{\mu_\varepsilon}(C_{3,j} > \delta\sqrt{j}) = 0$.

- **Step 2**. Let us now consider the term $C_{1,j} := \max_{k=0,1,\dots,m_j}|R_{k\ell_j}|$. By a union bound, Markov inequality and the uniform bound in (55), for any $\delta>0$, we can estimate

$$\mathbb{P}^{(\varepsilon)}_{\mu_\varepsilon}\left(\max_{k=0,1,\dots,m_j}|R_{k\ell_j}|\ge \delta\sqrt{j}\right) \le \frac{4\kappa^2 m_j}{\delta^2 j} = \mathcal{O}(j^{-1/5}) , \tag{58}$$

- **Step 3.** We control the remaining term

$$C_{2,j} := \max_{k=0,1,\dots,m_j-1}\sup_{u\in I^j_k}|A_u(f)-A_{k\ell_j}(f)| .$$

First we observe that

$$|A_u(f)-A_{k\ell_j}(f)| \le \int_{I^j_k}|f(\eta_s)|ds \le \|f\|_\infty \ell_j , \qquad \forall u\in I^j_k,\ k\le m_j-1.$$

Thus, $C_{2,j}\ge\delta\sqrt{j}$ implies that $\|f\|_\infty\lceil j^{1/5}\rceil \ge \delta\sqrt{j}$. In particular,

$$\mathbb{P}^{(\varepsilon)}_{\mu_\varepsilon}(C_{2,j}\ge 2\sqrt{j}) = 0 \tag{59}$$

for j large enough.

We can now conclude the proof. We consider a generic $T\ge 1$ and let j be such that $j\le T<j+1$. Since

$$\frac{\sup_{t\le T}|R_t|}{\sqrt{T}} \le \frac{\sup_{t\le j+1}|R_t|}{\sqrt{j+1}}\frac{\sqrt{j+1}}{\sqrt{j}} .$$

from (57), the arbitrariness of δ together with the three steps above, we get

$$\lim_{T\to\infty}\mathbb{P}^{(\varepsilon)}_{\mu_\varepsilon}\Big(T^{-1/2}\sup_{t\le T}|R_t|\ge\delta\Big) = 0$$

for any $\delta>0$, and therefore the thesis. ☐

Remark 7.3. *The above proof is an extension to the non–reversible case of the classic Kipnis–Varadhan approach. Lemma 7.1 is standard, but the control of the rest R_t provided in Lemma 7.2 does not follow from the estimates in* [26] *(note in particular that Lemma 1.4 there requires reversibility). We mention that an alternative strategy in the non–reversible setting is given by* [28, Thm. 2.32, Chp. 2]*, which on the other hand would require additional assumptions on the Markov process. For what concerns extensions to the non–reversible case of the classic Kipnis–Varadhan approach, we refer to* [28]*,* [34]*,* [44] *and references therein.*

8. A coupling

In this section we describe a coupling between the dynamic random environment, the unperturbed random walk and the perturbed random walk. To this aim we define $\lambda := \sup_\eta \sum_{y\in\mathbb{Z}^d}\big(r(y,\eta)+\max\{0,\hat{r}_\varepsilon(y,\eta)\}\big)$, which is finite due to (4) in Assumption 3. For each $\eta\in\Omega$ we fix two partitions

$$[0,1]=\big(\cup_{y\in\mathbb{Z}^d} I(y,\eta)\big)\cup J(\eta)\,,\qquad [0,1]=\big(\cup_{y\in\mathbb{Z}^d} I_\varepsilon(y,\eta)\big)\cup J_\varepsilon(\eta)\,,$$

where $I(y,\eta)$, $J(\eta)$, $I_\varepsilon(y,\eta)$, $J_\varepsilon(\eta)$ are Borel sets such that

$$|I(y,\eta)|=r(y,\eta)/\lambda\,,\qquad |I_\varepsilon(y,\eta)|=r_\varepsilon(y,\eta)/\lambda\,,$$
$$|I(y,\eta)\cap I_\varepsilon(y,\eta)|=\big[r(y,\eta)+\min\{0,\hat{r}_\varepsilon(y,\eta)\}\big]/\lambda$$

(above $|I|$ denotes the measure of the set I). The above partitions are chosen with the property that the characteristic function $(a,\eta)\mapsto \mathbb{1}\,(a\in I(y,\eta))$ is measurable for any $y\in\mathbb{Z}^d$, where $(a,\eta)\in[0,1]\times\Omega$. The same must be valid for $I_\varepsilon(y,\eta)$.

Let $\mathbb{P}^{\rm env}_\eta$ be the law of the dynamic random environment, i.e. the process with generator $L_{\rm env}$ starting at η. We denote by $(\sigma_t)_{t\in\mathbb{R}_+}$ a generic trajectory of this process. We build a Poisson point process $\mathcal{T}:=\{t_1<t_2<\cdots\}\subset\mathbb{R}_+$ with intensity λ on a suitable probability space with probability measure $P^{\rm Poisson}$. We then build a sequence $\mathcal{U}:=(U_k)_{k\geq 1}$ of i.i.d. uniform variables taking value in $[0,1]$ on another probability space with probability measure $P^{\rm uniform}$. We then consider the product probability space with probability measure $\mathcal{P}_\eta:=\mathbb{P}^{\rm env}_\eta\otimes P^{\rm Poisson}\otimes P^{\rm uniform}$.

We now consider the function $F\left((\sigma_t)_{t\in\mathbb{R}_+},\mathcal{T},\mathcal{U}\right)$ with value in $D(\mathbb{R}_+;\mathbb{Z}^d)$ associating with $(\sigma_t)_{t\in\mathbb{R}_+}$, $\mathcal{T}=\{t_1<t_2<\cdots\}$, $\mathcal{U}=(U_k)_{k\geq1}$ the path $(x_t)_{t\in\mathbb{R}_+}$ defined as follows. We set $x_s=0$ for all $s\in[0,t_1)$. Suppose in general that x_t has been defined for any $t\in[0,t_k)$, with jump times $t_1,t_2\ldots,t_{k-1}$. Set $z:=x_{t_k-}$ and $\zeta:=\sigma_{t_k-}$. If $U_k\in I(y,\tau_z\zeta)$ for some $y\in\mathbb{Z}^d$, then we set $x_t:=z+y$ for any $t\in[t_k,t_{k+1})$, otherwise set $x_t:=z$ for any $t\in[t_k,t_{k+1})$. Since $\lim_{n\to\infty}t_n=\infty$ $\mathcal{P}_\eta$–a.s., the definition of F is well posed $\mathcal{P}_\eta$–a.s.. By construction, sampling $\left((\sigma_t)_{t\in\mathbb{R}_+},\mathcal{T},\mathcal{U}\right)$ according to $\mathcal{P}_\eta$, the random path $\left((\sigma_t)_{t\in\mathbb{R}_+},F\left((\sigma_t)_{t\in\mathbb{R}_+},\mathcal{T},\mathcal{U}\right)\right)$ is the joint Markov process given by the dynamic random environment and the unperturbed random walk. In particular, sampling $\left((\sigma_t)_{t\in\mathbb{R}_+},\mathcal{T},\mathcal{U}\right)$ according to $\mathcal{P}_\eta$, the random path $\tau_{F(t)}\sigma_t$ has law $\mathbb{P}_\eta$, where $F(t)$ stands for the process $F\left((\sigma_t)_{t\in\mathbb{R}_+},\mathcal{T},\mathcal{U}\right)$ evaluated at time t. If in the above definitions, we replace "$U_k\in I(y,\tau_z\zeta)$" by "$U_k\in I_\varepsilon(y,\tau_z\zeta)$", we get a new function F_ε and, sampling $\left((\sigma_t)_{t\in\mathbb{R}_+},\mathcal{T},\mathcal{U}\right)$ according to $\mathcal{P}_\eta$, the random path $\tau_{F_\varepsilon(t)}\sigma_t$ has law $\mathbb{P}^{(\varepsilon)}_\eta$.

In the sequel, we denote by $\mathcal{E}_\eta$ the expectation corresponding to $\mathcal{P}_\eta$, and we adopt the convention that $(X_t)_{t\geq0}:=F\left((\sigma_t)_{t\in\mathbb{R}_+},\mathcal{T},\mathcal{U}\right)$ denotes the walker process in the unperturbed setting, while $(X^{(\varepsilon)}_t)_{t\geq0}:=F_\varepsilon\left((\sigma_t)_{t\in\mathbb{R}_+},\mathcal{T},\mathcal{U}\right)$ refers to the walker process in the perturbed setting. Given ν probability measure on Ω we define $\mathcal{P}_\nu := \int \nu(d\eta)\mathcal{P}_\eta$ and we write $\mathcal{E}_\nu$ for the expectation w.r.t. $\mathcal{P}_\nu$.

9. Proof of Theorem 1 (asymptotic stationary state and velocity)

9.1. **Connection with L^2–perturbation of Markov processes discussed in Section 3.** The operator $L_{\rm env} : \mathcal{D}(L_{\rm env}) \subset L^2(\mu) \to L^2(\mu)$ is the closure in $L^2(\mu)$ of the Markov generator $\mathcal{L}_{\rm env} : \mathcal{D}(\mathcal{L}_{\rm env}) \subset C(\Omega) \to C(\Omega)$, shortly $\big(L_{\rm env}, \mathcal{D}(L_{\rm env})\big) = \overline{\big(\mathcal{L}_{\rm env}, \mathcal{D}(\mathcal{L}_{\rm env})\big)}$ (see e.g. the proof of [32, Prop.4.1, Chp.IV]). Recall the definition (7) of the operator $\hat{L}_\varepsilon$ and set $L_{\rm jumps} f := \sum_{y\in\mathbb{Z}^d} r(y,\eta)\big[f(\tau_y\eta) - f(\eta)\big]$. As done for $\hat{L}_\varepsilon$, one can easily prove that $L_{\rm jump}$ is a bounded operator on $L^2(\mu)$.

Due to the previous observations, the $L^2(\mu)$–generator $L_{\rm ew}$ of the environment seen by the unperturbed walker is the closure of the associated $C(\Omega)$–generator $\mathcal{L}_{\rm env} + \mathcal{L}_{\rm jump}$ with domain $\mathcal{D}(\mathcal{L}_{\rm env})$. In particular, we have

$$L_{\rm ew} f = L_{\rm env} f + L_{\rm jump} f\,, \qquad\qquad f \in \mathcal{D}(L_{\rm ew}) = \mathcal{D}(L_{\rm env})\,. \tag{60}$$

We introduce the operator $L^{(\varepsilon)}_{\rm ew} : \mathcal{D}(L^{(\varepsilon)}) \subset L^2(\mu) \to L^2(\mu)$defined as

$$L^{(\varepsilon)}_{\rm ew} := L_{\rm ew} + \hat{L}_\varepsilon\,, \qquad \mathcal{D}(L^{(\varepsilon)}_{\rm ew}) := \mathcal{D}(L_{\rm ew}) = \mathcal{D}(L_{\rm env})\,. \tag{61}$$

As already observed $\big(L_{\rm env}, \mathcal{D}(L_{\rm env})\big)$ is the closure in $L^2(\mu)$ of $\big(\mathcal{L}_{\rm env}, \mathcal{D}(\mathcal{L}_{\rm env})\big)$. From this property it is simple to check that $(L^{(\varepsilon)}_{\rm ew}, \mathcal{D}(L^{(\varepsilon)}_{\rm ew}))$ is the closure in $L^2(\mu)$ of the $C(\Omega)$–generator $(\mathcal{L}_{\rm env} + \mathcal{L}_{\rm jump} + \hat{\mathcal{L}}^{(\varepsilon)}, \mathcal{D}(\mathcal{L}_{\rm env}))$ of the Feller process given by the environment seen by the perturbed walker (cf. Assumption 4). Recall that $S_\varepsilon(t)$ denotes the semigroup in $L^2(\mu)$ generated by $L^{(\varepsilon)}_{\rm ew}$. By applying Lemma 3.2 with $\mathcal{C}_\varepsilon := \mathcal{D}(\mathcal{L}_{\rm env})$, we get that $S_\varepsilon(t)$ satisfies the identity (23).

Due to the following proposition, we are in the setting of Section 3. Indeed, the unperturbed Markov process in Section 3 is the environment viewed from the unperturbed walker X_t, and $L^{(\varepsilon)}_{\rm ew}$ can be thought of as the perturbed form of $L_{\rm ew}$:

Proposition 9.1. *Assumption 7 is satisfied when the operators $L^{(\varepsilon)}_{\rm ew}$ and $L_{\rm ew}$ play the role of L_ε and L in* (22)*, respectively.*

Proof. By Assumption 5, μ is stationary for the environment seen by the unperturbed walker. For what concerns the Poincaré inequality, we observe that for any $f \in \mathcal{D}(L_{\rm ew})$ it holds

$$-(f, L_{\rm ew} f)_\mu = -(f, L_{\rm env} f)_\mu - (f, L_{\rm jump} f)_\mu \geq -(f, L_{\rm env} f)_\mu \geq \gamma \operatorname{Var}_\mu(f)\,, \tag{62}$$

since

$$-(f, L_{\rm jumps} f)_\mu = \frac{1}{2}\sum_y \int \mu(d\eta)\frac{r(y,\eta) + r(-y,\tau_y\eta)}{2}\left[f(\tau_y\eta) - f(\eta)\right]^2 \geq 0\,.$$

Due to Remark 3.1, μ is ergodic for the unperturbed environment viewed from the walker. Finally, as already pointed out, identity (23) is satisfied. $\square$

Having Theorem 5, Proposition 9.1 and Proposition 5.2, Items (i) and (iv) of Theorem 1 become trivial (for Item (iv) apply in particular (28) and (42)). Below we prove Items (ii) and (iii).

9.2. **Proof of Theorem 1–(ii).** By Item (i) we know that $\mu_\varepsilon \ll \mu$. We prove that $\mu \ll \mu_\varepsilon$ by means of the criterion given in Theorem 5–(iv). Fix $\eta \in \Omega$, $t > 0$ and $B \subset \Omega$ measurable. Recall (cf. Section 3) that $\mathbb{P}^{(\varepsilon)}_\eta$ $[\mathbb{P}_\eta]$ is the law of the environment viewed from the perturbed [unperturbed] walker, and $\mathbb{P}^{\rm env}_\eta$ is the law of the dynamic environment. Given $\underline{t} = (t_1, t_2, \dots, t_k)$ with $0 < t_1 < t_2 < \cdots < t_k \leq t$ and $\underline{y} = (y_1, y_2, \dots, y_k)$ we set

$$A_\varepsilon(\underline{t}, \underline{y}) := \mathbb{E}^{\rm env}_\eta \Big[\mathbb{1}(\tau_{y_1+\cdots+y_k}\sigma_t \in B) \cdot \prod_{i=1}^k r_\varepsilon(y_i, \tau_{y_1+y_2+\cdots+y_{i-1}}\sigma_{t_i}) \Big]$$

and we define $A(\underline{t}, \underline{y})$ similarly, with $r(\cdot,\cdot)$ instead of $r_\varepsilon(\cdot,\cdot)$.

By the construction of the process given in Section 8 we have

$$\mathbb{P}^{(\varepsilon)}_\eta(\eta_t \in B) = \sum_{k=0}^\infty \frac{e^{-\lambda t}(\lambda t)^k}{k!} \sum_{y_1, y_2, \dots, y_k} \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{k-1}}^t dt_k A_\varepsilon(\underline{t}, \underline{y}) \,, \tag{63}$$

and a similar formula relates $\mathbb{P}_\eta(\eta_t \in B)$ to $A(\underline{t}, \underline{y})$.

We first assume that (10) is satisfied. If $\mathbb{P}^{(\varepsilon)}_\eta(\eta_t \in B) = 0$, then by (63) $A_\varepsilon(\underline{t}, \underline{y}) = 0$ for almost every choice of $t_1, \dots, t_k$ and for each choice of $y_1, \dots, y_k$. Then the same holds for $A(\underline{t}, \underline{y})$ and by the analogous of (63) in the unperturbed case we conclude that $\mathbb{P}_\eta(\eta_t \in B) = 0$. Hence the criterion given by Theorem 5–(iv) is satisfied and $\mu \ll \mu_\varepsilon$.

Let us suppose now that there are subsets V, V_ε satisfying properties (a), (b), (c) in Item (ii). If $\mathbb{P}_\eta(\eta_t \in B) > 0$, by the analogous of (63) in the unperturbed case we conclude that there exist $\underline{t} = (t_1, t_2, \dots, t_k)$ and $\underline{y} = (y_1, y_2, \dots, y_k)$ such that $A(\underline{t}, \underline{y}) > 0$. In particular, $y_1, y_2, \dots, y_k \in V$ and $\mathbb{P}^{\rm env}_\eta(\tau_y \sigma_t \in B) > 0$ where $y = y_1 + \cdots + y_k$. For simplicity of notation we take $k = 1$ (the argument can be easily generalized). By condition (c) we can write $y_1 = z_1 + \cdots + z_r$ with $z_i \in V_\varepsilon$. By condition (b) and since $\mathbb{P}^{\rm env}_\eta(\tau_y \sigma_t \in B) > 0$, we conclude that $A_\varepsilon((t_1, \dots, t_r), (z_1, \dots, z_r)) > 0$ for each choice of $(t_1, t_2, \dots, t_r)$. This together with (63) implies that $\mathbb{P}^{(\varepsilon)}_\eta(\eta_t \in B) > 0$, hence by Theorem 5–(iv) we get $\mu \ll \mu_\varepsilon$.

9.3. **Proof of Theorem 1–(iii).** We refer to the construction of the random walk $(X^{(\varepsilon)}_t)_{t\geq 0}$ given in Section 8. It is convenient here to identify the Poisson point process $\mathcal{T} = \{t_1 < t_2 < t_3 < \cdots\}$ with the Poisson process $N = (N_t)_{t\geq 0}$ having $\mathcal{T}$ as set of jump times. Moreover, also for later uses, it is convenient to enlarge the random sequence $(U_k)_{k\geq 1}$ by adding U_0, with $(U_k)_{k\geq 0}$ i.i.d.. Further, we define the function $V : \mathbb{R}_+ \to [0,1]$ by setting $V_s := U_k$ for $s \in [t_k, t_{k+1})$, with the convention $t_0 := 0$. Without loss of generality we can assume that the product probability measure $\mathcal{P}_\eta$ introduced in Section 8 (with the modification due to U_0) is defined directly on the product measure space $\Theta := D(\mathbb{R}_+; \Omega) \otimes D(\mathbb{R}_+; [0,1]) \otimes D(\mathbb{R}_+; \mathbb{N})$, whose generic element is given by $(\sigma_t, V_t, N_t)_{t\geq 0}$. Next, for $(a, \sigma) \in [0,1] \times \Omega$ we introduce the measurable functions $h^{(\varepsilon)}_y(a, \sigma) := \mathbb{1}(a \in I_\varepsilon(y, \sigma))$ for $y \in \mathbb{Z}^d$, and $h^{(\varepsilon)}(a, \sigma) := \sum_{y \in \mathbb{Z}} y h^{(\varepsilon)}_y(a, \sigma)$.

Consider the filtration $\mathcal{F}_t$, $t \geq 0$, on Θ defined by

$$\mathcal{F}_t = \sigma(\sigma_s : s \geq 0) \vee \sigma(V_s, N_s : s \in [0,t]) \tag{64}$$

Due to [9, Thm. T25, App.A.2], $(\mathcal{F}_t)_{t\geq 0}$ is a right continuous filtration. We denote by $(\bar{\mathcal{F}}_t)_{t\geq 0}$ its completion w.r.t. $\mathcal{P}_\eta$, i.e. $\bar{\mathcal{F}}_t = \mathcal{F}_t \vee \mathcal{N}$ where $\mathcal{N}$ is the σ–algebra of all events on Θ with $\mathcal{P}_\eta$–zero measure. Due to [9, Thm. T25, App.A.2] $(\bar{\mathcal{F}}_t)_{t\geq 0}$ is right continuous, and therefore it is a filtration satisfying the so-called usual conditions.

Setting $\eta_s := \tau_{X_s^{(\varepsilon)}}\sigma_s$, the construction presented in Section 8 implies that

$$X_t^{(\varepsilon)} = \int_0^t h^{(\varepsilon)}(V_s, \eta_{s-})dN_s, \qquad \forall t \geq 0\,, \qquad \mathcal{P}_\eta\text{–a.s..} \tag{65}$$

We claim that

$$\hat{M}_t := X_t^{(\varepsilon)} - \int_0^t j^{(\varepsilon)}(\eta_s)ds\,, \qquad t \geq 0\,, \tag{66}$$

is a vector–valued martingale w.r.t to the filtered probability space $(\Theta, (\bar{\mathcal{F}}_t)_{t\geq 0}, \mathcal{P}_\eta)$. Indeed, this follows from [27, Theorem 9.12] since $F_n(t), m_n$ there are simply $1-e^{-\lambda t}$, and $j^{(\varepsilon)}$ (recall that $\sum_{y\in\mathbb{Z}^d}\int_0^t |y| r_\varepsilon(y,\cdot)$ is uniformly bounded by our assumptions, and this allows to check the hypothesis of [27, Theorem 9.12]).

Note that, due to Assumption 3, $j^{(\varepsilon)}(\eta) = \sum y r_\varepsilon(y,\eta)$ is a well defined bounded function.

Claim 1. *$\mathcal{P}_\eta$–a.s. it holds $\lim_{t\to\infty}\hat{M}_t/t = 0$, for μ_ε–a.e. η.*

Before proving the above claim, we conclude the proof of Theorem 1–(iii). Recall that the trajectory $(\eta_t := \tau_{F_\varepsilon(t)}\sigma_t)_{t\geq 0}$ sampled according to $\mathcal{P}_\eta$ has law $\mathbb{P}_\eta^{(\varepsilon)}$. We know that $\mathbb{P}_\mu^{(\varepsilon)}$–a.s. $t^{-1}\int_0^t ds\, j^{(\varepsilon)}(\eta_s)$ converges to $\mu_\varepsilon(j^{(\varepsilon)})$ (cf. Theorem 1 and Theorem 1–(i)). Hence for μ_ε–a.e. η we have that $t^{-1}\int_0^t ds\, j^{(\varepsilon)}(\eta_s) \to \mu_\varepsilon(j^{(\varepsilon)})$ w.r.t $\mathcal{P}_\eta$. This limit together with the above claim and with (66) allows to conclude that $X_t^{(\varepsilon)}/t \to \mu_\varepsilon(j^{(\varepsilon)})$ $P_{\eta,0}^{(\varepsilon)}$–a.s..

Proof of Claim 1. We fix $i \in \{1,\dots,d\}$ and let $\hat{M}_t^{(i)}$ be the i–th coordinate of $\hat{M}_t$. We point out that $\hat{M}_t$ (and therefore also $\hat{M}_t^{(i)}$) is a square integrable martingale. This follows from (66): since we have assumed that (4) holds with $n=2$, it is simple to check that $\mathcal{E}_\eta(|X_t^{(\varepsilon)}|^2) < +\infty$ for any $t \geq 0$, while $j^{(\varepsilon)}(\cdot)$ is uniformly bounded.

Due to (4) with $n=2$, the functions $m^{(\varepsilon)}(\eta) := \sum_y y_i r_\varepsilon(y,\eta)$ and $v^{(\varepsilon)}(\eta) := \sum_y y_i^2 r_\varepsilon(y,\eta)$ are uniformly bounded. Then, by [27, Theorem 9.14] (note that $F_n(s)$ and $A(t)$ in [27, Theorems 9.12, 9.13] are given by $1-e^{-\lambda s}$ and $\int_0^t j^{(\varepsilon)}(\eta_s)ds$, respectively), we have that

$$\langle \hat{M}^{(i)}\rangle_t = \int_0^t v^{(\varepsilon)}(\eta_s)ds\,.$$

As the dynamic environment is an ergodic process, $\mathcal{P}_\eta$–a.s. it holds $\langle \hat{M}^{(i)}\rangle_t/t \to \mu_\varepsilon(v^{(\varepsilon)}) \in (0,+\infty)$ (use (4) with $n=2$). At this point, by applying the LLN for square integrable martingales (cf. [31, Thm.1]), we conclude that $\hat{M}_t^{(i)}/t \to 0$ $\mathcal{P}_\eta$–a.s. for μ_ε–a.e. η. □

10. Proof of Theorem 2, Theorem 3 and Lemma 2.6

10.1. **Proof of Theorem 2.** Define $\mathcal{G}_n$ as the σ–algebra on Ω generated by $(\eta_x : |x|_\infty \leq n)$. Then the smallest σ–algebra containing each $\mathcal{G}_n$ is the standard Borel σ–algebra $\mathcal{G}$ on Ω.

Let $h \in L^1(\mu)$. By Lévy's upward theorem (cf. [46][page 134]), as $n\to\infty$ it holds

$$\mu(h|\mathcal{G}_n) \to \mu(h|\mathcal{G}) = h \text{ in } L^1(\mu). \tag{67}$$

We then claim that, for any local function f, it holds

$$\lim_{|x|\to\infty} \mu(h\tau_x f) = \mu(h)\mu(f)\,. \tag{68}$$

To prove our claim, we need to show that $\lim_{|x|\to\infty} \mu(h\tau_x[f-\mu(f)]) = 0$. Equivalently we need to show that $\lim_{|x|\to\infty} \mu(h\tau_x f) = 0$ if $\mu(f) = 0$. In particular, we can restrict the proof to the case f local function with zero mean. We can write

$$\mu(h\tau_x f) = \mu([h-\mu(h|\mathcal{G}_n)]\tau_x f) + \mu\left(\mu(h|\mathcal{G}_n)\tau_x f\right)\,. \tag{69}$$

By (67), given $\varepsilon > 0$ we can find n large enough that $\mu(|h-\mu(h|\mathcal{G}_n)|) \le \varepsilon$. Fix such n. Note that since μ is translation invariant we have $\mu(\tau_x f) = \mu(f) = 0$, which implies that $\mu\left(\mu(h|\mathcal{G}_n)\tau_x f\right) = \mathrm{Cov}_\mu(\mu(h|\mathcal{G}_n), \tau_x f)$. Then (69) gives

$$\begin{aligned} |\mu(h\tau_x f)| &\le \|f\|_\infty \mu(|h-\mu(h|\mathcal{G}_n)|) + |\mathrm{Cov}_\mu(\mu(h|\mathcal{G}_n), \tau_x f)| \\ &\le \|f\|_\infty \varepsilon + |\mathrm{Cov}_\mu(\mu(h|\mathcal{G}_n), \tau_x f)|\,. \end{aligned} \tag{70}$$

At this point, using (13), we conclude that

$$\limsup_{|x|\to\infty} |\mu(h\tau_x f)| \le \|f\|_\infty \varepsilon\,.$$

By the arbitrariness of ε we get our claim.

Having the above claim, Theorem 2 becomes immediate. Indeed, it is enough to take $h := d\mu_\varepsilon/d\mu$. Then (68) becomes equivalent to (14).

10.2. **Proof of Lemma 2.6.** Recall the coupling introduced in Section 8. Since the rates $r(\cdot,\eta)$ do not depend on η, for $\eta\in\Omega$, under $\mathcal{P}_\eta$, $X_\cdot$ and $\sigma_\cdot$ are independent. Therefore, fixed a local function f, $t\in\mathbb{R}_+$, $\eta\in\Omega$, we have

$$\begin{aligned} \mathbb{E}_\eta[f(\tau_x\eta_t)] &= \sum_{y\in\mathbb{Z}^d} \mathcal{E}_\eta\left[\mathbf{1}_{X_t=y} f(\tau_{x+y}\sigma_t)\right] = \sum_{y\in\mathbb{Z}^d} \mathcal{P}_\eta(X_t=y)\mathbb{E}^{\mathrm{env}}_\eta[f(\tau_{x+y}\sigma_t)] \\ &= \sum_{y\in\mathbb{Z}^d} \mathcal{P}_\eta(X_t=y)\mathbb{E}^{\mathrm{env}}_{\tau_x\eta}[f(\tau_y\sigma_t)] = \sum_{y\in\mathbb{Z}^d} \mathcal{E}_{\tau_x\eta}\left[\mathbf{1}_{X_t=y} f(\tau_y\sigma_t)\right] = \mathbb{E}_{\tau_x\eta}[f(\eta_t)]\,. \end{aligned}$$

10.3. **Proof of Theorem 3.** Recall the definition of the functions in (25) and that $\mathbb{E}_\eta$ denotes the expectation for the *environment seen by the unperturbed walker* starting from η, that is, the Markov process with generator (60). To shorten the notation, we set $f_x := f\circ\tau_x$ for all $x\in\mathbb{Z}^d$.

The proof of Theorem 3 is based on the following technical result:

Lemma 10.1. *If $S(t)(g\circ\tau_x) = (S(t)g)\circ\tau_x$ holds for any g local, $t\ge 0$, $x\in\mathbb{Z}^d$, then for all $n\ge 0$, $f:\Omega\to\mathbb{R}$ local function and $\eta\in\Omega$, it holds*

$$\begin{aligned} \hat{L}_\varepsilon S^{(n)}_\varepsilon(t) f(\eta) = \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n & \sum_{z\in B(R)^{n+1},\, \delta\in\{0,1\}^{n+1}} (-1)^{|\delta|} \\ \times\, \mathbb{E}_\eta & \left[\left(\prod_{i=1}^{n+1} \hat{r}_\varepsilon\left(z_i, \tau_{(\delta\cdot z)_{[i-1]}}\eta_{t-t_{i-1}}\right)\right) f_{(\delta\cdot z)_{[n+1]}}(\eta_t)\right]\,, \end{aligned} \tag{71}$$

where $|\delta| = \sum_{i=1}^{n+1}(1-\delta_i)$, $(\delta\cdot z)_{[i]} = \delta_1 z_1 + \ldots + \delta_i z_i$, $(\delta\cdot z)_{[0]} = 0$ and $t_0 = t$.

Formula (71) has to be thought with no time integration in the degenerate case $n=0$.

Proof. For simplicity of notation, as in (39), we set $g_{n+1}(t) := S_\varepsilon^{(n)}(t)f(\eta), n \geq 0$. The proof is done by induction on n and relies on the following two identities (based on the definition of $\hat{L}_\varepsilon$, $g_{n+1}(t)$ and Assumptions (i)[8] and (iii) in Theorem 3):

$$\hat{L}_\varepsilon g_1(t)(\eta) = \sum_{z \in B(R)} \hat{r}_\varepsilon(z,\eta)\mathbb{E}_\eta\left[f(\tau_z \eta_t) - f(\eta_t)\right] , \tag{72}$$

$$\hat{L}_\varepsilon g_{n+1}(t)(\eta) = \sum_{z_0 \in B(R)} \hat{r}_\varepsilon(z_0,\eta) \int_0^t dt_1\, \mathbb{E}_\eta \left[\hat{L}_\varepsilon g_n(t_1)\,(\tau_{z_0}\eta_{t-t_1}) - \hat{L}_\varepsilon g_n(t_1)\,(\eta_{t-t_1})\right] . \tag{73}$$

Trivially (72) corresponds to (71) for $n = 0$. Let us now assume (71) holds for $n-1$, where $n \geq 1$, and deduce that it holds for n. By the induction hypothesis we get

$$\mathbb{E}_\eta\left[\hat{L}_\varepsilon g_n(t_1)\,(\tau_{z_0}\eta_{t-t_1})\right] = \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \dots \int_0^{t_{n-1}} dt_n \sum_{z \in B(R)^n} \sum_{\delta \in \{0,1\}^n} (-1)^{|\delta|}$$
$$\times\, \mathbb{E}_\eta\Big[\mathbb{E}_{\tau_{z_0}\eta_{t-t_1}}\Big[\Big(\prod_{i=1}^n \hat{r}_\varepsilon\big(z_i, \tau_{(\delta\cdot z)_{[i-1]}}\eta_{t_1-t_i}\big)\Big) f_{(\delta\cdot z)_{[n]}}\big(\eta_{t_1}\big)\Big]\Big]. \tag{74}$$

By Assumption (i) in Theorem 3 and the Markov property applied at time $t - t_1$, we can rewrite the expectation in the r.h.s. of (74) as

$$\mathbb{E}_\eta\Big[\mathbb{E}_{\eta_{t-t_1}}\Big[\Big(\prod_{i=1}^n \hat{r}_\varepsilon\big(z_i, \tau_{z_0+(\delta\cdot z)_{[i-1]}}\eta_{t_1-t_i}\big)\Big) f_{z_0+(\delta\cdot z)_{[n]}}\big(\eta_{t_1}\big)\Big]\Big]$$
$$= \mathbb{E}_\eta\Big[\Big(\prod_{i=1}^n \hat{r}_\varepsilon\big(z_i, \tau_{z_0+(\delta\cdot z)_{[i-1]}}\eta_{t-t_i}\big)\Big) f_{z_0+(\delta\cdot z)_{[n]}}\big(\eta_t\big)\Big]$$

If we set $(z_1', z_2' \dots, z_{n+1}') := (z_0, z_1, \dots, z_n)$ and $(\delta_1', \delta_2', \dots, \delta_{n+1}') := (1, \delta_1, \delta_2, \dots, \delta_n)$, recalling the convention $t_0 := t$ we can write

$$\hat{r}_\varepsilon(z_0,\eta)\Big(\prod_{i=1}^n \hat{r}_\varepsilon\big(z_i, \tau_{z_0+(\delta\cdot z)_{[i-1]}}\eta_{t-t_i}\big)\Big) f_{z_0+(\delta\cdot z)_{[n]}}(\eta_t)$$
$$= \prod_{i=1}^{n+1} \hat{r}_\varepsilon\left(z_i', \tau_{(\delta'\cdot z')_{[i-1]}}\eta_{t-t_{i-1}}\right) f_{(\delta'\cdot z')_{[n+1]}}(\eta_t)\,.$$

Coming back to (74) the above observations imply that

$$\sum_{z_0 \in B(R)} \hat{r}_\varepsilon(z_0,\eta) \int_0^t dt_1\, \mathbb{E}_\eta\left[\hat{L}_\varepsilon g_n(t_1)\,(\tau_{z_0}\eta_{t-t_1})\right]$$

is given by the r.h.s. of (71) where the sum among δ is restricted to $\delta \in B(R)^{n+1}$ with $\delta_1 = 1$. By the same arguments, we get that

$$\sum_{z_0 \in B(R)} \hat{r}_\varepsilon(z_0,\eta) \int_0^t dt_1\, \mathbb{E}_\eta\left[\hat{L}_\varepsilon g_n(t_1)\,(\eta_{t-t_1})\right]$$

is given by the r.h.s. of (71) where the sum among δ is restricted to $\delta \in B(R)^{n+1}$ with $\delta_1 = 0$. To get the thesis is now enough to invoke (73). □

[8]Note that Assumption (i) can be restated as $\mathbb{E}_\eta\big(f(\tau_x\eta_t)\big) = \mathbb{E}_{\tau_x\eta}\big(f(\eta_t)\big)$ for any local function f and $x \in \mathbb{Z}^d$

We can now get our estimates for the convergence. To simplify the notation, in what follows we write $|\cdot|$ for the uniform norm $|\cdot|_\infty$. Unless otherwise stated, the constants do not depend on f.

Claim 2. *It is enough to show that for all n, t, x,*

$$\Big|\mu\Big(\hat{L}_\varepsilon S_\varepsilon^{(n)}(t) f_x\Big)\Big| \le C_0(f)\frac{(Ct)^n}{n!}e^{-\theta_2|x|+\theta_3 t} \tag{75}$$

for some $\theta_2 > 0$, $\theta_3, C, C_0(f) \in \mathbb{R}_+$ not depending on n, t, x (possibly depending on ϵ, γ).

Proof of Claim 2. We need to show that

$$\Big|\sum_{n\ge 0}\int_0^\infty dt\,\mu\Big(\hat{L}_\varepsilon S_\varepsilon^{(n)}(t) f_x\Big)\Big| \le C_1(f)e^{-\theta'|x|}, \tag{76}$$

since the l.h.s. equals $\mu_\varepsilon(f_x) - \mu(f)$ by (9). Notice that, in addition to (75), by (42) we can bound

$$\Big|\mu\Big(\hat{L}_\varepsilon S_\varepsilon^{(n)}(t) f_x\Big)\Big| \le \varepsilon e^{-\gamma t}\frac{(\varepsilon t)^n}{n!}\|f\|\,. \tag{77}$$

We can estimate

$$\Big|\sum_{n\ge 0}\int_0^\infty dt\,\mu\Big(\hat{L}_\varepsilon S_\varepsilon^{(n)}(t) f_x\Big)\Big| \le C_2(f)\sum_{n\ge 0}\int_0^\infty dt\,\Big(\frac{(Ct)^n}{n!}e^{-\theta_2|x|+\theta_3 t}\Big)\wedge\Big(e^{-\gamma t}\frac{(\varepsilon t)^n}{n!}\Big), \tag{78}$$

where $C_2(f) = C_0(f)\vee(\varepsilon\|f\|)$. Let $\alpha \in (0, \theta_2/(C+\theta_3))$. The sum in the right-hand side above can be estimated by

$$\begin{aligned}
&\sum_{n\ge 0}\int_0^{\alpha|x|} dt\,\frac{(Ct)^n}{n!}e^{-\theta_2|x|+\theta_3 t} + \sum_{n\ge 0}\int_{\alpha|x|}^\infty dt\,e^{-\gamma t}\frac{(\varepsilon t)^n}{n!}\\
&= \sum_{n\ge 0}\frac{C^n(\alpha|x|)^{n+1}}{(n+1)!}e^{-\theta_2|x|+\theta_3\alpha|x|} + \int_{\alpha|x|}^\infty dt\,e^{-(\gamma-\varepsilon)t}\\
&\le \frac{1}{C}e^{-(\theta_2-\alpha C-\alpha\theta_3)|x|} + \frac{1}{\gamma-\varepsilon}e^{-(\gamma-\varepsilon)\alpha|x|}\,.
\end{aligned}$$

Therefore we can indeed choose $\theta' > 0$ as in (76). $\square$

We now move to prove (75). We claim that, in view of Lemma 10.1, it is enough to show that there exist $\theta_2 > 0, C, \theta_3 \in \mathbb{R}_+$ such that $\forall n \ge 1$, $\forall t = t_0 > t_1 > \cdots > t_n$, $\forall z \in B(R)^n$, $\forall z_{n+1} \in B(R)$, $\forall \delta' \in \{0,1\}^n$, we have:

$$\begin{aligned}
\Big|\mathbb{E}_\mu\Big[\Big(\prod_{i=1}^{n+1} r_\varepsilon\big(z_i, \tau_{(\delta'\cdot z)_{[i-1]}}\eta_{t-t_{i-1}}\big)\Big)\Big(f_{x+(\delta'\cdot z)_{[n]}+z_{n+1}}(\eta_t) - f_{x+(\delta'\cdot z)_{[n]}}(\eta_t)\Big)\Big]\Big|&\\
\le C'(f)C^n e^{-\theta_2|x|+\theta_3 t}\,.&
\end{aligned} \tag{79}$$

To see why this is enough, consider the sum indexed by the $(n+1)$-uples $\delta \in \{0,1\}^{n+1}$ appearing in (71) and reindex it by gathering together the two terms sharing the same first n coordinates. To use (79), we set δ' to be the corresponding n-coordinate vector. The integrals and sums contribute at most a factor $\frac{t^n}{n!}(2(2R+1)^d)^{n+1}$ and therefore we get (75) from (79) (changing the value of C).

In order to prove (79), we abbreviate

$$\Pi = \prod_{i=1}^{n+1} r_\varepsilon\big(z_i, \tau_{(\delta\cdot z)_{[i-1]}}\eta_{t-t_{i-1}}\big), \qquad \Delta f_x = f_{x+(\delta\cdot z)_{[n]}+z_{n+1}}\big(\eta_t\big) - f_{x+(\delta\cdot z)_{[n]}}\big(\eta_t\big)\,.$$

To conclude we need to show that $\mathbb{E}_\mu\left[\Pi\Delta f_x\right] \le C'(f)C^n e^{-\theta_2|x|+\theta_3 t}$.

Let $\beta(x) = |x|/5$. Given $\sigma \in \{0,1\}^{\mathbb{Z}^d}$ we write $\tilde\sigma$ for the configuration obtained from σ by periodizing σ restricted to the box $B(2\beta(x))$ (for simplicity of notation we assume $\beta(x)$ to be integer). Recall that in Section 8 we have built the random walk $X.$ as a function $F\left(\sigma., \mathcal{T}, \mathcal{U}\right)$ of the environment trajectory $\sigma.$, Poisson times $\mathcal{T}$ (with parameter λ) and uniform random variables $\mathcal{U}$ (these last objects defined on a probability space with probability measure $\mathcal{P}_\mu$). Let $(\tilde X_s)_{s\ge 0} := F\left((\tilde\sigma_s)_{s\in\mathbb{R}_+}, \mathcal{T}, \mathcal{U}\right)$ and let N be the cardinality of $\mathcal{T}\cap[0,t]$. By the definition of F and since the jump rates have finite range R and support of size R, we have for any $u \le t$ that $|X_u| \le NR$ and $X_u = F\left(\sigma., \mathcal{T}, \mathcal{U}\right)$ depends on $\sigma.$ only through $\left(\sigma_{s|B((N+1)R)}\right)_{s\in[0,t]}$. Note that N is a Poisson variable with parameter λt and in particular $P(N > k) \le e^{-k+(e-1)\lambda t}$. Hence, taking $k = \beta(x)/R - 1$, for x large enough it holds $\mathcal{P}_\mu(\mathcal{G}) \ge 1 - e e^{-\beta(x)/R+(e-1)\lambda t}$ where $\mathcal{G} := \{X_s = \widetilde X_s \in B(\beta(x))\ \forall s \le t\}$.

Let us set

$$\widetilde\Pi = \prod_{i=1}^{n+1} r_\varepsilon\big(z_i, \tau_{(\delta\cdot z)_{[i-1]}+\widetilde X_{t-t_{i-1}}}\sigma_{t-t_{i-1}}\big),$$
$$\widetilde{\Delta f_x} = f_{x+(\delta\cdot z)_{[n]}+z_{n+1}+\widetilde X_t}\big(\sigma_t\big) - f_{x+(\delta\cdot z)_{[n]}+\widetilde X_t}\big(\sigma_t\big)$$

and let us introduce the event $\mathcal{B} := \{\tilde X_{t-t_1}, ..., \tilde X_{t-t_n}, \tilde X_t \in B(\beta(x))\}$. Since $\mathcal{G}\subset\mathcal{B}$ it holds $\mathcal{P}_\mu(\mathcal{B}) \ge 1 - e e^{-\beta(x)/R+(e-1)\lambda t}$.

Since Π, Δf_x, $\widetilde\Pi$, $\widetilde{\Delta f_x}$ are bounded uniformly in x (respectively by R^{n+1}, $2\|f\|_\infty$, R^{n+1}, $2\|f\|_\infty$), writing $\widetilde{\mathbf{X}} := (\tilde X_{t-t_1}, ..., \tilde X_{t-t_n}, \tilde X_t)$ and $y_0 = 0$, we can estimate

$$\begin{aligned}
|\mathbb{E}_\mu[\Pi\Delta f_x]| &\le |\mathcal{E}_\mu[\Pi\Delta f_x \mathbb{1}_{\mathcal{G}}]| + C(f)R^{n+1}e^{-\beta(x)/R+(e-1)\lambda t}\\
&= |\mathcal{E}_\mu[\widetilde\Pi\widetilde{\Delta f_x}\mathbb{1}_{\mathcal{G}}]| + C(f)R^{n+1}e^{-\beta(x)/R+(e-1)\lambda t}\\
&\le |\mathcal{E}_\mu[\widetilde\Pi\widetilde{\Delta f_x}\mathbb{1}_{\mathcal{B}}]| + 2C(f)R^{n+1}e^{-\beta(x)/R+(e-1)\lambda t}\\
&\le \Big|\sum_{\mathbf{y}\in B(\beta(x))^{n+1}} \mathcal{E}_\mu[\overline\Pi\,\overline{\Delta f_x}\mathbf{1}_{\widetilde{\mathbf{X}}=\mathbf{y}}]\Big| + 2C(f)R^{n+1}e^{-\beta(x)/R+(e-1)\lambda t}\\
&\le \Big|\sum_{\mathbf{y}\in B(\beta(x))^{n+1}} \mathbb{E}^{\mathrm{env}}_\mu[\overline\Pi\,\overline{\Delta f_x}\mathcal{P}_\mu(\widetilde{\mathbf{X}}=\mathbf{y}|(\sigma_s)_{s\le t})]\Big| + 2C(f)R^{n+1}e^{-\beta(x)/R+(e-1)\lambda t}\\
&\le \Big|\sum_{y_{n+1}\in B(\beta(x))} \mathbb{E}^{\mathrm{env}}_\mu\Big[\overline{\Delta f_x}\sum_{\mathbf{y}\in B(\beta(x))^n}\overline\Pi\,\mathcal{P}_\mu(\widetilde{\mathbf{X}}=(\mathbf{y},y_{n+1})|(\sigma_s)_{s\le t})\Big]\Big|\\
&\quad + 2C(f)R^{n+1}e^{-\beta(x)/R+(e-1)\lambda t},
\end{aligned}$$

where

$$\overline\Pi := \prod_{i=1}^{n+1} r_\varepsilon\big(z_i, \tau_{(\delta\cdot z)_{[i-1]}+y_{i-1}}\sigma_{t-t_{i-1}}\big)\,,$$
$$\overline{\Delta f_x} := f_{x+(\delta\cdot z)_{[n]}+z_{n+1}+y_{n+1}}\big(\sigma_t\big) - f_{x+(\delta\cdot z)_{[n]}+y_{n+1}}\big(\sigma_t\big)\,,$$

and $C(f) = 2e\|f\|_\infty$.

By definition, $\widetilde{X}$ depends only on $(\sigma_s)_{s\leq t}$ restricted to $B(2\beta(x))$ (and therefore the same holds for $\mathcal{P}_\mu(\widetilde{\mathbf{X}} = \mathbf{y}|(\sigma_s)_{s\leq t})$), $\overline{\Pi}$ depends only the process restricted to $B((n+1)R + \beta(x))$ and $\overline{\Delta f_x}$ on $B(x, (n+1)R + L + \beta(x)) \subset B(x, (n+1)R + 2\beta(x))$ (for x large, where $B(x, r) = x + B(r)$). We note that $B(x, (n+1)R + 2\beta(x))$ and $B((n+1)R + 2\beta(x))$ have uniform distance $|x| - 4\beta(x) - 2R(n+1)$, so that by finite speed propagation of the environment process, if $|x| - 4\beta(x) - 2R(n+1) \geq \kappa t$ (where κ is the constant appearing in the definition of finite speed propagation),

$$\begin{aligned}
&\mathbb{E}_\mu[\Pi\Delta f_x] \leq \Big| \sum_{y_{n+1}\in B(\beta(x))} \mathbb{E}^{\mathrm{env}}_\mu[\overline{\Delta f_x}]\mathbb{E}^{\mathrm{env}}_\mu\Big[\sum_{\mathbf{y}\in B(\beta(x))^n} \overline{\Pi}\,\mathcal{P}_\mu(\widetilde{\mathbf{X}} = (\mathbf{y}, y_{n+1})|(\sigma_s)_{s\leq t})\Big]\Big| \\
&\ + 2C(f)R^{n+1}e^{-\beta(x)/R+(e-1)\lambda t} + (2\beta(x)+1)R^{n+1}C(f)e^{-\theta(|x|-4\beta(x)-2R(n+1))} \\
&= 2C(f)R^{n+1}e^{-\beta(x)/R+(e-1)\lambda t} + (2\beta(x)+1)R^{n+1}C(f)e^{-\theta(|x|-4\beta(x)-2R(n+1))} \\
&\leq 2C(f)R^{n+1}e^{-|x|/5R+(e-1)\lambda t} + cRe^{2R\theta}C(f)(Re^{2R\theta})^n e^{-\theta|x|/6}\,,
\end{aligned} \tag{80}$$

where the equality follows from $\mathbb{E}^{\mathrm{env}}_\mu[\overline{\Delta f_x}] = 0$ (since μ is translation invariant), we have estimated $\|\sum_{\mathbf{y}\in B(\beta(x))^n} \overline{\Pi}\,\mathcal{P}_\mu(\widetilde{\mathbf{X}} = (\mathbf{y}, y_{n+1})|(\sigma_s)_{s\leq t})\|_\infty$ by R^n and the constant c is such that $(2u/5+1)e^{-\theta u/5} \leq e^{-\theta u/6}$.

On the other hand, if $|x| - 4\beta(x) - 2R(n+1) \leq \kappa t$, we can estimate

$$\begin{aligned}
\mathbb{E}_\mu[\Pi\Delta f_x] &\leq 2\|f\|_\infty R^{n+1} = 2\|f\|_\infty R^{2(n+1)}R^{-(n+1)} \\
&\leq 2\|f\|_\infty R^{2(n+1)}R^{-\frac{|x|-4\beta(x)}{2R}+\kappa t} \leq 2\|f\|_\infty R^2(R^2)^n e^{-|x|\frac{\ln R}{10R}+t\kappa\ln R}\,.
\end{aligned} \tag{81}$$

In both cases (80) and (81), we find an estimate of the form (79), which concludes the proof.

11. Proof of Theorem 4–(i)

Recall the notation of Sections 8 and 9.3. We introduce the probability measure $\mathcal{P}_{\mu_\varepsilon} := \int \mu_\varepsilon(d\eta)\mathcal{P}_\eta$ on the space Θ. We write Q for the image of $\mathcal{P}_{\mu_\varepsilon}$ induced by the map $\Theta \ni (\sigma_t, V_t, N_t)_{t\geq 0} \mapsto (\tau_{X^{(\varepsilon)}_t}\sigma_t, V_t, N_t)_{t\geq 0} \in \Theta$. Note that the projection of Q along the first coordinate is simply $\mathbb{P}^{(\varepsilon)}_{\mu_\varepsilon}$. To stress this property, we write $(\eta_t, V_t, N_t)_{t\geq 0}$ for a generic element of probability space (Θ, Q), since usually we set $\eta_t := \tau_{X^{(\varepsilon)}_t}\sigma_t$.

Given $t \geq 0$, we define $\mathcal{H}_t = \sigma(\eta_s, V_s, N_s \,:\, s \in [0, t])$ as σ–algebra on Θ. Then we write $\bar{\mathcal{H}}_t$ for the augmented filtration w.r.t. Q following [24, Def.7.2, Sec.2.7]. Since $(\eta_s, V_s, N_s)_{t\geq 0}$ is a strong Markov process, by [24, Prop.7.7, App.A] the filtration $(\bar{\mathcal{H}}_t)_{t\geq 0}$ on (Θ, Q) satisfies the usual conditions.

By the martingale representation in (66) and since $v(\varepsilon) = \mu_\varepsilon(j^{(\varepsilon)})$ (cf. Theorem 1–(iii)), the position of the walker centered with its asymptotic velocity can be written as

$$X^{(\varepsilon)}_t - v(\varepsilon)t = \hat{M}_t + \int_0^t ds\,[j^{(\varepsilon)}(\eta_s) - \mu_\varepsilon(j^{(\varepsilon)})] =: \hat{M}_t + A_t(f)\,, \tag{82}$$

where $(\hat{M}_t)_{t\geq 0}$ is a martingale w.r.t. $(\Theta, (\bar{\mathcal{H}}_t)_{t\geq 0}, Q)$ and $A_t(f)$ is the additive functional introduced in Section 7 associated with the function $f(\eta) := j^{(\varepsilon)}(\eta) - \mu_\varepsilon(j^{(\varepsilon)})$. Note that, by Assumption 3, the vector–valued function f is a bounded continuous function on Ω with $\mu_\varepsilon(f) = 0$. In particular, following the proof of Proposition 3.6, for each coordinate $i = 1, \dots, d$, we can find $g^i \in \mathcal{D}(\mathcal{L}_\varepsilon)$ such that $f^i = -\mathcal{L}_\varepsilon g^i$, thus leading to the decomposition

$$\int_0^t f^i(\eta_s) ds = \tilde{M}_t^{(i)} + R_t^i, \tag{83}$$

where $\tilde{M}_t^{(i)} = g^i(\eta_t) - g^i(\eta_0) - \int_0^t ds\, \mathcal{L}_\varepsilon g^i(\eta_s)$ is a martingale and $R_t^i = -g^i(\eta_t) + g^i(\eta_0)$ satisfies the conclusion of Lemma 7.2. As a consequence, we have that

$$X_t^{(\varepsilon)} - v(\varepsilon) t = \sum_{i=1}^d M_t^{(i)} e_i + R_t, \qquad M_t := \hat{M}_t + \tilde{M}_t \,. \tag{84}$$

Due to Lemma 7.2, to get the thesis we only need to apply [28, Thm. 2.29, Chp. 2], which will show the invariance principle for the martingale term. Due to the definition of $\hat{M}_t, \tilde{M}_t$ (see also (65) and (66)), the martingale clearly has stationary increments on (Θ, Q) and is square integrable (see the discussion after (66)). It remains to show that for any $i, j = 1, \dots, d$, $\langle M^{(i)}, M^{(j)} \rangle_n / n$ converge a.s. and in L^1 to $D_\varepsilon(i,j) = E[M_1^{(i)} M_1^{(j)}]$, denoting by E the expectation w.r.t. Q. By the parallelogram identity, it is enough to show that $\langle M^{(i)} \pm M^{(j)} \rangle_n / n$ converge a.s. and in L^1 to $\mathbb{E}_{\mu_\varepsilon}^{(\varepsilon)}[(M_1^{(i)} \pm M_1^{(j)})^2]$.

To this aim we set $\mathcal{M}_t := M_t^{(i)} \pm M_t^{(j)}$ and observe that

$$\mathcal{M}_{t+s} = \mathcal{M}_t + \mathcal{M}_s \circ \theta_s \qquad \forall t, s \geq 0 \,, \tag{85}$$

where θ_s is the time-translation on Θ at time s. Due to the martingale property, we have

$$E\big[\mathcal{M}_{t+s}^2 - \mathcal{M}_t^2 | \bar{\mathcal{H}}_t\big] = E\big[(\mathcal{M}_{t+s} - \mathcal{M}_t)^2 | \bar{\mathcal{H}}_t\big] = E\big[\mathcal{M}_s^2 | \bar{\mathcal{H}}_0\big] \circ \theta_t \,. \tag{86}$$

By the definition of $\mathcal{M}_t$ it follows easily that $E\big[\mathcal{M}_s^2 | \bar{\mathcal{H}}_0\big]$ is $\sigma(\eta_0)$–measurable. To simplify the notation, we write $F_s(\eta_0)$ for $E\big[\mathcal{M}_s^2 | \bar{\mathcal{H}}_0\big]$, thus allowing to write $E\big[\mathcal{M}_{t+s}^2 - \mathcal{M}_t^2 | \bar{\mathcal{H}}_t\big] = F_s(\eta_t)$. On the other hand, by [41, Thm. (31.2), Chp. VI.6.31], $\langle \mathcal{M} \rangle_t$ is the limit of Σ_S^t in the weak (L^∞) topology of $L^1(\Theta, Q)$ as the mesh of the partition S goes to zero, where $\Sigma_S^t := \sum_{i=1}^n E[\mathcal{M}_{t_i}^2 - \mathcal{M}_{t_i-1}^2 | \bar{\mathcal{H}}_{t_{i-1}}]$ for a partition $S = \{t_0 = 0 < t_1 < \dots < t_n = t\}$.[9]

By what we just proved, we can write $\Sigma_S^t = \sum_{i=1}^n F_{t_i - t_{i-1}}(\eta_{t_{i-1}})$. As a byproduct, we deduce that, given $t, s \geq 0$, it holds

$$\langle \mathcal{M} \rangle_{t+s} = \langle \mathcal{M} \rangle_t + \langle \mathcal{M} \rangle_s \circ \theta_t \qquad Q\text{–a.s.} \tag{87}$$

Moreover, we get that $\langle \mathcal{M} \rangle_t$ depends only on $(\eta_s)_{s \leq t}$, more precisely that $\langle \mathcal{M} \rangle_t = E[\langle \mathcal{M} \rangle_t | \mathcal{G}_t]$ where $\mathcal{G}_t$ is the σ–algebra on Θ generated by $\{\eta_s : s \in [0,t]\}$. Indeed, by definition of weak limit and since Σ_S^t is $\mathcal{G}_t$–measurable, we have

$$E\big[\langle \mathcal{M} \rangle_t \xi\big] = \lim_S E\big[\Sigma_S^t \xi\big] = \lim_S E\big[\Sigma_S^t E[\xi \,|\, \mathcal{G}_t]\big] = E\big[\langle \mathcal{M} \rangle_t E[\xi \,|\, \mathcal{G}_t]\big] \,, \tag{88}$$

for any bounded random variable ξ on Θ. Since $E\big[\langle \mathcal{M} \rangle_t E[\xi \,|\, \mathcal{G}_t]\big] = E\big[E\big[\langle \mathcal{M} \rangle_t \,|\, \mathcal{G}_t\big] \xi\big]$, we conclude that $E\big[\langle \mathcal{M} \rangle_t \xi\big] = E\big[E\big[\langle \mathcal{M} \rangle_t \,|\, \mathcal{G}_t\big] \xi\big]$ for any ξ as above, thus implying that $\langle \mathcal{M} \rangle_t = E[\langle \mathcal{M} \rangle_t | \mathcal{G}_t]$.

At this point, one can deduce that $\langle \mathcal{M} \rangle_n / n$ converges a.s. and in L^1 to $E(\mathcal{M}_1^2) = \mathbb{E}_{\mu_\varepsilon}^{(\varepsilon)}[(M_1(i) \pm M_1(j))^2]$ by the same arguments used in the proof of Lemma 7.1.

[9][41, Thm. (31.2), Chp. VI.6.31] is stated for the compensator of submartingale of class (D), on the other hand for any fixed $T > 0$ the process $\mathcal{M}_{t \wedge T}^2$ is uniformly integrable and therefore it is a submartingales of class (D) (combine [27, Thm. 7.32] and [41, Lemma (29.6), Chp. VI.6.29]).

Remark 11.1. *Note that for $1 \leq i \neq j \leq d$, the martingales $M_t^{(i)}$ and $M_t^{(j)}$ have common jumps if the rate $r_\varepsilon(y,\cdot)$ is positive for some $y \in \mathbb{Z}^d$ of the form $y = \sum_{i=1}^d c_i e_i$ with $c_i \neq 0 \neq c_j$. Hence, in general, they are not orthogonal, resulting into a non–diagonal diffusion matrix D_ε.*

12. Proof of Theorem 4–(ii)

It remains to show the non–degeneracy of D_ε under the extra hypotheses that L_{env} and L_{ew} are self–adjoint in $L_2(\mu)$, and that (4) holds with $n = 4$. We refer to the notation introduced in the previous section and in Section 8. One consequence of the previous proof is that $\lim_{t\to\infty} \frac{1}{t} Var_{\mu_\varepsilon}\left[X_t^{(\varepsilon)} \cdot e\right] = \lim_{t\to\infty} \frac{1}{t}\mathcal{E}_{\mu_\varepsilon}((M_t \cdot e)^2)$, where Var_{μ_ε} denotes the variance w.r.t $\mathcal{P}_{\mu_\varepsilon}$. On the other hand, it holds $\mathcal{E}_{\mu_\varepsilon}((M_t \cdot e)^2) = \mathcal{E}_{\mu_\varepsilon}(\langle M \cdot e\rangle_t) = t\langle e, D_\varepsilon e\rangle$ (for the last identity see the conclusion of the previous section). Hence, to prove the non–degeneracy of D_ε it is enough to show that $\lim_{t\to\infty} \frac{1}{t} Var_{\mu_\varepsilon}\left[X_t^{(\varepsilon)} \cdot e\right]$ is bounded away from zero.

Along this proof we will heavily use the coupling and the notation introduced in Section 8. We further define $\eta_t^{(\varepsilon)} := \tau_{X_t^{(\varepsilon)}}\sigma_t$ and $\eta_t := \tau_{X_t}\sigma_t$.

Denote by $(\mathcal{H}_t)_{t\geq 0}$ the filtration on Θ with $\mathcal{H}_t = \sigma(\eta_s, V_s, N_s : s \in [0,t])$ as in Section 9.3. Fixed a vector $e \in \mathbb{R}^d\setminus\{0\}$ and an integer $T > 0$, define the discrete-time martingale $(M_n^T)_{0\leq n\leq T}$ as (cf. (66))

$$\begin{aligned} M_n^T &:= \mathcal{E}_\eta\left[X_T^{(\varepsilon)} \cdot e \middle| \mathcal{H}_n\right] - \mathcal{E}_\eta\left[X_T^{(\varepsilon)} \cdot e\right] \\ &= X_n^{(\varepsilon)} \cdot e + \int_0^{T-n} S_\varepsilon(s) j_e^{(\varepsilon)}(\eta_n^{(\varepsilon)})\, ds - \int_0^T S_\varepsilon(s) j_e^{(\varepsilon)}(\eta)\, ds, \end{aligned}$$

with $j_e^{(\varepsilon)}(\eta) = \sum_{y\in\mathbb{Z}^d}(y\cdot e) r_\varepsilon(y,\eta), \eta \in \Omega$. Since

$$Var_{\mu_\varepsilon}\left[X_T^{(\varepsilon)} \cdot e\right] = \mathcal{E}_{\mu_\varepsilon}\left(Var_{\mu_\varepsilon}^{(\varepsilon)}\left[X_T^{(\varepsilon)} \cdot e \middle| \mathcal{H}_0\right]\right) + Var_{\mu_\varepsilon}\left(\mathcal{E}_{\mu_\varepsilon}\left[X_T^{(\varepsilon)} \cdot e \middle| \mathcal{H}_0\right]\right),$$

by using the above martingale, the stationarity of the perturbed process under μ_ε, and the semigroup property, we can estimate

$$\begin{aligned} Var_{\mu_\varepsilon}\left[X_T^{(\varepsilon)} \cdot e\right] &\geq \mathcal{E}_{\mu_\varepsilon}\left[\left(X_T^{(\varepsilon)} \cdot e - \mathcal{E}_\eta\left[X_T^{(\varepsilon)} \cdot e\right]\right)^2\right] = \sum_{n=1}^T \mathcal{E}_{\mu_\varepsilon}\left[(M_n^T - M_{n-1}^T)^2\right] \\ &= \sum_{n=1}^T \mathcal{E}_{\mu_\varepsilon}\left[\left(X_1^{(\varepsilon)} \cdot e + \int_0^{T-n} S_\varepsilon(t) j_e^{(\varepsilon)}(\eta_1^{(\varepsilon)})\, dt - \int_0^{T-n+1} S_\varepsilon(t) j_e^{(\varepsilon)}(\eta)\, dt\right)^2\right] \\ &= \sum_{n=1}^T \mathcal{E}_{\mu_\varepsilon}\left[\left(A_1^{(\varepsilon)} + B_{T-n}^{(\varepsilon)}\right)^2\right], \end{aligned} \tag{89}$$

where

$$\begin{aligned} A_1^{(\varepsilon)} &:= X_1^{(\varepsilon)} \cdot e - \int_0^1 S_\varepsilon(t) j_e^{(\varepsilon)}(\eta) dt \\ B_{T-n}^{(\varepsilon)} &:= \int_0^{T-n} dt \left(S_\varepsilon(t) j_e^{(\varepsilon)}(\eta_1^{(\varepsilon)}) - \mathbb{E}_\eta^{(\varepsilon)}\left[S_\varepsilon(t) j_e^{(\varepsilon)}(\eta_1^{(\varepsilon)})\right]\right). \end{aligned} \tag{90}$$

Note that in the derivation of the last equality in (89) we have used that

$$S_\varepsilon(t+1)j_e^{(\varepsilon)}(\eta) = \mathbb{E}_\eta^{(\varepsilon)}\left[S_\varepsilon(t)j_e^{(\varepsilon)}(\eta_1^{(\varepsilon)})\right].$$

We want to show that $A_1^{(\varepsilon)}$ and $B_{T-n}^{(\varepsilon)}$ are "ε–close" to their *unperturbed* counterparts A_1 and B_{T-n} defined as

$$\begin{aligned} A_1 &:= X_1\cdot e - \int_0^1 S(t)j_e(\eta)dt \\ B_{T-n} &:= \int_0^{T-n} dt\,\left(S(t)j_e(\eta_1) - \mathbb{E}_\eta\left[S(t)j_e(\eta_1)\right]\right), \end{aligned} \tag{91}$$

where

$$j_e(\eta) := \sum_{y\in\mathbb{Z}^d}(y\cdot e)r(y,\eta), \quad \eta\in\Omega, \tag{92}$$

is the unpertubed analogous of the function $j^{(\varepsilon)}$ in Theorem 1. Note that due to Assumption (4) with $n=4$ it holds $\|j_e\|_\infty < +\infty$. Having (89) the rest of the proof is divided in three main steps:

Claim 3. *There exists $\delta(\varepsilon)$ going to zero as $\beta(\varepsilon)\to 0$ (recall (19)) such that*

$$\mathcal{E}_{\mu_\varepsilon}\left[\left(A_1^{(\varepsilon)} + B_{T-n}^{(\varepsilon)}\right)^2\right] \geq \mathcal{E}_{\mu_\varepsilon}\left[(A_1+B_{T-n})^2\right] - \delta(\varepsilon) \tag{93}$$

for any T,n.

Claim 4. *There exists a positive constant C such that*

$$\mathcal{E}_{\mu_\varepsilon}\left[(A_1+B_{T-n})^2\right] \geq \mathcal{E}_{\mu}\left[(A_1+B_{T-n})^2\right] - C\frac{\varepsilon}{\gamma-\varepsilon} \tag{94}$$

for any T,n and $\varepsilon<\gamma$.

Claim 5. *It holds*

$$\lim_{T\to+\infty}\frac{\mathcal{E}_\mu\left[(X_T\cdot e)^2\right] - \mu\left(\left(\mathcal{E}_\eta\left[X_T\cdot e\right]\right)^2\right)}{T} > 0. \tag{95}$$

We postpone the proof of the above claims to Sections 12.1, 12.4 and 12.5, and explain how to conclude. First we note that, as in the derivation in (89), for the unperturbed processes we can write

$$\sum_{n=1}^{T}\mathcal{E}_\mu\left[(A_1+B_{T-n})^2\right] = \mathcal{E}_\mu\left[\left(X_T\cdot e - \mathcal{E}_\eta\left[X_T\cdot e\right]\right)^2\right]. \tag{96}$$

Therefore, by combining (89), Claim 3 and Claim 4, we get that

$$Var_{\mu_\varepsilon}\left[X_T^{(\varepsilon)}\cdot e\right] \geq \mathcal{E}_\mu\left[(X_T\cdot e)^2\right] - \mu\left(\left(\mathcal{E}_\eta\left[X_T\cdot e\right]\right)^2\right) - (C\varepsilon/(\gamma-\varepsilon)+\delta(\varepsilon))T. \tag{97}$$

Thus, by using (97) together with Claim 5 and choosing $\beta(\varepsilon)$ small enough, the non-degeneracy of the diffusion matrix is proven.

Before moving to the proofs of the above three claims we collect some technical facts that will be repeatedly used below.

Lemma 12.1. *There exists a function $F(c,n,t)$, where $c>0$, n is a positive integer and $t \geq 0$, such that* $\sup_{\eta\in\Omega}\mathcal{E}_\eta\left(|X_t^{(\varepsilon)}|^n\right) \leq F(c,n,t)$ *if* $\sum_{y\in\mathbb{Z}^d}|y|^n \sup_{\eta\in\Omega} r_\varepsilon(y,\eta) \leq c$.

Proof. We consider the extended Markov process on $\Omega\times\mathbb{Z}^d\times\mathbb{R}_+$ with Markov generator

$$\begin{aligned}\mathcal{L}_\varepsilon f(\eta,x,\ell) := L_{\rm env} f(\eta,x,\ell) + \sum_{y\in\mathbb{Z}^d} r_\varepsilon(y,\tau_x\eta)\big[f(\eta,x+y,\ell+|y|)-f(\eta,x,\ell)\big] \\ + \sum_{y\in\mathbb{Z}^d}\{\sup_{\zeta\in\Omega} r_\varepsilon(y,\zeta)-r_\varepsilon(y,\tau_x\eta)\}\big[f(\eta,x,\ell+|y|)-f(\eta,x,\ell)\big]\,.\end{aligned}\tag{98}$$

Note that $\mathcal{L}_\varepsilon$ acts as $L^{(\varepsilon)}_{\rm rwre}$ on functions f depending on η, x only, while on functions $f=f(\ell)$ it reads $\mathcal{L}_\varepsilon f(\ell)=\sum_v R_\varepsilon(v)\big[f(\ell+v)-f(\ell)\big]$, where v varies in $V:=\{|y| : y\in\mathbb{Z}^d\}$ and

$$R_\varepsilon(v) := \sum_{y\in\mathbb{Z}^d\,:\,|y|=v}\sup_{\eta\in\Omega} r_\varepsilon(y,\eta)\,.$$

Hence, the extended Markov process with generator $\mathcal{L}_\varepsilon$ gives a coupling between the joint process with generator $L^{(\varepsilon)}_{\rm rwre}$ and a jump process $(Z_t)_{t\geq 0}$ on $\mathbb{R}_+$ with jump probability rates $R_\varepsilon(\cdot)$. Moreover, by construction, $Z_t \geq |X_t^{(\varepsilon)}|$ for any time t if $Z_0 \geq |X_0^{(\varepsilon)}|$. Starting the extended Markov generator at $(\eta,0,0)$, we conclude that $\mathcal{E}_\eta\big(|X_t^{(\varepsilon)}|^n\big) \leq \mathbb{E}(Z_t^n)$.

It remains to bound $\mathbb{E}(Z_t^n)$. To this aim we define $\lambda_\varepsilon := \sum_v R_\varepsilon(v) \leq c$ and take a sequence $U_1, U_2, \dots$ of i.i.d. random variables taking value in V with $\mathbb{P}(U_i=v)=R_\varepsilon(v)/\lambda_\varepsilon$. Our main hypothesis implies that $\mathbb{E}[U_i^n] \leq \sum_{y\in\mathbb{Z}^d}|y|^n\sup_{\eta\in\Omega} r_\varepsilon(y,\eta) \leq c$. Taking an independent Poisson process $(N_t)_{t\geq0}$ of parameter λ_ε and setting $S_n := U_1+\cdots+U_n$, the process Z_t can be written as S_{N_t}. In particular, we have $\mathbb{E}(Z_t^n)=\sum_{k=0}^\infty \mathbb{P}(N_t=k)\mathbb{E}[(U_1+\cdots+U_k)^n]$. By Hölder inequality, it holds $(U_1+\cdots+U_k)^n \leq k^{n-1}(U_1^n+\cdots+U_k^n)$. Hence, we conclude that $\mathbb{E}(Z_t^n)\leq \mathbb{E}[U_i^n]\mathbb{E}[N_t^n]$ leading to the thesis. $\square$

Since the positivity of D_ε has to be proved for ε small enough, in the rest of this section we assume $\varepsilon \leq \gamma/2$ so that the term $1/(\gamma-\varepsilon)$ is uniformly bounded.

Lemma 12.2. *The expected values* $\mathcal{E}_\mu\big[A_1^4\big]$, $\mathcal{E}_{\mu_\varepsilon}\big[A_1^4\big]$, $\mathcal{E}^{(\varepsilon)}_{\mu_\varepsilon}\big[\big(A_1^{(\varepsilon)}\big)^4\big]$ *are bounded from above uniformly in ε. The expected values* $\mathcal{E}_\mu\big[B^4_{T-n}\big]$, $\mathcal{E}_{\mu_\varepsilon}\big[B^2_{T-n}\big]$, $\mathcal{E}^{(\varepsilon)}_{\mu_\varepsilon}\big[\big(B^{(\varepsilon)}_{T-n}\big)^4\big]$ *are bounded from above uniformly in ε, T, n.*

Proof. The term $\mathcal{E}_\mu\big[A_1^4\big]$ is bounded since $\int_0^1 S(t)j_e(\eta)dt$ is bounded in uniform norm (as j_e is bounded in uniform norm), and since $\mathcal{E}_\mu\big[(X_1\cdot e)^4\big]$ is bounded (as application of modified version of Lemma 12.1 with a suitable choice of the rates and due to our condition (4)). Similarly, one gets that $\mathcal{E}_{\mu_\varepsilon}\big[A_1^4\big]$ and $\mathcal{E}_{\mu_\varepsilon}\big[\big(A_1^{(\varepsilon)}\big)^4\big]$ are bounded from above uniformly in ε.

We now consider the term $\mathcal{E}_\mu\big[B^4_{T-n}\big]=\mathbb{E}_\mu\big[B^4_{T-n}\big]$. To this aim we first observe that, given $k\geq 1$ and generic numbers $a_1,a_2,\dots,a_k$, by Schwarz inequality it holds

$\big(\sum_{i=1}^k a_i\big)^2 \leq c\sum_{i=1}^k i^2 a_i^2$, where $c := \sum_{i=1}^\infty \frac{1}{i^2}$. By applying twice the above inequality we conclude that $\big(\sum_{i=1}^k a_i\big)^4 \leq c^3 \sum_{i=1}^k i^6 a_i^4$. This implies that

$$\big(\sum_{i=1}^k a_i\big)^4 \leq c^3 \big(\sup_{1\leq i\leq k} |a_i|\big)^2 \sum_{i=1}^k i^6 a_i^2 \,. \tag{99}$$

Let us come back to B_{T-n}. Note that in the definition of B_{T-n} we can replace j_e by $\bar{j} := j_e - \mu(j_e)$. Since j_e is uniformly bounded, we have $\sup_{t\geq 0} \|S(t)\bar{j}\|_\infty < \infty$. By applying (99) then we have

$$\mathbb{E}_\mu\Big[B_{T-n}^4\Big] \leq C\sum_{i=1}^{T-n} i^6 \mathbb{E}_\mu\Big\{\Big[\int_{i-1}^i \Big(S(t)\bar{j}(\eta_1) - \mathbb{E}_\eta\big[S(t)\bar{j}(\eta_1)\big]\Big)dt\Big]^2\Big\}\,. \tag{100}$$

Above and in what follows, C, C' denote positive universal constants (not depending from T, n, ε) that can change from line to line. By applying Schwarz inequality we have

$$\text{r.h.s. of (100)} \leq C\sum_{i=1}^{T-n} i^6 \mathbb{E}_\mu\Big\{\int_{i-1}^i \Big(S(t)\bar{j}(\eta_1)\Big)^2 dt + \int_{i-1}^i \mathbb{E}_\eta\Big[\Big(S(t)\bar{j}(\eta_1)\Big)^2\Big]dt\Big\}\,. \tag{101}$$

By stationarity and by the spectral gap of $L_{\rm ew}$ in $L^2(\mu)$ (cf. (62)), we then conclude that

$$\begin{aligned}\text{r.h.s. of (101)} &\leq C\sum_{i=1}^{T-n} i^6 \int \mu(d\eta)\Big\{\int_{i-1}^i \Big(S(t)\bar{j}(\eta)\Big)^2 dt\Big\} = C\sum_{i=1}^{T-n} i^6 \int_{i-1}^i \|S(t)\bar{j}\|^2 \\ &\leq C'\sum_{i=1}^{T-n} i^6 \int_{i-1}^i e^{-2\gamma t}dt \leq C'\sum_{i=1}^{T-n} i^6 e^{-2\gamma(i-1)} < +\infty\,.\end{aligned} \tag{102}$$

By combining (100), (101) and (102) we get the thesis, i.e. $\mathbb{E}_\mu\big[B_{T-n}^4\big]$ is bounded from above uniformly in T and n.

By similar arguments, considering now the perturbed process, one can prove that $\mathbb{E}_{\mu_\varepsilon}^{(\varepsilon)}\big[\big(B_{T-n}^{(\varepsilon)}\big)^4\big]$ is bounded from above uniformly in T, n and ε.

We now consider $\mathbb{E}_{\mu_\varepsilon}\big[B_{T-n}^2\big]$. By (32) and since both $\mu(f)$ and $\|f-\mu(f)\|$ are bounded by $\|f\|$ for any $f \in L^2(\mu)$, we can estimate

$$\mathbb{E}_{\mu_\varepsilon}\big[B_{T-n}^2\big] = \mu_\varepsilon\big(\mathbb{E}_\cdot\big[B_{T-n}^2\big]\big) \leq \big(1+\frac{\varepsilon}{\gamma-\varepsilon}\big)\|\mathbb{E}_\cdot\big[B_{T-n}^2\big]\|\,.$$

By Schwarz inequality, we have $\|\mathbb{E}_\cdot\big[B_{T-n}^2\big]\| \leq \int \mu(d\eta)\mathbb{E}_\eta\big[B_{T-n}^4\big] = \mathbb{E}_\mu\big[(B_{T-n})^4\big]$. Hence to conclude we invoke that $\mathbb{E}_\mu\big[(B_{T-n})^4\big]$ is bounded from above uniformly in T, n as just proven. □

12.1. **Proof of Claim 3.** Let us start with a simple computation showing that, to get (93), it is enough to prove that there exists $\delta(\varepsilon) \to 0$ as $\beta(\varepsilon) \to 0$ such that

$$\mathcal{E}_{\mu_\varepsilon}\Big[\Big(A_1^{(\varepsilon)} - A_1\Big)^2\Big] \;\leq\; \delta^2(\varepsilon)\,, \tag{103}$$

$$\mathcal{E}_{\mu_\varepsilon}\Big[\Big(B_{T-n}^{(\varepsilon)} - B_{T-n}\Big)^2\Big] \;\leq\; \delta^2(\varepsilon)\,. \tag{104}$$

Below C will denote a positive constant, independent from n, T, ε.

Since $a^2 - b^2 = (a-b)(a+b)$ we can bound

$$\left|\left(A_1^{(\varepsilon)} + B_{T-n}^{(\varepsilon)}\right)^2 - (A_1 + B_{T-n})^2\right| \\ \leq \left[\left|A_1^{(\varepsilon)} - A_1\right| + \left|B_{T-n}^{(\varepsilon)} - B_{T-n}\right|\right] \cdot \left|A_1^{(\varepsilon)} + B_{T-n}^{(\varepsilon)} + A_1 + B_{T-n}\right| .$$

Using the above bound and Schwarz inequality we conclude that

$$\left|\mathcal{E}_{\mu_\varepsilon}\left[(A_1^{(\varepsilon)} + B_{T-n}^{(\varepsilon)})^2 - (A_1 + B_{T-n})^2\right]\right| \\ \leq c\,\mathcal{E}_{\mu_\varepsilon}\left[(A_1^{(\varepsilon)} - A_1)^2\right]^{1/2} + c\,\mathcal{E}_{\mu_\varepsilon}\left[(B_{T-n}^{(\varepsilon)} - B_{T-n})^2\right]^{1/2} ,$$

where $c := \mathcal{E}_{\mu_\varepsilon}\left[\left|A_1^{(\varepsilon)} + B_{T-n}^{(\varepsilon)} + A_1 + B_{T-n}\right|^2\right]^{\frac{1}{2}}$. Due to Schwarz inequality and Lemma 12.2 we conclude that c is bounded uniformly in T, n, ε. In particular, to get Claim 3 it is enough to have (103) and (104).

Let us now prove (103). We set $c(\varepsilon) := \sup_\eta \sum_y |\hat{r}_\varepsilon(y,\eta)|$ and

$$E_n := \{\eta_{t_k}^{(\varepsilon)} = \eta_{t_k}\ \forall k < n \text{ and } t_n \in [0,1]\}, \qquad \text{for } n \geq 1. \tag{105}$$

We then observe that, writing $\zeta = \eta_{t_{n-1}}^{(\varepsilon)}$, it holds[10]

$$\begin{aligned} \mathcal{P}_\eta(\eta_{t_n}^{(\varepsilon)} \neq \eta_{t_n} | E_n) &= \sum_{y \in \mathbb{Z}^d} \mathcal{P}_\eta\big(U_n \in I_\varepsilon(y,\zeta) \Delta I(y,\zeta)\big) \\ &\leq \sum_{y \in \mathbb{Z}^d} (|I_\varepsilon(y,\zeta) + |I(y,\zeta)| - 2|I_\varepsilon(y,\zeta) \cap I(y,\zeta)|) \\ &= \lambda^{-1} \sum_{y \in \mathbb{Z}^d} |\hat{r}_\varepsilon(y,\zeta)| \leq \lambda^{-1} c(\varepsilon) . \end{aligned} \tag{106}$$

Hence we can estimate

$$\begin{aligned} \mathcal{P}_\eta\left(\exists s \in [0,1] \text{ s.t. } \eta_s^{(\varepsilon)} \neq \eta_s\right) &= \sum_{n=1}^\infty \mathcal{P}_\eta(\eta_{t_n}^{(\varepsilon)} \neq \eta_{t_n} | E_n) \mathcal{P}_\eta(E_n) \leq \frac{c(\varepsilon)}{\lambda} \sum_{n=1}^\infty \mathcal{P}_\eta(E_n) \\ &\leq \frac{c(\varepsilon)}{\lambda} \sum_{n=1}^\infty \mathcal{P}_\eta(t_n \in [0,1]) = \frac{c(\varepsilon)}{\lambda} \mathcal{E}_\eta(|\mathcal{T} \cap [0,1]|) = c(\varepsilon) . \end{aligned} \tag{107}$$

In particular, $\mathcal{P}_\eta(X_1^{(\varepsilon)} \neq X_1) \leq c(\varepsilon)$. By Schwarz inequality and Lemma 12.1, which allows with (4) to bound the forth moments of $X_1^{(\varepsilon)}, X_1$ uniformly in ε (for X_1 one has to slightly change the notation in the lemma), we get

$$\mathcal{E}_{\mu_\varepsilon}\left[(X_1^{(\varepsilon)} - X_1)^2\right] \leq C\, c(\varepsilon)^{1/2} . \tag{108}$$

We point out that $\|j_e^{(\varepsilon)} - j_e\|_\infty \leq \sup_\eta \sum_y |y|\,|\hat{r}_\varepsilon(y,\eta)| \leq \beta(\varepsilon)$. Note that $c(\varepsilon) \leq \beta(\varepsilon)$. Hence, given $t \in [0,1]$, using (107) we get

$$\left|S_\varepsilon(t) j_e^{(\varepsilon)}(\eta) - S(t) j_e(\eta)\right| = \left|\mathcal{E}_\eta\left[j_e^{(\varepsilon)}(\eta_t^{(\varepsilon)}) - j_e(\eta_t)\right]\right| \\ \leq \beta(\varepsilon) \mathcal{P}_\eta\big(\eta_t^{(\varepsilon)} = \eta_t\big) + (\|j_e^{(\varepsilon)}\|_\infty + \|j_e\|_\infty) \mathcal{P}_\eta\big(\eta_t^{(\varepsilon)} \neq \eta_t\big) \leq C\,\beta(\varepsilon) . \tag{109}$$

[10] Δ denotes the symmetric difference, i.e. $A \Delta B := (A \setminus B) \cup (B \setminus A)$

In particular, by (108) and (109), the l.h.s. of (103) is bounded by $C\beta(\varepsilon)+Cc(\varepsilon)^{1/2}$. This concludes the proof of (103).

In order to get (104) we abbreviate

$$b_t^{(\varepsilon)} := S_\varepsilon(t) j_e^{(\varepsilon)}\left(\eta_1^{(\varepsilon)}\right), \qquad b_t := S(t) j_e(\eta_1).$$

Then $B_{T-n}^{(\varepsilon)} = \int_0^{T-n}(b_t^{(\varepsilon)} - \mathbb{E}_\eta^{(\varepsilon)}(b_t^{(\varepsilon)}))dt$ and $B_{T-n} = \int_0^{T-n}(b_t - \mathbb{E}_\eta(b_t))dt$. In particular we can bound

$$\mathcal{E}_{\mu_\varepsilon}\left[\left(B_{T-n}^{(\varepsilon)} - B_{T-n}\right)^2\right] \le c\sum_{i=1}^{T-n} i^2 \int_{i-1}^{i} dt \mathcal{E}_{\mu_\varepsilon}\left[\left(b_t^{(\varepsilon)} - \mathbb{E}_\eta^{(\varepsilon)}\left[b_t^{(\varepsilon)}\right] - \left(b_t - \mathbb{E}_\eta\left[b_t\right]\right)\right)^2\right]. \tag{110}$$

At this point, to get (104) it is enough to show that there exists a constant $w(\varepsilon)$ going to zero as $\beta(\varepsilon)$ goes to zero such that

$$\mathcal{E}_{\mu_\varepsilon}\left[\left(b_t^{(\varepsilon)} - \mathbb{E}_\eta^{(\varepsilon)}\left[b_t^{(\varepsilon)}\right] - \left(b_t - \mathbb{E}_\eta\left[b_t\right]\right)\right)^2\right] \le \frac{w(\varepsilon)}{(\gamma-\varepsilon)^{3/2}} e^{-\frac{\gamma-\varepsilon}{2}t}, \qquad \forall t \ge 0. \tag{111}$$

Since given any $a, b \ge 0$ it holds $\min(a,b) \le \sqrt{ab}$, it is enough to show that the l.h.s. of (111) is bounded from above both by $w^2(\varepsilon)/C$ and by $Ce^{-(\gamma-\varepsilon)t}/(\gamma-\varepsilon)^3$. We start with the latter.

12.2. **The l.h.s. of** (111) **is bounded from above by** $Ce^{-(\gamma-\varepsilon)t}/(\gamma-\varepsilon)$**.** We observe that

$$\begin{aligned}\mathcal{E}_{\mu_\varepsilon}\Big[\Big(b_t^{(\varepsilon)} - \mathbb{E}_\eta^{(\varepsilon)}\big[b_t^{(\varepsilon)}\big]\Big)^2\Big] &\le 2\mathcal{E}_{\mu_\varepsilon}\Big[\Big(b_t^{(\varepsilon)} - \mu_\varepsilon(j_e^{(\varepsilon)})\Big)^2\Big] + 2\mathcal{E}_{\mu_\varepsilon}\Big[\Big(\mu_\varepsilon(j_e^{(\varepsilon)}) - \mathbb{E}_\eta^{(\varepsilon)}\big[b_t^{(\varepsilon)}\big]\Big)^2\Big]\\ &= 2\mu_\varepsilon\Big(\Big(S_\varepsilon(t)j_e^{(\varepsilon)} - \mu_\varepsilon(j_e^{(\varepsilon)})\Big)^2\Big) + 2\mu_\varepsilon\Big(\Big(S_\varepsilon(t+1)j_e^{(\varepsilon)} - \mu_\varepsilon(j_e^{(\varepsilon)})\Big)^2\Big)\\ &\le 4\|j_e^{(\varepsilon)} - \mu_\varepsilon(j_e^{(\varepsilon)})\|_\infty^2\Big(\frac{\gamma}{\gamma-\varepsilon}\Big)^3 e^{-(\gamma-\epsilon)t},\end{aligned} \tag{112}$$

where the equality follows from the semigroup property implying that $\mathbb{E}_\eta^{(\varepsilon)}(b_t^{(\varepsilon)}) = S_\varepsilon(t+1)(\eta)$ and from the invariance of μ_ε for the environment viewed by the perturbed walker. Moreover, the last inequality follows from (36).

On the other hand, we have

$$\mathcal{E}_{\mu_\varepsilon}\left[(b_t - \mathbb{E}_\eta\left[b_t\right])^2\right] \le 2\|j_e\|_\infty \mathbb{E}_{\mu_\varepsilon}\left[|b_t - \mathbb{E}_\eta\left[b_t\right]|\right] = 2\|j_e\|_\infty \mu_\varepsilon(f), \tag{113}$$

where $f(\eta) := \mathbb{E}_\eta\big[\big|b_t - \mathbb{E}_\eta\big[b_t\big]\big|\big]$. Now, thanks to (32), we can bound

$$\mu_\varepsilon(f) \le \mu(f) + \frac{\varepsilon}{\gamma-\varepsilon}\mu(f^2)^{\frac12} \le \frac{\gamma}{\gamma-\varepsilon}\mu(f^2)^{\frac12} \le \frac{\gamma}{\gamma-\epsilon}\mathcal{E}_\mu\Big[\Big(b_t - \mathbb{E}_\eta\Big[b_t\Big]\Big)^2\Big]^{1/2}.$$

In particular, we conclude that

$$\mathcal{E}_{\mu_\varepsilon}\left[(b_t - \mathbb{E}_\eta\left[b_t\right])^2\right] \le \frac{C}{\gamma-\epsilon}\mathcal{E}_\mu\left[(b_t - \mathbb{E}_\eta\left[b_t\right])^2\right]^{1/2}. \tag{114}$$

Reasoning as in (112) (now using directly (21) instead of (36)) we get that the square of the last factor in (114) is bounded by $4\|j_e\|^2 e^{-2\gamma t}$. In particular, (114) can be refined to

$$\mathcal{E}_{\mu_\varepsilon}\left[(b_t - \mathbb{E}_\eta\left[b_t\right])^2\right] \le \frac{C}{\gamma-\epsilon}e^{-\gamma t}. \tag{115}$$

As a byproduct of (110), (112) and (115) we get that the l.h.s. of (111) is bounded from above by $Ce^{-(\gamma-\varepsilon)t}/(\gamma-\varepsilon)$.

12.3. **The l.h.s. of** (111) **is bounded from above by** $o(1)$**.** We say that a quantity is $o(1)$ if it goes to zero as $\beta(\varepsilon)$ goes to zero. Let us write

$$
\begin{aligned}
b_t^{(\varepsilon)} - b_t = &\Big[S_\varepsilon(t)j_e^{(\varepsilon)}\left(\eta_1^{(\varepsilon)}\right) - S(t)j_e^{(\varepsilon)}\left(\eta_1^{(\varepsilon)}\right)\Big] + \Big[S(t)j_e^{(\varepsilon)}\left(\eta_1^{(\varepsilon)}\right) - S(t)j_e\left(\eta_1^{(\varepsilon)}\right)\Big]\\
&+ \Big[S(t)j_e\left(\eta_1^{(\varepsilon)}\right) - S(t)j_e\left(\eta_1\right)\Big]
\end{aligned}
\tag{116}
$$

Let us deal with the first term in the r.h.s. We can bound

$$
\begin{aligned}
&\mathcal{E}_{\mu_\varepsilon}\Big[\Big(S_\varepsilon(t)j_e^{(\varepsilon)}(\eta_1^{(\varepsilon)}) - S(t)j_e^{(\varepsilon)}(\eta_1^{(\varepsilon)})\Big)^2\Big] = \mu_\varepsilon\left(\Big(S_\varepsilon(t)j_e^{(\varepsilon)} - S(t)j_e^{(\varepsilon)}\Big)^2\right)\\
&\quad\le \|j_e^{(\varepsilon)}\|_\infty \mu_\varepsilon\Big(\,|S_\varepsilon(t)j_e^{(\varepsilon)} - S_\varepsilon^{(0)}(t)j_e^{(\varepsilon)}|\,\Big)\\
&\quad\le C\mu\Big(\,|S_\varepsilon(t)j_e^{(\varepsilon)} - S_\varepsilon^{(0)}(t)j_e^{(\varepsilon)}|\,\Big) + C\varepsilon(\gamma-\varepsilon)^{-1}\|S_\varepsilon(t)j_e^{(\varepsilon)} - S_\varepsilon^{(0)}(t)j_e^{(\varepsilon)}\|_\mu\\
&\quad\le C'(\gamma-\varepsilon)^{-1}\|S_\varepsilon(t)j_e^{(\varepsilon)} - S_\varepsilon^{(0)}(t)j_e^{(\varepsilon)}\|_\mu \le C''\varepsilon(\gamma-\varepsilon)^{-1} = o(1)\,.
\end{aligned}
$$

Indeed, the first identity follows from the invariance of $(\eta_t^{(\varepsilon)})_{t\ge 0}$ under μ_ε, the second inequality follows from (32) and (25), the third one from Schwarz inequality and the last one from (27)with $k=1$.

We move to the second term which is bounded in uniform norm by $\|S(t)(j_e^{(\varepsilon)} - j_e)\|_\infty \le \|j_e^{(\varepsilon)} - j_e\|_\infty \le \beta(\varepsilon) = o(1)$. On the other hand, using that $\|S(t)j_e\|_\infty$ is uniformly bounded in t and using (107), the $\mathcal{E}_{\mu_\varepsilon}$–second moment of the third term in the r.h.s. of (116) can be estimated by $C\mathcal{P}_\eta(\eta_1\neq\eta) \le C\,c(\varepsilon) = o(1)$.

As a byproduct of the above observations we conclude that $\mathcal{E}_{\mu_\varepsilon}\left[\big(b_t^{(\varepsilon)} - b_t\big)^2\right] = o(1)$. This also implies that

$$
\begin{aligned}
\mathcal{E}_{\mu_\varepsilon}\left[\Big(\mathbb{E}_\eta^{(\varepsilon)}\left[b_t^{(\varepsilon)}\right] - \mathbb{E}_\eta\left[b_t\right]\Big)^2\right] &= \mathcal{E}_{\mu_\varepsilon}\left[\Big(\mathcal{E}_\eta[b_t^{(\varepsilon)} - b_t]\Big)^2\right]\\
&\le \mathcal{E}_{\mu_\varepsilon}\left[\mathcal{E}_\eta\left[(b_t^{(\varepsilon)} - b_t)^2\right]\right] = \mathcal{E}_{\mu_\varepsilon}\left[(b_t^{(\varepsilon)} - b_t)^2\right] = o(1)\,.
\end{aligned}
$$

By Schwarz inequality we then conclude that the l.h.s. of (111) is bounded from above by $o(1)$.

12.4. **Proof of Claim 4.** Let $f_{T,n}(\eta) := \mathcal{E}_\eta\Big[\big(A_1 + B_{T-n}\big)^2\Big]$. Then (94) reads $\mu_\varepsilon(f_{T,n}) \ge \mu(f_{T,n}) - C\varepsilon/(\gamma-\varepsilon)$. This follows from (32) if we prove that $\mu(f_{T,n}^2)$ is bounded from above uniformly in T,n. By Schwarz inequality, it is enough to bound from above $\mathcal{E}_\mu\big[A_1^4\big]$ and $\mathcal{E}_\mu\big[B_{T-n}^4\big]$ uniformly in T,n. This follows from Lemma 12.2.

12.5. **Proof of Claim 5.** By standard techniques [43, 15] we have the following variational characterization of the diffusion coefficient of a symmetric walker in reversible environment:

$$
\langle e, D_0 e\rangle = \frac{1}{2}\inf_f\Big\{-2\mu\left(fL_{\rm env}f\right) + \sum_{y\in\mathbb{Z}^d}\mu\left(r(y,\eta)\left[y\cdot e + f(\tau_y\eta) - f(\eta)\right]^2\right)\Big\},
\tag{117}
$$

where the infimum is taken over local functions f on Ω and where e is any vector of the canonical basis.

In (117), by definition of the spectral gap, the first term is bounded from below by $2\gamma\text{Var}_\mu(f)$. On the other hand, using the inequality $(a+b)^2 \geq \beta a^2 - \frac{\beta}{1-\beta} b^2$ for $\beta < 1$, we get

$$\begin{aligned}
&\mu\left(r(y,\eta)\left[y\cdot e + f(\tau_y\eta) - f(\eta)\right]^2\right) \\
&\qquad \geq \beta\mu(r(y,\cdot))(y\cdot e)^2 - \frac{\beta}{1-\beta}\mu\left(r(y,\eta)\left[f(\tau_y\eta)-f(\eta)\right]^2\right) \\
&\qquad\qquad \geq \beta\mu(r(y,\cdot))(y\cdot e)^2 - 4\sup_\eta r(y,\eta)\frac{\beta}{1-\beta}Var_\mu(f).
\end{aligned}$$

Injecting this in (117) and choosing $\beta < 1$ so that

$$2\gamma - 4\frac{\beta}{1-\beta}\sum_{y\in\mathbb{Z}^d}\sup_\eta r(y,\eta) \geq 0,$$

we get $\langle e, D_0 e\rangle > 0$. Hence, we conclude that (cf. [15, Eq. (2.43)])

$$\lim_{T\to+\infty}\frac{1}{T}\mathcal{E}_\mu\left[\left(X_T\cdot e\right)^2\right] = \langle e, D_0 e\rangle > 0. \tag{118}$$

We claim that

$$\sup_{T\geq 0}\mu\left(\left(\mathcal{E}_\eta\left[X_T\cdot e\right]\right)^2\right) < +\infty \tag{119}$$

For simplicity we restrict the proof to T integer (indeed, to our final aim this would be enough, anyway one could extend the thesis to the general case). Due to the Markov property, we get

$$\mathcal{E}_\eta\Big[X_T\cdot e\Big] = \mathcal{E}_\eta\Big[\sum_{k=0}^{T-1}(X_{k+1}\cdot e - X_k\cdot e)\Big] = \mathcal{E}_\eta\Big[\sum_{k=0}^{T-1}\mathcal{E}_{\eta_k}(X_1\cdot e)\Big] = \sum_{k=0}^{T-1}\mathcal{E}_\eta\Big[\mathcal{E}_{\eta_k}(X_1\cdot e)\Big]\,. \tag{120}$$

Consider now the function $f(\eta) = \mathcal{E}_\eta(X_1\cdot e)$.

Since $S(t)f(\eta) = \mathcal{E}_\eta\Big[\mathcal{E}_{\eta_t}(X_1\cdot e)\Big]$, from (120) we get that

$$\mathcal{E}_\eta\Big[X_T\cdot e\Big] = \sum_{k=0}^{T-1}S(k)f(\eta)\,.$$

Note that $\mu(f) = 0$ by reversibility and that $f\in L^2(\mu)$ by Lemma 12.1 adapted to the unperturbed process and by condition (4). By the Poincaré inequality (21) we conclude that $\|S(t)f\| \leq e^{-\gamma t}\|f\|$. At this point we have

$$\begin{aligned}
\mu\Big(\mathbb{E}_\eta\Big[X_T\cdot e\Big]^2\Big) &= \mu\Big[\Big(\sum_{k=0}^{T-1}S(k)f(\eta)\Big)^2\Big] = \|\sum_{k=0}^{T-1}S(k)f\|^2 \leq \Big(\sum_{k=0}^{T-1}\|S(k)f\|\Big)^2 \\
&\leq \|f\|^2\Big(\sum_{k=0}^{T-1}e^{-\gamma k}\Big)^2 \leq \frac{\|f\|^2}{1-e^{-\gamma}}\,,
\end{aligned} \tag{121}$$

thus concluding the proof of (119). Trivially, Claim 5 follows as a byproduct of (118) and (119).

Appendix A. Miscellanea

Lemma A.1 and Lemma A.2 below have a standard derivation and therefore we omit their proof. Detailed proofs can be found in [4, Appendix A].

Lemma A.1. *Let Ω be a metric space and let ν be a Borel probability measure on Ω. Then:*

(i) *The subset $C_b(\Omega)$ of bounded continuous functions $f : \Omega \to \mathbb{R}$ is dense in $L^2(\nu)$.*

(ii) *Let h be a function in $L^2(\nu)$ such that $\nu(hf) \geq 0$ for any $f \in C_{b,+}(\Omega) := \{g \in C_b(\Omega) : g \geq 0\}$. Then, $h \geq 0$ ν–a.s..*

Lemma A.2. *The semigroup $S(t)$, $t \in \mathbb{R}_+$, defined at the beginning of Section 3 is strongly continuous.*

Lemma A.3. *In the same setting of Section 3, given a positive constant $\gamma > 0$,* (20) *is equivalent to* (21)*.*

The above lemma is usually proven in the reversible case. We give the proof to stress that it holds even without reversibility.

Proof. For any $f \in \mathcal{D}(L)$ the map $[0, +\infty) \ni t \to S(t)f \in L^2(\mu)$ is C^1, $S(t)f \in \mathcal{D}(L)$ and $\frac{d}{dt}S(t)f = LS(t)f$ [18, Chap. II, Sec. 1]. In particular, taking $f \in \mathcal{D}(L)$, by differentiating one gets

$$\frac{d}{dt}\|S(t)f\|^2 = \langle LS(t)f, S(t)f\rangle + \langle S(t)f, LS(t)f\rangle = 2\langle S(t)f, LS(t)f\rangle\,, \tag{122}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2(\mu)$ (note that we have used the symmetry of the scalar product: $\langle g, g'\rangle = \langle g', g\rangle$).

We first assume Poincaré inequality (20) to be satisfied and take $f \in \mathcal{D}(L)$ with $\mu(f) = 0$. By (122) and the Poincaré inequality, one gets

$$\frac{d}{dt}\|S(t)f\|^2 = 2\langle S(t)f, LS(t)f\rangle \leq -2\gamma\|S(t)f\|^2\,.$$

Note that we have used the stationarity of μ, implying that $\mu(S(t)f) = \mu(f) = 0$. Gronwall inequality then leads to $\|S(t)f\| \leq e^{-\lambda t}\|f\|$. In particular, (21) holds for any $f \in \mathcal{D}(L)$ with $\mu(f) = 0$, and therefore for any $f \in \mathcal{D}(L)$ (observe that constant functions are left invariant by $S(t)$). By density of $\mathcal{D}(L)$ in $L^2(\mu)$ one gets (21) for any $f \in L^2(\mu)$.

We now assume (21) to be satisfied and fix $f \in \mathcal{D}(L)$ with $\mu(f) = 0$. By (122) we have $\|S(t)f\|^2 = \|f\|^2 - 2t\langle f, -Lf\rangle + o(t)$ as $t \downarrow 0$. On the other hand, $e^{-2\gamma t}\|f\|^2 = \|f\|^2 - 2\gamma\|f\|^2 + o(t)$ as $t \downarrow 0$. Hence the Taylor expansion of (21) implies (20). $\square$

The following lemma extends the probabilistic interpretation of the semigroup $S_\varepsilon(t)$ given in (23).

Lemma A.4. *Consider the same assumptions of Theorem 5. Then, given $f \in L^2(\mu)$, it holds $S_\varepsilon(t)f(\eta) = \mathbb{E}^{(\varepsilon)}_\eta\big(f(\eta_t)\big)$ μ_ε–a.s.*

Proof. By Lemma A.1 there exists a sequence $(f_n)_{n\geq 1}$ in $C_b(\Omega)$ with $\|f_n - f\| \to 0$ as $n \to \infty$. Since $S_\varepsilon(t)$ is a bounded operator in $L^2(\mu)$, we get that $\|S_\varepsilon(t)f_n - S_\varepsilon(t)f\| \to 0$ as $n \to \infty$. In particular, at cost to extract a subsequence, we have

$S_\varepsilon(t)f_n(\eta) \to S_\varepsilon(t)f(\eta)$ for μ–a.e. η. Since $\mu_\varepsilon \ll \mu$ (by Theorem 5), we conclude that

$$S_\varepsilon(t)f_n(\eta) \to S_\varepsilon(t)f(\eta) \text{ for } \mu_\varepsilon\text{–a.e. } \eta\,. \tag{123}$$

On the other hand, by the stationarity of μ_ε for the perturbed dynamics, we have

$$\mu_\varepsilon\Big[\Big|\mathbb{E}^{(\varepsilon)}_\eta\big(f_n(\eta_t)\big)-\mathbb{E}^{(\varepsilon)}_\eta\big(f(\eta_t)\big)\Big|\Big] \le \mu_\varepsilon\Big[\mathbb{E}^{(\varepsilon)}_\eta\big(|f_n(\eta_t)-f(\eta_t)|\big)\Big] = \mathbb{E}^{(\varepsilon)}_{\mu_\varepsilon}[|f_n(\eta_t)-f(\eta_t)|]$$
$$= \mu_\varepsilon[|f_n-f|) = \mu\Big[\frac{d\mu_\varepsilon}{d\mu}|f_n-f|\Big] \le \|\frac{d\mu_\varepsilon}{d\mu}\|\cdot\|f_n-f\| \to 0\,.$$

We have shown that the map $\mathbb{E}^{(\varepsilon)}_{\cdot}\big(f_n(\eta_t)\big)$ converges to the map $\mathbb{E}^{(\varepsilon)}_{\cdot}\big(f(\eta_t)\big)$ in $L^1(\mu_\varepsilon)$. Hence, at cost to extract a subsequence, the convergence holds also μ_ε–a.s.. The thesis is then a byproduct of the last observation, of (123) and the identity (23) applied to $f_n \in C_b(\Omega)$ instead of f (which holds μ–a.s. and therefore μ_ε–a.s. since $\mu_\varepsilon \ll \mu$). □

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