UNIVERSITÉ PARIS DIDEROT - PARIS 7

ÉCOLE DOCTORALE DE SCIENCES MATHÉMATIQUES DE PARIS CENTRE

THÈSE DE DOCTORAT

Discipline : Mathématiques Appliquées

Présentée par

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DYNAMIQUES DE PARTICULES SUR RÉSEAUX AVEC CONTRAINTES CINÉTIQUES

KINETICALLY CONSTRAINED PARTICLE SYSTEMS ON A LATTICE

Sous la direction de **Thierry BODINEAU, Cristina TONINELLI**

Soutenue publiquement le 3 décembre 2013 devant le jury composé de

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Remerciements

Mes remerciements vont d'abord à mes directeurs, Cristina et Thierry. Merci pour votre disponibilité, votre écoute, votre enthousiasme pour la recherche, votre optimisme à toute épreuve, votre incroyable patience, vos conseils toujours pertinents. C'est peu de dire que cette thèse n'aurait pas été la même sans vous.

Je suis très heureuse qu'Ellen Saada ait acceptée de rapporter cette thèse. Organisatrice infatigable de conférences et séminaires, le peu de culture probabiliste acquise pendant ces trois années lui doit beaucoup. Et ces dernières semaines m'ont confirmé ce que je pensais depuis longtemps : pour elle l'attention aux autres et en particulier aux doctorants n'est pas un vain mot. I am very honoured and grateful that Frank den Hollander took the time to read my work and did the trip all the way from Leiden to attend my defence. Having never properly met him before, I appreciate all the more his patience. Je suis également très reconnaissante envers Olivier Garet, Giambattista Giacomin et Cyril Roberto d'avoir accepté de faire partie de mon jury. Il a été très enrichissant de côtoyer Giambattista au LPMA et de travailler (même brièvement) avec Cyril.

Nel corso degli ultimi anni, ho avuto la fortuna di lavorare con persone molto brave e gentili. Ringrazio qui Fabio Martinelli per la sua ospitalitá in Roma 3 all'inizio della tesi. Grazie a lui e a Nicoletta Cancrini di aver guidato i miei primi veri passi nel mondo della ricerca. Poi ho avuto il piacere di lavorare anche con Alessandra Faggionato e Luca Avena. Li ringrazio molto per la loro collaborazione e ospitalitá, sia a Roma che a Berlino.

Je dois remercier également les deux laboratoires qui m'ont accueillie ces trois (voire quatre) dernières années : le LPMA à Paris 7 et le DMA à l'ENS. Merci à tous leurs membres de faire de ces deux endroits des lieux agréables et propices au travail. Je voudrais remercier ici tout particulièrement Bénédicte, Nathalie, Zaïna et Valérie pour leur disponibilité, leur efficacité, leur bonne humeur et leur aide inestimable dans la jungle de l'administration. Un grand merci à Vivien également, pour ses lumières sur le monde mystérieux de la physique, ainsi qu'à Marie, guide parfait pour doctorant perdu.

Il faut aussi remercier ici les professeurs qui m'ont doucement amenée vers les probabilités. Jean Bertoin d'abord, qui m'a convaincue en première année d'ENS qu'il y avait là quelque chose qui méritait qu'on y regarde de plus près. Wendelin Werner ensuite, qui a achevé de me convaincre au deuxième semestre, puis m'a fait entrevoir les beautés de la physique statistique avant de m'orienter vers Thierry et Cristina : merci !

Ces années de thèse auraient été beaucoup moins plaisantes sans compagnie pour prendre le café, le thé... voire davantage. Un merci tout particulier à Igor et Stéphane pour leurs dépannages en maths et en LaTeX qui remontent à bien avant la thèse. Merci à Christophe pour son calme rassurant et la relecture de l'intro. Merci à ceux du LPMA : ceux qui sont partis – Laurent, Paul, Ennio, Christophe –, ceux qui restent – Nicolas et sa culture du sport, Thomas et ses jeux de société, Aser et ses exploits pâtissiers, Guillaume et ses pulls, Maud et son énergie zumbaesque, Marc-Antoine et ses critiques culturelles, Lorick et ses pattes de mouche au tableau, Sophie qui ne nous en veut pas trop de faire de l'aléatoire, Jiatu et ses propositions culinaires originales, Arturo et ses bulles, Anna, Huy, Noufel, Simone – et ceux qui sont passés. Merci à ceux du passage vert et d'au-dessus : Laure pour le travail au soleil, Augusto et Zhan pour les conseils littéraires, Pierre pour ses conseils geeks, Nicolas pour son infatigabilité, Ilaria, Jérémie et Bénédicte pour le thé du mercredi, Clément pour nos aventures en territoire polytechnique, Bastien pour ses étonnements sans fin, Gilles, Emilien, Pierre et Paul pour être descendus de temps en temps nous voir. Merci aux compagnons de conférence, Eric, Loren, Marie et bien d'autres.

Han, Claire, No, Antonin, Maks, Fatou, Guylène, Adrien, Matthieu, Marion, Nicolas, Marie... Je ne peux pas remercier tous ceux qui n'ont rien-à-voir-avec-ma-recherche-et-c'est-très-bien-comme-ça pour tous les moments où j'ai sorti la tête de l'eau avec vous. Mais sachez-le, si vous avez un doute, vous êtes là. Merci à ma famille, ceux qui sont là et ceux qui sont loin – et Yves, que j'aurais voulu voir ici.

Enfin et par-dessus tout, merci à Steve d'être toujours là, dans les moments d'enthousiasme comme dans les moments de panique.

Résumé

Dans cette thèse, je m'intéresse à des modèles stochastiques de particules sur réseaux qui suivent une dynamique de Glauber avec contraintes cinétiques (KCSM), et particulièrement aux modèles Est et FA-1f. Ces modèles sont apparus en physique pour l'étude des systèmes vitreux.

Dans ce document se trouve d'abord un résumé en français de son contenu. Puis viennent trois chapitres présentant le cadre dans lequel mes travaux s'inscrivent et montrant à la fois leurs contributions et à quelles notions et techniques ils font appel. Je centre ma présentation des KCSM sur les objets et résultats qui ont joué un rôle direct dans mes recherches. Mes articles sont regroupés en annexe avec éventuellement quelques extensions retranchées pour la publication.

Le premier chapitre est une introduction aux KCSM. Le deuxième chapitre présente des résultats hors équilibre pour les KCSM. J'expose d'abord des résultats de relaxation locale ; pour le modèle FA-1f il s'agit d'un travail commun avec N. Cancrini, F. Martinelli, C. Roberto et C. Toninelli. J'étudie ensuite la progression d'un front dans le modèle Est, et montre un théorème de forme ainsi qu'un résultat d'ergodicité pour le processus vu du front. Ce résultat repose sur la quantification de la relaxation locale du processus vu du front plutôt que sur des arguments classiques de sous-additivité.

Le dernier chapitre explore des questions liées à la dynamique des KCSM à basse température (soit à haute densité). Je rappelle des résultats asymptotiques sur le trou spectral des modèles Est et FA-1f et propose quelques heuristiques et conjectures. Je m'intéresse ensuite au comportement à basse température du coefficient de diffusion d'un traceur dans un KCSM, dans l'optique de donner des réponses rigoureuses à des questions posées dans la littérature physique.

Abstract

This thesis is about stochastic lattice models of particle systems with Glauber dynamics and kinetic constraints (KCSM), more specifically the East and FA-1f models. These models were introduced in physics for the study of glassy systems.

In this document one finds first a summary of its contents (in French), then three introductory chapters in which I present the context of my works and show both what what my contributions add to the picture and on which notions and techniques they rely. In my presentation of KCSM, I focus on objects and results that are directly related to my research. Finally my papers are assembled in the Appendix, in some cases with extensions that were cut off for publication.

The first chapter is an introduction to KCSM. The second chapter presents non-equilibrium issues for KCSM. First I give results about out-of-equilibrium local relaxation; in the FA-1f model it is a joint work with N. Cancrini, F. Martinelli, C. Roberto and C. Toninelli. Then I study the progression of a front in the East model and show a shape theorem as well as an ergodicity result for the process seen from the front. This result relies on quantifying the local relaxation of the process seen from the front rather than using classic sub-additivity arguments.

The last chapter explores low-temperature (or high density) dynamics of KCSM. I first recall asymptotic results about East and FA-1f spectral gaps and offer some heuristics and conjectures. I then focus on the low temperature behaviour of the diffusion coefficient of a tracer in a KCSM, so as to give rigorous answers to questions raised in the physics literature.

Table des matières

Les travaux présentés pour cette thèse sont les suivants.
The works presented for this thesis are the following.

- *Tracer diffusion in low temperature Kinetically Constrained Models*, O. Blondel, 20 pages, submitted, http://arxiv.org/abs/1306.6500.
- *Is there a breakdown of the Stokes-Einstein relation in Kinetically Constrained Models at low temperature ?*, O. Blondel, C. Toninelli, 5 pages, submitted, http://arxiv.org/abs/1307.1651.
- *Front progression in the East model*, O. Blondel, 35 pages, *Stochastic Processes and their Applications*, Vol. 123, Issue 9, Sept. 2013, p. 3430–3465, http://arxiv.org/abs/1212.4435.
- *Fredrickson-Andersen one-spin facilitated spin model out of equilibrium*, O. Blondel, N. Cancrini, F. Martinelli, C. Roberto, C. Toninelli, 21 pages, *Markov Processes Relat. Fields*, 19 :383-406, May 2013, http://arxiv.org/abs/1205.4584.

Résumé détaillé

Je présente ici un résumé en français des principaux résultats de ma thèse. Ils sont décrits avec plus de détails et de rigueur dans les chapitres suivants, et les preuves complètes se trouvent réunies en annexe.

Dans le cadre de ma thèse, je m'intéresse à des modèles stochastiques de particules sur réseaux (principalement $\mathbb{Z}^d$). Les systèmes que j'étudie suivent une dynamique de Glauber, à laquelle on ajoute des contraintes cinétiques. On parlera de modèles contraints cinétiquement (Kinetically Constrained Spin Models, KCSM). Plus précisément, les KCSM sont des processus de Markov à temps continu à dynamique non conservative (on peut aussi définir des modèles contraints conservatifs, mais ils n'interviendront pas ici). Ils sont définis sur un graphe G, qui sera ici $\mathbb{Z}^d$, et leur espace des configurations est $\Omega = \{0,1\}^{\mathbb{Z}^d}$. Cela revient à dire que dans une configuration $\omega \in \Omega$ chaque site de $\mathbb{Z}^d$ peut être occupé par une particule (auquel cas $\omega_x = 1$) ou vide ($\omega_x = 0$). Ces systèmes ont une mesure d'équilibre produit, donc sans interaction. La spécificité des KCSM est la présence de contraintes cinétiques : une création/destruction de particule ne peut avoir lieu que si la configuration remplit certaines contraintes locales. Ces contraintes se formulent habituellement par une condition du type : "il y a tant de sites vides autour de l'endroit où je veux modifier la configuration". Cela signifie que lorsqu'il y a "trop" (en un certain sens, dont dépend le modèle) de particules dans le voisinage d'un site, les taux de transition correspondant à une mise à jour du site s'annulent.

Dans cette thèse j'ai tout particulièrement étudié deux de ces modèles : le modèle Est et le modèle FA-1f. Dans le premier, qui vit sur $\mathbb{Z}$, la contrainte est satisfaite ssi le voisin à l'Est est vide. Dans le second (qui vit par exemple sur $\mathbb{Z}^d$), elle est satisfaite si au moins l'un des plus proches voisins est vide. Plus formellement, soient $p \in (0,1)$, qui sera appelé densité, et $q = 1-p$, qui sera appelé densité de zéros. Le générateur du modèle Est ou FA-1f à densité p est donné par

$$\mathcal{L}f(\eta) = \sum_{x\in\mathbb{Z}^d} c_x(\eta)\left(p(1-\eta_x) + q\eta_x\right)\left[f(\eta^x) - f(\eta)\right], \tag{1}$$

où f est une fonction locale, η^x est la configuration η retournée en x ; pour le modèle Est $d = 1$ et $c_x(\eta) = 1 - \eta_x$, et pour le modèle FA-1f, $c_x(\eta) = 1 - \prod_{y\sim x}(1-\eta_y)$. On peut construire ces processus de la façon suivante. A chaque site est attaché un processus de Poisson de paramètre 1, qui joue le rôle d'une horloge signalant les moments où une évolution est possible au niveau du site concerné. Lorsqu'une de ces horloges "sonne" en x, on regarde si la contrainte est vérifiée. Si ce n'est pas le cas, le système reste bloqué. Sinon, on réinitialise l'état du site x en y attachant une particule avec probabilité p et en le laissant vide avec probabilité q. Les modèles Est et FA-1f sont ergodiques à toute densité $p < 1$ et ont même un trou spectral strictement positif dans toute cette région (voir Section 1.3).

Le phénomène physique qui a motivé l'étude des KCSM est la transition liquide/verre ([FA84, FA85, JE91, RS03]). Ce sujet soulève beaucoup de questions et provoque toujours de vifs débats dans la communauté physique. Une des difficultés de cette étude est que, si le verre apparaît solide à échelle humaine, il ne présente aucune régularité microscopique : sur la base d'une seule photo, verre et liquide sont indistinguables. Le verre est donc un matériau intrigant, présentant à la fois des caractéristiques solides et des caractéristiques liquides. Une explication qui a été avancée pour comprendre le phénomène de la transition vitreuse est la suivante. Lorsqu'on refroidit rapidement un liquide (ou qu'on augmente rapidement sa densité), les particules qui le composent n'ont pas le temps de s'organiser pour former la structure qu'aurait un solide. On obtient donc ainsi un système à haute densité et sans aucune structure. Mais si la densité est très élevée, localement les particules sont bloquées : il n'y a pas assez d'espace autour d'elles pour qu'un mouvement soit possible, de sorte que le temps de relaxation du système devient extrêmement élevé et l'équilibre inatteignable sur une échelle de temps observable. On parle de solide amorphe.

L'introduction de contraintes dans la dynamique vise à reproduire ce blocage géométrique et fait effectivement apparaître un certain nombre de phénomènes observés dans l'étude des systèmes vitreux : temps de relaxation qui divergent plus vite qu'une loi de puissance, relaxation spatialement hétérogène, phénomènes de vieillissement... D'un point de vue plus mathématique, elle fait perdre des propriétés de monotonie qui apparaissent classiquement dans de nombreux systèmes de particules en interaction et l'annulation des taux de transitions due aux contraintes entraîne l'existence de plusieurs mesures invariantes, ce qui demande l'introduction de techniques inédites pour l'étude de ces modèles. Les KCSM (et particulièrement les modèles Est et FA-1f) ont été abondamment étudiés dans la littérature physique, en particulier numériquement. Cependant la rapide divergence des temps de relaxation rend les estimations de résultats asymptotiques à basse température difficiles à réaliser et peu fiables. Par exemple, des estimées sur le temps de relaxation ainsi que la prédiction d'une séparation des échelles de temps dans le modèle Est et d'une violation fractionnaire de la relation de Stokes-Einstein se sont révélées fausses avec une étude mathématique rigoureuse ([CMRT08, CFM12, Blo13b]).

Le premier problème auquel j'ai été confrontée concernait la relaxation hors équilibre du modèle FA-1f. Plus précisément, il s'agit de la relaxation à l'équilibre quand on part d'une mesure initiale loin de l'équilibre. Habituellement, la relaxation hors équilibre est étudiée à l'aide de la constante log-Sobolev, son inverse contrôlant la vitesse de relaxation. Cependant, elle est infinie pour les modèles qui nous intéressent. Il faut donc développer des outils spécifiques pour analyser ce régime. Pour le modèle Est, un outil spécifique (le zéro distingué) permet de résoudre ce problème (voir Section 2.1.3). Pour les modèles où on ne peut pas définir un outil du même genre, la question reste largement ouverte. En collaboration avec Nicoletta Cancrini, Fabio Martinelli, Cyril Roberto et Cristina Toninelli, nous avons élaboré une stratégie permettant de traiter le cas du modèle FA-1f pourvu que la densité ne soit pas trop élevée, et que la configuration initiale compte assez de vides. C'est l'objet de l'article [BCM+13], Annexe A, présenté plus en détail en Section 2.1.2, et dont le résultat central est le théorème qui suit (qui peut être énoncé plus généralement sur des graphes à croissance polynomiale ou pour des modèles dits non-coopératifs – cf Section 1.1.3). Ce théorème est le premier résultat de relaxation hors équilibre d'un KCSM fondamentalement différent du modèle Est.

Théorème 0.0.1 *Considérons le modèle FA-1f sur $\mathbb{Z}^d$ à densité p. Soit ν une mesure de probabilité initiale sur Ω. Soit μ la mesure produit sur Ω de densité p. On suppose*

1. $p < 1/2$
2. $\sup_{x\in\mathbb{Z}^d} \nu\left(\theta^{d(x,\{\textit{zéros de }\eta\})}\right) < \infty$ *pour un certain* $\theta > 1$.

Alors pour toute fonction locale f il existe une constante $0 < c < \infty$ telle que

$$|\mathbb{E}_\nu [f(\eta(t))] - \mu(f)| \leqslant c\|f\|_\infty \begin{cases} e^{-t/c} & si\ d = 1 \\ e^{-\left(\frac{t}{c\log t}\right)^{1/d}} & si\ d > 1. \end{cases} \tag{2}$$

La preuve repose sur un résultat général permettant de contrôler la relaxation vers l'équilibre d'une chaîne de Markov grâce à une chaîne auxiliaire (appelée chaîne chapeau, ou "hat chain" dans l'article) qui est définie en supprimant les transitions menant hors d'un certain ensemble $\mathcal{A}$. Les termes qui interviennent dans ce contrôle mettent en jeu le trou spectral et la constante log-Sobolev de la chaîne auxiliaire ainsi que la probabilité pour la chaîne de départ de sortir de l'ensemble $\mathcal{A}$. L'idée est de choisir $\mathcal{A}$ de façon à ce que la constante log-Sobolev de la chaîne auxiliaire soit suffisamment petite et que la chaîne de départ reste dans $\mathcal{A}$ avec bonne probabilité.

Je me suis ensuite intéressée au modèle Est, qui est connu pour présenter un comportement complexe. Mon angle d'attaque était le suivant : lorsqu'on observe l'évolution du système sous la dynamique du modèle Est au cours du temps, on observe la formation de "bulles" de particules (voir Figure 1.2). Celles-ci sont véritablement des structures dynamiques (puisque la mesure d'équilibre est sans interaction), dont l'apparition est directement liée à la contrainte qui empêche certaines transitions. Pour comprendre un peu mieux la forme de ces bulles, j'ai effectué en quelque sorte un zoom sur le bord de l'une d'elle (voir Figure B.1), ce qui revient à étudier la progression d'un zéro ne voyant que des particules dans la direction ouest : le front. Plus précisément, on part d'une configuration sur $\mathbb{Z}$ dans laquelle tous les sites à gauche de l'origine sont occupés par des particules et l'origine est vide, et on appelle X_t la position du zéro le plus à gauche à l'instant t. Les principaux résultats de [Blo13a], présenté en Annexe B et en Section 2.2, sont un théorème de forme pour la progression du front et un résultat d'ergodicité du processus vu du front.

Théorème 0.0.2 *1. Il existe une constante $v < 0$ telle que pour toute configuration initiale η dans laquelle tous les sites à gauche de l'origine sont occupés par des particules et l'origine par un zéro*

$$\frac{X_t}{t} \underset{t\to\infty}{\longrightarrow} v \quad \textit{en probabilité.} \tag{3}$$

2. Le processus vu du front admet une unique mesure invariante ν, et le processus vu du front partant de η converge en loi vers ν pour toute configuration initiale η comme ci-dessus.

La difficulté fondamentale de ce travail vient du fait que la dynamique Est n'est pas attractive, ce qui interdit l'utilisation des arguments de sous-additivité qui sont habituellement centraux dans la preuve de théorèmes de forme. Plutôt que de la sous-additivité, j'utilise en fait un résultat de relaxation loin du front (Théorème 2.2.2) qui signifie essentiellement que sous de bonnes hypothèses, à distance L derrière le front, la loi de la configuration vue du front à l'instant t est à distance au plus $O(e^{-\epsilon L})$ de la mesure produit de densité p, au sens de la variation totale. Pour démontrer ce dernier théorème, je suis amenée à utiliser de façon assez fine des résultats de relaxation hors équilibre et le zéro distingué, qui est un outil central dans l'étude du modèle Est. Une fois ce résultat établi, on peut séparer le processus vu du front en deux parties : l'une éloignée et bien connue car proche de la mesure d'équilibre ; l'autre proche et mal comprise, mais moralement de petite taille. Comme cette dernière partie est finie, en attendant suffisamment longtemps tout événement finira par s'y produire. Tout le défi est donc de gérer l'interaction entre les deux parties, sachant qu'un déplacement du front (qui ne dépend que des sites proches du front) a des répercussions à l'infini sur la configuration vue du front puisqu'il translate toute la configuration.

Enfin, les derniers résultats présentés pour cette thèse (en Annexes C et D et en Section 3.2) concernent le comportement d'un traceur qu'on injecte dans un KCSM. Plus précisément, dans un environnement dynamique donné par un KCSM à l'équilibre, on ajoute une particule qui essaie de suivre une marche aléatoire simple, mais n'est autorisée à sauter qu'entre deux sites vides. En utilisant des méthodes classiques, on peut montrer que la trajectoire du traceur convenablement renormalisée converge vers un mouvement brownien avec coefficient de diffusion $D > 0$ qui peut s'exprimer à l'aide d'une formule variationnelle. L'environnement ne sent pas le traceur, mais comprendre le déplacement de celui-ci permet d'avoir des informations sur le système dans lequel il vit. On cherche en particulier des informations sur le comportement asymptotique de D à basse température, c.-à-d. quand $q \to 0$. En fait, comme je l'ai déjà mentionné, la question qui intéresse les physiciens est celle d'une éventuelle violation de la relation de Stokes-Einstein à basse température. Cette relation met en jeu deux quantités : le temps de relaxation de l'environnement τ (inverse du trou spectral) et le coefficient de diffusion du traceur. Elle décrit de manière adéquate ce qui se passe dans les liquides homogènes et prend la forme

$$D \approx \tau^{-1}. \tag{4}$$

Dans de nombreux systèmes vitreux, cette relation n'est plus satisfaite, $D\tau$ augmentant de plusieurs ordres de grandeur quand la température décroît. Une relation qui coïncide bien avec les observations correspond à une violation fractionnaire de la relation de Stokes-Einstein qui prend la forme

$$D \approx \tau^{-\xi}, \quad \xi < 1. \tag{5}$$

Cette violation est interprétée comme un marqueur d'hétérogénéités dynamiques, un phénomène fondamental dans les systèmes vitreux. Il est donc crucial de vérifier quels modèles proposés pour l'étude de ces systèmes présentent une telle violation. En particulier, mon objectif principal était de vérifier si des prédictions physiques faites dans [JGC04, JGC05] sur la base de simulations en une dimension pour les modèles Est et FA-1f à basse température étaient correctes. Ces prédictions sont les suivantes. Dans le modèle FA-1f, $D \sim q^2$ quand $q \to 0$ en toute dimension (rappelons que q est la densité de zéros), et au vu de résultats sur le trou spectral (rappelés en Section 3.1) cela conduit à $\xi = 2/3$ en dimension 1 et $\xi = 1$ pour $d \geq 2$. Dans le modèle Est, les auteurs prédisent $D \approx \tau^{-\xi}$ pour $\xi \approx 0.73$.

La prédiction pour le modèle FA-1f s'avère correcte, et je montre en fait plus généralement pour le modèle "k-zéros", dans lequel la contrainte est d'avoir au moins k sites vides à distance entre 1 et k, le théorème suivant (notons que pour $k = 1$ on retrouve le modèle FA-1f).

Théorème 0.0.3 *Il existe $C > 0$ ne dépendant que de la dimension tel que, si $u \in \mathbb{R}^d$ et $\|u\| = 1$, alors*

$$C^{-1} q^{k+1} \leqslant u.Du \leqslant C q^{k+1}. \tag{6}$$

La preuve de la borne inférieure passe par une comparaison avec une dynamique auxiliaire décrite en Section 3.2, celle de la borne supérieure par le choix d'une fonction test appropriée dans la formule variationnelle évoquée plus haut. Ces techniques se généralisent à tout modèle non-coopératif (voir la définition en Section 1.1.3), même si le théorème n'est énoncé que dans le cadre "k-zéros".

Dans le modèle Est en revanche, je montre le théorème suivant.

Théorème 0.0.4 *Il existe $C, \alpha > 0$ des constantes telles que*

$$C^{-1} q^2 \,\mathrm{gap} \leqslant D \leqslant C q^{-\alpha} \,\mathrm{gap}, \tag{7}$$

où gap *désigne le trou spectral du modèle Est.*

Il a par ailleurs été montré que gap tendait vers 0 plus vite que toute loi de puissance quand $q \to 0$ ([AD02]) et par conséquent le théorème ci-dessus est incompatible avec la conjecture physique 5 qui impliquerait $D \approx \text{gap}^{\xi}$, $\xi < 1$. La borne inférieure est un résultat général, valable pour tout KCSM, qui me permet en particulier de montrer que $D > 0$ dans la région d'ergodicité. La borne supérieure est établie d'une part en analysant les barrières d'énergies que le traceur doit franchir pour avancer, et surtout en remarquant que ce sont les mêmes que celles qui gouvernent le comportement asymptotique du trou spectral quand $q \to 0$. D'autre part on utilise le fait que l'environnement vu du traceur a un trou spectral supérieur à celui de l'environnement pour trouver des décorrélations dans la dynamique.

Chapter 1

Introduction

1.1 Definition and first properties of KCSMs

I start with the definition of the models I studied during my thesis. I discuss physical motivations for such a definition in the next section.

I am interested in Kinetically Constrained Spin Models, which I will from now on denote by KCSMs. They are interacting particle systems on a discrete space $\Omega = \{0,1\}^V$, where V is the set of vertices of a given graph (which for us will be $\mathbb{Z}^d$). They have (non-conservative) Glauber dynamics: the transitions are creations/destructions of particles rather than jumps of particles from one site to the other. KCSM are called so because of the following specificity: to request the satisfaction of a local constraint in order to create or destroy a particle at a site x. The constraint is of the type "there are enough zeros around x". Moreover, the transition rates are chosen so that the detailed balance condition is satisfied w.r.t. the product Bernoulli measure of density $p \in (0,1)$ on Ω. KCSMs are therefore reversible w.r.t. this measure. The introduction of constraints in the dynamics leads to untypical behaviours as well as mathematical challenges which call for original methods.

A great number of different KCSMs can be defined by changing the underlying graph and the constraints. The behaviours induced by the choice of different constraints can be qualitatively quite different, which gives room for plenty of investigation. The results of this thesis mainly concern the two most famous and most studied KCSMs: the East model and the FA-1f model (and more generally non-cooperative models). These two models display very different behaviours, especially at low temperature. In the following paragraphs I start with a rather general description of KCSMs before giving more details about the specificities of the East and FA-1f models.

In the whole document, $p \in (0,1)$ denotes the equilibrium density parameter of the model; $q = 1 - p$ will be called density of zeros or vacancies. The vertices or sites of the underlying graph will typically be called x, y or z; the configurations in $\Omega = \{0,1\}^V$ will have names such as ω, η or σ. ω_x will denote the occupation variable of site x in the configuration ω, $\omega_x = 1$ meaning that x is occupied by a particle or filled, $\omega_x = 0$ that x is empty or that there is a zero at x. $\eta(t)$ will denote the state of the system at time t if the initial configuration was η. Measures on Ω will be called μ, ν or π.

1.1.1 General description of KCSMs

I keep this paragraph as little technical as possible, which possibly makes it a bit informal at times. I consider only the case where the system lives on $\mathbb{Z}^d$, *i.e.* the state space of the processes

I consider is $\Omega = \{0,1\}^{\mathbb{Z}^d}$. KCSMs are Markov processes on Ω that are defined by (1) a set of constraints and (2) the density parameter $p \in (0,1)$. At rate one each site tries to update its occupation variable, becoming filled with probability p and empty with probability q, but is only allowed to do so if the local constraint is satisfied. The constraints are encoded by a collection $(c_x(\eta))_{x\in\mathbb{Z}^d,\eta\in\Omega}$ with values in $\{0,1\}$. $c_x(\eta) = 1$ if the constraint at x is satisfied in configuration η and $c_x(\eta) = 0$ else. In turn, this collection has to satisfy a number of conditions to define a KCSM:

- the constraints are finite range: $c_x(\eta)$ depends only on a fixed finite neighbourhood of x;
- the constraint at x does not depend on the current configuration *at site* x ($c_x(\eta)$ does not depend on η_x);
- the constraints are translation invariant;
- if η' has more zeros than η ($\eta' \leqslant \eta$ pointwise) and the constraint at x is satisfied in η, then the constraint at x is also satisfied in η' (more vacant sites help satisfy constraints).

These constraints being fixed, the generator of the process is

$$\mathcal{L}f(\eta) = \sum_{x\in\mathbb{Z}^d} c_x(\eta)(p(1-\eta_x) + q\eta_x)\left[f(\eta^x) - f(\eta)\right], \tag{1.1}$$

where η^x is the configuration η flipped at x defined by

$$\eta^x_y = \begin{cases} 1-\eta_x & \text{if } y = x \\ \eta_y & \text{if } y \neq x. \end{cases} \tag{1.2}$$

In other words, provided the constraint is satisfied, an empty site is filled with rate p and an occupied site is emptied with rate q. With general and classic arguments (see [Lig85]) one can define the Markov process associated to this generator. In a more elementary and constructive way, starting from a configuration $\eta \in \Omega$, one can also describe the dynamics of the system using the following graphical representation. With every $x \in \mathbb{Z}^d$ independently we associate a Poisson process with parameter 1 that will be called the (Poisson) clock at x. The process can then be constructed in the following way.

- Check the constraint: if the clock at site x rings at time t, look at the constraint at x in $\eta(t)$, the configuration at time t.
- If $c_x(\eta(t)) = 1$ the constraint is satisfied and the occupation variable at site x is replaced by a Bernoulli variable of parameter p independent of all the rest. The ring at time t is said to be a *legal ring*.
- If $c_x(\eta(t)) = 0$, the constraint is not satisfied and the system is left unchanged.

In the sequel I will write

$$P_t f(\eta) = \mathbb{E}_\eta\left[f(\eta(t))\right] \tag{1.3}$$

for a function f in the domain of $\mathcal{L}$, where $\eta(t)$ is the configuration obtained at time t by the graphical construction starting from η and $\mathbb{E}$ is the expectation w.r.t. the Poisson clocks and Bernoulli variables. Abusively I will also write $\mathbb{E}_\pi\left[f(\eta(t))\right]$ for $\pi\left(\mathbb{E}_\eta\left[f(\eta(t))\right]\right)$, where π is a probability measure on Ω.

One remark about the above construction: it may seem ill-defined. Indeed, in order to construct the process by hand using this recipe, one needs to know in which state the system is at any time of ring. Thus one would *a priori* have to look for the first time a clock rings, make the corresponding update, then look for the following ring and so on. Yet this first ring does not exist, nor does the "following clock ring", because the times of first ring on every site form an infinite family of i.i.d. exponential variables. This difficulty is resolved by a classical argument which shows that the above recipe gives indeed a rigorous construction of a continuous time process. It is sometimes known as Harris' percolation argument ([Har72, Lig04]) and I write it only in the case when $c_x(\eta)$ depends only on x's nearest neighbours. Let us first construct the process up to a fixed time t small enough. Then by Markov property we can iterate the construction to get a process in infinite time. For $t > 0$ small enough, almost surely the sites whose clock has rung before t do not percolate so that around the origin there is a chain of sites that are fixed up to time t (and similarly at every point in $\mathbb{Z}^d$). Consequently we can partition $\mathbb{Z}^d$ into finite clusters whose boundaries are fixed up to time t: the connected components of the percolation cluster of parameter $1 - e^{-t}$. It is then possible to define up to time t the process on the finite clusters, using the sites that have not rung as boundary conditions. The juxtaposition of these well-defined finite volume dynamics gives the desired process up to time t.

Of course (and I already used this fact) the graphical construction also allows to define a process on $\{0,1\}^\Lambda$, $\Lambda \subset \mathbb{Z}^d$: all we have to do is fix boundary conditions. For instance, if $d = 1$, $\Lambda = \{0, \dots L\}$ and $c_x(\eta) = 1 - \eta_{x+1}$ (this is the constraint in the East model, which we will study in more details in the next paragraph) and we fix as boundary condition $\eta_{L+1} = 1$, the process will be frozen up to the first zero to the left of $L+1$ in the initial configuration. Indeed, if $\eta_{|\{l+1,\dots L\}} \equiv 1$ and $\eta_l = 0$, at no time can a constraint be satisfied in $\{l, \dots L\}$ and the process will therefore evolve as a system on $\{0, \dots l-1\}$ with zero boundary condition, meaning that $l-1$ is never constrained.

Remark 1.1.1 *In the physics and mathematics literature one can find another convention for the definition of the models: the roles of zeros and ones can be reversed (the terminology would then rather be that an occupied site corresponds to an excitation). This choice is of course indifferent for the mathematical study but can be confusing to the unaware reader.*

A special measure is associated in a natural way to KCSMs with parameter p: μ the product Bernoulli measure with density p on Ω. In the sequel μ will be referred to as the equilibrium measure.

Property 1.1.2 *Let μ (resp. μ_Λ, $\Lambda \subset \Omega$) the product Bernoulli measure with density p on Ω (resp. on $\{0,1\}^\Lambda$). The process described by* (1.1) *(resp. the process on Λ with zero boundary condition) is reversible w.r.t. μ (resp. μ_Λ).*

Proof

We can easily check the detailed balance condition for the rates $r(\eta \to \eta^x) = p(1-\eta_x) + q\eta_x$. But the additional constraint factor $c_x(\eta)$ does not depend on η_x by assumption: $c_x(\eta) = c_x(\eta^x)$. Therefore detailed balance is still satisfied when the constraint is added, this fact relying crucially on the non-dependence of the constraint at x on the state of site x itself. ✓

μ is not the only invariant or even reversible measure for the process. For instance the measure $\delta_{\mathbf{1}}$ that only gives weight to the entirely occupied configuration is also reversible. Depending on the choice of the constraints there can be other more complicated examples of invariant measures. At the end of Section 2.1.3 I describe thoroughly the invariant measures of the East model.

Another general property of KCSMs is that of *finite speed of propagation*. Essentially it says that information propagates at most linearly in time: up to time t the origin does not see what

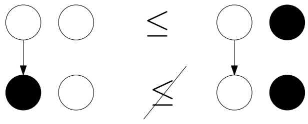


Figure 1.1: Occupied sites are denoted by black disks, empty sites by white disks. On the first line are represented part of two configurations on $\mathbb{Z}$, the one on the right being above the other. On the second line we see the outcome of the following transition: the first clock rings on the left site and the associated Bernoulli variable is 1. The constraint requests that the nearest neighbour on the right be empty (this is the constraint of the East model). This transition does not preserve the order between the configurations.

happens outside a ball of size of order t. In particular, if one looks at what happens to the origin up to time t it makes little difference to consider the infinite volume dynamics or the dynamics restricted to a ball of size of order t with given boundary condition. This property is crucial in [BCM+13] where we use this reduction to finite volume dynamics, as well as in [Blo13a] where it allows to find independence between remote events. More precisely the property can be written as follows. Let $B(x, r)$ be the ball centred at x with radius r for the graph distance.

Proposition 1.1.3 *Let l be a positive integer. There exists a constant $\overline{v} < \infty$ such that for any $t > 0$, on an event of probability at least $1-e^{-t}$ which does not depend on the initial configuration, the process restricted to $B(0, l)$ up to time t depends only on the initial configuration, Poisson clocks and Bernoulli variables inside $B(0, \overline{v}t)$.*

Indeed in order for the exterior of $B(0, \overline{v}t)$ and $B(0, l)$ to communicate before time t, there has to be a path linking the boundary of $B(0, \overline{v}t)$ and that of $B(0, l)$ on which every clock has rung before time t (at least when the constraint is nearest-neighbour). An estimate on the number of jumps of a Poisson process allows to conclude.

Remark 1.1.4 *Contrary to other classic interacting particle systems KCSMs are not attractive. Put another way, using the same Poisson clocks and Bernoulli variables to construct the processes started from two configurations $\eta' \leqslant \eta$ does not imply that $\eta'(t) \leqslant \eta(t)$: the order between configurations is not conserved by the dynamics. This is not surprising once one notices that if a configuration has more zeros than another, it is allowed to create more particles. See Figure 1.1 for an example of a non order-preserving transition. In the same spirit, we cannot compare the realizations of a KCSM with two different densities: if $\eta(p, t)$ is the evolution at time t starting from η with density p and if we take $p' < p$, we do not know how to construct simultaneously the processes $(\eta(p, t))_{t\geq 0}$ and $(\eta(p', t))_{t\geq 0}$ in such a way that $\eta(p', t) \leqslant \eta(p, t)$ for all $t > 0$.*

1.1.2 The East model and its distinguished zero

The East model is a one-dimensional KCSM ($\Omega = \{0, 1\}^{\mathbb{Z}}$). In this specific model, the constraint is satisfied at x iff the neighbour on the right of x (or to the East) is vacant[1]. Namely

$$c_x(\eta) = 1 - \eta_{x+1}. \tag{1.4}$$

See Figure 1.2 for an illustration of the East process at different densities.

[1]In the literature the orientation is often reversed so that the constraint looks at the left neighbour.

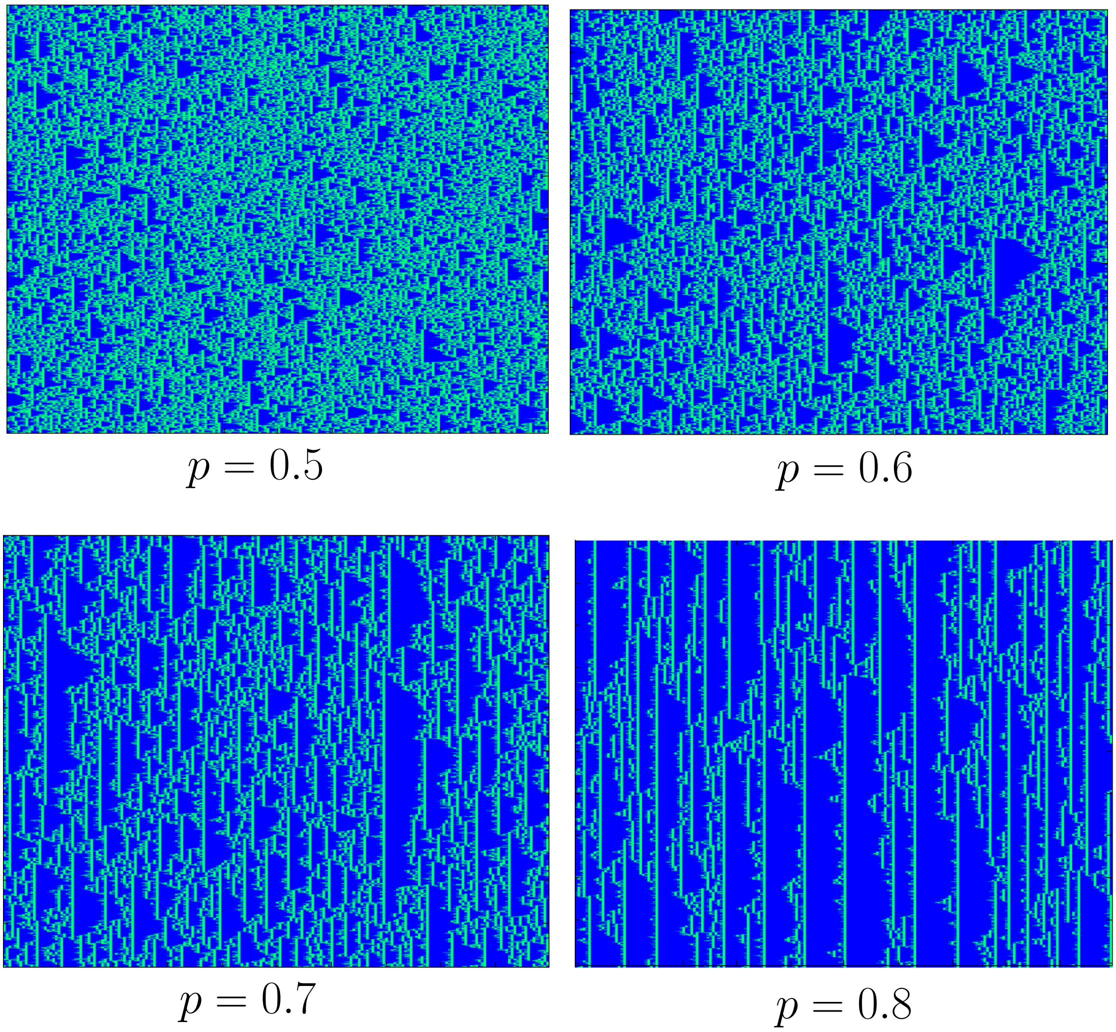


Figure 1.2: Simulations of the East model at different densities p, courtesy of Arturo Leos Zamorategui. The space $\mathbb{Z}$ is represented horizontally and time vertically. Occupied sites are in dark blue and vacant sites in bright green. The system is at equilibrium: for every panel the initial configuration is chosen with law $\mu \sim \mathcal{B}(p)$. 200 sites are represented up to time $t = 2000$.

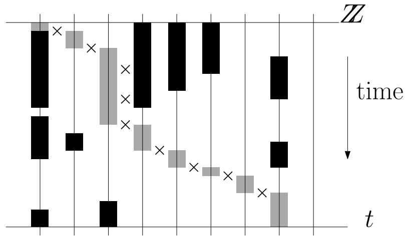


Figure 1.3: In grey, a trajectory of a distinguished zero up to time t; time goes downwards, sites are highlighted in black at the times when they are occupied. The crosses represent the times when the distinguished zero tries to jump to the right, *i.e.* the clock rings at the site occupied by the distinguished zero.

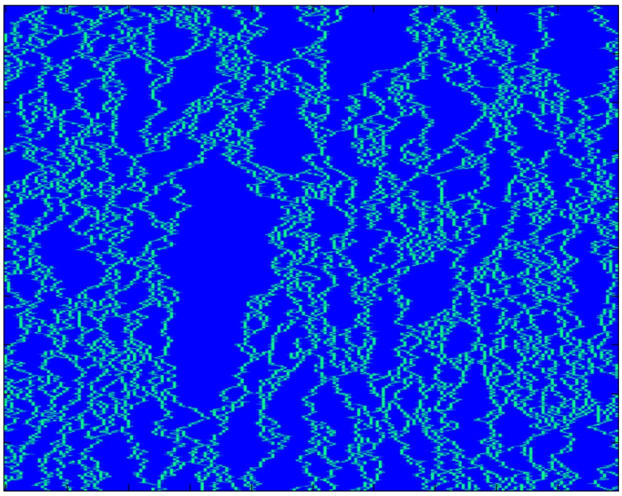
The reader is referred to [FMRT12b] for a recent review of the results obtained for this model. These are far more consistent than what is known for other KCSMs. Indeed, the East model has two specificities that allow to grasp some properties more easily than in other KCSMs: it is one-dimensional and *oriented*. More precisely, the orientation property means that in the above graphical construction the evolution of the process at a given site $x \in \mathbb{Z}$ can be expressed exclusively in terms of the initial configuration, Poisson clocks and Bernoulli variables *on x and on the right of x*. This is of course not the same as saying that there is independence between the dynamics on two disjoint blocks –which is false– but in some sense the dependence goes only one way. This characteristic gives the possibility to define an object which turns out to be quite powerful when studying the East model, the distinguished zero (introduced in [AD02]).

Let us start from an initial configuration η with a zero at $z \in \mathbb{Z}$, *i.e.* $\eta_z = 0$. We decide to make this zero distinguished and let $\xi_0 = z$. The distinguished zero remains fixed until t_0 the time of the first *legal ring* at z: let $\xi_s = z$ for $s < t_0$. Notice that several non-legal rings can occur at z before t_0 but since the constraint is not satisfied, $\eta_{\xi_s}(s) = \eta_z(s) = 0$ for all $s < t_0$. Furthermore notice that at time t_0, since the ring at z is legal by definition, there is a zero at $z+1$ which we can make our new distinguished zero: let $\xi_{t_0} = z+1$. Now we can iterate the procedure: wait for t_1 the first legal ring at $z+1$ happening after t_0 and let $\xi_s = z+1$ for $t_0 \leqslant s < t_1$, $\xi_{t_1} = z+2$, and so on. In this way we define a càdlàg trajectory $(\xi_s)_{s\geq 0}$ on $\mathbb{Z}$ always jumping to the right and always sitting on a zero of the system: $\eta_{\xi_s}(s) = 0$ for all $s \geq 0$. See Figure 1.3 for an example of a distinguished zero trajectory.

The distinguished zero is an interesting object because it gives a conditional decoupling property similar to that of orientation. In fact, conditional on the trajectory $(\xi_s)_{s\geq 0}$, the dynamics on the left of this trajectory does not depend on the dynamics on the right. In particular the following conditional stationarity property is true: if initially there is equilibrium to the left of the distinguished zero, when it moves to the right it leaves equilibrium on its left.

Proposition 1.1.5 (Lemma 4 in [AD02], Lemma 3.5 in [CMST10]) *Let Λ a subset of $\mathbb{Z}$ of the form $\Lambda = \{x_-, \ldots, x_+\}$ or $\Lambda = \{\ldots, x_+ - 1, x_+\}$. Assume the initial configuration is sampled after μ_Λ on Λ and it has a zero at $x_+ + 1$. Make this zero distinguished and let $\xi_0 = x_+ + 1$. Conditional on the trajectory of the distinguished zero $(\xi_s)_{s\geq 0}$, at any time $s \geq 0$ the law of $\eta(s)$ on $\Lambda_s := \Lambda \cup \{x_+ + 1, \ldots, \xi_s - 1\}$ is given by μ_{Λ_s}.*

Indeed, as long as the distinguished zero does not jump the system on its left behaves like a system with a zero boundary condition and equilibrium is conserved. Moreover when the distinguished zero jumps to the right, the occupation variable of the site it just left is updated to a Bernoulli(p) independently from the rest.



$p = 0.8$

Figure 1.4: Simulation of the FA-1f model in dimension 1 at density $p = 0.8$. The setting is the same as in Figure 1.2 and the picture is also the work of Arturo Leos Zamorategui.

The distinguished zero will be a fundamental tool in Sections 2.1.3, B.4 and C.5.

1.1.3 FA-1f and other non-cooperative models

Apart from the East model, the most famous KCSM is the Fredrickson-Andersen model FA-1f (for "Fredrickson-Andersen one-spin facilitated model"). In that case the constraint is satisfied at x iff x has at least one vacant nearest neighbour. Put another way, for the FA-1f model

$$c_x(\eta) = 1 - \prod_{y \sim x} \eta_y. \tag{1.5}$$

See Figure 1.4 for an illustration in dimension 1.

This model has more symmetries than the East model: it is invariant by permutation of the base vectors $\{e_1, -e_1, \dots, e_d, -e_d\}$. It is also *non-cooperative*, in the sense that it is enough to have a fixed finite set of zeros in the initial configuration to be able to empty any site in $\mathbb{Z}^d$ using only flips authorized by the constraints. For FA-1f in fact, one zero is enough. More formally, one can give the following definition of non-cooperative models.

Definition 1.1.6 *A KCSM is non-cooperative if there exists a finite set S of $\mathbb{Z}^d$ with the following property: if $\eta_y = 0$ for all $y \in S$, then for any $x \in \mathbb{Z}^d$ there exists a finite sequence $\eta^{(0)}, \dots, \eta^{(n)} \in \Omega$ such that $\eta^{(0)} = \eta$, $\eta_x^{(n)} = 0$ and $\eta^{(i+1)} = \left(\eta^{(i)}\right)^{x_i}$ for some $x_i \in \mathbb{Z}^d$ where $c_{x_i}\left(\eta^{(i)}\right) = 1$. A set S satisfying the previous condition is called a* seed. *Models that are not non-cooperative are called cooperative.*

The name "non-cooperative" comes from the fact that in such a model there is no need for an infinite number of zeros to cooperate to be able to empty any arbitrary site. On the contrary, the East model is cooperative: if there are only finitely many zeros in the initial configuration, no site on the right of the rightmost zero can ever have its constraint satisfied. The generalization of the FA-1f model, FA-jf, for which the constraint requires at least j empty nearest neighbours, is also cooperative for $j = 2, \dots, 2d$. Note that the seed S in the definition is not unique; in particular, since the constraints are translation invariant, any translation of S is again a seed.

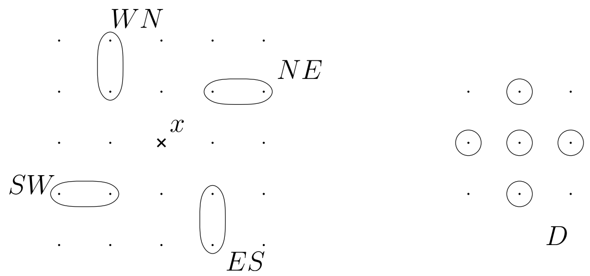


Figure 1.5: The windmill constraint requests that one of the circled groups of two sites in the left panel be empty. In the right panel, a seed for the windmill model.

A great number of non-cooperative models can be defined. However the general belief is that their behaviour should not be qualitatively different from that of FA-1f. In [Blo13b] I studied more precisely some of them: the k-zeros models and the windmill model. In the first case, the constraint is to have at least k zeros among the sites at distance between 1 and k. A group of k neighbouring zeros is enough to empty the whole lattice. In the second case, which is a model on $\mathbb{Z}^2$, we request that one of the circled groups of two sites in Figure 1.1.3 be empty; a possible seed for the windmill model is a group of five zeros in shape of a diamond (right of Figure 1.1.3).

1.2 Physical motivations

KCSMs appeared in the physics literature in the context of the study of the glass transition. This area of research offers a huge and varied literature that goes much beyond the scope of KCSM. I do not pretend to give here a general presentation of the phenomena and questions related to the glass transition. Rather, I explain some elements that are relevant to understand the definition of KCSMs and the physical relevance of the questions addressed mathematically in the sequel. The reader wishing for a more sound discussion on glass transition is referred to [Cav09] or [BB11].

The understanding of the liquid/glass transition and of the glass material itself is still a challenge for physicists. The most remarkable feature of the glass is that it is solid on human scale but does not show any microscopic regularity; it can be called an amorphous solid. In fact glass and liquid are so far undistinguishable on the basis of a single picture. When a liquid is cooled down to form a glass, one cannot easily characterize the transition point as the one where a significant change in the microscopic structure occurs. In particular the very definition of a critical temperature separating the liquid and glass phases is not trivial. Several remarkable phenomena have been observed in this context, of which I cite only a few: fast divergence of relaxation times close to the transition, aging ([Bou00]), dynamical heterogeneities – namely the occurrence of spatially correlated regions of high and low mobility that persist for a finite lifetime and that grow in size as one approaches the glass transition (see [Ber11a] for a recent experimental capture of this feature)– and decoupling of the relaxation time and diffusion coefficient ([EEH+12, CE96, CS97, SBME03]), etc. One of the major challenges of the study of glassy systems is to come up with models that capture these phenomena and make an interpretation of their microscopic origin possible.

In this context the definition of KCSMs comes from the specific point of view of facilitation ([Gla60, CG10]). The general idea is that when a liquid is cooled down rapidly, its density increases and geometric constraints cause jamming at a microscopic level and therefore a dramatic slowdown of the dynamics, such that a crystal can never form. Facilitation then suggests that the system can only evolve through some more mobile or less jammed regions that can unjam neighbouring regions. These in turn become mobile and so on. Along these lines, in KCSMs a coarse-grained mobile region is modelled by a site that favours transitions (*i.e.* an empty site in the above terminology) and a jammed region by a blocking site (or occupied). The philosophy or belief that one finds behind the definition of KCSMs is that facilitation alone can explain most of the phenomena observed in liquids close to the glass transition and that by contrast the thermodynamic properties of the system play little role. In KCSMs this last feature is in fact pushed to the point where there is no interaction between particles under the equilibrium measure since μ is product (see however [CMRT09] for results on constrained models with a weak interaction). From this point of view, KCSMs have the advantage that their simple description gives access to a detailed study of how the microscopic interactions can cause divergence of the relaxation times ([AD02, CMRT08, CMRT07, CMST10, BT13]), aging ([FMRT12a]) and dynamical heterogeneities ([BT12]). However, simulations of KCSMs are not easy to implement efficiently enough, which led to a few inaccurate predictions concerning their behaviour, especially at low temperature. On the other hand, some features of KCSMs such as degenerate transition rates and non-attractiveness induce several challenges for their mathematical study. Let us finally stress that KCSMs are also relevant in the study of other complex disordered systems where facilitation plays a role such as colloidal suspensions, foams, emulsions... For a more detailed and backed up discussion on the role of KCSMs in the study of the glass transition, see [RS03, TGS].

The first class of models introduced in the physics literature gave as a constraint "to have at least j mobile neighbours" ([FA84, FA85]), $j = 1, ...d$. For $j = 1$ we recover the FA-1f model introduced above. This one is used to model strong liquids *i.e.* systems in which the relaxation time displays Arrhenius behaviour near the transition, meaning that a good fit for the relaxation time is $\exp(cst/T)$, T being a reduced temperature, which corresponds to $q^{-\alpha}$ (see (1.7) below). The other FA-jf models are cooperative models with more relation to fragile liquids, in which the relaxation time is super-Arrhenius (diverging faster than any polynom in q). In [JE91], the authors introduce the East model, in which the breakdown of symmetry creates cooperative effects in the dynamics and a clear hierarchical structure in the dynamics at low temperature that is studied mathematically with detail in [FMRT12c, FMRT12a]. Let me make a quick comment about the East model: the asymmetry of the constraint and the one-dimensionality may not seem very natural, but experiments (see for instance [DDK+98, AGS+03, WCL+00]) suggest that in real systems facilitation occurs in a "stringlike" fashion, in a single direction. Simulations on atomistic models (with more natural interaction assumptions) confirm the spontaneous emergence of this phenomenon ([KHG+11]).

So far we considered only non-conservative models. A family of conservative models has also been introduced, called Kinetically Constrained Lattice Gases (KCLG). In this class of models, transitions correspond to jumps of a particle from one site to another, the jumps being authorized only if a specific constraint is satisfied around the initial and final site of the jump. In this category, one can list for instance the Kob-Andersen models (KA) or Triangular Lattice Gases (TLG). Their study is also very rich, has been given much attention and has many connections with KCSMs, but I only focused on the latter in this thesis so I will make no further reference to KCLGs.

In the sequel, I will use without distinction the terms "high density" and "low temperature". Indeed, the equilibrium density p and the reduced temperature of the system T are linked by the following relation

$$p = \frac{1}{1 + e^{-\beta}}, \tag{1.6}$$

where $\beta = 1/T$ is the inverse temperature. In terms of the density of zeros, this reads

$$q = \frac{1}{1 + e^{\beta}}, \tag{1.7}$$

therefore $q \to 0$ when $T \to 0$ and at low temperatures $T \sim (\log(1/q))^{-1}$.

1.3 Ergodicity and equilibrium relaxation

In Chapter 2, I will come back to the question of relaxation when the system starts out of equilibrium, which can correspond for instance to a rapid cool-down (Section 2.1) or a closer investigation of the structure of the dynamics (Section 2.2). In Chapter 3, I review what can be said about characteristic times (relaxation time, persistence time and diffusion coefficient) at low temperature, *i.e.* when $q \to 0$. Dynamical heterogeneity, characterized by the existence of active and less active regions which increase in size as $q \to 0$, will be a transversal point of interest during these investigations. Before turning to these issues, let us see how to identify the ergodicity regime of KCSMs and what can be said about equilibrium relaxation.

1.3.1 Bootstrap percolation and ergodicity

Due to the presence of degenerate transition rates, classic arguments to show ergodicity of the process ([Lig85]) do not apply to KCSMs. In [CMRT08], the authors identify the ergodicity region of KCSMs and characterize it in terms of sub-critical regime of a certain *bootstrap percolation*, associated to the specific constraints of the model. Let us first define this percolation. Fix a KCSM with constraints $(c_x(\eta))_{\eta\in\Omega, x\in\mathbb{Z}}$ and define the bootstrap map $B : \Omega \to \Omega$ by

$$B(\eta)_x = \begin{cases} 0 & \text{if } \eta_x = 0 \text{ or } c_x(\eta) = 1 \\ 1 & \text{else.} \end{cases} \tag{1.8}$$

In words, the effect of this application is to empty the sites everywhere it is allowed by the constraints. If we apply it infinitely many times starting from μ, we get a limit distribution on Ω. The criterium for ergodicity is then whether or not this limit distribution is $\delta_{\mathbf{0}}$, the distribution that charges only the empty configuration.

Proposition 1.3.1 *[CMRT08] The KCSM with constraints $(c_x(\eta))_{\eta\in\Omega, x\in\mathbb{Z}}$ at density p is ergodic iff, starting from μ the Bernoulli(p) product measure, almost surely the origin is emptied after finitely many iteration of the bootstrap procedure.*

Here ergodic means for instance that if $f \in L^2(\mu)$ is invariant for the equilibrium dynamics (in the sense that for all $t > 0$ $\mathbb{E}_\eta[f(\eta(t))] = f(\eta)$ $d\mu(\eta)$-a.s.), then it is constant μ-a.s., *i.e.* 0 is a simple eigenvalue of $\mathcal{L}$. Equivalently,

$$Var\left(\mathbb{E}_\eta\left[f(\eta(t))\right]\right) \underset{t\to\infty}{\longrightarrow} 0 \qquad \forall f \in L^2(\mu). \tag{1.9}$$

Let us look at this proposition in more detail. If there is a positive probability that the origin is still occupied after infinitely many applications of B, it means that μ-a.s. there is somewhere in the lattice a blocked cluster, *i.e.* a collection of occupied sites such that the constraint is satisfied on none of these sites, even if the rest of the configuration is empty. For instance, in the East model a blocked cluster would appear if the set $\{L, L+1, \ldots\}$ was entirely occupied. It is not difficult to see that if a blocked cluster exists with positive probability the model is not ergodic. Conversely, in [CMRT08] the authors showed that if every site can be emptied through the bootstrap procedure almost surely, there is ergodicity. The main idea is to reconstruct a given flip using a finite sequence of allowed flips, which is possible if there is no blocked cluster. Bootstrap percolation has been studied in several settings, which allows to identify the ergodicity regime for a large class of models (see for instance [Sch92]). Note that blocked clusters can be more complex to identify than in the case of the East or FA-1f model. For instance, in the FA-2f model on $\mathbb{Z}^2$, the constraint requires at least two empty nearest-neighbours. One of many possible blocked clusters is then an occupied double line.

Corollary 1.3.2 *All non-cooperative models, as well as the East model, are ergodic at every density $p < 1$.*

Indeed for the East model at any density $p < 1$ there are μ-almost surely infinitely many zeros on the right of the origin, which means no blocked cluster. For non-cooperative models, take S as in Definition 1.1.6. μ-a.s. there is somewhere on the lattice a translation of S that is empty, which by Definition 1.1.6 means there is no blocked cluster.

Note that another corollary of Proposition 1.3.1 is that if a KCSM is ergodic at a density p, then it is also ergodic at any density $p' < p$. The remarkable thing is that, due to lack of attractiveness, I do not know a direct proof of this fact even though it looks rather natural since more zeros should facilitate relaxation. In consequence, for every KCSM there exists a critical density $p_c \in [0, 1]$ such that the model is ergodic when $p < p_c$ and non-ergodic when $p > p_c$. Corollary 1.3.2 implies that $p_c = 1$ for the East model and non-cooperative models. In turn, for other models $p_c \in (0, 1)$. For instance, the North-East model is a KCSM on $\mathbb{Z}^2$ where the constraint is satisfied iff the North and East neighbours are empty. In that case, p_c coincides with the critical parameter for oriented percolation on $\mathbb{Z}^2$.

Let us now turn to the question of the speed of convergence in (1.9).

1.3.2 Spectral gap

Let us recall the definition of the spectral gap.

Definition 1.3.3 *The spectral gap is the infimum of all non-zero eigenvalues of $-\mathcal{L}$. Equivalently*

$$\text{gap} = \inf_{f \text{ non cst}} \frac{\mathcal{D}(f)}{Var(f)}, \tag{1.10}$$

where the variance is taken w.r.t. μ and

$$\mathcal{D}(f) = -\mu(f\mathcal{L}f) = \frac{1}{2}\sum_{x\in\mathbb{Z}^d} \mu\left(c_x(\eta)(p(1-\eta_x) + q\eta_x)\left[f(\eta^x) - f(\eta)\right]^2\right) \tag{1.11}$$

is the Dirichlet form associated to $\mathcal{L}$ and μ.

The spectral gap is a non-negative quantity which is the best constant such that

$$Var\left(\mathbb{E}_\eta\left[f(\eta(t))\right]\right) \leqslant e^{-2t\,\text{gap}} Var(f) \quad \forall f \in L^2(\mu) \tag{1.12}$$

In particular, if gap > 0, the decay in (1.9) is exponential and the typical time is given by gap^{-1}. Thus the inverse of the spectral gap will be called the *relaxation time* in accordance to the physical denomination. Let me informally state a theorem on the question of the positivity of the spectral gap (see [CMRT08] for a precise statement).

Theorem 1.3.4 *[CMRT08] For a large class of KCSMs, including East and non-cooperative models, the spectral gap is positive in the whole ergodic regime (except maybe at the critical density).*

Note that this result was already known for East since the paper [AD02]. The proof in [CMRT08] relies on a bisection-renormalization technique designed for the setting of KCSMs, but later used in other contexts. A similar result was also shown in [CMRT08] for the persistence time, *i.e.* the typical time during which the origin doesn't flip. More precisely, it was shown that the quantity $\mathbb{P}_\mu\left(\eta_0(s) = \eta_0 \quad \forall s \leqslant t\right)$ decays exponentially fast as soon as the spectral gap is positive. The typical scale of this decay is the persistence time. In Section 3 we will see that even though the spectral gap and inverse of persistence time are positive in the ergodicity regime, close to the critical density they quickly become very small. A consequence of this fact is that simulations are not always reliable. For instance, physics papers predicted a stretched exponential decay of the persistence function.

A corollary of (1.12) is exponential decay of correlations on time scale gap^{-1}. Namely, if $f, g \in L^2(\mu)$, then by Cauchy-Schwarz inequality

$$|\mu\left(fP_t g\right) - \mu(f)\mu(g)| \leqslant \sqrt{Var(f)Var(g)}e^{-t\,\mathrm{gap}}. \tag{1.13}$$

Chapter 2

Out-of-equilibrium dynamics

2.1 Out-of-equilibrium relaxation

The previous chapter presented results about the equilibrium dynamics of KCSMs, *i.e.* when the initial configuration is chosen with the equilibrium distribution μ. A natural question is what happens when the initial distribution is different. In particular, what can we say about the dynamics of a KCSM started from μ' a product Bernoulli measure with parameter $p' \neq p$? Physically, this corresponds to changing abruptly the temperature of the system, a setting which has been investigated in several numerical and experimental works. Mathematically, very few results are available in this direction.

2.1.1 Preliminary remarks

In order to study equilibrium dynamics, as seen above, the spectral gap is the relevant constant to consider. Indeed, for KCSM, the L^2 behaviour of the dynamics started from equilibrium is controlled by the spectral gap. The constant that usually allows to get similar results on dynamics started far from equilibrium is the log-Sobolev constant

$$\alpha = \sup_{f \text{ non cst}} \frac{Ent_\mu(f)}{\mathcal{D}(f)}, \tag{2.1}$$

where $Ent_\mu(f) = \mu(f \log f) - \mu(f)\log(\mu(f))$ denotes the entropy of f. Roughly speaking, if $\alpha < \infty$, then using the hypercontractivity property we can show exponential relaxation with speed $1/\alpha$ to μ starting from any initial distribution ([ABC+00]; see also Appendix A). Unfortunately, for KCSMs we have no hope of α being finite, be it only because starting from the entirely occupied configuration, the system can never evolve. In fact, for the East model it was shown in detail (see [FMRT12b], Section 3.3) that even weaker log-Sobolev-like constants are infinite in infinite volume.

Relevant questions concerning out-of-equilibrium relaxation are therefore: under what conditions on the density and the initial configuration is there out-of-equilibrium relaxation for a given KCSM? One would expect that if the initial configuration or distribution has enough zeros in some sense (at the very least there should be no blocked clusters) and if the density p is in the ergodic regime, then

$$|\mathbb{E}_\eta \left[f(\eta(t)) \right] - \mu(f)| \underset{t \to +\infty}{\longrightarrow} 0 \quad \text{for any local function } f. \tag{2.2}$$

In fact, very few results are known in this direction. For the East model, as we will see in Section 2.1.3, the distinguished zero saves the day and allows to show out-of-equilibrium relaxation at any density under minimal hypotheses on the initial configuration and with exponential decay. An analogue of the distinguished zero can also be defined for an oriented model on the binary tree: the AD model (see Section 4 in [CMST10]), which allows to prove similar results for this model. For FA-1f and other non-cooperative models, this is no longer possible and we can only show out-of-equilibrium relaxation for small enough densities (this is the object of Section 2.1.2). There is currently no other result in this direction for more general models, with the exception of a perturbative result (when the initial configuration is close to μ) in one dimension which I now state.

Theorem 2.1.1 *[CMST10, Theorem 1] Let $\mathcal{L}$ be the generator of a KCSM in dimension $d = 1$ with positive spectral gap. Then there exist constants $\lambda < \infty$, $m > 0$ such that if ν is a probability measure on Ω satisfying*

$$\sup_l \max_{\eta_{-l},\dots,\eta_l} e^{-\lambda l} \frac{\nu(\eta_{-l}\dots,\eta_l)}{\mu(\eta_{-l}\dots,\eta_l)} < \infty \tag{2.3}$$

and f a local function, then there exists $C_f < \infty$ such that

$$\int d\nu(\eta) \left|\mathbb{E}_\eta\left[f(\eta(t))\right] - \mu(f)\right| \leqslant C_f e^{-mt}. \tag{2.4}$$

2.1.2 Local relaxation in the FA-1f model

In the following paragraph, we will see that for the East model we can answer the question of the out-of-equilibrium relaxation in an almost optimal way with the help of the distinguished zero. The models in which one can define an analogue of the distinguished zero are the only ones for which available results are so complete. The only other class of models for which we can give a partial answer is that of non-cooperative models. This is a joint work with Nicoletta Cancrini, Fabio Martinelli, Cyril Roberto and Cristina Toninelli ([BCM+13]), which has been accepted for publication in *Markov Processes and Related Fields* and is presented in Appendix A. The main result is the following (Theorem A.2.1), stated here in lesser generality (only on $\mathbb{Z}^d$, while the techniques allow to treat any graph with polynomial growth). The proof could also be adapted to more general non-cooperative models.

Theorem 2.1.2 *Consider the FA-1f model on $\mathbb{Z}^d$ with density p. Let ν be a probability measure on Ω. Assume*

1. $p < 1/2$
2. $\sup_{x\in\mathbb{Z}^d} \nu\left(\theta^{d(x,\{zeros\ of\ \eta\})}\right) < \infty$ *for some* $\theta > 1$

Then for any local function f there is a constant $0 < c < \infty$ such that

$$\left|\mathbb{E}_\nu\left[f(\eta(t))\right] - \mu(f)\right| \leqslant c\|f\|_\infty \begin{cases} e^{-t/c} & \text{if } d = 1 \\ e^{-\left(\frac{t}{c\log t}\right)^{1/d}} & \text{if } d > 1 \end{cases} \tag{2.5}$$

We make two hypotheses in this theorem: one concerning the density and the other concerning the zeros of the initial configuration. The latter means that the maximal distance between a site in $\mathbb{Z}^d$ and the nearest zero in the initial configuration has a non-trivial exponential moment. This is satisfied if for instance ν is Bernoulli with parameter $p' \neq p$, $p' < 1$ (which is physically particularly relevant), and also if ν is a Dirac on a configuration with zeros separated by uniformly bounded distances. The main improvement to this theorem would be to get rid of the first hypothesis and extend it to any $p < 1$. In fact, in dimension 1 we were able to push the condition to $p < 4/5$ (at least). I will comment a bit further on what we would need to control better in order to have the result for any $p < 1$. The second possible room for improvement is we only get stretched exponential relaxation in dimension $d > 1$. We do not expect this to be the correct behaviour and would rather predict exponential relaxation in all dimensions. Let me now try to explain the strategy we follow. All details, along with additional results, can be found in Appendix A.

The first step is to reduce the dynamics essentially to finite volume using finite speed of propagation. For $t > 0$ let $\Lambda = B(0, \overline{v}t)$. Let $\eta^\Lambda(t)$ be the configuration obtained at time t starting from η with the FA-1f dynamics in volume Λ with zero boundary condition. The finite speed of propagation property implies that it is enough to show the theorem with $\eta^\Lambda(t)$ instead of $\eta(t)$ if $\overline{v}$ is large enough (recall Proposition 1.1.3). From now on, we work with the FA-1f dynamics in volume Λ with zero boundary condition.

As I said above, the log-Sobolev constant is infinite in infinite volume, but one can also consider the log-Sobolev constant associated to our new dynamics in finite volume Λ with zero boundary condition

$$\alpha_\Lambda = \sup_{f \text{ non cst}} \frac{Ent_\mu(f)}{\mathcal{D}_\Lambda(f)}, \tag{2.6}$$

where the supremum is taken over non constant functions on $\{0,1\}^\Lambda$, and $\mathcal{D}_\Lambda(f)$ is the Dirichlet form associated to the dynamics in volume Λ with zero boundary condition. This constant satisfies the following inequality for f with support in Λ ([HS87, SZ92])

$$\left|\mathbb{E}_\nu\left[\eta^\Lambda(t)\right] - \mu(f)\right| \leqslant \|f\|_\infty e^{-t\,\mathrm{gap}_\Lambda/2}\left(\frac{1}{p \wedge q}\right)^{|\Lambda|\exp(-2t/\alpha_\Lambda)}, \tag{2.7}$$

where gap_Λ denotes the spectral gap of the FA-1f dynamics in volume Λ with zero boundary condition (which is larger than gap the spectral gap in infinite volume). This is something, but α_Λ scales like $|\Lambda|$ so that when $|\Lambda|$ grows faster than t (which is our case), the above inequality is not very useful. We need something a bit more clever.

Now cut Λ into smaller boxes of diameter $\epsilon\,(t/\log t)^{1/d}$ (ϵt for $d = 1$) and consider the event $\mathcal{A}_t$ that for any $s \leqslant t$, there are at least two zeros in each of the smaller boxes. On this event the process $\left(\eta^\Lambda(s)\right)_{s\leqslant t}$ is the same as $\left(\hat{\eta}^\Lambda(s)\right)_{s\leqslant t}$ obtained with the following dynamics: start with a configuration with at least two zeros in each of the smaller boxes and make it evolve with the FA-1f dynamics in volume Λ with zero boundary condition, suppressing the updates which lead to a configuration that does not have at least two zeros in each box.

In Section A.3 we design a general argument to control the relaxation to equilibrium of a Markov chain in terms of an auxiliary chain (the "hat chain") defined by suppressing the jumps that lead out of a certain set $\mathcal{A}$. The main control terms involve the spectral gap and log-Sobolev constant of the auxiliary chain and the probability of the original chain escaping $\mathcal{A}$. This is Proposition A.3.1.

$$|\mathbb{E}_\nu(f(X_t))| \leq |\hat{\pi}(f)| + 4||f||_\infty \mathbb{P}_\nu(\mathcal{A}_t^c) + ||f||_\infty \exp\left\{-\hat{\gamma}\frac{t}{2} + e^{-\frac{2t}{\hat{\alpha}}}\log\frac{1}{\hat{\pi}^*}\right\}, \tag{2.8}$$

where $(X_s)_{s\geq 0}$ is the original chain with equilibrium distribution π on state space S, $\hat{\pi}$ is the equilibrium distribution of the hat chain, $\hat{\gamma}$ and $\hat{\alpha}$ are respectively the spectral gap and log-Sobolev constant of the hat chain, $\mathcal{A}_t = \{X_s \in \mathcal{A},\ \forall s \leq t\}$ and $\hat{\pi}^* := \min_{x\in S}\hat{\pi}(x)$. In order to use this inequality efficiently, one has to find a set $\mathcal{A}$ with two properties: that the log-Sobolev constant of the auxiliary chain be suitably small and that the original chain stays in $\mathcal{A}$ with good probability. In the context of out-of-equilibrium relaxation for FA-1f, the special set $\mathcal{A}$ is $\{$*there are at least two zeros in each of the smaller boxes*$\}$. The log-Sobolev constant of the hat chain is then showed to be of order the size of the smaller boxes: $\epsilon^d t/\log t$ (ϵt for $d=1$). For ϵ small enough, this takes care of the last term in (A.10).

What remains is to control the probability of escaping the special set $\mathbb{P}_\nu(\mathcal{A}_t^c)$. This is the object of Section A.4, and it is where all the assumptions in Theorem 2.1.2 are needed. Let us see why in dimension 1. We want to show that with high probability the maximal distance between two zeros is small enough (of order ϵt) up to time t. Consider the FA-1f dynamics on $\mathbb{N}$ with *occupied* boundary condition (the boundary site being -1). Call ζ_t the position at time t of the zero closest to the origin. It is enough for our purpose to show that $u(t) = \mathbb{E}_\nu\left[\theta^{\zeta_t}\right]$ grows subexponentially for some $\theta > 1$. Roughly speaking, computing the derivative of u gives

$$\begin{aligned} u'(t) &\simeq \mathbb{E}_\nu\left[\theta^{\zeta_t}\left(q\left(\frac{1}{\theta}-1\right) + p\left(1-\eta_{\zeta_t+1}(t)(\theta-1)\right)\right)\right] && (2.9)\\ &\lesssim u(t)(\theta-1)\left(p-q/\theta\right), && (2.10)\end{aligned}$$

where the best control we are able to get on the first line is $\eta_{\zeta_t+1}(t) \geq 0$. We need $p<q$ to have $p-q/\theta \leqslant 0$ for some $\theta>1$. In dimension 1, we can improve this condition by computing further derivatives of u.

2.1.3 Relaxation in the East model

In the East model, contrary to FA-1f, we have the special help of the distinguished zero, which turns out to be a powerful tool in the study of out-of-equilibrium dynamics. The following result was proved in [CMST10] (the statement below is not exactly the one given in the paper, but can easily be extracted from the proof of Theorem 3.1).

Theorem 2.1.3 *[CMST10] Let η be a configuration with a zero in $z\in\mathbb{Z}$ and f a local function with support in $\{x_-,\dots,x_+\}$, $x_+<z$. Then we have for any density $p<1$*

$$\left|\mathbb{E}_\eta\left[f(\eta(t))\right]-\mu(f)\right| \leqslant \sqrt{Var(f)}\left(\frac{1}{p\wedge q}\right)^{z-x_-} e^{-t\,\mathrm{gap}}. \qquad (2.11)$$

In particular, there is relaxation to equilibrium in the sense of (2.2) with exponential decay as soon as the initial configuration has an infinite number of zeros to the right of the origin. Building on this result, the authors are also able to prove in [CMST10] that a result of the form (2.4) holds in the East model for ν the Bernoulli product measure with parameter $p'<1$ ([CMST10, Theorem 3.2]).

Let me give a sketch of the techniques used to prove Theorem 2.1.3 before stating a result that extends it. The main idea is to distinguish the zero that is initially in z (recall Section 1.1.2) and notice that between the jumps of the trajectory $(\xi_t)_{t\geq 0}$ the system in $\{x_-,\dots,\xi_t-1\}$ follows the East dynamics in finite volume with boundary condition zero. Suppose that the initial configuration in $\{x_-,\dots,z-1\}$ is at equilibrium $\mu_{\{x_-,\dots,z-1\}}$. Then between the jumps of the distinguished zero we can use the spectral gap of East in finite volume with boundary condition zero to relax. Since the spectral gap of East in finite volume with zero boundary condition is larger than gap (the spectral gap in infinite volume), combining the relaxations between two jumps of the distinguished zero, we can show exponential decay in mean w.r.t. $\mu_{\{x_-,\dots,z-1\}}$ (any distribution at equilibrium in $\{x_-,\dots,z-1\}$). The additional factor $1/(p\wedge q)^{z-x_-}$ appearing in Theorem 2.1.3 comes from the cost of changing the initial configuration in $\{x_-,\dots,z-1\}$ and putting it at equilibrium.

This is a very useful result which I used several times in [Blo13a]. Its main restriction is that the prefactor $1/(p \wedge q)^{z-x_-}$ depends on the support of f and the distance between zeros in the initial configuration, and that it can be quite large. Therefore Theorem 2.1.3 only gives relevant information for times t large enough to overcome this term. An even more obvious restriction is the requirement that the initial zero be outside the support of f, which is essential in the proof. During my work on a shape theorem for the East model (Appendix B), I was led to improve the statement of Theorem 2.1.3 in order to apply it to functions with large support (growing with t for instance). This is Proposition B.4.3 in Appendix B, although it is stated a bit differently there to fit the later use I make of it in the paper.

Proposition 2.1.4 *Let $\eta \in \Omega$ be such that it has a zero at $z > 0$ and f be a bounded function with support in $\mathbb{N}$. Then*

$$\left|\mathbb{E}_\eta\left[f(\eta(t)) - \mu_{\{0,\dots,z-1\}}(f)(\eta(t))\right]\right| \leqslant \sqrt{2}\|f\|_\infty \left(\frac{1}{p \wedge q}\right)^z e^{-t\,\mathrm{gap}}, \tag{2.12}$$

where $\mu_{\{0,\dots,z-1\}}$ denotes the mean w.r.t. the Bernoulli product measure with parameter p on $\{0,\dots,z-1\}$, so that $\mu_{\{0,\dots,z-1\}}(f)(\eta)$ is a function of $\eta_{|\{z,z+1,\dots\}}$.

The additional difficulty here with respect to Theorem 2.1.3 is that the support of f does not need to be on the left of the distinguished zero and therefore $(\xi_t)_{t\geq 0}$ depends on $(f(\eta(t)))_{t\geq 0}$. To overcome this difficulty, I need to define carefully a conditioning by the whole dynamics on the right of a distinguished zero instead of just conditioning on its trajectory. See Step 1 in the proof of Proposition B.4.3 for the details. Then I check that the computations used in [CMST10] to prove Theorem 2.1.3 can be carried through.

It may not seem obvious why this allows to show relaxation on a volume growing with time. In fact, this proposition is designed to be used iteratively. Start with a configuration η with zeros in $z_1, z_2, \dots$ and let f be a local function with support in, say, $\{0,\dots,z_n - 1\}$. Apply Proposition 2.1.4 with $z = z_1$. It shows that $\mathbb{E}_\eta\left[f(\eta(t))\right]$ is close to $\mathbb{E}_\eta\left[\mu_{\{0,\dots,z_1-1\}}(f)(\eta(t))\right]$ with error at most $\sqrt{2}\|f\|_\infty \left(\frac{1}{p\wedge q}\right)^{z_1} e^{-t\,\mathrm{gap}}$. Now notice that $\mu_{\{0,\dots,z_1-1\}}(f)$ is a function with support in $\{z_1,\dots,z_n-1\}$. Apply again the proposition with this function and $z = z_2$, and iterate. The final difference between $\mathbb{E}_\eta\left[f(\eta(t))\right]$ and $\mu(f)$ is now at most

$$\sqrt{2}\|f\|_\infty e^{-t\,\mathrm{gap}} \sum_{i=1}^{n} \left(\frac{1}{p \wedge q}\right)^{z_i - z_{i-1}},$$

where we defined $z_0 = 0$. The thing to notice here is that the sum above does not grow exponentially with the size of the support of f, but only in the maximal initial distance between two zeros in the support of f and linearly in the size of the support of f. This is a substantial improvement if the initial configuration has enough zeros.

To conclude this section, let us notice that Theorem 2.1.3 allows to characterize all invariant measures for the East model. For $n \in \mathbb{Z}$, call $\mu.\mathbf{1}_n$ the product measure on Ω such that $\mu.\mathbf{1}_n(\eta_x)$ is 1 if $x > n$, 0 if $x = n$ and p if $x < n$, *i.e.* a configuration with law $\mu.\mathbf{1}_n$ is at equilibrium on the left of n, has a zero in n and is entirely occupied on the right of n. Then the only invariant measures for East are $\mu, \delta_1, \mu.\mathbf{1}_n, n \in \mathbb{Z}$ and convex combinations of these. To prove this, consider ν an invariant measure for East. ν is characterized by its effect on local functions, and for any local function f and any time t, $\nu(f) = \nu\left(\mathbb{E}_\eta\left[f(\eta(t))\right]\right)$ since ν is invariant. On the event that η has infinitely many zeros on the right of the origin, $\mathbb{E}_\eta\left[f(\eta(t))\right] \underset{t\to+\infty}{\longrightarrow} \mu(f)$ because of Theorem 2.1.3. On the event that $\eta \equiv 1$, $\mathbb{E}_\eta\left[f(\eta(t))\right] = \delta_1(f)$. And on the event that the rightmost zero in η is at n, again Theorem 2.1.3 allows to say that $\mathbb{E}_\eta\left[f(\eta(t))\right] \underset{t\to+\infty}{\longrightarrow} \mu.\mathbf{1}_n(f)$.

2.2 Bubbles and front

So far we have studied out-of-equilibrium relaxation locally, in the sense that we took an out-of-equilibrium initial condition, picked any finite window and tried to prove that soon enough the system in the window is close to equilibrium. Now I would like to report works and open problems about a somewhat different non-equilibrium issue, very much related to dynamical heterogeneity.

Let us take a look again at the dynamics represented in Figures 1.2 and 1.4. The structures that we see being more and more present when density increases are often referred to as bubbles. Notice that these are strictly dynamical features: recall that the simulations presented here are at equilibrium, so that at a given time (along a horizontal line) there is no interaction between the sites. What happens is that zeros cannot appear in the middle of ones, so that when a long string of ones forms (which is bound to happen somewhere under the product measure) it can only disappear if zeros coming from its endpoints manage to join. These bubbles are regions of zero activity: no flip is allowed inside them, whereas other regions are more active and sites flip back and forth. The activity and dynamics of the system are clearly heterogeneous. Understanding the shape of the bubbles is therefore a crucial issue in the understanding of KCSM dynamics.

A naturally related problem is the following. Consider for instance the East dynamics and start with a configuration fully occupied on the negative half-line. This effectively puts the system out of equilibrium. Now consider the left-most zero in the system. It will move exactly as if it were following the border of a bubble (see Figure B.1). Therefore we would like to describe its motion, which is what I studied in [Blo13a], where the left-most zero is called the front. Before stating my results I can actually prove, let me mention other problems this question is related to.

In [GJL+09, GJL+07] it was proved that the large deviations of the activity (*i.e.* the number of flips) exhibit a non-equilibrium phase transition. This is due to the fact that the constraints allow to devise a strategy that freezes the dynamics for a time t in a box of size N at a cost which is subexponential in Nt: start from an all-occupied configuration and prevent the boundary sites from ringing. In [BT12] the authors analyse the finite size effects around this transition and evidence coexistence between active and inactive phases. Heuristically and in vague terms, in the regime where the activity is forced to be smaller than typical, the dominant inactive phase should be separated by an interface similar to a front and the cost of maintaining this inactive phase should be related to a surface tension of the interface (Figure 2 in [BT12]). This picture is supported by further analysis and numerical simulations ([BLT12]).

Another question with much similarities with the front progression was raised in [KL06] concerning the North-East model. Recall that in this two-dimensional model the constraint requests that both the East and North neighbours be empty (although the orientation is reversed in [KL06]). This model is ergodic for $p < p_c$, where p_c is the critical parameter of oriented percolation in $\mathbb{Z}^2$. Consider the dynamics in the south-west quadrant with zero boundary condition and initial configuration entirely filled. For a time $t \geq 0$ let R_t be the union of unit squares centred around sites that have flipped at least once by time t. The conjecture of [KL06] is that influence propagates at linear rate. More precisely that for $p < p_c$ there is a deterministic shape $S \subset \mathbb{Z}^2$ such that $R_t/t \underset{t\to\infty}{\longrightarrow} S$ (see Figure 1 in [KL06]). A first step in the direction of proving this conjecture could be to show that the mixing time on $\{-n, \dots, -1\}^2$ with zero boundary condition is of order n. A result in this direction was proved in [CM12], establishing that the mixing time is at most of order $n \log n$, but the conjecture remains open.

Let me now present the results of [Blo13a]. Start from a configuration entirely occupied on the negative half-line, with a zero at the origin and an arbitrary configuration on the positive half-line, and run the East dynamics described by the graphical representation. The left-most zero of the system, which we call the front, makes only nearest-neighbour jumps. Indeed, zeros cannot appear in the middle of ones so the only way the front can move to the left is when the site on its left flips. In the other direction, in order for the front to move to the right, the site where it sits has to flip, which requires having a zero immediately to its right, so that the jump is only of length one. Call X_t the position of the front at time t. What we are looking for is a shape theorem for the border of a bubble: we expect X_t to grow linearly with a negative coefficient, and this is what I showed in Theorem B.6.1.

Theorem 2.2.1 *There exists $v < 0$ such that for any initial configuration η occupied on the negative half-line with a zero at the origin,*

$$\frac{1}{t} X_t \underset{t\to+\infty}{\longrightarrow} v \quad \textit{in } \mathbb{P}_\eta\textit{-probability.} \tag{2.13}$$

The proof of this kind of result usually relies on subadditivity. Let me sketch the proof in the case of the contact process on $\mathbb{Z}$ ([Dur80]). Recall that in this model the ones represent infected sites and zeros healthy sites. The transition rates of a flip at site x in configuration η are the following

$$1 \to 0 \quad \text{with rate} \quad 1 \tag{2.14}$$

$$0 \to 1 \quad \text{with rate} \quad \lambda \sum_{y\sim x} \eta_y. \tag{2.15}$$

In words, an infected site is cured at rate 1 and a healthy site is infected at a rate proportional to the number of infected neighbours. The proportionality constant λ is a parameter of the model. The process can be constructed using a graphical construction: attach to every site a "curing" Poisson clock with parameter one, and to every oriented edge an "infection" clock with parameter λ and construct the process in the natural way. When a curing clock rings at x, this site becomes healthy and when an infection clock rings on the oriented edge (x, y), if x is infected at that time, y becomes infected. In particular, one can define the basic coupling between two contact processes started from η, η' by constructing $(\eta(t))_{t\geq 0}, (\eta'(t))_{t\geq 0}$ with the same clocks. The contact process has the crucial property that it is attractive, contrary to KCSMs. Starting from two configurations $\eta \leqslant \eta'$ (*i.e.* every site infected in η is also infected in η'), the basic coupling gives two ordered processes: $\forall t \geq 0 \quad \eta(t) \leqslant \eta'(t)$.

Now consider $\eta^{\mathbf{1}}(t)$ a contact process started from a configuration healthy on the negative half-line and infected on the origin and the positive half-line and call $X^{\mathbf{1}}_t$ the position of the left-most infected site at time t[1]. See Figure 2.1. $\mathbf{1}$ denotes the initial configuration, with x infected iff $x \geq 0$. Let $0 < s < t$ and consider the configuration at time s. It has a left-most infected site at $X^{\mathbf{1}}_s$ by definition, and on its right a mixture of infected and healthy sites. We do not know much about this configuration, but it is certainly below (has fewer infected sites) than $\mathbf{1}$ translated by $X^{\mathbf{1}}_s$. Consider now the basic coupling between the contact process started from $\mathbf{1}$ and the one started from the configuration $\eta^{\mathbf{1}}(s)$ translated by $X^{\mathbf{1}}_s$. Attractiveness shows that $X^{\mathbf{1}}_t - X^{\mathbf{1}}_s \geq \tilde{X}^{\mathbf{1}}_{t-s}$, where $\tilde{X}^{\mathbf{1}}_t$ has the same distribution as $X^{\mathbf{1}}_t$ (an infection started from all non-negative sites infected propagates further than an infection started from any other configuration with negative sites healthy). This is the central hypothesis of the subadditive ergodic theorem, which gives immediately the convergence of $X^{\mathbf{1}}_t/t$ when $t \to \infty$.

[1] I apologize for the somewhat reversed convention on the roles of zeros and ones with respect to the East model setting.

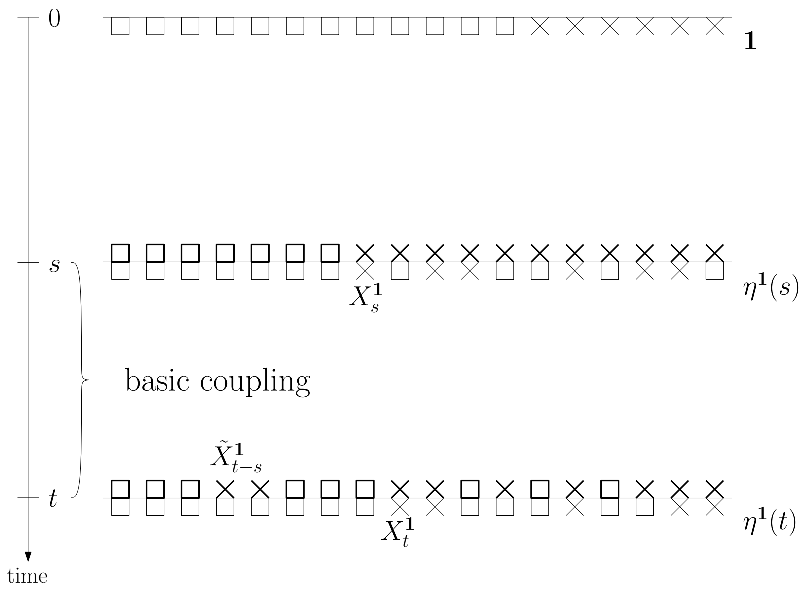


Figure 2.1: Infected sites are represented by crosses, healthy sites by squares. The configurations represented below the lines at times $0, s, t$ are respectively $\mathbf{1}, \eta^{\mathbf{1}}(s), \eta^{\mathbf{1}}(t)$. At time s, the configuration $\mathbf{1}$ translated by $X_s^{\mathbf{1}}$ is represented above the line and dominates $\eta^{\mathbf{1}}(s)$ by definition of $X_s^{\mathbf{1}}$. Between times s and t, the basic coupling is applied to the processes started from the two configurations. At time t, the configuration $\eta^{\mathbf{1}}(t)$ (represented below the line) is dominated by the one which started from $\mathbf{1}$ translated by $X_s^{\mathbf{1}}$. In particular, the left-most infected site of the latter, $\tilde{X}_{t-s}^{\mathbf{1}}$, is on the left of $X_t^{\mathbf{1}}$.

The key argument above is that at an intermediate time s, we can start anew from a *known* configuration and still be able to compare the original evolution between time s and t with the evolution in time $t-s$ started from this known configuration. The ingredient that makes this strategy work is attractiveness. Other shape theorems develop much more elaborate and difficult arguments (see for instance [KRS12] for a recent review). However, to my knowledge, they all use some kind of monotonicity of the processes in the initial configuration.

Note that another way of looking at the front is to consider it as a tagged zero in the East process. A number of results concerning the limiting behaviour of a tagged particle have been established, mostly when the initial distribution is invariant for the process seen from the particle ([Fer96]).

In the context of the East model, due to the lack of attractiveness, I could not use the strategy described for the contact process. Instead I had to understand better the *process seen from the front*. Since order of initial configurations gives me no information on the later order of processes with the East dynamics, I need to find another kind of reference to use as a base point or a fresh start to find independence in the process. In fact, the argument of stochastic domination is replaced by the quantification of the return to a reference measure.

To begin with, I prove that far from the front the configuration is almost at equilibrium (see Theorems B.4.4, B.4.7 for a precise statement).

Theorem 2.2.2 *Let $0 < L < M$ be integers, $t > 0$. If t is sufficiently large w.r.t. M or if the initial configuration has enough zeros, then the distribution of the configuration at distance between L and M on the right of the front is at total variation distance from $\mu_{\{L,\dots,M\}}$ at most of order $e^{-\epsilon(L\wedge t)}$, $\epsilon > 0$.*

The proof of this theorem heavily uses the results of Section 2.1.3, first to establish that the front moves at least with linear speed, then to ensure that with high probability there are "many" zeros behind the front ("many" being quantified). Finally Proposition 2.1.4 gives the tool needed to make use of these zeros and relax to equilibrium in short time over a long distance. The precise statement of this theorem is quite heavy because very quantitative. The upside is that it allows for explicit estimates that are rather important for proving the main results of the paper.

Using Theorem 2.2.2 I design a coupling between the processes seen from the front at time t with two different initial configurations (Definition B.5.3 and Theorem B.5.2) so as to prove that their laws converge to the same limit. The issue encountered here is of the same kind as in the papers [KS01, KPS02], although the authors work in a different setting. Roughly speaking (a more detailed discussion can be found in the Introduction of [Blo13a], Appendix B), in both cases we have a distant and large part of our system which we control rather well. In my case, the control is given by Theorem 2.2.2. The remaining part, which for me is the configuration near the front, is unruly but finite or small in some sense. Therefore if one waits long enough, this smaller part should end up doing whatever we want it to do. In fact, taken separately, the far and large part as well as the small and close part are well behaved. However both parts are coupled and the tricky point is the interplay between the two: one has to control the two parts in the same construction. To illustrate the difficulty, let me point out that a jump of the front depends only on the configuration close to it. However it translates the whole configuration seen from the front and therefore has repercussions infinitely far from the front. The construction of the coupling I designed to take care of this issue is rather lengthy, so I do not detail it here (see Section B.5). Let me just say that this construction does not use the orientation of the East model, but would transpose immediately to another one-dimensional system for which we could prove an analogue of Theorem 2.2.2. In particular one gets the following result (Theorem B.5.1)

Theorem 2.2.3 *The process seen from the front is ergodic, in the sense that it has a unique invariant measure ν and the distribution of the configuration seen from the front at time t converges weakly to ν for any initial configuration.*

We know little about ν (which is a measure on Ω such that ν-a.s. the left-most zero is at the origin), except that far on the right of the origin it looks like μ, in the sense of Proposition B.5.5.

Theorem 2.2.3 is not quite enough to show the law of large numbers for X_t since we do not assume the initial distribution to be invariant. Moreover the convergence to the invariant measure established in the proof of Theorem 2.2.3 is quite slow, although it could possibly be improved by a cleverer control on the part close to the front. Finally, the coupling is between the configurations seen from the front at a fixed time, and does not give a coupling of the processes on a time interval. The proof of the law of large numbers relies in fact directly on Theorem 2.2.2 and orientation: we can define auxiliary fronts whose distribution is almost that of a front started with equilibrium on its right (see Figure B.8) and such that they stay on the right of the original front. Theorem 2.2.2 furthermore gives independence between subfamilies of auxiliary fronts (represented by different dash styles in Figure B.8), so that we can design a rather classic proof of a law of large numbers for these auxiliary fronts.

Chapter 3

Low temperature dynamics

I now turn to the analysis of the low-temperature behaviour of KCSM, *i.e.* asymptotic results when $q \to 0$. To fix ideas, recall that when q is small, the typical volume one has to consider in order to find a zero is of the kind $\{x, x+1, \ldots, x+1/q^{1/d}\}^d$. Also, at a site where the constraint is satisfied, zeros appear typically in time $1/q \gg 1$ and disappear in time $1/p \approx 1$. Therefore in the FA-1f and East models zeros tend to be isolated at low temperature.

3.1 Asymptotics for the spectral gap at low temperature

As the temperature goes to 0, so does the spectral gap. Equilibrium relaxation still takes place with exponential decay but the time scales involved diverge because zeros become fewer and thus it is increasingly costly to satisfy a constraint. A simple test function proving this fact is $f(\eta) = \eta_0$, which verifies $\mathcal{D}(f)/Var(f) = \mu(c_0)$, where c_0 is the constraint at 0. Therefore the variational definition of gap (1.10) implies that gap $\leqslant \mu(c_0)$, which goes to zero as soon as the constraint c_0 requires at least a zero in a finite neighbourhood of the origin. This however is a poor lower bound for the divergence of the relaxation time. Indeed, mechanisms more complex than the mere rarefaction of zeros are involved in this divergence, which I explain in the next paragraphs in the cases of the East and non-cooperative models.

3.1.1 Asymptotics for East and energy barriers

In the East model, the cooperative nature of the dynamics plays a crucial role in the divergence of the relaxation time and causes it to be super-Arrhenius, *i.e.* faster than any polynomial in $1/q$. More precisely, it was showed in [AD02, CMRT08] that for all $\delta > 0$, q small enough and $c_\delta > 0$ some positive constant

$$c_\delta \exp\left(-\frac{\log(1/q)^2}{2\log 2 - \delta}\right) \leqslant \text{gap} \leqslant \exp\left(-\frac{\log(1/q)^2}{2\log 2 + \delta}\right). \tag{3.1}$$

Physicists had conjectured the form of the relaxation time $\tau = \text{gap}^{-1} \approx \exp\left(-\left(\log(1/q)\right)^2/cst\right)$ ([RS03, SE99]). However the right value of the constant was first thought to be $\log 2$, based on a heuristic which I explain below. In fact, in [AD02] the authors found the correct upper bound and a lower bound of the form $\exp\left(-\left(\log(1/q)\right)^2/\log 2\right)$. In [CMRT08] the authors refine the lower bound and get the correct constant $2\log 2$. In the Appendix of [CMST10] is an analysis of the extra factor 2 being due to an entropic factor neglected in previous heuristics. The rigorous proofs of (3.1) are a bit involved. Rather than giving an idea of the proof, I would like to explain the heuristics behind the fact that gap $\approx \exp\left(-\left(\log(1/q)\right)^2/(2\log 2)\right)$.

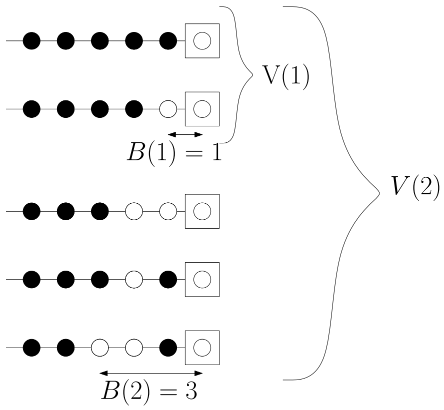


Figure 3.1: An example of the game for $n = 1, 2$. The fixed zero at the origin is squared. All transitions between successive lines are allowed by the East dynamics on the negative half-line with zero boundary condition. In particular $|V(1)| = 2, B(1) = 1, |V(2)| = 5$ and $B(2) = 3$.

The heuristic arguments are based on combinatorial considerations which were proved rigorously in [CDG01]. In the sequel I make no distinction between $1/q$ and its integer part. Consider $\Lambda = \{\dots, -2, -1\}$ the negative half-line and fix a zero at the origin. Let n be a positive integer and play the following game: you have n zeros at your disposal. You are allowed to add them and remove them as you wish *while respecting the East constraint*, *i.e.* only if there is a zero on the right of the site you wish to fill or empty. There can never be more than n zeros on the negative half-line (the zero at the origin is free: it is a boundary condition). The goal of the game is to bring a zero as far left as possible. To match the notations in [CDG01], call $B(n)$ the maximal length you can travel to the left with n zeros, and $V(n)$ the set of configurations you can reach while always respecting the rules. Figure 3.1 shows what you can do for $n = 1, 2$. The results that will be important for us are the following.

Theorem 3.1.1 *[CDG01]*

$$B(n) = 2^n - 1 \tag{3.2}$$

and there are constants $0 < c_1 \leqslant c_2 < 1$ *such that for* n *large enough*

$$2^{\binom{n}{2}} n! c_1^n \leqslant |V(n)| \leqslant 2^{\binom{n}{2}} n! c_2^n \ . \tag{3.3}$$

An element of the proof of Theorem 3.1.1 explaining the form of $B(n)$ is a description of the best strategy for the game. With n zeros, what you need to do is send as far as you can a single zero. Then use it as an anchor from which you send another zero as far as possible using the remaining $n - 1$ zeros, and so on.

Let us see what this implies for the low-temperature East dynamics. At equilibrium, the typical distance between two zeros is $1/q$. Heuristically, the relaxation time in infinite volume should be of the same order as the relaxation time on scale $1/q$ (see Lemma 3.1.3 for a rigorous statement). Consider therefore the East dynamics on $\Lambda_q = \{0, \dots, 1/q\}$ with zero boundary condition and initial configuration equal to zero at the origin and fully occupied on $\{1, \dots, 1/q\}$. The relaxation time is roughly the typical time it takes to turn the zero initially at the origin into a one, or equivalently to turn the one initially at 1 into a zero (there is equivalence because it is very costly for the system to maintain a zero at the origin once the constraint is satisfied there; see [CFM12, Theorem 1] for a precise statement). The rules of our game and Theorem 3.1.1 imply that in order to do this, the system will have to create at least $n := \lfloor \log_2(1/q) \rfloor$ extra zeros and the system on Λ_q will have to visit a configuration in $V(n)$ with n zeros. In other words, the subset of $V(n)$ of configurations with n zeros is a bottleneck for the East dynamics on scale $1/q$. The probability of a configuration with n zeros is of order $q^n = \exp\left(-\log(1/q)^2/\log 2\right)$, which accounts for the strong divergence of gap^{-1}, since the system has to cross this energy barrier to relax. However, as [CMST10] pointed out, one also has to take into account the entropic factor of the number of possible visited configurations, which is bounded from above by $|V(n)|$, *i.e.* $2^{\binom{n}{2}} \approx e^{\log(1/q)^2/(2\log 2)}$ to leading order. This accounts for the extra factor 2 in (3.1).

Another consequence of this combinatorial result is a hierarchical structure in the relaxation of the East model. Roughly speaking, at low temperature and starting with an initial configuration with too many zeros, the system should try to make zeros disappear. However, in order for a given zero to disappear, the first zero on its right has to send extra zeros to the left in order for the constraint to be satisfied. The system has to cross an energy barrier that increases with the distance to the first zero on the right (more precisely with the $\log_2$ of that distance). Therefore, the typical behaviour one should observe is that zeros for which the constraint is satisfied are erased first, then those requiring the creation of an additional zero (i.e zeros with a zero at distance 2 on the right), then those requiring the creation of two additional zeros (i.e zeros with a zero at distance 3 or 4 on the right) and so on. This rough picture was made rigorous in [FMRT12a], where the authors show that the East model at low temperature is well approximated in a very strong sense by a certain hierarchical coalescent process. The properties of the hierarchical coalescent process derived in [FMRT12c] allow the authors to describe a plateau behaviour as well as aging in the East dynamics at low temperature. Let me give an idea of what this means, with much lesser precision and generality than in the paper. Make the system start for instance from a Bernoulli product measure of parameter $1/2$ and consider the behaviour of $\mathbb{P}_{1/2}\left(\eta_0(t)=0\right)$. For any $\epsilon > 0$, in the limit $q \to 0$, this probability remains constant during periods of time of the form $[1/q^{k(1+\epsilon)}, 1/q^{(k+1)(1-\epsilon)}]$ and decreases only during periods of time of the form $[1/q^{k(1-\epsilon)}, 1/q^{k(1+\epsilon)}]$ (see [FMRT12a, Figure 1]). The other main result of this paper is that the auto-correlation function displays aging: the quantity $C(t,s) = \mathbb{E}_{1/2}\left[\eta_0(s)\eta_0(t)\right] - \mathbb{E}_{1/2}\left[\eta_0(s)\right]\mathbb{E}_{1/2}\left[\eta_0(t)\right]$ depends on both t and s, and not just on their difference.

Let me now give more recent statements that refine (3.1). In [CFM12] the authors studied more precisely the dynamics on scale $L \leqslant d/q$ for d any positive constant. Call $T_{\mathrm{rel}}(L)$ the relaxation time (*i.e.* inverse of the spectral gap) on $\{1,\ldots,L\}$ with zero boundary condition.

Theorem 3.1.2 *[CFM12, Theorem 2] Let $d > 0$. There exist $\alpha, \alpha' > 0$ depending only on d such that for any $L \leqslant d/q$*

$$\frac{n!}{q^n 2^{\binom{n}{2}}} q^{\alpha} \leqslant T_{\mathrm{rel}}(L) \leqslant \frac{n!}{q^n 2^{\binom{n}{2}}} q^{-\alpha'}, \quad n = \lceil \log_2(L) \rceil. \tag{3.4}$$

The proof of the lower bound exhibits a better bottleneck than $V(n)$, which significantly reduces the entropic factor. A rigorous description of this bottleneck would be too lengthy to fit here (see [CFM12, Section 5.2]). Let me just say that its definition is again based on the idea that, in a low temperature dynamics (that tends to erase all zeros), the zeros with a zero immediately on their right should disappear first, then on a longer time scale the zeros that have a zero at distance 2 on their right, and so on. Among other results on time-scale separation and dynamical heterogeneity, the authors also show that there is no time scale separation on length $1/q$: more precisely, $\forall d > 0$, $T_{\mathrm{rel}}(1/q) \sim T_{\mathrm{rel}}(d/q)$ ([CFM12, Theorem 4]).

To complete the picture on the relaxation times in the East model at low temperature, here is a result comparing the relaxation time in infinite volume with the one on the typical equilibrium scale $1/q$. This was an intermediate result in [Blo13b] (Lemma C.5.5).

Lemma 3.1.3
$$T_{\mathrm{rel}}(1/q) \leqslant \mathrm{gap}^{-1} \leqslant Cq^{-C}T_{\mathrm{rel}}(1/q), \tag{3.5}$$
for some constant C not depending on q.

Given the order of magnitude of $T_{\mathrm{rel}}(1/q)$, this is saying that the relaxation time in infinite volume is essentially given by the relaxation time on the typical equilibrium scale. The lower bound is trivial: $T_{\mathrm{rel}}(1/q)$ is monotone in L. The proof of the upper bound uses a refinement of the bisection argument used to show the lower bound in (3.1) together with Theorem 3.1.2 and Theorem 4 in [CFM12].

Let me finally point out that all along I have given results only for the relaxation time. In [CFM12], the authors show that for the dynamics in volume $O(1/q)$ with zero boundary condition it makes no difference to consider the relaxation time, the mixing time or the hitting time: the three are equivalent. The hitting time here is the expectation of the first time the zero at the origin is erased when the initial configuration is entirely occupied except at the origin.

3.1.2 Asymptotics for FA-1f and conjectures for non-cooperative models

For the FA-1f model the picture is very different and no longer governed by the crossing of energy barriers that were caused by the cooperative character of the East model. Let me first state a rigorous result on the relaxation time, then explain what the picture should be and give a conjecture on the relaxation time of non-cooperative models.

Theorem 3.1.4 *[CMRT08, Theorem 6.4] There exists a constant C depending only on the dimension d such that for all $q \in (0,1)$*

$$C^{-1}q^3 \leqslant \mathrm{gap} \leqslant Cq^3 \quad \textit{for } d=1 \tag{3.6}$$
$$C^{-1}q^2/\log(1/q) \leqslant \mathrm{gap} \leqslant Cq^2 \quad \textit{for } d=2 \tag{3.7}$$
$$C^{-1}q^2 \leqslant \mathrm{gap} \leqslant Cq^{1+2/d} \quad \textit{for } d \geq 3 \tag{3.8}$$

What does the FA-1f model at low temperature look like? Imagine a single zero in an entirely filled configuration. On time scale shorter than $1/q$, nothing should happen since it is the typical time that the system has to wait before it can create an extra zero on either side of the original one. On time scales between $1/q$ and $1/q^2$ the system should not have the time to create two extra zeros simultaneously. Therefore on these time scales, what can happen is that a zero is created on either side of the original one with equal probability, and either the new zero or the original one is erased in time $O(1)$. Consequently, an approximate description of the FA-1f dynamics up to time $t = 1/q^2$ starting from a single zero should be that of a random walk with diffusion coefficient of order q. Starting with more zeros (say, at equilibrium) we should see coalescing random walks diffusing at rate q. On time scales larger than $1/q^2$, branching should occur since we wait long enough to allow the creation of three neighbouring zeros and the deletion of the middle one, which leads to two isolated zeros. This picture is unfortunately far from being rigorously established. Let us see however what we can expect about the relaxation time if this is true.

In dimension 1, as in the East model, the relaxation time should be governed by the time it takes for two zeros initially at equilibrium distance to communicate. The equilibrium distance is $1/q$, each zero diffuses at rate q, therefore the relaxation time should be $1/q^2 \times 1/q$, which is indeed showed rigorously in Theorem 3.1.4. In higher dimension, there is typically one zero in a box of side $1/q^{1/d}$. If the same argument as in dimension 1 was true, we would get a spectral gap of order $q^{1+2/d}$. However the working conjecture in the physics literature is that gap $\approx q^2$ when $d \geq 2$ ([JMS06]). In [BT13] we propose a more general conjecture, valid for all non-cooperative models (though stated in lesser generality in the paper). Before stating it, let me introduce two more relevant quantities.

For a given non-cooperative model, there is a minimal cardinality for a set S to be a seed (recall Definition 1.1.6). Let us call this number k. From now on, a seed will always have cardinality k. In FA-1f for instance, $k = 1$. For the k-zeros model, consistently $k = k$ (!). For the windmill model, $k = 5$ (the cardinality of the diamond in Figure 1.1.3; one has to check that indeed no set of less than 4 zeros satisfies the conditions of Definition 1.1.6). Given the definition of non-cooperative models, it is possible to shift an initial minimal set of zeros in a sea of ones through a finite number of flips. In particular, one can try to minimize the number of extra zeros present at any time during the different steps of the shift. Call m the minimal number of extra zeros needed to allow the initial seed to move around the lattice. For FA-1f, $m = 1$: to move the initial seed (which is just one zero), the strategy that costs the fewest zeros is to create an extra zero near the original one, then delete the original one, and then start again until we reach the desired position. For the k-zeros model also, it is not difficult to see that $m = 1$. For the windmill model, it is less obvious that $m = 1$, but Figure C.8 shows a strategy allowing to move the seed using only one extra zero at all times.

For an example in which $m > 1$, consider the model on $\mathbb{Z}$ in which $c_x(\eta) = 1-(1-\eta_{x+1})(1-\eta_{x-2})$. In words, the constraint is satisfied if either the East neighbour or the neighbour two steps on the left is empty. In that case it is not difficult to check that $k = 1$ and $m = 2$.

Conjecture 3.1.5 *Consider a non-cooperative model with k, m as above. Then we expect*

$$\text{gap} \sim q^{2k+m} \quad \textit{for} \quad d = 1 \tag{3.9}$$
$$\text{gap} \sim q^{k+m} \quad \textit{for} \quad d \geq 2. \tag{3.10}$$

The argument in dimension 1 is the same as for FA-1f: the typical distance between two seeds is of order $1/q^k$, and each seed diffuses at rate q^m, so we have to wait a time $1/q^{2k} \times 1/q^m$ for two seeds to communicate. In fact, to generalize it to higher dimension, we have to formulate it in terms of cover time. Our general argument goes as follows. In a ball of radius $r = 1/q^{k/d}$ there is typically one seed that diffuses at rate q^m. The relaxation time in this ball (which should be the same as in infinite volume, as was showed rigorously for FA-1f in [CMRT08]) is expected to be the time necessary for a positive fraction of the ball to have been visited by a seed, since updates are possible only in contact with a seed. Rigorous results on cover times (see [Ald83, DPRZ04]) imply that when $d \geq 2$ we have to wait a time r^d divided by the diffusion coefficient of the random walk, therefore $1/q^{k+m}$. In turn when $d = 1$ we have to wait a time $r^2/q^m = 1/q^{2k+m}$.

3.2 Diffusion coefficient and Stokes-Einstein relation

The relaxation time is one way of quantifying the mobility of a system. Another method is to probe the system by injecting a tracer particle and study its diffusion coefficient. This is the object of the paper [Blo13b] and its corresponding physical letter [BT13]. Consider a KCSM at equilibrium (the environment) and add a tracer at the origin. The environment does not see the tracer and evolves according to the KCSM dynamics. The tracer attempts to perform a simple symmetric random walk, but is only allowed to jump between two empty sites. More formally, the generator of the process seen from the tracer is given by

$$\mathcal{L}'f(\eta) = \mathcal{L}f(\eta) + \sum_{y\sim 0}(1-\eta_0)(1-\eta_y)[f(\eta_{y+\cdot}) - f(\eta)], \tag{3.11}$$

where $\mathcal{L}$ is the generator of the KCSM, the sum is taken over the origin's nearest neighbours and $\eta_{y+\cdot}$ denotes the configuration η translated by y ($(\eta_{y+\cdot})_x = \eta_{y+x}$). The second term corresponds to the jumps of the tracer. The process seen from the tracer is still reversible w.r.t. μ. We consider only KCSM in the ergodic regime. Classic results of martingale approximation (see [KV86, DMFGW89, Spo90]) adapt easily to show that, properly rescaled, the trajectory of the tracer is a Brownian motion with a diffusion coefficient given by a variational formula depending on the KCSM.

Proposition 3.2.1

$$\lim_{\epsilon\to 0} \epsilon X_{\epsilon^{-2}t} = \sqrt{2D} B_t, \tag{3.12}$$

where X_t is the position of the tracer at time t, B_t is the standard d-dimensional Brownian motion, the convergence holds in the sense of weak convergence of path measures on $D\left([0,\infty),\mathbb{R}^d\right)$ and the diffusion matrix D is given by

$$\begin{aligned} u.Du \;=\; \frac{1}{2}\inf_f \Bigg\{ &\sum_{y\in\mathbb{Z}^d} \mu\left(c_y(\eta)((1-q)(1-\eta_y)+q\eta_y)\left[f(\eta^y)-f(\eta)\right]^2\right) \\ &+ \sum_{i=1}^{d}\sum_{\alpha=\pm 1} \mu\left((1-\eta_0)(1-\eta_{\alpha e_i})\left[\alpha u_i + f(\eta_{\alpha e_i+\cdot}) - f(\eta)\right]^2\right)\Bigg\}, \end{aligned} \tag{3.13}$$

where $u\in\mathbb{R}^d$ and the infimum is taken over local functions f on Ω.

The main results in [Blo13b] are asymptotics for D when $q\to 0$. However I start by using the variational formula (3.13) to establish general bounds on D, showing in particular that the convergence in (3.12) is non-degenerate as soon as the spectral gap of the environment is positive.

Proposition 3.2.2 *There exists $c>0$ such that for any normed vector $u\in\mathbb{R}^d$*

$$cq^2\,\text{gap} \leqslant u.Du \leqslant q^2, \tag{3.14}$$

where gap *denotes the spectral gap of the environment.*

The upper bound appears immediately by taking a constant test function. For the lower bound, notice that the first sum in 3.13 can be bounded from below by $2\,\text{gap}\,Var(f)$. Then we can informally argue that if $Var(f)$ is small, the second sum should be of order q^2 (the difference $f(\eta_{\alpha e_i+\cdot}) - f(\eta)$ should play little role). Else, the spectral gap gives a lower bound.

Let us turn now to more precise asymptotic results for D. In fact the diffusion of a tracer in KCSM was studied in several physics papers, in particular [JGC04, JGC05]. The focus of these papers is the question of a breakdown of the Stokes-Einstein relation. In homogeneous liquids, it is well established in the physics community (see [HM06a] for instance) that the temperature T_l of the liquid, the relaxation time τ and the diffusion coefficient D of a tracer satisfy the Stokes-Einstein relation

$$D \propto T_l\tau^{-1}. \tag{3.15}$$

On the contrary, in many glassy systems it has been observed experimentally that this relation is violated. Indeed the diffusion coefficent does not decrease as fast as the relaxation time increases: $D\tau$ can increases by 2-3 orders of magnitude while approaching the glass transition. A good fit is (see for instance [EEH+12], [CE96], [CS97], [SBME03])

$$D \sim \tau^{-\xi}, \quad \xi < 1. \tag{3.16}$$

This violation is a celebrated landmark of dynamical heterogeneities. Indeed, the decoupling among diffusion coefficient and relaxation time should be due to the fact that diffusion is dominated by the fastest regions whereas structural relaxation is dominated by the slowest regions. In [JGC04, JGC05], the authors study the diffusion of a tracer in the East and FA-1f model in one dimension with numerical simulations.

For the FA-1f model, they find that

$$D \sim q^2, \tag{3.17}$$

a result which they expect to hold in every dimension. Therefore there should be violation of the Stokes-Einstein relation of the form (3.16) with $\xi = 2/3$ in dimension 1, and no violation in higher dimension (recall Theorem 3.1.4 and Conjecture 3.1.5). In [Blo13b] I confirm (3.17) in all dimensions and show more generally the following result.

Theorem 3.2.3 *For the k-zeros model, there is a constant $C > 0$ depending only on the dimension and k such that for every normed vector $u \in \mathbb{R}^d$*

$$C^{-1}q^{k+1} \leqslant u.Du \leqslant Cq^{k+1}. \tag{3.18}$$

The theorem is stated only for the k-zeros model; however the techniques used in the proof adapt to any non-cooperative model (I sketch the proof of the lower bound for the windmill model in Section C.7). The general statement should therefore be

$$C^{-1}q^{k+m} \leqslant u.Du \leqslant Cq^{k+m}, \tag{3.19}$$

for some $C > 0$, where k, m are the quantities introduced to state Conjecture 3.1.5. I will give below a consistent heuristic for the $k+m$ exponent. Taking (3.19) for granted and assuming Conjecture 3.1.5 is valid, one gets for any non-cooperative model a violation of Stokes-Einstein of the form (3.16) with $\xi = (k+m)/(2k+m) < 1$ in dimension 1, and no violation in higher dimension. The fact that this behaviour should be shared by all non-cooperative models is consistent with the belief that they should not be qualitatively much different from FA-1f.

For the East model, simulations are much harder because of the fast divergence of both τ and D^{-1}. In [JGC04] the authors predict a violation of the Stokes-Einstein relation of the form (3.16) with $\xi \approx 0.73$. Instead I show

Theorem 3.2.4 *There exist constants $C, \alpha > 0$ such that in the East model*

$$C^{-1}q^2 \,\mathrm{gap} \leqslant D \leqslant Cq^{-\alpha} \,\mathrm{gap}\,. \tag{3.20}$$

Because of the fast divergence of gap^{-1} (recall (3.1)), this is incompatible with a fractional Stokes-Einstein violation of the form (3.16). This fast divergence is also what prevented the authors in [JGC04] from running simulations with a parameter small enough to observe the asymptotic behaviour. It seems however likely that there is a weaker decoupling between D and gap of the kind $D \approx q^{-2}\,\mathrm{gap}$. Let me now sketch the proof of the two theorems.

For non-cooperative models, the asymptotic behaviour of D is extracted from the variational formula (3.13). The lower bound is derived by introducing a one-dimensional auxiliary dynamics with positive diffusion coefficient $\overline{D}$ not depending on q, which satisfies the following inequality w.r.t. D

$$cq^{k+1}\overline{D} \leqslant D\,. \tag{3.21}$$

In [Blo13b] I write the proof in the 3-zeros case because 3 is the lowest k for which the proof does not display any significant simplification with respect to higher k. Here, for the sake of simplicity, let me describe the auxiliary dynamics when $k = 1$ and show where the exponent

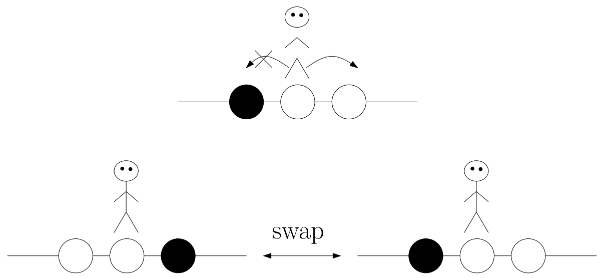


Figure 3.2: Transitions of the auxiliary dynamics. Up, the tracer jumps with rate one to an empty neighbour. Below, the neighbours of the tracer swap their occupation variable.

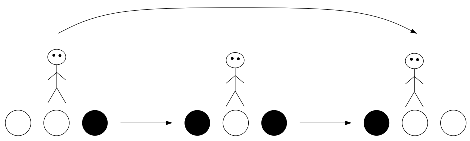


Figure 3.3: Reconstructing a swap with flips authorized under the FA-1f constraint, without extra zeros.

2 comes out. For the auxiliary dynamics, start with a configuration on $\mathbb{Z}$ with a zero at the origin and at least a zero among $\{\pm 1\}$. Let the tracer sit initially at the origin. The dynamics is designed in such a way that the tracer is always on a zero and has always a zero on its right or left. The rules for the tracer are unchanged: it jumps to its right/left with rate 1, provided the landing site is empty. The environment is frozen everywhere with the exception of the neighbours of the tracer. These two sites swap their occupation variables with rate 1. See Figure 3.2. It is not difficult to check that this dynamics seen from the tracer is reversible w.r.t. the measure $\mu^{(1)} = \mu\left(\cdot|\eta_0 = \eta_1\eta_{-1} = 0\right)$, which is μ conditioned on having a zero at the origin and a zero at -1 or 1. A variational formula can be established for the diffusion coefficient of the tracer in this auxiliary dynamics and can be compared with (3.13) to establish (3.21). The factor q^2 comes from the conditioning to go from μ to $\mu^{(1)}$ and the key idea to establish (3.21) is that one can reconstruct the swap using only flips allowed by the FA-1f dynamics *without using more zeros than initially available*, so that no extra factor q comes out when making this comparison (see Figure 3.3). It is not difficult (and it was proved for more general auxiliary dynamics in [Spo90]) to show that $\overline{D} > 0$ and does not depend on q, which concludes the proof of the lower bound.

For the upper bound, the trick is to find an appropriate test function to evaluate the functional in the r.h.s. of (3.13). To find an upper bound on $e_1.De_1$, take as a test function the maximal coordinate on the first direction of a site belonging to the cluster of zeros of the origin. The idea behind this choice is that the cluster of zeros of the origin is the maximal region the tracer could span if the environment was frozen. Moreover, it cancels the second line in (3.13). Note that since we are interested in $q \to 0$, the site percolation with parameter q is very subcritical. Therefore, even rough estimates on the functional in (3.13) are enough to establish the desired bound. The order q^{k+1} is in fact given by $\mu(c_x(\eta)((1-q)(1-\eta_y) + q\eta_y))$ and the extreme subcriticality allows to control that not too many of those terms are involved.

To conclude on the case of non-cooperative models, let me give a heuristic explaining the exponent $k+m$ in the statement (3.19). This corresponds to Remark C.4.2. Let me explain it in the case of the k-zeros model. In this model, a group of k zeros diffuses with rate q, and the typical equilibrium distance between two such groups is $1/q^k$. Therefore, let us consider the fraction of time spent in 0 before time T by a group of k zeros performing a random walk with diffusion coefficient q on $\{-1/(2q^k), ..., 1/(2q^k)\}$. That is Tq^k, which is also the time during which the group of k zeros is in contact with the tracer sitting at the origin. During that time, the tracer diffuses with the group, *i.e.* with rate q. In the end, the diffusion coefficient of the tracer should therefore be of order $Tq^k \times q/T = q^{k+1}$.

Let me now turn to a discussion of (3.20) (Theorem C.5.2 in Appendix C). The lower bound follows immediately from (3.14). The upper bound, which is the one contradicting the numerical prediction, relies on two arguments. The first one is very connected with Section 3.1.1. It consists in saying that if the tracer initially has a sequence of $1/q$ occupied sites on its right and wants to go beyond it, it will have to wait until a zero has travelled through the sequence, which requires a time of order gap^{-1}. This means it has to cross the same energy barrier that governs the behaviour of the spectral gap (see Figure 3.4). For a more precise statement see Lemma C.5.6, whose proof relies on the precise estimates of Lemma 3.1.3. That being said, the further the tracer wants to travel to the right, the more sequences of $1/q$ occupied sites it will encounter, *i.e.* the more barriers it will have to cross. This, along with the symmetry in the model, allows to show that the expected square displacement of the tracer at time roughly gap^{-1} is not larger than a power of $1/q$ (Proposition C.5.4). Now comes the second argument: the process seen from the tracer has a spectral gap which is larger than the spectral gap of the environment (Lemma C.5.7). Therefore, if we cut the trajectory of the tracer in pieces of time length roughly gap^{-1}, the correlation between two pieces of the trajectory decreases exponentially fast in the time separating the two pieces. All this put together yields (3.20).

To conclude, let me state some open questions related to this work, some of which are discussed in the conclusion of [BT13]. There is a multi-dimensional analog of the East model, in which the constraint is satisfied at x if either of the sites $x + e_i$, $i = 1, \ldots, d$ is empty. This model is still cooperative, the spectral gap should go to zero faster than any power of q and we still expect that (3.16) does not hold,at variance with the physicists predictions from numerical simulations([JGC04, JGC05]). However our techniques do not adapt immediately and it is not obvious for instance whether the persistence and relaxation time remain of the same order. There are also many other cooperative models in which the barrier structure is very different, such as the FA-2f model (the constraint is to have at least two empty nearest neighbours). Among these we could possibly find KCSM in which a relation of the form (3.16) holds.

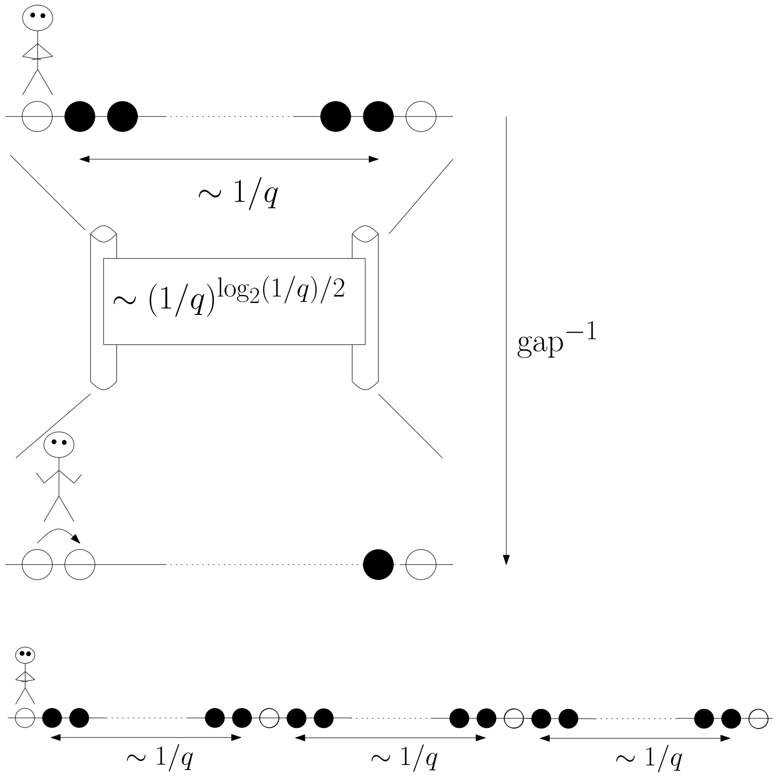


Figure 3.4: On top, the tracer typically faces a string of $1/q$ occupied sites on its right. Before jumping to the right, it needs to wait for the system to create a zero on its right, which has a cost roughly $(1/q)^{\log_2(1/q)/2}$, which is also approximately gap^{-1}. The typical time before crossing this barrier is therefore both gap^{-1} and the typical time before the tracer can jump. Bottom line: to travel to the right the tracer will face many more barriers.

Appendix A

Out of equilibrium relaxation in the FA-1f model

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This article is published in *Markov Processes and Related Fields*, Vol. 19, p. 383-406, 2013.

Contents

We consider the Fredrickson and Andersen one spin facilitated model (FA1f) on an infinite connected graph with polynomial growth. Each site with rate one refreshes its occupation variable to a filled or to an empty state with probability $p \in [0,1]$ or $q = 1-p$ respectively, provided that at least one of its nearest neighbours is empty. We study the non-equilibrium dynamics started from an initial distribution ν different from the stationary product p-Bernoulli measure μ. We assume that, under ν, the distance between two nearest empty sites has exponential moments. We then prove convergence to equilibrium when the vacancy density q is above a proper threshold $\bar{q} < 1$. The convergence is exponential or stretched exponential, depending on the growth of the graph. In particular it is exponential on $\mathbb{Z}^d$ for $d = 1$ and stretched exponential for $d > 1$. Our result can be generalized to other *non cooperative* models.

A.1 Introduction

Fredrickson-Andersen one spin facilitated model (FA1f) [FA84, FA85] belongs to the class of interacting particle systems known as Kinetically Constrained Spin Models (KCSM), which have been introduced and very much studied in the physics literature to model liquid/glass transition and more generally glassy dynamics (see [RS03, TGS] and references therein). A configuration for a KCSM is given by assigning to each vertex x of a (finite or infinite) connected graph $\mathcal{G}$ its occupation variable $\eta_x \in \{0,1\}$, which corresponds to an empty or filled site respectively. The evolution is given by Markovian stochastic dynamics of Glauber type. With rate one each site refreshes its occupation variable to a filled or to an empty state with probability $p \in [0,1]$ or $q = 1-p$ respectively, provided that the current configuration satisfies an a priori specified local constraint. For FA1f the constraint at x requires at least one of its nearest neighbours to be empty. Note that a single empty site is sufficient to ensure irreducibility of the chain. KCSM in which a finite subset of empty sites is able to move around and empty the whole space are called *non-cooperative* and are in general easier to analyze than cooperative ones. Note also that (and this is a general feature of KCSM) the constraint which should be satisfied to allow creation/annihilation of a particle at x does not involve η_x. Thus FA1f dynamics satisfies detailed balance w.r.t. the Bernoulli product measure at density p, which is therefore an invariant reversible measure for the process. Key features of FA1f model and more generally of KCSM are that a completely filled configuration is blocked (for generic KCSM other blocked configurations may occur) - namely all creation/destruction rates are identically equal to zero in this configuration -, and that due to the constraints the dynamics is not attractive, so that monotonicity arguments valid for *e.g.* ferromagnetic stochastic Ising models cannot be applied. Due to the above properties the basic issues concerning the large time behavior of the process are non-trivial.

In [CMRT08] it has been proved that the model on $\mathcal{G} = \mathbb{Z}^d$ is ergodic for any $q > 0$ with a positive spectral gap which shrinks to zero as $q \to 0$ corresponding to the occurrence of diverging mixing times. A key issue both from the mathematical and the physical point of view is what happens when the evolution does not start from the equilibrium measure μ. The analysis of this setting usually requires much more detailed information than just the positivity of the spectral gap, *e.g.* boundedness of the logarithmic Sobolev constant or positivity of the entropy constant uniformly in the system size. The latter requirement certainly does not hold (see Section 7.1 of [CMRT08]) and even the basic question of whether convergence to μ occurs remains open in the infinite volume case. Of course, due to the existence of blocked configurations, convergence to μ cannot hold *uniformly* in the initial configuration and one could try to prove it a.e. or in mean w.r.t. a proper initial distribution $\nu \neq \mu$.

From the point of view of physicists, a particularly relevant case (see *e.g.* [LMS+07]) is when ν is a product Bernoulli(p') measure with $p' \neq p$ and $p' \neq 1$). In this case the most natural guess is that convergence to equilibrium occurs for any local (*i.e.* depending on finitely many occupation variables) function f *i.e.*

$$\lim_{t\to\infty} \int d\nu(\eta)\mathbb{E}_\eta\big(f(\eta_t)\big) = \mu(f) \tag{A.1}$$

where η_t denotes the process started from η at time t and that the limit is attained exponentially fast.

The only case of KCSM where this result has been proved [CMST10] (see also [FMRT12b]) is the East model, that is a one dimensional model in which the constraint at x requires the neighbour to the right of x to be empty. The strategy used to prove convergence to equilibrium for

East model in [CMST10] relies however heavily on the oriented character of the East constraint and cannot be extended to FA1f model. We also recall that in [CMST10] a perturbative result has been established proving exponential convergence for any one dimensional KCSM with finite range jump rates and positive spectral gap (thus including FA1f at any $q > 0$), provided the initial distribution ν is "not too far" from the reversible one (*e.g.* for ν Bernoulli at density $p' \sim p$).

Here we prove convergence to equilibrium for FA1f on a infinite connected graph $\mathcal{G}$ with polynomial growth (see the definition in sec. A.2.1 below) when the equilibrium vacancy density q is above a proper threshold $\bar{q}$ (with $\bar{q} < 1$) and the starting measure ν is such that the distance between two nearest empty sites has exponential moments. That includes in particular any non-trivial Bernoulli product measure with $p' \neq p$ but also the case in which ν is the Dirac measure on a fixed configuration with infinitely many empty sites and such that the distance between two nearest empty sites is uniformly bounded. The derived convergence is either exponential or stretched exponential depending on the growth of the graph. In the particular case $\mathcal{G} = \mathbb{Z}^d$, we can prove exponential relaxation only for $d = 1$. If $d > 1$ we get a stretched exponential behavior. Although our results can be generalized to other *non cooperative* KCSM (see Section A.6 for a specific example and [CMRT08] for the general definition of this class) we consider here only the FA1f case to let the paper be more readable.

We finish with a short road map of the paper. In Section A.2 we introduce the notations and give the main result, Theorem A.2.1, which is proved in Section A.5. The strategy to derive this result can be summarized as follows. We first replace $\mathbb{E}_\nu(f(\eta_t))$ with a similar quantity but computed w.r.t the FA1f finite volume process (actually a finite state, continuous time Markov chain evolving in a finite ball of radius proportional to time t around the support of f). This first reduction is standard and it follows easily from the so-called *finite speed of propagation*. Then we show that, with high probability, only the evolution of a restricted chain inside a suitable ergodic component matters. This reduction is performed via a general result on Markov processes which we derive in Section A.3. The ergodic component is chosen in such a way that the log-Sobolev constant for the restricted chain is much smaller than t. This second reduction is new and it is at this stage that the restriction on q appears and that all the difficulties of the non-equilibrium dynamics appear. Its implementation requires the estimate of the spectral gap of the process restricted to the ergodic component (see Section A.6) and the study of the persistence of zeros out of equilibrium (see Section A.4).

A.2 Notation and Result

A.2.1 The graph

Let $\mathcal{G} = (V, E)$ be an infinite, connected graph with vertex set V, edge set E and graph distance $d(\cdot, \cdot)$. Given $x \in V$ the set of neighbors of x will be denoted by $\mathcal{N}_x$. For all $\Lambda \subset V$ we call $\mathrm{diam}(\Lambda) = \sup_{x,y \in \Lambda} d(x, y)$ the diameter of Λ and $\partial\Lambda = \{x \in V \setminus \Lambda \colon d(x, \Lambda) = 1\}$ its (outer) boundary. Given a vertex x and an integer r, $B(x, r) = \{y \in V \colon d(x, y) \leq r\}$ denotes the ball centered at x and of radius r. We introduce the growth function $F \colon \mathbb{N} \setminus \{0\} \to \mathbb{N} \cup \{\infty\}$ defined by

$$F(r) = \sup_{x \in V} |B(x, r)|$$

where $|\cdot|$ denotes the cardinality. Then we say that $\mathcal{G}$ has (k, D)-*polynomial growth* if $F(r) \leq k\, r^D$ for all $r \geq 1$, with k and D two positive constants. An example of such a graph is given by the d-dimentional square lattice $\mathbb{Z}^d$ that has $(3^d, d)$-polynomial growth (with the constant 3^d certainly not optimal).

A.2.2 The probability space

The configuration space is $\Omega = \{0,1\}^V$ equipped with the Bernoulli product measure μ of parameter p. Similarly we define Ω_Λ and μ_Λ for any subset $\Lambda \subset V$. Elements of Ω (Ω_Λ) will be denoted by Greek letters η, ω, σ (η_Λ, ω_Λ, σ_Λ) etc. Furthermore, we introduce the shorthand notation $\mu(f)$ to denote the expected value of f and $\mathrm{Var}(f)$ for its variance (when it exists).

A.2.3 The Markov process

The interacting particle model that will be studied here is a Glauber type Markov process in Ω, reversible w.r.t. the measure μ. It can be informally described as follows. Each vertex x waits an independent mean one exponential time and then, provided that the current configuration σ is such that one of the neighbors of x (i.e. one site $y \in \mathcal{N}_x$) is empty, the value $\sigma(x)$ is refreshed with a new value in $\{0,1\}$ sampled from a Bernoulli p measure and the whole procedure starts again.

The generator $\mathcal{L}$ of the process can be constructed in a standard way (see *e.g.* [Lig85]). It acts on local functions as

$$\mathcal{L}f(\sigma) = \sum_{x \in V} c_x(\sigma)[q\sigma(x) + p(1-\sigma(x))][f(\sigma^x) - f(\sigma)] \tag{A.2}$$

where $c_x(\sigma) = 1$ if $\prod_{y \in \mathcal{N}_x} \sigma(y) = 0$ and $c_x(\sigma) = 0$ otherwise (namely the constraint requires at least one empty neighbor), σ^x is the configuration σ flipped at site x, $q \in [0,1]$ and $p = 1-q$. It is a non-positive self-adjoint operator on $\mathbb{L}^2(\Omega, \mu)$ with domain $Dom(\mathcal{L})$, core $\mathcal{D}(\mathcal{L}) = \{f : \Omega \to \mathbb{R} \text{ s.t. } \sum_{x \in V} \sup_{\sigma \in \Omega} |f(\sigma^x) - f(\sigma)| < \infty\}$ and Dirichlet form given by

$$\mathcal{D}(f) = \sum_{x \in V} \mu\left(c_x \,\mathrm{Var}_x(f)\right), \quad f \in Dom(\mathcal{L}).$$

Here $\mathrm{Var}_x(f) \equiv \int d\mu(\omega(x)) f^2(\omega) - \left(\int d\mu(\omega(x)) f(\omega)\right)^2$ denotes the local variance with respect to the variable $\omega(x)$ computed while the other variables are held fixed. To the generator $\mathcal{L}$ we can associate the Markov semigroup $P_t := e^{t\mathcal{L}}$ with reversible invariant measure μ. We denote by σ_t the process at time t starting from the configuration σ. Also, we denote by $\mathbb{E}_\eta(f(\eta_t))$ the expectation over the process generated by $\mathcal{L}$ at time t and started at configuration η at time zero and, with a slight abuse of notation, we let

$$\mathbb{E}_\nu(f(\sigma_t)) := \int d\nu(\eta) \mathbb{E}_\eta(f(\eta_t))$$

and let $\mathbb{P}_\nu$ be the distribution of the process started with distribution ν at time zero.

For any subset $\Lambda \subset V$ and any configuration $\eta \in \Omega$

$$\mathcal{L}^\eta_\Lambda f(\sigma) = \sum_{x \in \Lambda} c^\eta_{x,\Lambda}(\sigma)[q\sigma(x) + p(1-\sigma(x))][f(\sigma^x) - f(\sigma)] \tag{A.3}$$

where $c^\eta_{x,\Lambda}(\sigma) = c_x(\sigma_\Lambda \eta_{\Lambda^c})$ where $\sigma_\Lambda \eta_{\Lambda^c}$ is the configuration equal to σ on Λ and equal to η on Λ^c. When η is the empty configuration we write simply $c_{x,\Lambda}$ and $\mathcal{L}_\Lambda$. We also let σ^Λ_t be the configuration at time t of the process starting from σ_Λ with empty boundary condition.

A.2.4 Main Result

In order to state our main theorem, we need some notations. For any vertex $x \in V$, and any configuration $\sigma \in \Omega$, let

$$\xi^x(\sigma) = \min_{y \in V:\ \sigma(y)=0} \{d(x,y)\}$$

be the distance of x from the set of empty sites of σ.

Theorem A.2.1 *Let $q > 1/2$. Assume that the graph $\mathcal{G}$ has (k,D)-polynomial growth and $f\colon \Omega \to \mathbb{R}$ is a local function with $\mu(f) = 0$. Let ν be a probability measure on Ω such that $\kappa := \sup_{x\in V} \mathbb{E}_\nu(\theta_o^{\xi^x}) < \infty$ for some $\theta_o > 1$. Then, there exists a positive constant $c = c(q,k,D,\kappa,|\mathrm{supp(f)}|)$ such that*

$$|\mathbb{E}_\nu(f(\sigma_t))| \le c||f||_\infty \begin{cases} e^{-t/c} & \text{if } D = 1 \\ e^{-[t/(c\log t)]^{1/D}} & \text{if } D > 1. \end{cases} \qquad \forall t \ge 2.$$

Remark A.2.2 *We expect that our results hold also for $0 < q \le \frac{1}{2}$. This needs a more precise control of the behavior of $\xi_t^x = \xi^x(\sigma_t)$. In dimension one we can obtain a better threshold by calculating further time derivatives of $u(t) = \mathbb{E}_\eta(\theta^{\xi_t})$, see Proposition A.4.1 below.*

Remark A.2.3 *Observe that if ν is a Dirac mass on some configuration η, the condition reads $\sup_{x\in V} \theta_o^{\xi^x(\eta)} < \infty$. This encodes the fact that η has infinitely many empty sites and that, in addition, the distance between two nearest empty sites is uniformly bounded. This condition is different from the case of the East model in [CMST10] where the condition on the initial configuration was the presence of an infinite number of zeros.*

Remark A.2.4 *If one considers the case in which ν is the product of Bernoulli-p' on $\mathcal{G}$, one has that, for all $\theta < 1/p'$ and all $x \in \mathcal{G}$,*

$$\mathbb{E}_\nu(\theta^{\xi^x}) = \sum_{k=0}^{\infty} \theta^k \mathbb{P}_\nu(\xi^x = k) \le \sum_{k=0}^{\infty} \theta^k (p')^{|B(x,k)|} \le \sum_{k=0}^{\infty} (\theta p')^k = \frac{1}{1-p'\theta}.$$

Hence, $\kappa \le \frac{1}{1-p'\theta_o}$ for $\theta_o \in (1, 1/p')$. In particular Theorem A.2.1 applies to any initial probability measure, product of Bernoulli-p' on $\mathcal{G}$, with $p' \in [0,1)$.

Remark A.2.5 *Note that graphs with polynomial growth are amenable. We stress anyway that there exist amenable graphs which do not satisfy our assumption. This is due to Proposition A.5.1 below that gives a useless bound in the case of amenable graphs with intermediate growth (i.e. faster than any polynomial but slower than any exponential, see [Gri91]). The same happens to any graph with exponential growth (such as for example any regular n-ary tree ($n \ge 2$)).*

A.3 A preliminary result on Markov processes

We prove here a general result which relates the behavior of a Markov process on a finite space to that of a restricted Markov process. This result, which might be of independent interest, will be a key tool in our analysis. Indeed we will use it in the proof of Theorem A.2.1 to reduce the evolution of the FA1f process on a large volume to the same process on smaller sets on a properly defined ergodic component.

We start by recalling some basic notions on continous time Markov chains which will be used in the following. Let S be a finite space and $Q = (q(x,y))_{x,y\in S}$ be a transition rate matrix, namely a matrix such that for any $x, y \in S$ it holds

$$q(x,y) \geq 0 \quad \text{for } x \neq y \quad \text{and} \sum_{y\in S} q(x,y) = 0.$$

Recall that Q defines a continuous time Markov chain $(X_t)_{t\geq 0}$ on S as follows [Lig85]. If $X_t = x$, then the process stays at x for an exponential time with parameter $c(x) = -q(x,x)$. At the end of that time, it jumps to $y \neq x$ with probability $p(x,y) = q(x,y)/c(x)$, stays there for an exponential time with parameter $c(y)$, etc. Assume that $(X_t)_{t\geq 0}$ is reversible with respect to a probability measure π. Then, we define the spectral gap $\gamma(Q)$ and the log-Sobolev constant $\alpha(Q)$ of the chain as

$$\gamma(Q) := \inf_{f\colon f\neq \text{const}} \frac{\sum_{x,y} \pi(x)p(x,y)(f(y)-f(x))^2}{2\,\mathrm{Var}_\pi(f)} \tag{A.4}$$

$$\alpha(Q) := \sup_{f\colon f\neq \text{const}} \frac{2\mathrm{Ent}_\pi(f^2)}{\sum_{x,y} \pi(x)p(x,y)(f(y)-f(x))^2} \tag{A.5}$$

where $\mathrm{Ent}_\pi(f) = \pi(f\log f) - \pi(f)\log \pi(f)$ denotes the entropy of f. Let $(P_t)_{t\geq 0}$ be the semigroup of the Markov chain. Then

$$\mathrm{Var}_\pi(P_t f) \leqslant e^{-2tc}\,\mathrm{Var}_\pi(f) \qquad \forall f \tag{A.6}$$

is equivalent to $\gamma \geq c$. On the other hand the positivity of the log-Sobolev constant is equivalent to the following hypercontractivity property [Gro75]

$$||P_t f||_{L^r(\pi)} \leqslant ||f||_{L^2(\pi)} \tag{A.7}$$

$\forall t \geq 0$ and $\forall r \leqslant 1 + e^{\frac{4t}{\alpha}}$. We refer to [ABC+00] for an introduction of these notions.

We are now ready to introduce the restricted Markov chain. Fix $\mathcal{A} \subset S$ and set

$$\hat{\mathcal{A}} = \mathcal{A} \cup \{y \notin \mathcal{A}\colon q(x,y) > 0 \text{ for some } x \in \mathcal{A}\}. \tag{A.8}$$

Let $(\hat{X}_t)_{t\geq 0}$ be a continuous time Markov chain (which we will call the *hat chain*) on $\hat{\mathcal{A}}$ with transition rate matrix $\hat{Q} = (\hat{q}(x,y))_{x,y\in\hat{\mathcal{A}}}$ which satisfies

$$\hat{q}(x,y) = q(x,y) \qquad\qquad \forall (x,y) \in \mathcal{A} \times \hat{\mathcal{A}} \tag{A.9}$$

and assume that the process is reversible with respect to a measure $\hat{\pi}$. We denote by $\hat{\gamma}$ and $\hat{\alpha}$ the spectral gap and log-Sobolev constant of the hat chain, namely $\hat{\gamma} := \gamma(\hat{Q})$ and $\hat{\alpha} := \alpha(\hat{Q})$.

Proposition A.3.1 *Let $(X_t)_{t\geq 0}$, $(\hat{X}_t)_{t\geq 0}$, π, $\hat{\pi}$, $\hat{\gamma}$ and $\hat{\alpha}$ as above. Then, for all initial probability measure ν on S and all $f\colon S \to \mathbb{R}$ with $\pi(f) = 0$, it holds for any $t \geq 0$*

$$|\mathbb{E}_\nu(f(X_t))| \leq |\hat{\pi}(f)| + 4||f||_\infty \mathbb{P}_\nu(\mathcal{A}_t^c) + ||f||_\infty \exp\left\{-\hat{\gamma}\frac{t}{2} + e^{-\frac{2t}{\hat{\alpha}}}\log\frac{1}{\hat{\pi}^*}\right\} \tag{A.10}$$

where $\mathcal{A}_t = \{X_s \in \mathcal{A},\ \forall s \leq t\}$ and $\hat{\pi}^* := \min_{x\in S}\hat{\pi}(x)$.

Remark A.3.2 *The standard argument (see [HS87, SZ92]) using the log-Sobolev constant would lead to*

$$|\mathbb{E}_\nu(f(X_t))| \leq ||f||_\infty \exp\left\{-\gamma\frac{t}{2} + \exp\{-\frac{2t}{\alpha}\}\log\frac{1}{\pi^*}\right\}.$$

with $\gamma = \gamma(Q)$ and $\alpha = \alpha(Q)$. The difference in Proposition A.3.1 comes from the fact that we deal with the hat chain. This can be useful if the choice of $\hat{\mathcal{A}}$ and $\hat{Q}$ are done properly so that the log-Sobolev constant $\hat{\alpha}$ is smaller than α and/or the spectral gap $\hat{\gamma}$ is larger then γ. This will be the case for the application of the above result in the proof of Theorem A.2.1 for which, fixed t, we will have to consider a state space which depends on t and the corresponding chain will have $\alpha \simeq t^d$ and $\log(1/\pi) \simeq t^d$. Hence the standard argument gives

$$|\mathbb{E}_\nu(f(X_t))| \leq ||f||_\infty \exp\left\{-\gamma\frac{t}{2} + ct^d\right\}.$$

and therefore does not prove decay in t. We will instead devise a hat chain for which $\hat{\gamma} \geq c > 0$ and $\hat{\alpha}$ is much smaller than t so that the dominant term in $\exp\left\{-\hat{\gamma}\frac{t}{2} + \exp\{-\frac{2t}{\hat{\alpha}}\}\log\frac{1}{\hat{\pi}^*}\right\}$ is given by the gap term $\hat{\gamma}t$. The price to pay are the first two extra terms in (A.10) that we will analyze separately.

Proof

Fix a probability measure ν and a function f with $\pi(f) = 0$ and let $g = f - \hat{\pi}(f)$. Then

$$|\mathbb{E}_\nu(f(X_t))| \leq |\hat{\pi}(f)| + ||g||_\infty \mathbb{P}_\nu(\mathcal{A}_t^c) + |\mathbb{E}_\nu(g(X_t)\mathbb{1}_{\mathcal{A}_t})|. \tag{A.11}$$

We now concentrate on the last term in (A.11). By definition of the chains $(X_t)_{t\geq 0}$ and $(\hat{X}_t)_{t\geq 0}$ one has

$$\mathbb{E}_\nu(g(X_t)\mathbb{1}_{\mathcal{A}_t}) = \int d\nu(x)\mathbb{E}_x(g(\hat{X}_t)\mathbb{1}_{\{\hat{X}_s\in\mathcal{A},\,\forall s\leq t\}}).$$

Hence, by Hölder inequality, we have

$$\begin{aligned}|\mathbb{E}_\nu(g(X_t)\mathbb{1}_{\mathcal{A}_t})| &= |\int_{\hat{\mathcal{A}}} d\nu(x)\mathbb{E}_x(g(\hat{X}_t)(1 - \mathbb{1}_{\{\hat{X}_s\in\mathcal{A},\,\forall s\leq t\}^c})| \\ &\leq |\hat{\pi}(h\,\hat{P}_t g)| + 2||f||_\infty \mathbb{P}_\nu(\mathcal{A}_t^c) \\ &\leq ||h||_{L^\beta(\hat{\pi})}||\hat{P}_t g||_{L^{\beta'}(\hat{\pi})} + 2||f||_\infty \mathbb{P}_\nu(\mathcal{A}_t^c)\end{aligned}$$

where for any $x \in \hat{\mathcal{A}}$ we let $h(x) = \nu(x)/\hat{\pi}(x)$ and $\beta,\ \beta' \geq 1$, that will be chosen later, are such that $1/\beta + 1/\beta' = 1$. To bound the previous expression take $\beta' = 1 + e^{\frac{2t}{\hat{\alpha}}}$. Using (A.7) and (A.6) we obtain

$$||\hat{P}_t g||_{L^{\beta'}(\hat{\pi})} = ||\hat{P}_{\frac{t}{2}}\hat{P}_{\frac{t}{2}}g||_{L^{\beta'}(\hat{\pi})} \leq ||\hat{P}_{\frac{t}{2}}g||_{L^2(\hat{\pi})} \leq e^{-\hat{\gamma}\frac{t}{2}}||g||_{L^2(\hat{\pi})} \leq e^{-\hat{\gamma}\frac{t}{2}}||f||_\infty.$$

On the other hand

$$||h||_{L^\beta(\hat{\pi})} \leq \left(\int h d\hat{\pi}\right)^{\frac{1}{\beta}} ||h||_\infty^{\frac{\beta-1}{\beta}} = ||h||_\infty^{\frac{1}{\beta'}} \leq \exp\{e^{-\frac{2t}{\hat{\alpha}}}\log||h||_\infty\}$$

and the proof is completed since $||h||_\infty \leq \frac{1}{\hat{\pi}^*}$. ✓

A.4 Persistence of zeros out of equilibrium

In this section we study the behavior of the minimal distance from a fixed site to the nearest site at which one finds a vacancy. The result that we obtain will be a key tool for the proof of our main theorem A.2.1.

For any $\sigma \in \{0,1\}^V$ and any $x \in V$ define $\xi^x(\sigma)$ as the minimal distance at which one finds an empty site starting from x,

$$\xi^x(\sigma) = \min_{y \in V \colon \sigma(y)=0} \{d(x,y)\}$$

with the convention that $\min \emptyset = +\infty$, ($\xi^x(\sigma) = 0$ if $\sigma(x) = 0$).

Proposition A.4.1 *Consider the FA1f process on a finite set $\Lambda \subset V$ with generator $\mathcal{L}_\Lambda$. Then, for all $x \in \Lambda$, all $\theta \geq 1$, all $q \in (\frac{\theta}{\theta+1}, 1]$ and all initial configuration η, it holds*

$$\mathbb{E}_\eta \left(\theta^{\xi^x(\sigma_t^\Lambda)} \right) \leq \theta^{\xi^x(\eta)}\, e^{-\lambda t} + \frac{q}{q(\theta+1)-\theta} \qquad \forall t \geq 0,$$

where $\lambda = \frac{\theta^2-1}{\theta}(q - \frac{\theta}{\theta+1})$.

Proof

Fix $\theta > 1$, $q > 0$ and $x \in \Lambda$. To simplify the notation we drop the superscript x from ξ^x and set $\xi_t = \xi(\sigma_t^\Lambda)$ in what follows. Recall that σ_t^Λ is defined with empty boundary condition so that $\xi_t \leq d(x, \Lambda^c)$. Let $u(t) = \mathbb{E}_\eta(\theta^{\xi_t})$ and observe that

$$\frac{d}{dt} u(t) = \mathbb{E}_\eta(\mathcal{L}_\Lambda \theta^{\xi_t}).$$

To calculate the expected value above we distinguish two cases: **(i)** $\xi_t = 0$, **(ii)** $\xi_t \geq 1$.

Case **(i):** assume that $\xi_t = 0$. Then

$$(\mathcal{L}_\Lambda \theta^{\xi_t}) \mathbb{1}_{\xi_t=0} = \theta^{\xi_t} c_x(\sigma_t^\Lambda) p(\theta-1) \mathbb{1}_{\xi_t=0}. \tag{A.12}$$

Case **(ii)**. Define $E(\sigma) = \{y \in V \colon d(x,y) = \xi(\sigma) \text{ and } \sigma(y) = 0\}$ and $F(\sigma) = \{y \in V \colon d(y, E) = 1 \text{ and } d(x,y) = \xi(\sigma) - 1\}$. Then one argues that ξ_t can increase by 1 only if there is exactly one empty site in the set E, and that it can always decrease by 1 by a flip (which is legal by construction) on each site of F (see Figure A.1).

Hence

$$\begin{aligned}(\mathcal{L}_\Lambda \theta^{\xi_t}) \mathbb{1}_{\xi_t \geq 1} &= \theta^{\xi_t} \left[p(\theta-1) \sum_{y \in E} c_y(\sigma_t^\Lambda) \mathbb{1}_{|E|=1} + q|F|(\frac{1}{\theta} - 1) \right] \mathbb{1}_{\xi_t \geq 1} \\ &\leq \theta^{\xi_t} [p(\theta-1) - q\frac{\theta-1}{\theta}] + [q\frac{\theta-1}{\theta} - p(\theta-1)] \mathbb{1}_{\xi_t=0} \end{aligned} \tag{A.13}$$

Summing up (A.12) and (A.13) we end up with

$$\mathcal{L}_\Lambda \theta^{\xi_t} \leqslant \frac{\theta-1}{\theta} \left(\theta^{\xi_t}(p\theta - q) + q \right).$$

Therefore, since $p = 1-q$,

$$u'(t) \leq \frac{\theta-1}{\theta} \left((p\theta - q) u(t) + q \right) = -\lambda u(t) + q \frac{\theta-1}{\theta}$$

and the expected result follows. ✓

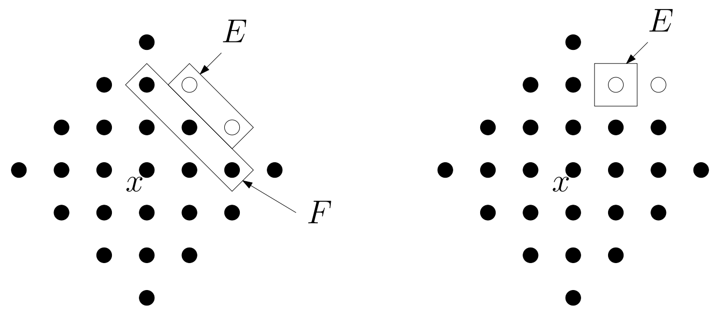

Figure A.1: On the graph $\mathcal{G} = \mathbb{Z}^2$, two examples of configurations for which $\xi^x = 3$. On the left ξ^x cannot increase since $|E| \geq 2$, it can decrease by a flip (legal thanks to the empty sites in E) in any points of F. On the right ξ^x can either increase or decrease.

A.5 Proof of the main theorem

In this section we prove Theorem A.2.1. We will first reduce the study of the evolution of the process from infinite volume to a finite ball of radius proportional to t thanks to finite speed of propagation. Then by using Proposition A.3.1 we reduce to the study of a restricted process on smaller sets on some ergodic component so that the log-Sobolev constant of the restricted process is much smaller than t (recall Remark A.3.2). In order to estimate the probability that the process gets out the ergodic component (namely to bound the second term in (A.10)) we will use Proposition A.4.1 which allows to upper bound the probability of a region to be completely filled.

Proof of Theorem A.2.1

Throughout the proof c denotes some positive constant $c = c(q, k, D, \kappa, |\text{supp(f)}|)$ which may change from line to line.

Fix $t \geq 2$ and a local function f. Let $x \in V$ and r integer be such that $\text{supp(f)} \subset B(x, r)$. Standard arguments using finite speed of propagation (see *e.g.* [Mar99]) prove that for any initial measure ν on Ω it holds

$$|\mathbb{E}_\nu(f(\sigma_t) - f(\sigma_t^\Lambda))| \leq c\|f\|_\infty e^{-t}$$

where $\Lambda = B(x, r + 100t)$ and we recall that σ_t^Λ is the configuration at time t of the process starting from σ_Λ evolving on the finite volume Λ with empty boundary condition and c is some positive constant depending on $|\text{supp(f)}|$. Hence,

$$|\mathbb{E}_\nu(f(\sigma_t))| \leq |\mathbb{E}_\nu(f(\sigma_t^\Lambda))| + c\|f\|_\infty e^{-t}. \tag{A.14}$$

Let $\Lambda_1, \Lambda_2, \ldots, \Lambda_n \subset \Lambda$ be connected sets such that $\cup_i \Lambda_i = \Lambda$ and $\Lambda_i \cap \Lambda_j = \emptyset$ for all $i \neq j$. Such a decomposition will be called a *connected partition* of Λ. The following holds

Proposition A.5.1 *For any $\Lambda \subset V$ and any f local, with $\text{supp(f)} \subset \Lambda$ and $\mu(f) = 0$ there exists a constant $c = c(q, |\text{supp(f)}|)$ such that for any connected partition $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ of Λ, for any initial probability measure ν on Ω, it holds that*

$$|\mathbb{E}_\nu(f(\sigma_t^\Lambda))| \leq c||f||_\infty \Bigg(ne^{-qm} + t|\Lambda| \sup_{s\in[0,t]} \mathbb{P}_\nu(\sigma_s^\Lambda \notin \mathcal{A}) + |\Lambda|e^{-t/3}$$
$$+ \exp\left\{-\frac{t}{c} + c|\Lambda|e^{-t/(cM)}\right\}\Bigg)$$

provided that $ne^{-qm} < 1/2$ *where*
$m := \min\{|\Lambda_1|, \dots, |\Lambda_n|\}$, $M := \max\{|\Lambda_1|, \dots, |\Lambda_n|\}$ *and* $\mathcal{A}$ *is the set of configurations containing at least two empty sites in each* Λ_i*, namely*

$$\mathcal{A} = \bigcap_{i=1}^{n} \{\sigma \in \Omega_\Lambda \ s.t. \ \sum_{x\in\Lambda_i} (1 - \sigma(x)) \geq 2\}. \tag{A.15}$$

We postpone the proof of this Proposition to the end of this section.

Observe that for any positive integer $\ell \leqslant t$, there exists[1] a connected partition $\Lambda_1, \dots, \Lambda_n$ of Λ, and vertices $x_1, \dots, x_n \in V$, such that for any i, $B(x_i, \ell) \subset \Lambda_i \subset B(x_i, 6\ell)$. Then, take $\ell = \epsilon[t/\log t]^{1/D}$ if $D > 1$ and $\ell = \epsilon t$ if $D = 1$ for some $\epsilon > 0$ that will be chosen later and observe that, with this choice,

$$M = \max(|\Lambda_1|, \dots, |\Lambda_n|) \leq k6^D \ell^D$$

(since $\mathcal{G}$ has (k, D)-polynomial growth). Furthermore

$$m = \min(|\Lambda_1|, \dots, |\Lambda_n|) \geq \ell$$

Since $n \leq |\Lambda| \leq ct^D$, Equation (A.14) and Proposition A.5.1 guarantee that

$$|\mathbb{E}_\nu(f(\sigma_t))| \leq c||f||_\infty t|\Lambda| \sup_{s\in[0,t]} \mathbb{P}_\nu(\sigma_s^\Lambda \notin \mathcal{A}) + c||f||_\infty \begin{cases} e^{-t/c} & \text{if } D = 1 \\ e^{-[t/(c\log t)]^{1/D}} & \text{if } D > 1 \end{cases}$$

provided ϵ is small enough.

It remains to study the first term of the latter inequality. We partition each set Λ_i into two connected sets Λ_i^+ and Λ_i^- (i.e. $\Lambda_i = \Lambda_i^+ \cup \Lambda_i^-$ and $\Lambda_i^+ \cap \Lambda_i^- = \emptyset$) such that for some $x_i^+, x_i^- \in V$, $B(x_i^\pm, \ell/4) \subset \Lambda_i^\pm$ (the existence of such vertices are left to the reader). The event $\{\sigma_s^\Lambda \notin \mathcal{A}\}$ implies that there exists one index i such that at least one of the two halves Λ_i^+, Λ_i^- is completely filled. Assume that it is for example Λ_i^+, i.e. assume that for any $x \in \Lambda_i^+$, $\sigma_s^\Lambda(x) = 1$. This implies that $\xi^{x_i^+}(\sigma_s^\Lambda) \geq \ell/4$. Hence, thanks to a union bound, Markov's inequality, and Proposition A.4.1, there exists $\theta > 1$ such that

$$\begin{aligned}
\mathbb{P}_\nu(\sigma_s^\Lambda \notin \mathcal{A}) &\leq 2n\mathbb{P}_\nu(\xi^{x_i^+}(\sigma_s^\Lambda) \geq \ell/4) \\
&\leq 2n\theta^{-\ell/4}\mathbb{E}_\nu(\theta^{\xi^{x_i^+}(\sigma_s^\Lambda)}) \\
&\leq cn\theta^{-\ell/4} \\
&\leq c \begin{cases} e^{-t/c} & \text{if } D = 1 \\ e^{-[t/(c\log t)]^{1/D}} & \text{if } D > 1 \end{cases}
\end{aligned}$$

where we used the definition of ℓ, the assumption $\sup_{x\in V} \mathbb{E}_\nu(\theta_o^{\xi^x}) < \infty$ and the fact that $n \leq |\Lambda| \leq ct^D$. This ends the proof.

[1]One can construct $\Lambda_1, \dots, \Lambda_n$, $x_1, \dots, x_n$ as follows. Recall that $\Lambda = B(x, r + 100t)$. Order (arbitrarily) the sites $y_1, y_2, \dots, y_N$ of $\{z \in \Lambda\colon B(z, \ell) \subset \Lambda$ and $d(z, x) = 2i(\ell + 1) - 1$ for some $i \geq 1\}$ and perform the following algorithm: set $x_1 = x$, $i_0 = 0$, and for $k \geq 1$ set $x_{k+1} = y_{i_k}$ with $i_k := \inf\{j \geq i_{k-1}+1 : B(y_j, \ell) \cap (\cup_{i=1}^k B(x_i, \ell)) = \emptyset\}$. Such a procedure gives the existence of n sites $x_1, \dots, x_n$ such that $B(x_i, \ell) \cap B(x_j, \ell) = \emptyset$, for all $i \neq j$, $B(x_i, \ell) \subset \Lambda$ for all i and any site $y_k \notin A := \cup_{i=1}^n B(x_i, \ell)$ is at distance at most 5ℓ from A. Now attach each connected component C of A^c to any (arbitrarily chosen) nearest ball $B(x_i, \ell)$, $i \in \{1, \dots, n\}$, with which C is connected, to obtain all the Λ_i with the desired properties.

✓

We are now left with proving Proposition A.5.1.

Proof of Proposition A.5.1

Fix $\Lambda \subset V$, f local, with supp(f) $\subset \Lambda$ and $\mu(f) = 0$. Fix a connected partition $\Lambda_1, \Lambda_2, \dots, \Lambda_n$ of Λ and an initial probability measure ν on Ω.

Our aim is to apply Proposition A.3.1. We let $S = \Omega_\Lambda$ and $(X_t)_{t\geq 0} = (\sigma_t^\Lambda)_{t\geq 0}$. The corresponding transition rates are, $\forall \sigma,\, \eta \in \Omega_\Lambda$,

$$q(\sigma,\eta) = \begin{cases} c_{x,\Lambda}(\sigma)[q\sigma(x) + p(1-\sigma(x))] & \text{if } \eta = \sigma^x \\ -\sum_{x\in\Lambda} q(\sigma,\sigma^x) & \text{if } \eta = \sigma \\ 0 & \text{otherwise} \end{cases}$$

We define $\mathcal{A}$ as in (A.15), namely the set of configurations in Ω_Λ such that there exist at least two empty sites in each set Λ_i, and $\hat{\mathcal{A}}$ as in (A.8). Next we define $(\hat{X}_t)_{t\geq 0}$ on $\hat{\mathcal{A}}$ via the rates $\hat{q}(\sigma,\eta) = q(\sigma,\eta)$ $\forall \sigma,\, \eta \in \hat{\mathcal{A}}$. In words, $(\hat{X}_t)_{t\geq 0}$ corresponds to a modification of the FA1f process in which the moves that would cause the process to leave $\hat{\mathcal{A}}$ are suppressed. Let $\pi = \mu_\Lambda$ and $\hat{\pi}(\cdot) = \mu_\Lambda(\cdot\,|\,\hat{\mathcal{A}})$. It is immediate to verify that $(X_t)_{t\geq 0}$ and $(\hat{X}_t)_{t\geq 0}$ are reversible with respect to π and $\hat{\pi}$, respectively. By construction the above processes satisfy the property (A.8). Thus, thanks to Proposition A.3.1, we have

$$|\mathbb{E}_\nu(f(\sigma_t^\Lambda))| \leq |\hat{\pi}(f)| + ||f||_\infty \left(4\mathbb{P}_\nu(\mathcal{A}_t^c) + \exp\left\{-\hat{\gamma}\frac{t}{2} + e^{-\frac{2t}{\hat{\alpha}}}\log\frac{1}{\hat{\pi}^*}\right\}\right). \tag{A.16}$$

We now study each term of the last inequality separately.

If we recall that $\mu_\Lambda(f) = \mu(f) = 0$ and using a union bound, we have

$$|\hat{\pi}(f)| = \frac{|\mu_\Lambda(f(1-\mathbb{1}_{\hat{\mathcal{A}}^c}))|}{\mu_\Lambda(\hat{\mathcal{A}})} \leq ||f||_\infty \frac{\mu_\Lambda(\hat{\mathcal{A}}^c)}{\mu_\Lambda(\hat{\mathcal{A}})} \leq ||f||_\infty \frac{ne^{-qm}}{1-ne^{-qm}}. \tag{A.17}$$

We now deal with the term $\mathbb{P}_\nu(\mathcal{A}_t^c)$.

Let $\mathcal{I}_t$ be the event that there exists a site in Λ with more than $2t$ rings in the time interval $[0,t]$. Then, by standard large deviations of Poisson variables and a union bound, there exists a universal positive constant d such that $\mathbb{P}_\nu(\mathcal{A}_t^c \cap \mathcal{I}_t) \leq d|\Lambda|e^{-t/3}$. Furthermore, using a union bound on all the rings on the event $\mathcal{I}_t^c$, we have

$$\mathbb{P}_\nu(\mathcal{A}_t^c \cap \mathcal{I}_t^c) \leq 2t|\Lambda| \sup_{s\in[0,t]} \mathbb{P}_\nu(\sigma_s^\Lambda \notin \mathcal{A}).$$

We deduce that

$$\mathbb{P}_\nu(\mathcal{A}_t^c) \leq c|\Lambda| \left(t \sup_{s\in[0,t]} \mathbb{P}_\nu(\sigma_s^\Lambda \notin \mathcal{A}) + e^{-t/3}\right). \tag{A.18}$$

Next we analyse the log-Sobolev constant $\hat{\alpha}$ and the spectral gap constant $\hat{\gamma}$. For that purpose, let us introduce a new process $(\tilde{X}_t)_{t\geq 0}$ on $\hat{\mathcal{A}}$ via the rates, $\forall \sigma,\, \eta \in \hat{\mathcal{A}}$,

$$\tilde{q}(\sigma,\eta) = \begin{cases} c^\omega_{x,\Lambda_{i(x)}}(\sigma)[q\sigma(x) + p(1-\sigma(x))] & \text{if } \eta = \sigma^x \\ -\sum_{x\in\Lambda} \tilde{q}(\sigma,\sigma^x) & \text{if } \eta = \sigma \\ 0 & \text{otherwise.} \end{cases}$$

where ω is the entirely filled configuration (i.e. such that $\omega(x) = 1$ for all $x \in V$) and $i(x)$ is such that $x \in \Lambda_{i(x)}$. In words $(\tilde{X}_t)_{t\geq 0}$ corresponds to n independent FA1f processes inside the boxes Λ_i each evolving with filled boundary conditions on the ergodic component of the configurations with at least one zero, namely on $\hat{\Omega}_{\Lambda_i}$ where we set for any $A \subset \Lambda$

$$\hat{\Omega}_A = \{\sigma \in \Omega_\Lambda \text{ s.t. } \exists x_i \in A \text{ with } \sigma(x_i) = 0\}. \tag{A.19}$$

Note that $\hat{\mathcal{A}} = \bigcap_{i=1}^n \hat{\Omega}_{\Lambda_i}$ thus the FA1f constraint and the filled boundary condition on each box indeed guarantee that $(\tilde{X}_t)_{t\geq 0}$ does not exit $\hat{\mathcal{A}}$ and $\hat{\pi}$ is a reversible measure also for $(\tilde{X}_t)_{t\geq 0}$. Furthermore, since the occupied boundary conditions imply that for any $\sigma \in \Omega_\Lambda$ it holds $c^{\omega}_{x,\Lambda_{i(x)}}(\sigma) \leqslant c_{x,\Lambda}(\sigma)$ (a zero which is present in $\sigma_{\Lambda_{i(x)}}\omega_{\Lambda^c_{i(x)}}$ is also present in σ) , the following inequalities holds between the spectral gap and log-Sobolev constant of the hat and tilde process $\hat{\alpha} \leq \tilde{\alpha}$ and $\hat{\gamma} \geq \tilde{\gamma}$, where $\hat{\alpha} := \alpha(\hat{Q})$, $\tilde{\alpha} := \alpha(\tilde{Q})$, $\hat{\gamma} := \gamma(\hat{Q})$ and $\tilde{\gamma} = \gamma(\tilde{Q})$ (see (A.4) and (A.5)). Observe now that $\tilde{X}_t$ restricted to each $\hat{\Omega}_{\Lambda_i}$ is ergodic and reversible with respect to $\hat{\mu}_i = \mu_{\Lambda_i}(\cdot \,|\, \hat{\Omega}_{\Lambda_i})$. Thus by the well-known tensorisation property of the Poincaré and the log-Sobolev inequalities (see *e.g.* [ABC+00, Chapter 1]), we conclude that $\tilde{\gamma} = \min(\tilde{\gamma}_1, \dots, \tilde{\gamma}_n)$ and $\tilde{\alpha} = \max(\tilde{\alpha}_1, \dots, \tilde{\alpha}_n)$ with $\tilde{\gamma}_i$ and $\tilde{\alpha}_i$ the spectral gap and log-Sobolev constant of an FA1f process on Λ_i with filled boundary condition on the ergodic component with at least one zero which, using (A.4) and (A.5), can be expressed as $\tilde{\gamma}_{\Lambda_i}$ (A.20) and $\tilde{\alpha}_{\Lambda_i}$ (A.21) respectively. Then, Proposition A.5.2 below shows that $\tilde{\gamma} \geq c$ and $\tilde{\alpha} \leqslant c|\Lambda_i|$. Hence, for c as in Proposition A.5.2 it holds

$$\exp\left\{-\hat{\gamma}\frac{t}{2} + \exp\{-\frac{2t}{\hat{\alpha}}\}\log\frac{1}{\hat{\pi}^*}\right\} \leqslant \exp\left\{-\frac{t}{c} + c|\Lambda|e^{-t/(cM)}\right\}.$$

By collecting this inequality together with (A.16), (A.17) and (A.18) we end the proof. ✓

Proposition A.5.2 ([CMRT09]) *Let $A \subset V$ be connected and $\hat{\mu}_A(\cdot) = \mu_A(\cdot \,|\, \hat{\Omega}_A)$. Let ω be the entirely filled configuration (i.e. such that $\omega(x) = 1$ for all $x \in V$) Then, there exists a constant $c = c(q)$ such that*

$$\tilde{\gamma}_A := \inf_{f: f \neq const.} \frac{\sum_{x\in A} \hat{\mu}_A(\ c^{\omega}_{x,A} \operatorname{Var}_x(f))}{\operatorname{Var}_{\hat{\mu}_A}(f)} \geq c \tag{A.20}$$

and

$$\tilde{\alpha}_A := \sup_{f: f \neq const.} \frac{\operatorname{Ent}_{\hat{\mu}_A}(f)}{\sum_{x\in A} \hat{\mu}_A(c^{\omega}_{x,A} \operatorname{Var}_x(f))} \leq c|A|. \tag{A.21}$$

Proof

The first part on the spectral gap is proved in [CMRT09, Theorem 6.4 page 336]. In Section A.6 we give an alternative proof which gives a better bound for small q and can be extended to non cooperative models different from FA1f.

The second part easily follows from the standard bound [DSC96, SC97]

$$\hat{\alpha}_A \leq \hat{\gamma}_A^{-1} \log \frac{1}{\hat{\mu}^*_A}$$

where $\hat{\mu}^*_A := \min_{\sigma \in \hat{\Omega}_A} \hat{\mu}_A(\sigma) \geq \exp\{-c|A|\}$. ✓

A.6 Spectral gap on the ergodic component

In this section we estimate the spectral gap of the process FA1f on a finite volume with occupied boundary conditions on the ergodic component of configurations with at least one zero. This result has been used in the proof of Theorem A.2.1 as a key tool to prove Proposition A.5.1. This has been done in [CMRT08, CMRT09]. We present here an alternative proof based on the ideas of [MT12] that, on the one hand, gives a somehow more precise bound for very small q and, on the other hand, can be generalized to *non cooperative* models different from FA1f on some ergodic component (not necessarily the largest one). The remaining of the proof of Theorem A.2.1 for these models carries over along the same lines as for FA1f. An example of non cooperative model different from FA1f is the following. Each vertex x waits an independent mean one exponential time and then, provided that the current configuration σ is such that at least two of the sites at distance less or equal to 2 are empty ($\sum_{y\in\hat{\mathcal{N}}_x}(1-\sigma(y))\geq 2$, where $\hat{\mathcal{N}}_x=\{y\colon d(x,y)\leq 2\}$), the value $\sigma(x)$ is refreshed with a new value in $\{0,1\}$ sampled from a Bernoulli p measure and the whole procedure starts again. For simplicity we deal with the FA-1f model.

For every $\Lambda\subset V$ finite recall that $\hat\Omega_\Lambda$ is the set of configurations with at least one zero (A.19) and $\hat\mu_\Lambda(\cdot)=\mu_\Lambda(\cdot\,|\,\hat\Omega_\Lambda)$. By using (A.4) the spectral gap $\tilde\gamma_\Lambda$ for the dynamics on $\hat\Omega_\Lambda$ with filled boundary conditions can be expressed as

$$\tilde\gamma_\Lambda=\inf_{f\colon f\neq const.}\frac{\sum_{x\in\Lambda}\hat\mu_\Lambda(\hat c_x\,\mathrm{Var}_x(f))}{\mathrm{Var}_{\hat\mu_\Lambda}(f)} \tag{A.22}$$

where the infimum runs over all non constant functions $f\colon\hat\Omega_\Lambda\to\mathbb{R}$, $\mathrm{Var}_x(f):=\mathrm{Var}_{\mu_{\{x\}}}(f)$, and $\hat c_x(\sigma):=c^{\omega}_{x,\Lambda}(\sigma)$ with ω the entirely filled configuration, i.e. $\omega(x)=1$ for all $x\in V$. We are now ready to state the result on the spectral gap.

Theorem A.6.1 *Let $\mathcal{G}=(V,E)$ be a graph with (k,D)-polynomial growth. Then there exists a positive constant $C=C(k,D)$ such that for any connected set $\Lambda\subset V$*

$$\tilde\gamma_\Lambda\geq C\,\frac{q^{D+4}}{\log(2/q)^{D+1}}$$

The proof of Theorem A.6.1 is divided in two steps. At first we bound from below the spectral gap of the hat chain in Λ by the spectral gap of the FA1f model (not restricted to the ergodic component), on all subsets of V with minimal boundary condition. Then we study such a spectral gap following the strategy of [MT12].

We need some more notations. Given $A\subset V$, $z\in\partial A$ and $x\in A$ define $c^z_{x,A}(\sigma)=c^{\omega^{(z)}}_{x,A}(\sigma)$, $\sigma\in\Omega$, where $\omega^{(z)}$ is the entirely filled configuration, except at site z where it is 0: $\omega^{(z)}(x)=1$ for all $x\neq z$ and $\omega^{(z)}(z)=0$. The corresponding generator $\mathcal{L}_A^{\omega^{(z)}}$ will be simply denoted by $\mathcal{L}^z_A$. It corresponds to the FA1f process in A with minimal boundary condition.

The first step in the proof of Theorem A.6.1 is the following result.

Proposition A.6.2 *For any finite connected $\Lambda\subset V$ with $8p^{\mathrm{diam}(\Lambda)/3}<\frac12$ it holds*

$$\tilde\gamma_\Lambda\geq\frac{1}{48}\inf_{\substack{A\subset V,\text{connected}\\ z\in\partial A}}\mathrm{gap}(\mathcal{L}^z_A).$$

Observe that, combining [CMRT09, Theorem 6.1] and [CMRT08, Theorem 6.1] for any set A and any site z, we had $\mathrm{gap}(\mathcal{L}^z_A)\geq cq^{\log_2(1/q)}$ for some universal positive constant c. Hence, for the FA1f process, we had the lower bound

$$\tilde\gamma_\Lambda\geq cq^{\log_2(1/q)}.$$

We present below an alternative strategy (based on [MT12]) which can be applied to other non-cooperative models and gives a more accurate bound for the FA1f process when q is small.

Proof

Consider a non constant function $f\colon \hat{\Omega}_\Lambda \to \mathbb{R}$ with $\hat{\mu}(f) = 0$ and define $\tilde{f}\colon \Omega_\Lambda \to \mathbb{R}$ as

$$\tilde{f}(\sigma) = \begin{cases} f(\sigma) & \text{if } \sigma \in \hat{\Omega}_\Lambda \\ 0 & \text{otherwise} \end{cases}$$

We divide[2] Λ into two disjoint connected subsets A and B such that their diameter is larger then $|\Lambda|/3$.

Thank to Lemma A.6.5 below (our hypothesis implies that $\max(1-\mu(c_A), 1-\mu(c_B)) < 1/16$)

$$\mathrm{Var}_{\hat{\mu}_\Lambda}(f) \le 24\, \hat{\mu}_\Lambda[c_B \,\mathrm{Var}_{\mu_A}(\tilde{f}) + c_A \,\mathrm{Var}_{\mu_B}(\tilde{f})]$$

where $c_A = \mathbb{1}_{\hat{\Omega}_A}$ and $c_B = \mathbb{1}_{\hat{\Omega}_B}$ and $\hat{\Omega}_A$ and $\hat{\Omega}_B$ are defined in (A.19).

Consider the first term. Define the random variable

$$\zeta := \sup_{x \in B}\{d(A, x)\colon \sigma(x) = 0\}$$

where by convention the supremum of the empty set is ∞. The function c_B guarantees that $\zeta \in \{1, 2, \cdots, \mathrm{diam}(\Lambda)\}$. Following the strategy of [CMRT08] we have

$$\begin{aligned} \hat{\mu}_\Lambda[c_B \,\mathrm{Var}_{\mu_A}(\tilde{f})] &= \frac{1}{\mu_\Lambda(\hat{\Omega}_\Lambda)} \sum_{n\ge 1} \mu_\Lambda[\mathbb{1}_{\zeta=n} \,\mathrm{Var}_{\mu_A}(\tilde{f})] \\ &\le \frac{1}{\mu_\Lambda(\hat{\Omega}_\Lambda)} \sum_{n\ge 1} \mu_\Lambda[\mathbb{1}_{\zeta=n} \,\mathrm{Var}_{\mu_{A_n}}(\tilde{f})] \end{aligned}$$

where $A_n = \{x \in \Lambda\colon d(A, x) \le n-1\}$ and we used the convexity of the variance (which is valid since the event $\{\zeta = n\}$ does not depend, by construction, on the value of the configuration σ_{A_n} inside A_n). The indicator function above $\mathbb{1}_{\zeta=n}$ guarantees the presence of a zero on the boundary ∂A_n of the set A_n. Order (arbitrarily) the points of ∂A_n and call Z the (random) position of the first empty site on ∂A_n. Then, for all $n \ge 1$,

$$\begin{aligned} \mu_\Lambda[\mathbb{1}_{\zeta=n} \,\mathrm{Var}_{\mu_{A_n}}(\tilde{f})] &= \sum_{z \in \partial A_n} \mu_\Lambda[\mathbb{1}_{\zeta=n}\mathbb{1}_{Z=z} \,\mathrm{Var}_{\mu_{A_n}}(\tilde{f})] \\ &\le \sum_{z\in\partial A_n} \mathrm{gap}(\mathcal{L}^z_{A_n})^{-1} \sum_{y \in A_n} \mu_\Lambda[\mathbb{1}_{\zeta=n}\mathbb{1}_{Z=z}\mu_{A_n}(c^z_{y,A_n} \,\mathrm{Var}_y(\tilde{f}))] \\ &\le \gamma \sum_{z\in\partial A_n}\sum_{y\in A_n} \mu_\Lambda[\mathbb{1}_{\zeta=n}\mathbb{1}_{Z=z} c^z_{y,A_n} \,\mathrm{Var}_y(\tilde{f})] \end{aligned}$$

where we used the fact that the events $\{\zeta = n\}$ and $\{Z = z\}$ depend only on $\sigma^c_{A_n}$, and where $\gamma := \sup \mathrm{gap}(\mathcal{L}^z_A)^{-1}$, the supremum running over all connected subset A of V and all $z \in \partial A$.

[2] To construct A and B take two points x, y such that $d(x, y) = \ell := \mathrm{diam}(\Lambda)$ and define $A_0 = \{z \in \Lambda\colon d(x, z) \le \ell/3\}$ and $B_0 = \{z \in \Lambda\colon d(y, z) \le \ell/3\}$. Attach to A_0 all the connected components of $\Lambda \setminus (A_0 \cup B_0)$ connected to A_0 to obtain A, then attach all the remaining connected components of $\Lambda \setminus (A_0 \cup B_0)$ to B_0 to obtain B.

Now observe that $\mathbb{1}_{\zeta=n}\mathbb{1}_{Z=z}c^z_{y,A} \leq \mathbb{1}_{\zeta=n}\mathbb{1}_{Z=z}\hat{c}_y$ for any $y \in A_n$. Hence,

$$\begin{aligned}\hat{\mu}_\Lambda[c_B \operatorname{Var}_{\mu_A}(\tilde{f})] &\leq \frac{\gamma}{\mu_\Lambda(\hat{\Omega}_\Lambda)} \sum_{n\geq 1}\sum_{z\in\partial A_n}\sum_{y\in A_n} \mu_\Lambda[\mathbb{1}_{\zeta=n}\mathbb{1}_{Z=z}\hat{c}_y \operatorname{Var}_y(\tilde{f})] \\ &\leq \frac{\gamma}{\mu_\Lambda(\hat{\Omega}_\Lambda)} \sum_{y\in\Lambda}\sum_{n\geq 1}\sum_{z\in\partial A_n} \mu_\Lambda[\mathbb{1}_{\zeta=n}\mathbb{1}_{Z=z}\hat{c}_y \operatorname{Var}_y(\tilde{f})] \\ &= \gamma\sum_{y\in\Lambda}\hat{\mu}_\Lambda[\hat{c}_y \operatorname{Var}_y(\tilde{f})] = \gamma\sum_{y\in\Lambda}\hat{\mu}_\Lambda[\hat{c}_y \operatorname{Var}_y(f)].\end{aligned}$$

The same holds for $\hat{\mu}_\Lambda[c_A \operatorname{Var}_{\mu_B}(\tilde{f})]$, leading to the expected result. ✓

The second step in the proof of Theorem A.6.1 is a careful analysis of $\operatorname{gap}(\mathcal{L}^z_A)$ for any given connected set $A \subset V$ and $z \in \partial A$.

Proposition A.6.3 *Let $\mathcal{G} = (V, E)$ be a graph with (k, D)-polynomial growth. Then, there exists a universal constant $C = C(k, D)$ such that for any connected set $A \subset V$, and any $z \in \partial A$, it holds*

$$\operatorname{gap}(\mathcal{L}^z_A) \geq C\frac{q^{D+4}}{\log(2/q)^{D+1}}.$$

We postpone the proof of Proposition A.6.3 to end the proof of Theorem A.6.1.

Proof of Theorem A.6.1

The result follows at once combining Proposition A.6.2 and Proposition A.6.3. ✓

In order to prove Proposition A.6.3, we need a preliminary result on the spectral gap of some auxiliary chain, and to order the points of A in a proper way, depending on z. Let $N := \max_{x\in A} d(x, z)$, for any $i = 1, 2, \ldots, N$, we define

$$A_i := \{x \in A \colon d(x, z) = i\} = \{x^{(i)}_1, \ldots, x^{(i)}_{n_i}\}$$

where $x^{(i)}_1, \ldots, x^{(i)}_{n_i}$ is any chosen order. Then we say that for any $x, y \in A$, $x \leq y$ if either $d(x, z) > d(y, z)$ or $d(x, z) = d(y, z)$ and x comes before y in the above ordering. Then, we set $A_x = \{y \in A \colon y \geq x\}$ and $\tilde{A}_x = A_x \setminus \{x\}$.

Lemma A.6.4 *Fix a connected set $A \subset V$, and $z \in \partial A$. For any $x \in A$ and $\sigma \in \Omega$, let $E_x \subset \Omega_{\tilde{A}_x}$, $\Delta_x = \operatorname{supp}(\mathbb{1}_{E_x})$ and $\tilde{c}_x(\sigma) = \mathbb{1}_{E_x}(\sigma_{\tilde{A}_x})$. Assume that*

$$\sup_{x\in A}\mu(1 - \tilde{c}_x)\sup_{x\in A}|\{y \in A \colon \Delta_y \cup \{y\} \ni x\}| < \frac{1}{4}.$$

Then, for any $f : \Omega_A \to \mathbb{R}$ it holds

$$\operatorname{Var}_{\mu_A}(f) \leq 4\sum_{x\in A}\mu_A(\tilde{c}_x \operatorname{Var}_x(f)).$$

Proof

We follow [MT12]. In all the proof, to simplify the notations, we set $\operatorname{Var}_B = \operatorname{Var}_{\mu_B}$, for any B. First, we claim that

$$\operatorname{Var}_A(f) \leqslant \sum_{x\in A}\mu_A(\operatorname{Var}_{A_x}(\mu_{\tilde{A}_x}(f))). \tag{A.23}$$

Take $x = x^{(N)}_{n_N}$, by factorization of the variance, we have

$$\mathrm{Var}_A(f) = \mu_A(\mathrm{Var}_{\tilde{A}_x}(f)) + \mathrm{Var}_A(\mu_{\tilde{A}_x}(f)).$$

The claim then follows by iterating this procedure, removing one site at a time, in the order defined above.

We analyze one term in the sum of (A.23) and assume, without loss of generality, that $\mu_{A_x}(f) = 0$. We write $\mu_{\tilde{A}_x}(f) = \mu_{\tilde{A}_x}(\tilde{c}_x f) + \mu_{\tilde{A}_x}((1-\tilde{c}_x)f)$ so that

$$\mu_A[\mathrm{Var}_{A_x}(\mu_{\tilde{A}_x}(f))] \le 2\mu_A[\mathrm{Var}_{A_x}(\mu_{\tilde{A}_x}(\tilde{c}_x f))] + 2\mu_A[\mathrm{Var}_{A_x}(\mu_{\tilde{A}_x}((1-\tilde{c}_x)f))]. \tag{A.24}$$

Observe that, by convexity of the variance and since $\tilde{c}_x$ does not depend on x, the first term of the latter can be bounded as

$$\mu_A[\mathrm{Var}_{A_x}(\mu_{\tilde{A}_x}(\tilde{c}_x f))] = \mu_A[\mathrm{Var}_x(\mu_{\tilde{A}_x}(\tilde{c}_x f))] \le \mu_A[\tilde{c}_x \,\mathrm{Var}_x(f)].$$

Now we focus on the second term of (A.24). Note that $\mu_{\tilde{A}_x}[(1-\tilde{c}_x)f)] = \mu_{\tilde{A}_x}[(1-\tilde{c}_x)\mu_{\tilde{A}_x\setminus\Delta_x}(f))]$. Set $\delta := \sup_{x\in A}\mu(1-\tilde{c}_x)$. Hence, bounding the variance by the second moment and using Cauchy-Schwarz inequality, we get

$$\begin{aligned}\mathrm{Var}_{A_x}(\mu_{\tilde{A}_x}((1-\tilde{c}_x)f)) &\le \mathrm{Var}_{A_x}\left(\mu_{\tilde{A}_x}[(1-\tilde{c}_x)\mu_{\tilde{A}_x\setminus\Delta_x}(f)]\right)\\ &\le \mu_{A_x}\left(\mu_{\tilde{A}_x}[(1-\tilde{c}_x)\mu_{\tilde{A}_x\setminus\Delta_x}(f)]^2\right)\\ &\le \delta\left(\mathrm{Var}_{A_x}(\mu_{\tilde{A}_x\setminus\Delta_x}(f))\right)\end{aligned}$$

From all the previous computations (and using (A.23)) we deduce that

$$\mathrm{Var}_A(f) \le 2\sum_{x\in A}\mu_A(\tilde{c}_x\,\mathrm{Var}_x(f)) + 2\delta\sum_{x\in A}\mu_A\left(\mathrm{Var}_{A_x}(\mu_{\tilde{A}_x\setminus\Delta_x}(f))\right).$$

Hence if one proves that

$$\sum_{x\in A}\mu_A\left(\mathrm{Var}_{A_x}(\mu_{\tilde{A}_x\setminus\Delta_x}(f))\right) \le \sup_{y\in A}|\{x\in A\colon \Delta_x\cup\{x\}\ni y\}|\,\mathrm{Var}_A(f) \tag{A.25}$$

the result follows. We now prove (A.25). Using (A.23), we have

$$\mathrm{Var}_{A_x}(g) \leqslant \sum_{y\in A_x}\mu_{A_x}\left(\mathrm{Var}_{A_y}(\mu_{\tilde{A}_y}(g))\right) = \sum_{y\in\Delta_x\cup\{x\}}\mu_{A_x}\left(\mathrm{Var}_{A_y}(\mu_{\tilde{A}_y}(g))\right)$$

where $g = \mu_{\tilde{A}_x\setminus\Delta_x}(f)$ and we used that $\mathrm{supp}(g)\subset A\setminus(\tilde{A}_x\setminus\Delta_x)$. It follows that

$$\mu_A\left(\mathrm{Var}_{A_x}(g)\right)\sum_{y\in\Delta_x\cup\{x\}}\mu_A\left(\mathrm{Var}_{A_y}(\mu_{\tilde{A}_y}(g))\right) \le \sum_{y\in\Delta_x\cup\{x\}}\mu_A\left(\mathrm{Var}_{A_y}(\mu_{\tilde{A}_y}(f))\right)$$

since, by Cauchy-Schwarz,

$$\begin{aligned}\mu_A\left(\mathrm{Var}_{A_y}(\mu_{\tilde{A}_y}(g))\right) &= \mu_A\left(\left[\mu_{\tilde{A}_x\setminus\Delta_x}\left(\mu_{\tilde{A}_y}(f)-\mu_{A_y}(f)\right)\right]^2\right)\\ &\le \mu_A\left(\mathrm{Var}_{A_y}(\mu_{\tilde{A}_y}(f))\right).\end{aligned}$$

This ends the proof. ✓

Proof of Proposition A.6.3

Our aim is to apply Lemma A.6.4. Let us define the events E_x, for $x \in A$. Fix an integer ℓ that will be chosen later and set $n = \ell \wedge d(x,z)$. Let $(x_1, x_2, \ldots, x_n)$ be an arbitrarily chosen ordered collection satisfying $d(x_i, x_{i+1}) = 1$, $d(x_i, x) = i$ and $d(x_i, z) = d(x,z) - i$ for $i = 0, \ldots, n$, with the convention that $x_0 = x$, and set $E_x = \{\sigma \in \Omega \colon \sum_{i=1}^n (1 - \sigma(x_i)) \geq 1\}$, i.e. E_x is the event that at least one of the site of $\Delta_x = \{x_1, x_2, \ldots, x_n\}$ is empty. Note that by construction $\Delta_x \subset A \cup \{z\}$ and is connected. Moreover for any x such that $d(x,z) \leq \ell$, $E_x = \Omega$ so that $\tilde{c}_x \equiv 1$. Since $|\Delta_x| \leqslant k\ell^D$ for any $x \in A$, the assumption of Lemma A.6.4 reads

$$p^\ell (1 + k\ell^D) < 1/4$$

which is satisfied if one chooses $\ell = \frac{c}{q} \log \frac{2}{q}$ with $c = c(k, D)$ large enough. Hence for any $f : \Omega_A \to \mathbb{R}$ it holds

$$\operatorname{Var}_{\mu_A}(f) \leq 4 \sum_{x \in A} \mu_A(\tilde{c}_x \operatorname{Var}_x(f)).$$

and we are left with the analysis of each term $\mu_A(\tilde{c}_x \operatorname{Var}_x(f))$ for which we use a path argument. Fix $x \in A$ and the collection $(x_1, x_2, \ldots, x_n)$ introduced above. Given a configuration σ such that $\tilde{c}_x(\sigma) = 1$, denote by ξ the (random) distance between x and the first empty site in the collection $(x_1, x_2, \ldots, x_n)$: i.e. $\xi(\sigma) = \inf\{i \colon \sigma(x_i) = 0\}$. Then we write

$$\begin{aligned} \mu_A(\tilde{c}_x \operatorname{Var}_x(f)) &= \sum_{i=1}^n \mu_A(\tilde{c}_x \mathbb{1}_{\xi = i} \operatorname{Var}_x(f)) \\ &= pq \sum_{i=1}^n \sum_{\sigma : \xi(\sigma) = i} \mu_A(\sigma)(f(\sigma^x) - f(\sigma))^2 \end{aligned}$$

where the sum is understood to run over all σ such that $\tilde{c}_x(\sigma) = 1$ (and $\xi(\sigma) = i$).

Fix $i \in \{1, \ldots, n\}$. For any $\sigma \in \Omega$ such that $\xi(\sigma) = i$, we construct a path of configurations $\gamma_x(\sigma) = (\sigma_0 = \sigma, \sigma_1, \sigma_2, \ldots, \sigma_{4i-5} = \sigma^x)$ from σ to σ^x, of length $4i - 5 \leq 4\ell$. The idea behind the construction is to bring an empty site from x_i, step by step, toward x_1, make the flip in x and going back, keeping track of the initial configuration σ. For any j, σ_{j+1} can be obtained from σ_j by a legal flip for the FA1f process. Furthermore σ_j differs from σ on at most three sites (possibly counting x). More precisely, define $T_k(\sigma) := \sigma^{x_k}$ for any k and σ, and

$$\sigma_j = \begin{cases} T_{i-k-1}(\sigma) & \text{if } j = 2k+1, \text{ and } k = 0, 1, \ldots, i-2 \\ T_{i-k} \circ T_{i-k-1}(\sigma) & \text{if } j = 2k, \text{ and } k = 1, \ldots, i-2 \\ T_1(\sigma^x) & \text{if } j = 2i - 2 \\ T_{k-i+2} \circ T_{k-i+3}(\sigma^x) & \text{if } j = 2k+1, \text{ and } k = i-1, \ldots, 2i-4 \\ T_{k-i+2}(\sigma^x) & \text{if } j = 2k, \text{ and } k = i, \ldots, 2i-3. \end{cases}$$

See Figure A.2 for a graphical illustration of such a path.

Denote by $\Gamma_x(\sigma) = \{\sigma_0, \sigma_1, \ldots, \sigma_{4i-6}\}$ (i.e. the configurations of the path $\gamma_x(\sigma)$ except the last one σ^x). For any $\eta = \sigma_j \in \Gamma_x(\sigma)$, $j \geq 1$, let $y = y(x, \eta) \in \{x, x_1, x_2, \ldots, x_\ell\}$ be such that

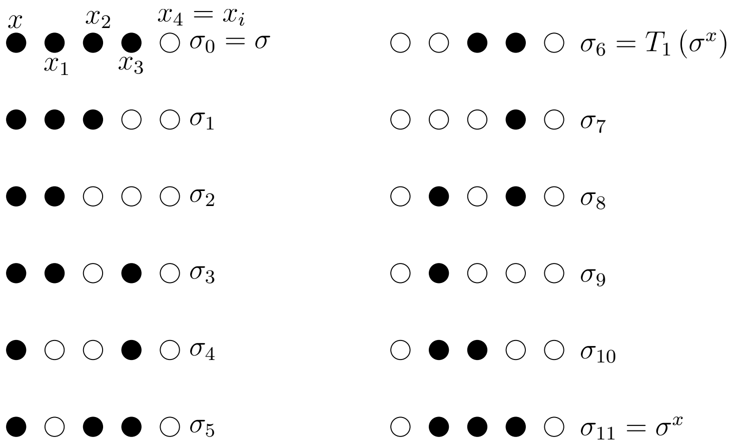

Figure A.2: Illustration of the path from σ to σ^x for a configuration σ satisfying $\xi(\sigma) = x_4$. Here $i = 4$ and the length of the path is $4i - 5 = 11$.

$\eta = \sigma^y_{j-1}$. Then, by Cauchy-Schwarz inequality,

$$\begin{aligned}(f(\sigma^x) - f(\sigma))^2 &= \left(\sum_{\eta \in \Gamma_x(\sigma)} (f(\eta^y) - f(\eta)) \right)^2 \leq 4\ell \sum_{\eta \in \Gamma_x(\sigma)} (f(\eta^y) - f(\eta))^2 \\ &\leq \frac{4\ell}{pq} \sum_{\eta \in \Gamma_x(\sigma)} c_y(\eta) \operatorname{Var}_y(f)(\eta).\end{aligned}$$

Hence,

$$\mu_A(\tilde{c}_x \operatorname{Var}_x(f)) \leq 4\ell K \sum_\eta \mu_A(\eta) c_y(\eta) \operatorname{Var}_y(f)$$

where

$$K = \sup_{\eta \in \Omega, x \in A} \left\{ \sum_\sigma \sum_{i=1}^{\ell} \frac{\mu_A(\sigma)}{\mu_A(\eta)} \mathbb{1}_{\xi(\sigma)=i} \mathbb{1}_{\Gamma_x(\sigma) \ni \eta} \right\} \leq \frac{8}{q^3}.$$

Indeed $\mu_A(\sigma)/\mu_A(\eta) \leq \frac{p^2}{q^2} \max(\frac{p}{q}, \frac{q}{p})$ since any $\eta \in \Gamma_x(\sigma)$ has at most two extra empty sites with respect to σ and differs from σ in at most three sites, and we used a computing argument.

Recall that $y = y(x, \eta)$. It follows from the latter that

$$\begin{aligned}\operatorname{Var}_{\mu_A}(f) &\leq \frac{128\ell}{q^3} \sum_{x \in A} \sum_\eta \mu_A(\eta) c_y(\eta) \operatorname{Var}_y(f) \\ &\leq \frac{128\ell}{q^3} K' \sum_{u \in A} \sum_\eta \mu_A(\eta) c_u(\eta) \operatorname{Var}_u(f)\end{aligned}$$

where

$$K' = \sup_\eta \sum_{x \in A} \mathbb{1}_{y(x,\eta)=u} \leq \sup_{u \in A} |B(u, \ell)|.$$

The result follows since the graph has polynomial growth. ✓

In Proposition A.6.2 we used the following lemma.

Lemma A.6.5 *Take $\Lambda, A, B \subset V$ such that $\Lambda = A \cup B$ and $A \cap B = \emptyset$. Define $c_A = \mathbb{1}_{\hat{\Omega}_A}$ and $c_B = \mathbb{1}_{\hat{\Omega}_B}$ where $\hat{\Omega}_A$ and $\hat{\Omega}_B$ are defined in (A.19). Assume that $\max(1-\mu(c_A), 1-\mu(c_B)) < 1/16$. Then, for all $f\colon \hat{\Omega}_\Lambda \to \mathbb{R}$ with $\hat{\mu}_\Lambda(f) = 0$ it holds*

$$\operatorname{Var}_{\hat{\mu}_\Lambda}(f) \leq 24\hat{\mu}_\Lambda[c_B \operatorname{Var}_{\mu_A}(\tilde{f}) + c_A \operatorname{Var}_{\mu_B}(\tilde{f})]$$

where $\tilde{f}\colon \Omega_\Lambda \to \mathbb{R}$ is defined as

$$\tilde{f}(\sigma) = \begin{cases} f(\sigma) & \text{if } \sigma \in \hat{\Omega}_\Lambda \\ 0 & \text{otherwise} \end{cases}$$

Proof

Recalling the variational definition of the variance we have

$$\begin{aligned}\operatorname{Var}_{\hat{\mu}_\Lambda}(f) &= \inf_{m\in\mathbb{R}} \hat{\mu}_\Lambda(|f-m|^2)\\ &\leq \frac{1}{\mu_\Lambda(\hat{\Omega}_\Lambda)} \inf_{m\in\mathbb{R}} \mu_\Lambda((f\mathbb{1}_{\hat{\Omega}_\Lambda} - m)^2)\\ &= \frac{1}{\mu_\Lambda(\hat{\Omega}_\Lambda)} \operatorname{Var}_{\mu_\Lambda}(\tilde{f}).\end{aligned}$$

Observe now that, by construction, $\mu_\Lambda(\tilde{f}) = 0$ and $(1-c_A)(1-c_B)\tilde{f} = 0$ so that we can apply Lemma A.6.6 below and obtain

$$\operatorname{Var}_{\hat{\mu}_\Lambda}(f) \leq \frac{24}{\mu_\Lambda(\hat{\Omega}_\Lambda)} \mu_\Lambda[c_B \operatorname{Var}_{\mu_A}(\tilde{f}) + c_A \operatorname{Var}_{\mu_B}(\tilde{f})]$$

and the result follows. ✓

The next Lemma might be heuristically seen as a result on the spectral gap of some constrained blocks dynamics (see [CMRT08]). Such a bound can be of independent interest.

Lemma A.6.6 *Let $\Lambda = A \cup B$ with A, $B \subset V$ satisfying $A \cap B = \emptyset$. Define μ_A and μ_B two probability measures on $\{0,1\}^A$ and $\{0,1\}^B$ respectively, and $\mu = \mu_A \otimes \mu_B$. Take c_A, $c_B\colon \{0,1\}^\Lambda \to [0,1]$ with support in A and B respectively. For any function g on $\{0,1\}^\Lambda$ such that $(1-c_A)(1-c_B)g = 0$ it holds*

$$\begin{aligned}\operatorname{Var}_\mu(g) \leq{}& 12\mu[c_B^2 \operatorname{Var}_{\mu_A}(g) + c_A^2 \operatorname{Var}_{\mu_B}(g)]\\ &+ 8\max(1-\mu(c_A), 1-\mu(c_B)) \operatorname{Var}_\mu(g).\end{aligned}$$

Proof

Fix g on $\{0,1\}^\Lambda$ such that $(1-c_A)(1-c_B)g = 0$ and assume without loss of generality that $\mu(g) = 0$. First we write

$$\begin{aligned}g &= c_B(g-\mu_A(g)) + (1-c_B)c_A(g-\mu_B(g)) + (1-c_B)c_A\mu_B(g)\\ &\quad - (1-c_B)c_A\mu_A(g) + (1-c_B)(1-c_A)(g-\mu_A(g)) + \mu_A(g)\\ &= c_B(g-\mu_A(g)) + (1-c_B)c_A(g-\mu_B(g)) + (1-c_B)c_A\mu_B(g) + c_B\mu_A(g)\end{aligned}$$

where we used the first hypothesis on g, $(1-c_A)(1-c_B)g = 0$, and we arranged the terms. Therefore since we assumed $\mu(g) = 0$ and c_A, $c_B \in [0,1]$

$$\begin{aligned}\mathrm{Var}_\mu(g) = \mu(g^2) &\le 4\mu(c_B^2(g-\mu_A(g))^2) + 4\mu(c_A^2(g-\mu_B(g))^2)\\ &\quad + 4\mu(\mu_B(g)^2) + 4\mu(\mu_A(g)^2)\\ &= 4\mu[c_B^2 \mathrm{Var}_{\mu_A}(g) + c_A^2 \mathrm{Var}_{\mu_B}(g)]\\ &\quad + 4\mu(\mu_B(g)^2) + 4\mu(\mu_A(g)^2).\end{aligned}$$

We now treat the fourth term in the latter inequality.

$$\begin{aligned}[\mu_A(g)]^2 &= [\mu_A(g) - \mu(g)]^2 = [\mu_A(g-\mu_B(g))]^2\\ &= [\mu_A(c_A[g-\mu_B(g)]) + \mu_A([1-c_A][g-\mu_B(g)])]^2\\ &\le 2\mu_A(c_A^2[g-\mu_B(g)]^2) + 2\mu_A((1-c_A)^2)\mu_A([g-\mu_B(g)]^2)\end{aligned}$$

If we average with respect to μ we have

$$\mu(\mu_A(c_A^2[g-\mu_B(g)]^2)) = \mu(c_A^2 \mathrm{Var}_{\mu_B}(g))$$

and, using Cauchy-Schwarz inequality and $x^2 \le x$ for $x \in [0,1]$,

$$\begin{aligned}\mu(\mu_A((1-c_A)^2)\mu_A([g-\mu_B(g)]^2)) &= \mu_A((1-c_A)^2)\mu([g-\mu_B(g)]^2)\\ &\le (1-\mu(c_A))\mathrm{Var}_\mu(g),\end{aligned}$$

so that

$$\mu(\mu_A(g)^2) \le 2\mu(c_A^2 \mathrm{Var}_{\mu_B}(g)) + 2(1-\mu(c_A))\mathrm{Var}_\mu(g).$$

An analogous calculation for $\mu(\mu_B(g)^2)$ allows to conclude the proof. ✓

Appendix B

Front progression in the East model

This article is published in *Stochastic Processes and their Applications*, Vol. 123, Issue 9, Sept. 2013, p. 3430-3465, http://dx.doi.org/10.1016/j.spa.2013.04.014.

Contents

The East model is a one-dimensional, non-attractive interacting particle system with Glauber dynamics, in which a flip is prohibited at a site x if the right neighbour $x+1$ is occupied. Starting from a configuration entirely occupied on the left half-line, we prove a law of large numbers for the position of the left-most zero (the front), as well as ergodicity of the process seen from the front. For want of attractiveness, the one-dimensional shape theorem is not derived by the usual coupling arguments, but instead by quantifying the local relaxation to the non-equilibrium invariant measure for the process seen from the front. This is the first proof of a shape theorem for a kinetically constrained spin model.

B.1 Introduction

The East model belongs to the class of kinetically constrained spin models (KCSM), which have been introduced in the physics literature to model glassy dynamics ([JE91], see [RS03, TGS] for physics reviews). KCSM are Markov processes on the space of configurations on a graph. In the case of the East model, the graph is $\mathbb{Z}$ and the state space is $\{0,1\}^{\mathbb{Z}}$. Zeros and ones correspond to empty and occupied sites respectively. The evolution is given by a Glauber dynamics: each site refreshes its state with rate one, to a zero or to a one respectively with probability q and $p = 1 - q$, provided the current configuration satisfies a specific constraint. For the East model, the constraint imposes that the right neighbour of the to-be-updated site be empty (see [FMRT12b] for a recent mathematical review). The constraint required to update site x is independent of the state of x, so that the product Bernoulli measure of parameter p (the equilibrium density of ones) is reversible for the process, so the thermodynamics of the system is trivial (see Figure B.1). In turn, the difficulty of the study of KCSM is concentrated on their dynamical features. In comparison with other interacting particle systems, KCSM are challenging models from a mathematical point of view, mainly because they are not attractive, due to the presence of constraints in the dynamics. In particular, usual coupling arguments cannot be used (see the original methods developed in [CMRT08] for instance). They also admit blocked configurations (where all flip rates are zero), and several invariant measures. In the East model, as in other KCSM, the dynamical constraint induces the creation of “bubbles" (see Figure B.1). They correspond to frozen zones where no flip can happen. This kind of dynamical heterogeneities is also observed in supercooled liquids. One open issue in the study of KCSM is thus to determine the shape of those “bubbles". Inspired by this consideration, we study a system evolving according to the East dynamics, started with a configuration entirely occupied in the negative half-line (see Figure B.1). The system is then out of equilibrium. This can be understood as a blow-up of the system on the boundary of a bubble. Our results deal with the behaviour of the leftmost zero –which we will refer to as the front–, as well as the distribution of the configuration that it sees. More precisely, at any time we can consider the configuration obtained by shifting the current configuration so that the front sits in zero. This yields what we call the process seen from the front. Note that zeros cannot appear in the middle of ones, so we can understand the front as a tagged void.

The results of this paper are a law of large numbers for the position of the front (Theorem B.6.1) and the ergodicity of the process seen from the front (Theorem B.5.1), namely the uniqueness of its invariant measure and the convergence towards it.

Shape theorems have been studied in a number of contexts (see [Dur80] or [KRS12]). Most of the time, some kind of attractiveness or monotonicity is needed in a crucial way to use a subadditive argument. As we have already mentioned, we have no such property in the East model, so we have to devise a new argument to get a shape theorem. For want of attractiveness, we use an argument of relaxation to equilibrium behind the front. The natural two-dimensional counterpart of the East model is the North-East model. For that model also, a limit shape was conjectured in [KL06], which would be the natural 2D extension of our result, but it seems far from being proven yet.

Invariant measures for systems seen from a tagged particle have also been studied, for instance in the context of the simple exclusion process, where product Bernoulli measures are stationary for the system seen from the tagged particle. This is not the case for us. In fact, our work is also related to the study of stationary measures for infinite dimensional processes started out of equilibrium, which is also the object of [KS01] in a different setting. The issue is to control the interplay between an infinite dimensional, well-behaved part, and a finite dimensional part that generates a lot of noise.

Guideline through the main results

Let us give an overview of the strategy designed to prove our results.

Classic proofs in the study of front progression or invariant measures for interacting particle systems usually rely on the basic (or standard) coupling between two appropriate processes. In the East model, since there is no attractiveness, the basic coupling is useless. We establish here

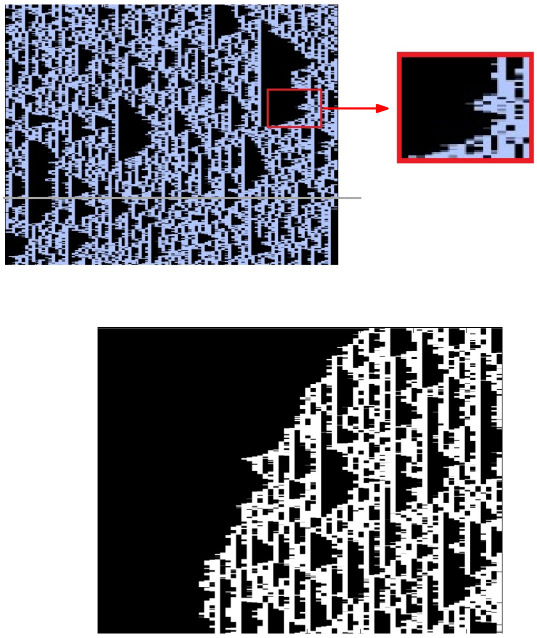

Figure B.1: A simulation of the East dynamics. Ones are black, space is horizontal, time goes downward. In the first picture, the system is at equilibrium with density $p = 1/2$: along the grey line (or at any time), the law of the configuration is just given by independent Bernoulli variables of parameter $1/2$. On the right, a blow-up of the system on the border of a “bubble". Below, a simulation of the front dynamics run with parameter $p = 1/2$.

a more elaborate coupling result (Theorem B.4.7), which is the key result on which both proofs rely: the law of large numbers (Theorem B.6.1), and the ergodicity of the process seen from the front (Theorem B.5.1). One difficulty in our study is that there is no explicit expression to describe the behaviour of the configuration near the front. Somehow, we get round this issue by proving a quite detailed result of relaxation far behind the front. Namely, Theorem B.4.7 says that, starting from any configuration with a leftmost zero, after enough time, the distribution of the configuration at a distance L behind the front is exponentially (in L) close to the equilibrium measure in terms of total variation distance. The proof of this result is the object of section B.4. Let us get back later to the methods we use to derive this result, and see now how we can use it to prove the law of large numbers for the position of the front in section B.5 and the ergodicity of the system seen from the front in section B.6.

The proof of the ergodicity of the process seen from the front is actually contained in the coupling result of Theorem B.5.2. Starting from any two configurations, we are able to construct a coupling between the configurations seen from the front at time t such that with probability going to 1 they agree on a distance arbitrarily large. The construction of this coupling is inspired by [KS01] and [KPS02]. In those works, the authors study a random dynamical system. Define recursively u^k, an infinite dimensional vector on a Hilbert space $\mathcal{H}$, by:

$$u^k = S\left(u^{k-1}\right) + \eta_k$$

with η a random noise with independent coordinates and S an operator with “good" properties. In particular, S contracts quite strongly the last coordinates. The authors make use of this fact to construct a coupling that brings together two trajectories started from different points. Let $\mathcal{H} = \mathcal{H}_N \bigoplus \mathcal{H}_N^{\perp}$, where $\mathcal{H}_N$ is the subspace generated by the first N coordinates. On the one hand, the contraction property of the operator guarantees that the dynamics is well-behaved on the infinite dimensional subspace $\mathcal{H}_N^\perp$. On the other hand, the projection on $\mathcal{H}_N$ is a stochastic finite dimensional system, which is easier to study. The delicate issue is to understand the coupling between both parts. In our system, the first N particles behind the front could be interpreted as the analogous of $\mathcal{H}_N$, and $\mathcal{H}_N^\perp$ would represent all the particles beyond distance N. The result of Theorem B.4.7 gives us a good control on what is happening far from the front, and the idea behind the construction of our coupling is that the part immediately behind the front is finite. The difficulty is to control the two parts together. To this end, we design an iterative construction in the spirit of [KS01]. In this procedure, until the coupling is successful, each step first brings together the "infinite" parts (far from the front) with good probability. Then we use the fact that the remaining parts (close to the front) are finite and thus have a positive probability of agreeing after some time.

Not much is known on the structure of the invariant measure constructed in this way, or on the speed of convergence towards it. This means in particular that further arguments are needed to implement a form of subadditive theorem and to prove the law of large numbers. As if we wanted to use the classic proof using the subadditive ergodic theorem, we cut the trajectories into smaller bits (see Figure B.8). Then, on each bit, we go back a distance L, look for the first zero on the right to play the role of the front and erase the zeros on its left. Thanks to the orientation of the East model, the original front is always on the left of the new ones. Those changes induce a small correction if L is chosen correctly. And now, thanks to the local equilibrium result derived in Theorem B.4.7, all the new fronts have almost the distribution of a front with initial configuration chosen with the equilibrium (product) measure on the right. Moreover, if we treat separately the terms corresponding to the different dash styles in Figure B.8 (with a well chosen, but fixed, number of dash styles), they are almost independent in a certain sense. With these almost iid variables, we can use the classic proof of the law of large numbers for variables with a fourth moment.

Let us get back to the key Theorem B.4.7, which is proved in section B.4. First, we prove a local relaxation result using the tool of the distinguished zero introduced in [AD02]. The distinguished zero (see Figure B.2) can be understood as a zero boundary condition moving to the right. It leaves equilibrium on its left, and in particular a number of zeros. A careful conditioning by the entire dynamics on the right of this moving boundary allows us to average locally at large time an evoluted function that may have infinite support. This being established, it remains to keep track of enough zeros to be able to distinguish a pertinent one at the right time. We distinguish the front –which is a particular zero– at different times, and use ballistic bounds on the front motion to guarantee that it will leave a number of zeros appropriately distributed behind it (see figures B.5 and B.6). We then distinguish one of these (which the previous study guarantees is not too far) to apply the above relaxation result. We get Theorem B.4.4 that tells us that on the site at distance L from the front, the distribution is Bernoulli with error at most $e^{-\epsilon L}$. Then Theorem B.4.7 is basically an iteration of this result.

B.2 Model

B.2.1 Setting and notations

The space of configurations for the East model on $\mathbb{Z}$ (resp. on $\Lambda \subset \mathbb{Z}$) is $\Omega = \{0,1\}^{\mathbb{Z}}$ (resp. $\Omega_\Lambda = \{0,1\}^{\Lambda}$). For $\omega \in \Omega$, we write $\omega = (\omega_x)_{x\in\mathbb{Z}}$, ω_x denoting the state of site x in the configuration ω. If $\omega_x = 1$ (resp. $\omega_x = 0$), we say that x is occupied (resp. empty) in the configuration ω. If $\omega \in \Omega$, we let $\omega_{|\Lambda} \in \Omega_\Lambda$ be the configuration restricted to Λ, defined by $\omega_{|\Lambda} = (\omega_x)_{x\in\Lambda}$.

For $x \in \mathbb{Z}$, by $\omega_{x+\cdot}$ we mean the translated configuration that takes value ω_{x+y} on the site y. ω^x is the configuration ω flipped at site x:

$$\omega^x_y = \begin{cases} \omega_y & \text{if } y \neq x \\ 1-\omega_x & \text{if } y = x \end{cases}$$

We are interested in the sets of configurations "left-occupied" (with a finite number of zeros on the negative half-line)

$$LO = \{\omega \in \Omega \mid \exists\, y < \infty \;\; \omega_{|(-\infty,y)} \equiv 1\}$$

and, for $x \in \mathbb{Z}$,

$$LO_x = \{\omega \in LO \mid \omega_{|(-\infty,x)} \equiv 1,\ \omega_x = 0\}$$

For any $\omega \in LO$, let us define $X(\omega) = x$ if $\omega \in LO_x$. $X(\omega)$ is the position of the front (or the left-most zero) in the configuration ω.

Fix $p \in (0,1)$ and let $q = 1-p$. p will be the density of occupied sites of the equilibrium distribution of our dynamics. Let μ (resp., for $\Lambda \subset \mathbb{Z}$, μ_Λ) be the product Bernoulli measure of density p on Ω (resp. on Ω_Λ). Define $\tilde{\mu}$ the product measure on Ω such that

$$\tilde{\mu}(\omega_x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ p & \text{if } x > 0 \end{cases} \tag{B.1}$$

Note that for functions f with support in $\mathbb{N}^*$, $\mu(f) = \tilde{\mu}$.

The East dynamics on Ω is a Markov process defined by the following generator: for any local function f, $\omega \in \Omega$,

$$\mathcal{L}f(\omega) = \sum_{x\in\mathbb{Z}} (1-\omega_{x+1})(p(1-\omega_x) + (1-p)\omega_x)\,[f(\omega^x) - f(\omega)]$$

$(P_t)_{t\geq 0}$ will be the associated semi-group, and $\omega(t)$ the configuration at time t starting from ω. That process is reversible w.r.t. μ, which is in particular an invariant distribution (so we refer to p as the equilibrium density, and to μ as the equilibrium measure). Also note that LO is a stable set for the East dynamics.

The dynamics can also be described as follows, and we will often use this description in the sequel: attach independently to each $x \in \mathbb{Z}$ a Poisson process of parameter one, and independently a countable infinite collection of independent, mean p Bernoulli variables. The Poisson processes can be understood as clocks: when the Poisson process attached to site x jumps, site x has an opportunity to flip. It then looks at the site on its right, $x+1$ (the East neighbour). If this neighbour is occupied in the current configuration ω ($\omega_{x+1} = 1$), nothing happens. If it is empty ($\omega_{x+1} = 0$), the ring is called *legal*, and the occupation state of site x is refreshed with the result of an unused Bernoulli variable, namely $\omega_x \to 1$ (resp. $\omega_x \to 0$) with probability p (resp. q).

The rigorous construction of this process in infinite volume is standard (see for instance [Lig85]).

In the following, $\mathbb{P}$ and $\mathbb{E}$ will refer to the law of the Poisson clocks and Bernoulli variables, so that we will write:

$$\nu(P_t f) = \mathbb{E}_\nu\left[f(\omega(t))\right]$$

for any initial measure ν, and abbreviate to $\mathbb{E}_\omega$ when ν is Dirac in ω.

One can also construct the dynamics in $\Lambda \subset \mathbb{Z}$, using the same construction. To this purpose, we should specify a boundary condition on the right border of Λ. In particular, if Λ is connected, the boundary condition can be zero or one. Only the zero boundary condition guarantees ergodicity of the process in Λ.

Zeros will play a special role in our proofs, since they are what allows flips in the dynamics. For a given configuration, we will be particularly interested in the following collection of zeros separated at least by a distance L. We define recursively the locations of these zeros, for any $L \in \mathbb{N}^*$ and for any $\omega \in LO$:

$$\begin{aligned} Z_0^L(\omega) &= X(\omega) \\ Z_{i+1}^L(\omega) &= \inf\{x \geq Z_i^L(\omega) + L \mid \omega_x = 0\} \qquad\qquad (\inf \emptyset = +\infty) \end{aligned} \tag{B.2}$$

We are going to study the behaviour of $X(\omega(t))$, but we will also be interested in the behaviour of the configuration behind the front. To this effect, we introduce the following notations.

For $\omega \in LO$, $L \in \mathbb{N}$, define the configurations $\theta_L\omega, \theta\omega \in LO_0$ in the following way:

$$(\theta_L\omega)_x = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ \omega_{X(\omega)+L+x} & \text{if } x > 0 \end{cases}$$

and $\theta\omega = \theta_0\omega$.

Let us also recall the definition of the spectral gap. For f in the domain of $\mathcal{L}$, let $\mathcal{D}(f) = -\mu(f, \mathcal{L}f)$ be the Dirichlet form of $\mathcal{L}$. Then the spectral gap of the East dynamics is

$$\text{gap} = \inf \frac{\mathcal{D}(f)}{Var_\mu(f)}$$

where the infimum is taken on f in the domain of $\mathcal{L}$ non constant (with $Var_\mu(f) > 0$). Recall that

$$Var_\mu(P_t f) \leqslant e^{-2t\,\text{gap}} Var_\mu(f)$$

In particular, if the spectral gap is positive, the reversible measure μ is mixing for P_t, with exponentially decaying correlations. The gap corresponds to the inverse of the relaxation time.

Moreover, for $\Lambda \subset \mathbb{Z}$, the spectral gap of the process restricted to Λ with zero boundary condition satisfies (see [CMRT08]):

$$\text{gap}_\Lambda^\circ \geq \text{gap}$$

B.2.2 Former useful results

The first result to recall is:

Proposition B.2.1 *([AD02, CMRT08]) For any $p \in (0, 1)$*

$$\text{gap} > 0 \tag{B.3}$$

Now we recall a tool introduced in [AD02], which we will use extensively: the distinguished zero[1].

[1] In [AD02] and many other papers, notably in physics, the roles of zeros and ones are reverted, so that the authors speak of a distinguished particle. The orientation of the constraint (to the right or to the left) is also subject to variations in the literature.

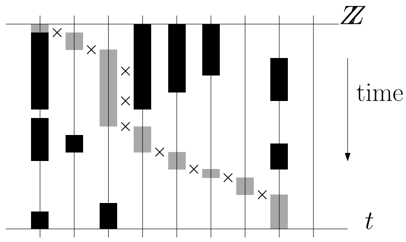

Figure B.2: In grey, a trajectory of a distinguished zero up to time t; time goes downwards, sites are highlighted in black at the times when they are occupied. The crosses represent times when the distinguished zero tries to jump to the right, i.e. clock rings at the site occupied by the distinguished zero.

Definition B.2.2 *Consider $\omega \in \Omega$ a configuration with $\omega_x = 0$ for some $x \in \mathbb{Z}$. Define $\xi(0) = x$. Call $T_1 = \inf\{t \geq 0 \mid \text{the clock in } x \text{ rings and } \omega_{x+1}(t) = 0\}$, the time of the first legal ring at x. Let $\xi(s) = x$ for $s < T_1$, $\xi(T_1) = x + 1$ and start again to define recursively $(\xi(s))_{s \geq 0}$.*

Notice that for any $s \geq 0$, $\omega_{\xi(s)}(s) = 0$, and that $\xi : \mathbb{R}^+ \to \mathbb{Z}$ is almost surely càdlàg and increasing by jumps of 1. See Figure B.2 for an illustration.

This distinguished zero has an important property: as it moves forward, it leaves equilibrium on its left (see Lemma 4 in [AD02] or Lemma 3.5 of [CMST10]). This property leads to Theorem 3.1 of [CMST10], which will be useful; we restate it with the explicit bound obtained in the proof. Later we will give an improved version of this result, valid also for f with infinite support (see Proposition B.4.3).

Proposition B.2.3 *([CMST10]) Let f be a function with support in $[x_-, x_+]$, $\omega \in \Omega$ with $\omega_{x_0} = 0$, $x_0 > x_+$. Assume $\mu(f) = 0$. Then*

$$|\mathbb{E}_\omega [f(\omega(t))]| \leqslant \sqrt{Var_\mu(f)} \left(\frac{1}{p \wedge q}\right)^{x_0 - x_-} e^{-t \,\mathrm{gap}} \tag{B.4}$$

B.3 Preliminary results

Ultimately, we want to show that the front moves ballistically. But let us start with some easy bounds.

Lemma B.3.1 – Finite speed of propagation
For $x, y \in \mathbb{Z}$, $t > 0$, define the event:

$$F(x, y, t) = \{\text{before time } t, \text{ there is a sequence of successive rings linking } x \text{ to } y\} \tag{B.5}$$

This means that (assuming for instance $x < y$) there is a ring at x, then at $x + 1$, then at $x + 2$, and so on up to y, all before time t. Only on this event can information be transmitted from x to y before time t. Then there are universal constants K_1, K_2 such that:

$$\mathbb{P}\left(F(x, y, t)\right) \leqslant K_1 e^{-K_2 |x-y| \left(\ln \frac{|x-y|}{t} - 1\right)} \tag{B.6}$$

In particular, if $|x - y| \geq \overline{v} t$ for $\overline{v}$ a constant large enough,

$$\mathbb{P}\left(F(x, y, t)\right) \leqslant e^{-|x-y|}$$

Proof

This just follows from a simple estimate of the probability for a Poisson process of parameter 1 to have at least $|x-y|$ instances in time t. ✓

Lemma B.3.2 *There exist constants $0 < \underline{v} < 1 < \overline{v} < \infty$ and $\gamma > 0$ depending only on q such that for any $\omega \in LO_0$, for any $t > 0$,*

$$\mathbb{P}_\omega\left(X(\omega(t)) \in [\![-\overline{v}t, -\underline{v}t]\!]\right) \geq 1 - e^{-\gamma t} \tag{B.7}$$

Proof

Let us split the proof and prove separately that with great probability $X(\omega(t))$ is bigger than $-\overline{v}t$ and smaller than $-\underline{v}t$.

• We choose $\overline{v}$ as in Lemma B.3.1 and notice that $X(\omega(t)) < -\overline{v}t$ implies $F(0, -\overline{v}t, t)$, so

$$\mathbb{P}_\omega\left(X(\omega(t)) < -\overline{v}t\right) \leqslant e^{-\overline{v}t}$$

• To bound the probability of $X(\omega(t)) > -\underline{v}t$, we use the method of the distinguished zero. Let $x_0 = 0$ be the distinguished zero at time 0. Let $l > 1$; write $V(t,l) = [\![-\underline{v}t - l, -\underline{v}t - 1]\!]$. Notice that for $\eta \in \Omega$, $X(\eta) > -\underline{v}t$ implies $\eta_{|V(t,l)} \equiv 1$. Consider the centered function:

$$f_{t,l}(\eta) = \mathbf{1}_{\{\eta_{|V(t,l)} \equiv 1\}} - p^l.$$

Thanks to Proposition B.2.3, we get for any $s > 0$

$$|\mathbb{E}_\omega\left[f_{t,l}(\omega(s))\right]| \leqslant \left(\frac{1}{p \wedge q}\right)^{\underline{v}t+l} e^{-s\mathrm{gap}} \mathrm{Var}_\mu(f)^{1/2}.$$

So, taking $l = \underline{v}t$ and $s = t$, we have:

$$\mathbb{P}_\omega(X(t) > -\underline{v}t) \leqslant \mathbb{P}_\omega\left(\omega_{|V(t,\underline{v}t)}(t) \equiv 1\right) \leqslant p^{\underline{v}t} + \left(\frac{1}{p \wedge q}\right)^{2\underline{v}t} e^{-t\mathrm{gap}} \mathrm{Var}_\mu(f)^{1/2}.$$

Hence the result by taking $\underline{v}$ small enough.

✓

Remark B.3.3 *From now on, $\underline{v}$, $\overline{v}$ and γ will denote fixed constants that satisfy (B.7) and*

$$\underline{v} < \frac{\mathrm{gap}}{\ln \frac{1}{p \wedge q}}$$

(for technical reasons that appear in the proof of Theorem B.4.4).

Let us also give right now a bound on any moment of the front progression:

Lemma B.3.4 *For any $r \in \mathbb{N}^*$, $t > 0$, $\omega \in LO_0$, there exists a constant $K < \infty$ depending only on r such that*

$$\mathbb{E}_\omega\left[|X(\omega(t))|^r\right] \leqslant K t^r \tag{B.8}$$

Proof

We bound $X(\omega(t))$ by two processes.

1. $(Y_1(t))_{t\geq 0}$ is a process that jumps only to the left with rate q, i.e. $(-Y_1(t))_{t\geq 0}$ is a Poisson process of parameter q.

2. $(Y_2(t))_{t\geq 0}$ is a process that jumps only to the right with rate p, i.e. a Poisson process of parameter p.

Using the graphical construction, we can construct the three processes so that $\mathbb{P}$-a.s., for all $t \geq 0$ and for all $\omega \in LO_0$:

$$Y_1(t) \leqslant X(\omega(t)) \leqslant Y_2(t)$$

✓

B.4 Decorrelation behind the front

The heart of the problem is to describe the configurations behind the front. In this section, we prove that far enough from the front the distribution is very close to μ the product of Bernoulli(p) (the equilibrium measure of the East process).

B.4.1 Presence of voids behind the front

First we show that the front generates zeros during its progression. In the next proposition, we choose the front –which is a particular zero– to be the distinguished zero at an intermediate time $t-s$ to deduce a local relaxation at time t around the position $X(\omega(t-s))$ (the front at time $t-s$).

Proposition B.4.1 *Let f be a local function with support contained in $[\![-x_-, -x_+]\!]$ such that $1 \leqslant x_+ \leqslant x_-$. Assume $\mu(f) = 0$. Then for any $\omega \in LO_0$*

$$\left|\mathbb{E}_\omega\left[f\left(\omega_{X(\omega(t-s))+\cdot}(t)\right)\right]\right| \leqslant Var_\mu(f)^{1/2}\left(\frac{1}{p\wedge q}\right)^{x_-} e^{-sgap}, \tag{B.9}$$

where we recall that $\omega_{X(\omega(t-s))+\cdot}(t)$ is the configuration at time t centered around the position that the front had reached at the intermediate time $t-s$.

Proof

We use again the distinguished zero technique. First of all, thanks to the Markov property applied at time $t-s$:

$$\left|\mathbb{E}_\omega\left[f\left(\omega_{X(\omega(t-s))+\cdot}(t)\right)\right]\right| = \left|\mathbb{E}_\omega\left[\mathbb{E}_{\omega(t-s)}\left[f\left(\sigma_{X(\sigma(0))+\cdot}(s)\right)\right]\right]\right|,$$

where in the r.h.s, $\sigma(s)$ denotes the configuration obtained when the dynamics runs during time s starting from the configuration $\omega(t-s)$. But for any $\sigma \in LO$, by choosing $x_0 = X(\sigma)$ and applying Proposition B.2.3, we get:

$$\left|\mathbb{E}_\sigma\left[f\left(\sigma_{X(\sigma)+\cdot}(s)\right)\right]\right| \leqslant \mathrm{Var}_\mu(f)^{1/2}\left(\frac{1}{p\wedge q}\right)^{x_-} e^{-s\mathrm{gap}}$$

Hence the result.

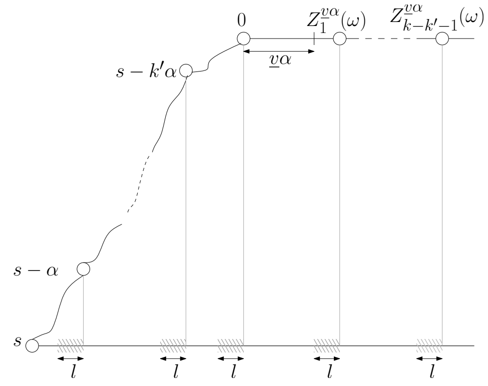

Figure B.3: The event $\mathcal{Z}$ is such that the configuration at time s has a zero in each shaded box. The positions of these boxes depend on the initial configuration and on the dynamics up to the intermediate times $s-i\alpha$ (via the positions of the front at times $s-i\alpha$).

✓

From this, we can easily infer the following corollary, stating that the front has left zeros behind. Namely, in boxes centered either around the front at intermediate times, or around zeros in the initial configuration, there are zeros with good probability (see Figure B.3).

Corollary B.4.2 *Let $\alpha > 0$, $k \in \mathbb{N}^*$. Define $k' = \max\{i \geq 0 \mid s - i\alpha \geq 0\}$. Choose $l < \underline{v}\alpha$ and $\omega \in LO_0$ such that in the initial configuration ω, there are at least $k-k'$ zeros at distance at least $\underline{v}\alpha$ from each other, i.e. $Z_{k-k'}^{\underline{v}\alpha}(\omega) < \infty$ (see (B.2)).*

Consider the event (see Fig. B.3)

$$\begin{aligned}\mathcal{Z} &= \{\forall\, 1 \leqslant i \leqslant k'\ \exists\, x \in [\![X(\omega(s-i\alpha)) - l, X(\omega(s-i\alpha)) - 1]\!]\ s.t.\ \omega_x(s) = 0 \\ &\quad \text{and } \forall k' < i \leqslant k+1\ \exists\, x \in \left[\!\left[Z_{i-k'-1}^{\underline{v}\alpha}(\omega) - l, Z_{i-k'-1}^{\underline{v}\alpha}(\omega) - 1\right]\!\right]\ s.t.\ \omega_x(s) = 0\Big\}\end{aligned}$$

Then we have:

$$\mathbb{P}_\omega(\mathcal{Z}^c) \leqslant (k+1)\left(p^l + \frac{p^{l/2}}{(p\wedge q)^l} e^{-\alpha \text{gap}}\right) \tag{B.10}$$

Proof

Write

$$\begin{aligned}\mathcal{Z}^c &= \left(\bigcup_{i=1}^{k'} \{\forall\, x \in [\![X(\omega(s-i\alpha)) - l, X(\omega(s-i\alpha)) - 1]\!]\quad \omega_x(s) = 1\}\right) \\ &\quad \bigcup \left(\bigcup_{i=k'+1}^{k+1} \{\forall\, x \in [\![Z_{i-k'-1}^{\underline{v}\alpha}(\omega) - l, Z_{i-k'-1}^{\underline{v}\alpha}(\omega) - 1]\!]\quad \omega_x(s) = 1\}\right)\end{aligned}$$

For $i = 1, \cdots, k'$, we bound the probability of

$$\{\forall\, x \in [\![X(\omega(s-i\alpha)) - l, X(\omega(s-i\alpha)) - 1]\!]\quad \omega_x(s) = 1\}$$

by applying Proposition B.4.1 to the centered function

$$f = \prod_{x=-l}^{-1} \omega_x - p^l$$

$$\begin{aligned}\mathbb{P}_\omega\left(\forall\, x \in [\![X(\omega(s-i\alpha)) - l, X(\omega(s-i\alpha)) - 1]\!]\ \omega_x(s) = 1\right) &= \mathbb{E}_\omega\left[f\left(\omega_{X(\omega(s-i\alpha)) + \cdot}(s)\right)\right] + p^l \\ &\leqslant p^l + \frac{p^{l/2}}{(p \wedge q)^l} e^{-i\alpha\, \mathrm{gap}} \\ &\leqslant p^l + \frac{p^{l/2}}{(p \wedge q)^l} e^{-\alpha\, \mathrm{gap}}\end{aligned}$$

For $i = k'+1, \cdots, k+1$, we use the same function and Proposition B.2.3 with $Z^{v\alpha}_{i-k'-1}(\omega)$ as the distinguished zero to bound the probability of

$$\left\{\forall\, x \in [\![Z^{v\alpha}_{i-k'-1}(\omega) - l, Z^{v\alpha}_{i-k'-1}(\omega) - 1]\!] \quad \omega_x(s) = 1\right\}.$$

✓

B.4.2 Relaxation to equilibrium on the left of a distinguished zero

We state here an extension of theorem 3.1 in [CMST10] (Proposition B.2.3), which holds for functions with infinite support. It is a result of local relaxation to equilibrium on the left of a zero present in the initial configuration. In this section, we consider the East dynamics on $\mathbb{N}^*$, without any notion of front.

Proposition B.4.3 *Let $\omega \in \Omega_{\mathbb{N}^*}$ be the initial data, such that $\omega_z = 0$ for some $z > 1$, and f a bounded function on $\Omega_{\mathbb{N}^*}$. Then*

$$\left|\mathbb{E}_\omega\left[f(\omega(t))\right] - \mathbb{E}_\omega\left[\mu_{\{1\}}(f)(\omega(t))\right]\right| \leqslant \sqrt{2}\|f\|_\infty \left(\frac{1}{p \wedge q}\right)^z e^{-t\, \mathrm{gap}},$$

where $\mu_{\{1\}}(f)$ denotes the function on $\Omega_{\mathbb{N}^* \setminus \{1\}}$ which is f averaged w.r.t the Bernoulli measure μ, only on site 1.

Proof

Step 1: Conditioning on the right of a distinguished zero

First, we need to define carefully a conditioning by "what happens on the right of a distinguished zero". For this, we use the description of the dynamics in terms of Poisson clocks and coin tosses introduced in section B.2.1. Thanks to the orientation of the dynamics (the flip rates depend only on the configuration on the right), the evolution of any given site is only a function of the Poisson clocks and coin tosses happening on its right and on itself. Here, we want to exploit this same idea, but with a site that is moving: the distinguished zero.

Initially the distinguished zero is located at z. Fix $t > 0$ and ω as in the statement of the theorem, and call $\mathfrak{C}$ the set of collections $(\mathcal{T}_x, \mathcal{B}_x)_{x \geq z}$ with $\mathcal{T}_x = (\tau^x_1, \ldots, \tau^x_{n_x})$ and $\mathcal{B}_x = (b^x_1, \ldots, b^x_{n_x - 1})$ satisfying the following conditions (see Fig. B.4 for an example). Keep in mind that in the graphical representation, it is the collection of variables which characterizes the dynamics on the right of the distinguished zero. In fact, $\mathcal{T}_x$ should be thought of as the sequence

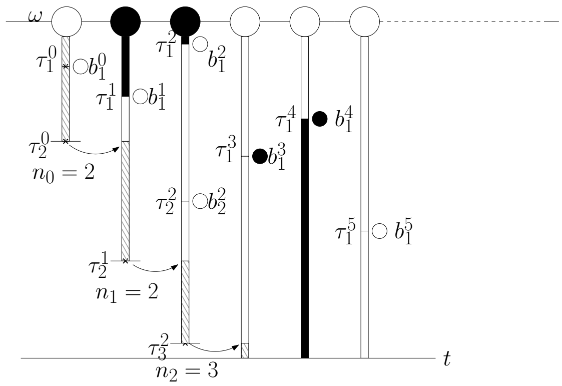

Figure B.4: An example of part of a collection in $\mathfrak{C}$. Time goes downward, up to time t. The small circles represent the outcome of the coin flips. The position of the distinguished zero is dashed, and the times at which it jumps depicted by an arrow.

of clock rings happening at site x until the distinguished zero jumps to $x+1$, and $\mathcal{B}_x$ as the results of the coin flips at those times, except the very last one. When we define a random variable in $\mathfrak{C}$, it will contain exactly the information on the trajectory of the distinguished zero up to time t and what happens on its right, and no information on the evolution of the system on its left.

Here are the conditions to be in $\mathfrak{C}$:

- all τ_i^x are distinct
- $\forall x \geq z$, $0 < \tau_1^x < \tau_2^x ... < \tau_{n_x}^x$
- $\exists\ x \geq z$ such that $\tau_{n_z}^z < \tau_{n_{z+1}}^{z+1} ... < \tau_{n_x}^x \leqslant t = \tau_{n_{x+1}}^{x+1} = \tau_{n_{x+2}}^{x+2}$ ($\tau_{n_x}^x$ is the infimum between t and the time at which the distinguished zero jumps from x to $x+1$).
- for any $x \geq z$, there exists $y > x$ such that $n_y = 0$ (i.e. $\mathcal{T}_y = \mathcal{B}_y = \varnothing$ —this will mean that there is no clock ring at site y before time t)
- $\forall\ x \geq z$, $\forall\ i = 1, ..., n_x - 1$, $b_i^x \in \{0, 1\}$ (the collection doesn't include the information of the value of the coin flip associated to a time when the distinguished zero jumps; this is an important condition for the sequel).

For the next conditions, up to time $\tau_{n_z}^z$ (the first time of jump), run the dynamics described in section B.2.1 in the volume $\mathbb{N}^* \backslash \{1, \cdots z-1\}$, starting from configuration ω and using the τ_i^x as clock rings and the b_i^x as coin tosses. The fourth condition ensures that this dynamics is actually a juxtaposition of finite volume dynamics (the sites with $n_y = 0$ play the role of boundary conditions), and the first condition ensures that these finite volume dynamics are well defined. So at any time $s \leqslant \tau_{n_z}^z$, the collection determines a well defined value $\omega_{z+1}(s)$ to the occupation variable in site $z+1$. We request that:

- for any $i < n_z$, we have $\omega_{z+1}(\tau_i^z) = 1$, and $\omega_{z+1}(\tau_{n_z}^z) = 0$ (i.e. $\tau_{n_z}^z$ is the first legal ring at z: the distinguished zero jumps from z to $z+1$ at $\tau_{n_z}^z$).

Now in the same way, run the deterministic East dynamics given by the collection up to time $\tau_{n_{z+1}}^{z+1}$ (the second time of jump), but now only in the volume $\mathbb{N}^* \backslash \{1, \cdots z\}$. Request that:

- for any $i < n_{z+1}$ such that $\tau_i^{z+1} > \tau_{n_z}^z$, $\omega_{z+2}(\tau_i^{z+1}) = 1$, and $\omega_{z+2}(\tau_{n_{z+1}}^{z+1}) = 0$ ($\tau_{n_{z+1}}^{z+1}$ is the first legal ring at $z+1$ after the distinguished zero has jumped on $z+1$; it is the time when the distinguished zero jumps from $z+1$ to $z+2$).

Repeat the process up to time t and add the corresponding conditions on the elements of $\mathfrak{C}$. Now with all our conditions, $\mathfrak{C}$ is the set of all possible evolutions of an East dynamics, on the right of a distinguished zero starting at z up to time t. From the description above, we see that the set of clock rings and coin tosses happening at the right of the distinguished zero starting from z in the configuration ω is almost surely a random variable that takes its values in $\mathfrak{C}$. Call this random variable $\mathcal{C}$. Note that, given $\mathcal{C}$, we can easily define the corresponding trajectory $(\xi(s))_{s \leqslant t}$ of the distinguished zero, as well as the configuration reached on its right at any time $s \leqslant t$.

Step 2: Relaxation on the left of a distinguished zero

We now adapt the proof of Theorem 3.1 in [CMST10] to the case where f doesn't have a finite support. For any $\mathcal{C} \in \mathfrak{C}$, $s \leqslant t$, let $\xi(s)$ be the position of the distinguished zero at time s, $V_s = [\![1, \xi(s) - 1]\!]$, $\sigma(s)$ the configuration reached in $\mathbb{N}^* \backslash V_s$ at time s when the evolution on the right of the distinguished zero is given by $\mathcal{C}$. For simplicity, call $t_1 = \tau_{n_z}^z, t_2 = \tau_{n_{z+1}}^{z+1}, ...t_k$ the times of jumps of the distinguished zero in $\mathcal{C}$. Also call $\tilde{f} = f - \mu_{\{1\}}(f)$. For any $\xi > 1$, for any $\sigma \in \Omega_{\mathbb{N}^* \backslash [\![1,\xi]\!]}$, it holds $\mu_{[\![1,\xi]\!]}(\tilde{f})(\sigma) = 0$. Then, for a given $\mathcal{C} \in \mathfrak{C}$, let $g_{\mathcal{C},t}$ be the function on $\{0,1\}^{V_t}$ defined by:

$$g_{\mathcal{C},t}(\eta) := \tilde{f}(\eta \cdot \sigma(t)), \qquad \eta \in \{0,1\}^{V_t} \tag{B.11}$$

where V_t the interval on the left of the distinguished zero at time t and $\sigma(t)$ the configuration on $\Omega_{\mathbb{N}^*}$ at time t are parameters fixed by $\mathcal{C}$ as above; $\eta \cdot \sigma(t)$ denotes the configuration on $\Omega_{\mathbb{N}^*}$ given by η on V_t and $\sigma(t)$ elsewhere. This function is defined on a finite volume: the dynamics on the infinite part on the right of the distinguished zero appears only through the configuration at time t, which is part of the parameter $\mathcal{C}$. The trick of introducing this function allows us to treat separately the dynamics on the left of the distinguished zero, and thus to reproduce the proof of Theorem 3.1 in [CMST10]. Recall that for $\mathcal{C}$ fixed, the evolution of the distinguished zero (in particular V_t and $t_1 < t_2 ... < t_k < t$ the times of jump before t) is also fixed, as well as $\sigma(t)$.

$$\begin{aligned}
\mathbb{E}_\omega\left[\tilde{f}(\omega(t))|\mathcal{C}\right] &= \mathbb{E}_{\omega_{V_0}}\left[g_{\mathcal{C},t}(\omega_{V_t}(t))\,\middle|(\xi(s))_{s \leqslant t}\right] \\
&= \sum_{\sigma \in \Omega_{V_0}} \sum_{\sigma' \in \{0,1\}} \mathbb{E}_{\omega_{V_0}}\left[\mathbf{1}_{\omega_{V_0}(t_1)=\sigma} \mathbf{1}_{\omega_z(t_1)=\sigma'} g_{\mathcal{C},t}(\omega_{V_t}(t))\,\middle|(\xi(s))_{s \leqslant t}\right] \\
&= \sum_{\sigma \in \Omega_{V_0}} \sum_{\sigma' \in \{0,1\}} P_{t_1}^{V_0,\circ}(\omega_{V_0}, \sigma)\,\mu(\sigma')\mathbb{E}_{\sigma \cdot \sigma'}\left[g_{\mathcal{C},t}((\sigma \cdot \sigma')_{V_t}(t - t_1))\,\middle|(\xi(s))_{t_1 \leqslant s \leqslant t}\right],
\end{aligned}$$

where $\left(P_s^{V_0,\circ}\right)_{s \geq 0}$ denotes the semigroup associated to the East dynamics restricted to V_0 with empty boundary condition. The first equality comes from the fact that when $\mathcal{C}$ is fixed, $\tilde{f}$ only depends on $\omega_{V_t}(t)$, whose distribution is entirely determined by the trajectory of the distinguished zero $(\xi_s)_{s \leqslant t}$, which in turn is entirely determined by $\mathcal{C}$. The third equality is an application of the Markov property at time $t_1 < t$. $\sigma.\sigma'$ here is the configuration that is equal to σ on V_0 and to σ' on $\{\xi(0)\} = V_{t_1} \backslash V_0$.

Thanks to the variational formula for the spectral gap, it is not difficult to see ([CMRT08], Lemma 2.11) that gap $\leqslant$ gap$(V_0, \circ)$. This is not surprising: relaxation should be faster in a box with a fixed zero boundary condition than it is on the entire line.

$$\begin{aligned}
&Var_{\mu_{V_0}}\left(\mathbb{E}_\omega\left[\tilde{f}(\omega(t))|\mathcal{C}\right]\right)\\
&\quad\leqslant e^{-2t_1\,\mathrm{gap}}Var_{\mu_{V_0}}\left(\sum_{\sigma'\in\{0,1\}}\mu(\sigma')\mathbb{E}_{\sigma\cdot\sigma'}\left[g_{\mathcal{C},t}\left(\sigma\cdot\sigma'\right)_{V_t}(t-t_1))\left|(\xi(s))_{t_1\,\leqslant\, s\,\leqslant\, t}\right.\right]\right)\\
&\quad\leqslant e^{-2t_1\,\mathrm{gap}}Var_{\mu_{V_{t_1}}}\left(\mathbb{E}_\sigma\left[g_{\mathcal{C},t}\left(\sigma\right)_{V_t}(t-t_1))\left|(\xi(s))_{t_1\,\leqslant\, s\,\leqslant\, t}\right.\right]\right),
\end{aligned}$$

by convexity of the variance. Then we can follow the same steps (using the Markov property at time t_2-t_1) to show that:

$$\begin{aligned}
&Var_{\mu_{V_{t_1}}}\left(\mathbb{E}_\sigma\left[g_{\mathcal{C},t}\left(\sigma\right)_{V_t}(t-t_1))\left|(\xi(s))_{t_1\,\leqslant\, s\,\leqslant\, t}\right.\right]\right)\\
&\qquad\leqslant e^{-2(t_2-t_1)\,\mathrm{gap}}Var_{\mu_{V_{t_2}}}\left(\mathbb{E}_\sigma\left[g_{\mathcal{C},t}\left(\sigma\right)_{V_t}(t-t_1))\left|(\xi(s))_{t_2\,\leqslant\, s\,\leqslant\, t}\right.\right]\right)
\end{aligned}\tag{B.12}$$

We can then iterate the procedure to get:

$$\begin{aligned}
Var_{\mu_{V_0}}\left(\mathbb{E}_\omega\left[\tilde{f}(\omega(t))|\mathcal{C}\right]\right) &\leqslant e^{-2t\,\mathrm{gap}}Var_{\mu_{V_t}}\left(g_{\mathcal{C},t}\left(\sigma\right)\right)\\
&\leqslant 2\|f\|_\infty^2 e^{-2t\,\mathrm{gap}},
\end{aligned}\tag{B.13}$$

where the last inequality is just an estimate on $Var_{\mu_{V_t}}\left(g_{\mathcal{C},t}\left(\sigma\right)\right)$ using its infinite norm (since conditionally on $\mathcal{C}$, $g_{\mathcal{C},t}$ is just a bounded function). We also have:

$$\mathbb{E}_{\mu_{V_0}}\left[g_{\mathcal{C},t}(\omega_{V_t}(t))\left|(\xi(s))_{s\,\leqslant\, t}\right.\right]=\mu_{V_t}\left(g_{\mathcal{C},t}\right)=0\tag{B.14}$$

The first equality comes from the property that the distinguished zero leaves equilibrium on its left (Lemma 4 in [AD02] or Lemma 3.5 in [CMST10]), and the second from the definition of $\tilde{f}$. So that

$$\begin{aligned}
\left|\mathbb{E}_\omega\left[\tilde{f}\right]\right| &\leqslant \mathbb{E}_\omega\left[\left|\mathbb{E}_{\omega_{V_0}}\left[\tilde{f}(\omega(t))\middle|\mathcal{C}\right]\right|\right]\\
&\leqslant \left(\frac{1}{p\wedge q}\right)^z\mathbb{E}_\omega\left[\int d\mu_{V_0}(\eta)\left|\mathbb{E}_\eta\left[g_{\mathcal{C},t}(\eta(t))|\,\mathcal{C}\right]\right|\right]\\
&\leqslant \left(\frac{1}{p\wedge q}\right)^z\mathbb{E}_\omega\left[\left\{\int d\mu_{V_0}(\eta)\left(\mathbb{E}_\eta\left[g_{\mathcal{C},t}(\eta(t))|\,\mathcal{C}\right]\right)^2\right\}^{1/2}\right]\\
&\leqslant \left(\frac{1}{p\wedge q}\right)^z\mathbb{E}_\omega\left[Var_{\mu_{V_0}}\left(\mathbb{E}_\eta\left[g_{\mathcal{C},t}(\eta(t))|\,\mathcal{C}\right]\right)^{1/2}\right]\\
&\leqslant \sqrt{2}\|f\|_\infty\left(\frac{1}{p\wedge q}\right)^z e^{-t\,\mathrm{gap}},
\end{aligned}$$

where the second inequality comes from the change of measure $\delta_{\omega_{V_0}}\to\mu_{V_0}$ on Ω_{V_0}, the third uses Cauchy-Schwarz inequality, the fourth uses (B.14) and the last one (B.13). ✓

B.4.3 Decorrelation behind the front at finite distance

In this section we prove the central coupling result of this paper (Theorem B.4.7). We refer to [LPW09] or [Kuk06] for classic results about total variation distance and maximal (or optimal) coupling. We start by showing that the configuration on a single site at distance L from the front is very close to being at equilibrium (a Bernoulli distribution), under appropriate assumptions that lead to consider three cases (see Remark B.4.5 below about this distinction). This result for a single site will then be iterated to get our main coupling result, Theorem B.4.7.

Theorem B.4.4 *Fix f a bounded function with support in $\mathbb{N}^*$, $t > 0$, $L \in \mathbb{N}^*$ and $\omega \in LO_0$.*
Define the quantities

$$\alpha = \alpha(L,t) = \frac{\text{gap}}{6\overline{v}(2\underline{v}+\overline{v})\ln\frac{1}{p\wedge q}} L \wedge 3\overline{v}t =: c_1(L \wedge 3\overline{v}t) \tag{B.15}$$

$$l = l(L,t) = \lfloor \underline{v}\alpha \rfloor \tag{B.16}$$

$$s = s(L,t) = \begin{cases} \left(t - \frac{L}{3\overline{v}}\right) \vee \alpha & \text{if } L < 3\overline{v}t \\ 0 & \text{else} \end{cases} \tag{B.17}$$

$$k = k(L,t) = \left\lfloor \frac{L - \underline{v}(t-s)}{\underline{v}\alpha} \right\rfloor + 2, \tag{B.18}$$

where $\underline{v}, \overline{v}$ have been introduced in Remark B.3.3. Note that α, l, s, k depend on p through the choice of $\underline{v}$, but since we work at fixed p, this dependence plays no role in the proof, so we ignore it in the notation.

There are constants $\epsilon > 0$, $K < \infty$ depending only on p such that:

1. *If $\lfloor \frac{s}{\alpha} \rfloor \geq k$ (for instance, if $L < \frac{\underline{v}}{1+2\underline{v}c_1} t$),*

$$\left|\mathbb{E}_\omega\left[f\left(\theta_L\omega(t)\right)\right] - \mathbb{E}_\omega\left[\mu_{\{1\}}(f)\left(\theta_L\omega(t)\right)\right]\right| \leqslant K\|f\|_\infty e^{-\epsilon L} \tag{B.19}$$

2. *If $\lfloor \frac{s}{\alpha} \rfloor < k$ and $L < 3\overline{v}t$ (for instance, if $\frac{\underline{v}}{3c_1\underline{v}+1} t \leqslant L < 3\overline{v}t$) and ω satisfies the following condition (see the definition (B.2)):*

$$\forall i = 1, ..., k - \left\lfloor \frac{s}{\alpha} \right\rfloor \quad Z_i^{\underline{v}\alpha}(\omega) - Z_{i-1}^{\underline{v}\alpha}(\omega) < \overline{v}\alpha \tag{B.20}$$

Then we also have:

$$\left|\mathbb{E}_\omega\left[f\left(\theta_L\omega(t)\right)\right] - \mathbb{E}_\omega\left[\mu_{\{1\}}(f)\left(\theta_L\omega(t)\right)\right]\right| \leqslant K\|f\|_\infty e^{-\epsilon L} \tag{B.21}$$

3. *If $3\overline{v}t \leqslant L$ and*

$$\forall i = 1, ..., k \quad Z_i^{\underline{v}\alpha}(\omega) - Z_{i-1}^{\underline{v}\alpha}(\omega) < \overline{v}\alpha \tag{B.22}$$

then

$$\left|\mathbb{E}_\omega\left[f\left(\theta_L\omega(t)\right)\right] - \mathbb{E}_\omega\left[\mu_{\{1\}}(f)\left(\theta_L\omega(t)\right)\right]\right| \leqslant K\|f\|_\infty \frac{L}{3\overline{v}t} e^{-\epsilon 3\overline{v}t} \tag{B.23}$$

Proof

Let us assume $\|f\|_\infty \leqslant 1$. Let us use the Markov property at time s –defined in (B.17) to write:

$$\mathbb{E}_\omega\left[f\left(\theta_L\omega(t)\right)\right] = \mathbb{E}_\omega\left[\mathbb{E}_{\omega(s)}\left[f\left(\theta_L\sigma(t-s)\right)\right]\right],$$

where $\sigma(t-s)$ here denotes the configuration obtained at time $t-s$ starting from $\omega(s)$. Thanks to Lemma B.3.2, we have:

$$\mathbb{E}_\omega\left[f\left(\theta_L\omega(t)\right)\right] = \sum_{y=\underline{v}(t-s)}^{\overline{v}(t-s)} \mathbb{E}_\omega\left[\mathbb{E}_{\omega(s)}\left[\mathbf{1}_{X(\sigma(t-s))-X(\sigma(0))=-y} f\left(\sigma_{X(\sigma(0))-y+L+\cdot}(t-s)\right)\right]\right] + O\left(e^{-\gamma(t-s)}\right)$$

Notice that we have chosen s so that:

$$\overline{v}(t-s) \leqslant L - 2\overline{v}(t-s). \tag{B.24}$$

This guarantees that the probability for information to travel from the support of the function we are looking at and the front in time $t-s$ is very small. More precisely, the probability that there is a sequence of successive clock rings linking $X(\omega(s))+L-y$ and $\max_{u \leqslant t-s} X(\sigma(u))$ (recall Lemma B.3.1) during $[s,t]$ is no bigger than $O\left(e^{-(t-s)}\right)$ (by finite speed of propagation). On the event that this sequence doesn't exist, the two functions appearing in the expectation are independent, since they depend on disjoint sets of clock rings and coin tosses. Indeed, $f\left(\sigma_{X(\sigma(0))-y+L+\cdot}(t-s)\right)$ depends only on those attached to sites on the right of $X(\omega(s))+L-y$, which can only influence the dynamics on the left of $\max_{u \leqslant t-s} X(\sigma(u))$ if a sequence of successive clock rings links $X(\omega(s))+L-y$ and $\max_{u \leqslant t-s} X(\sigma(u))+1$. Writing $p(\eta,y,s) = \mathbb{P}_\eta\left(X(\eta(s))-X(\eta)=-y\right)$, we thus have:

$$\begin{aligned}\mathbb{E}_\omega\left[f\left(\theta_L\omega(t)\right)\right] &= \sum_{y=\underline{v}(t-s)}^{\overline{v}(t-s)} \mathbb{E}_\omega\left[p(\omega(s),y,t-s)\mathbb{E}_{\omega(s)}\left[f\left(\sigma_{X(\sigma(0))-y+L+\cdot}(t-s)\right)\right]\right] \\ &\quad + O\left(e^{-\gamma(t-s)}+e^{-(t-s)}\right)\end{aligned}$$

Now we use Corollary B.4.2 to guarantee the presence of enough zeros at time s. Note that in the case 3 of Theorem B.4.4, we already request the presence of a number of zeros ($s=0$ and condition B.22 concerns the initial configuration). In the cases 1 and 2, let us consider the event (see figure B.3):

$$\begin{aligned}\mathcal{Z} &= \left(\bigcap_{i\in\{1,\ldots,\lfloor \frac{s}{\alpha}\rfloor \wedge k\}} \left\{\exists\, x \in [\![X\left(\omega(s-i\alpha)\right)-l, X\left(\omega(s-i\alpha)\right)-1]\!] \text{ s.t. } \omega_x(s)=0\right\}\right) \\ &\quad \bigcap \left(\bigcap_{i\in\{0,\ldots,k-(\lfloor \frac{s}{\alpha}\rfloor\wedge k)\}} \left\{\exists\, x \in [\![Z_i^{\underline{v}\alpha}(\omega(0))-l, Z_i^{\underline{v}\alpha}(\omega(0))-1]\!] \text{ s.t. } \omega_x(s)=0\right\}\right).\end{aligned}$$

Thanks to Corollary B.4.2, we have:

$$\mathbb{P}_\omega\left(\mathcal{Z}^c\right) \leqslant (k+1)\left(p^l + \frac{p^{l/2}}{(p\wedge q)^l}e^{-\alpha \text{gap}}\right).$$

So that in the cases 1 and 2:

$$\begin{aligned}\mathbb{E}_\omega\left[f\left(\omega_{X(\omega(t))+L+\cdot}(t)\right)\right] &= \sum_{y=\underline{v}(t-s)}^{\overline{v}(t-s)} \mathbb{E}_\omega\left[p(\omega(s),-y,t-s)\mathbf{1}_{\mathcal{Z}}\mathbb{E}_{\omega(s)}\left[f\left(\sigma_{X(\sigma(0))+y+L+\cdot}(t-s)\right)\right]\right] \\ &\quad + O\left(e^{-\gamma(t-s)}+e^{-(t-s)}+(k+1)\left(p^l+\frac{p^{l/2}}{(p\wedge q)^l}e^{-\alpha\text{gap}}\right)\right) \quad \text{(B.25)}\end{aligned}$$

Now we know that at time s, on the event $\mathcal{Z}$, there are zeros at random positions. The easy bounds obtained in Lemma B.3.2 let us control these positions. Namely, if we let

$$k' = \left\lfloor \frac{s}{\alpha}\right\rfloor \wedge k,$$

on an event B such that

$$\mathbb{P}_\omega\left(B^c\right) \leqslant (k'+1)e^{-\gamma\alpha},$$

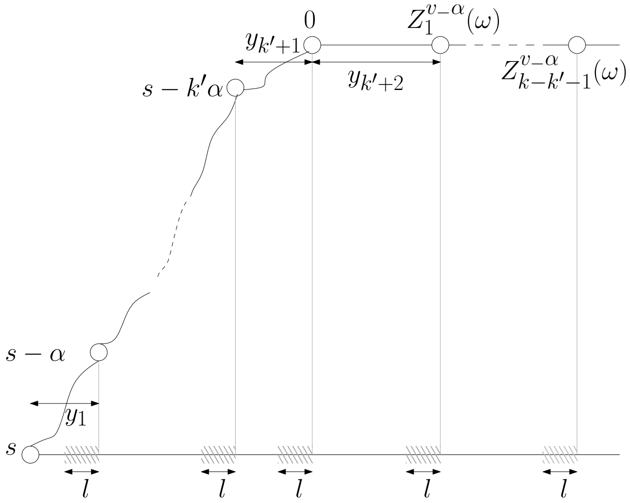

Figure B.5: On the event $\mathcal{Z}\cap B$, there is a zero in the shaded boxes and $\underline{v}\alpha \leqslant y_i \leqslant \overline{v}\alpha$. The occurrence of a zero in each box is obtained by the relaxation from a distinguished zero which was either present at time 0 or generated by the front motion.

we know that for all $i = 1, ...k'$, if $y_i = X(\omega(s-i\alpha)) - X(\omega(s-(i-1)\alpha))$,

$$\underline{v}\alpha \leqslant y_i \leqslant \overline{v}\alpha.$$

B is the event that during one of the k' intervals of length α of the form $[s-(i-1)\alpha, s-i\alpha]$, or during $[0, s-i\alpha]$, the dynamics is such that the front moves more or less than what is predicted by Lemma B.3.2. Moreover, in cases 2 and 3, if we let $y_{k'+1+i} = Z_i^{\underline{v}\alpha}(\omega) - Z_{i-1}^{\underline{v}\alpha}(\omega)$, our conditions guarantee that also

$$\underline{v}\alpha \leqslant y_{k'+1+i} \leqslant \overline{v}\alpha.$$

Therefore, on the event $\mathcal{Z}\cap B$, there are k boxes of length l behind the front, each containing a zero, and whose right ends are spaced at least by $\underline{v}\alpha$, and at most by $\overline{v}\alpha$ (see Figure B.5).

Remark B.4.5 *Notice that the distinction between cases 1 and 2 ($k' = k$ or $k' < k$) happens for $L \approx \underline{v}t$, which is natural, considering that our first construction block is Lemma B.3.2: roughly, for $L \lesssim \underline{v}t$, at distance L front the front at time t, we neglect the possibility of not being in the negative half-line, and we only need the zeros left by the passage of the front. For $L \gtrsim \underline{v}t$, we start taking into account the possibility that the front hasn't moved further than $-\underline{v}t$, and that at distance L from the front we can land in the positive half-line, and so we also need zeros from the initial configuration.*

From now on, we study the term $\mathbb{E}_\sigma\left[f\left(\sigma_{X(\sigma)+y+L+\cdot}(t-s)\right)\right]$ that appears in (B.25), with $\omega(s) = \sigma$ and $y, y_1, \ldots, y_k$ fixed as above. We have chosen s, α, l, k such that –since $3\overline{v}c_1 \leqslant 1$:

$$\overline{v}\alpha \leqslant L - \overline{v}(t-s) \tag{B.26}$$
$$k\underline{v}\alpha - l > L - a(t-s) \tag{B.27}$$

The two conditions ensure that $X(\sigma)+L-y$ lies between two of the zeros guaranteed by $\mathcal{Z}\cap B$ in cases 1 and 2, and by condition (B.22) in case 3, where $s = 0$ (see Figure B.6). Let us call $\xi(0)$ the first zero on the right of $X(\sigma)+L-y$. $\mathcal{Z}\cap B$ guarantees that $|X(\sigma)+L-y-\xi(0)| \leqslant \overline{v}\alpha + 2l$.

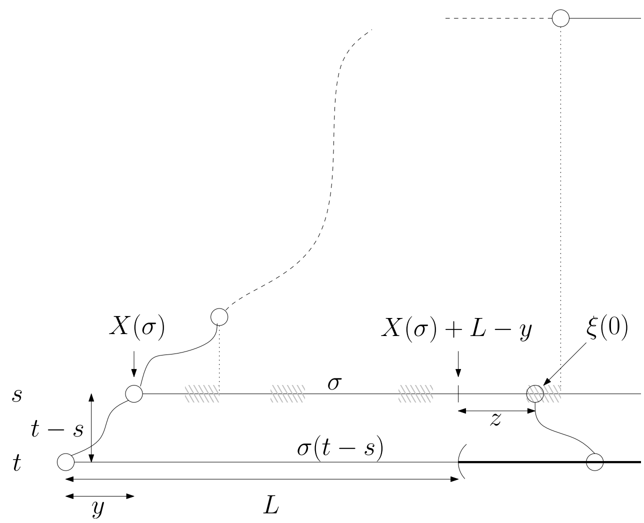

Figure B.6: σ is the configuration at time s. We are on the event $\mathcal{Z}\cap B$, so that as in Figure B.5 the shaded boxes at time s contain at least a zero. We also assumed (B.26) and (B.27), so that at time s, there are shaded boxes on both sides of $X(\sigma)+L-y$, which is therefore at distance z at most $\overline{v}+2l$ from the first zero on its right $\xi(0)$. The bolded half-line on the right of the parenthesis at time t is the part of the configuration at time t which plays a role in $f\left(\sigma_{X(\sigma)+y+L+\cdot}(t-s)\right)$. The line between times s and t starting from $\xi(0)$ represents the motion of the distinguished zero.

We will make $\xi(0)$ the distinguished zero. We apply proposition B.4.3 with the following choice $z=\xi(0)-(X(\sigma)+L-y)$:

$$\mathbb{E}_\sigma\left[\left(f-\mu_{\{1\}}(f)\right)\left(\sigma_{X(\sigma)-y+L+\cdot}(t-s)\right)\right] \leqslant \sqrt{2}\left(\frac{1}{p\wedge q}\right)^{\overline{v}\alpha+2l} e^{-(t-s)\,\mathrm{gap}}$$

So that we have:

$$\begin{aligned}
\mathbb{E}_\omega\left[f\left(\theta_L\omega(t)\right)\right] = & \sum_{y=\underline{v}(t-s)}^{\overline{v}(t-s)} \mathbb{E}_\omega\left[p(\omega(s),y,t-s)\mathbf{1}_{\mathcal{Z}\cap B}\mathbb{E}_{\omega(s)}\left[\mu_{\{1\}}(f)\left(\sigma_{X(\sigma(0))-y+L+\cdot}(t-s)\right)\right]\right] \\
& + O\Big(e^{-\gamma(t-s)}+e^{-(t-s)}+(k+1)\left(p^l+\tfrac{p^{l/2}}{(p\wedge q)^l}e^{-\alpha\mathrm{gap}}\right) \\
& \quad + (k'+1)e^{-\gamma\alpha}+\left(\tfrac{1}{p\wedge q}\right)^{\overline{v}\alpha+2l} e^{-(t-s)\,\mathrm{gap}}\Big) \\
= & \ \mathbb{E}_\omega\left[\mu_{\{1\}}(f)\left(\theta_L\omega(t)\right)\right]+O\Big(e^{-\gamma(t-s)}+e^{-(t-s)}+(k+1)\left(p^l+\tfrac{p^{l/2}}{(p\wedge q)^l}e^{-\alpha\mathrm{gap}}\right) \\
& \quad + (k'+1)e^{-\gamma\alpha}+\left(\tfrac{1}{p\wedge q}\right)^{\overline{v}\alpha+2l} e^{-(t-s)\,\mathrm{gap}}\Big)
\end{aligned}$$

by the same approximations as before.

Now the reader just needs to check that α,s,k,l have been chosen to satisfy the theorem (see Remark B.3.3). ✓

This theorem was the first step towards the following result. It states that the law of the configuration “far from the front" and the equilibrium measure are close in terms of total variation distance (see [LPW09] or [Kuk06]). Of course, if we start from a general distribution, this can't be true for the law of the configuration on an entire right half-line: for instance, if the initial configuration has a finite number of zeros, this property is preserved through the dynamics, and is not compatible with being close to a product measure in infinite volume. This means that for a general initial configuration (case 1 of the theorem below), the configuration far from the front at time t can only look like the equilibrium measure up to some length depending on t and the zeros that were present in the initial configuration. This restriction does not hold in case 2, when we start from $\tilde{\mu}$ (B.1), since far enough from the front, the law of the configuration at time t will be "exactly" the product Bernoulli measure μ.

First let us define the property that the initial configuration should satisfy for us to apply the theorem.

Definition B.4.6 *Let L_0, M be two natural integers, $t > 0$.*

We say that a configuration $\omega \in LO$ satisfies the hypothesis $H(L_0, M, t)$ if

$$\forall L = L_0, ..., L_0 + M \qquad \forall i = 1, ..., k(L,t) - \left\lfloor \frac{s(L,t)}{\alpha(L,t)} \right\rfloor \qquad Z_i^{\underline{v}\alpha(L,t)}(\omega) - Z_{i-1}^{\underline{v}\alpha(L,t)}(\omega) < \overline{v}\alpha(L,t), \tag{B.28}$$

where k, α, s are those defined in Theorem B.4.4.

Note that if L_0 is small enough (for instance $L_0 < \frac{\underline{v}}{1+2\underline{v}c_1} t$), the condition can be rewritten:

$$\forall L = \left\lfloor \frac{\underline{v}}{1+2\underline{v}c_1} t \right\rfloor, ..., L_0 + M, \ \forall i = 1, ..., k(L,t) - \left\lfloor \frac{s(L,t)}{\alpha(L,t)} \right\rfloor, \ Z_i^{\underline{v}\alpha(L,t)}(\omega) - Z_{i-1}^{\underline{v}\alpha(L,t)}(\omega) < \overline{v}\alpha(L,t)$$

Theorem B.4.7 *Let L_0, M be two natural integers. For $\omega \in LO_0$ (resp. π), $t > 0$, we denote by $\nu^{\omega}_{t,L_0,M}$ (resp. $\nu^{\pi}_{t,L_0,M}$) the distribution of the configuration seen from the front at time t, restricted to $[\![L_0+1, L_0+M]\!]$ (namely $(\theta_{L_0}\omega(t))_{|[\![1,M]\!]}$) when $\omega(0) = \omega$ (resp. $\omega(0) \sim \pi$). Recall the definition of $\tilde{\mu}$ (B.1): it is the product measure with only ones on the negative half-line, a zero in 0, and independent Bernoulli(p) variables on the positive half-line.*

1. *If ω satisfies $H(L_0, M, t)$, then there exist constants $\epsilon > 0$, $K < \infty$ depending only on p such that:*

$$\|\nu^{\omega}_{t,L_0,M} - \tilde{\mu}_{|[\![1,M]\!]}\|_{TV} \leqslant K \left(e^{-\epsilon L_0} + \sum_{i=(L_0-3\overline{v}t)\vee 1}^{(L_0+M-3\overline{v}t)\vee 0} \frac{3\overline{v}t + i}{3\overline{v}t} e^{-\epsilon 3\overline{v}t} \right) \tag{B.29}$$

2. $$\left\| \tilde{\mu} - \nu^{\tilde{\mu}}_{t,L_0,\infty} \right\|_{TV} \leqslant K e^{-\epsilon L_0} \tag{B.30}$$

Remark B.4.8 *$H(L_0, M, t)$ is always satisfied if t is large enough (bigger than $c(L_0 + M)$ for some constant c). Indeed, if that is the case, in the proof we only use the result of Theorem B.4.4 in the setting of case 1. Namely, we never use the zeros of the initial condition: the zeros generated by the front are enough.*

Proof of Theorem B.4.7

1. We want to show that for any f function on $\Omega_{[\![1,M]\!]}$ such that $\|f\|_\infty \leqslant 1$, we have:

$$\left|\mathbb{E}_\omega\left[f\left((\theta_{L_0}\omega(t))_{|[\![1,M]\!]}\right)\right]-\tilde{\mu}(f)\right| \leqslant K\left(e^{-\epsilon L_0}+\sum_{i=(L_0-3\overline{v}t)\vee 1}^{(L_0+M-3\overline{v}t)\vee 0}\frac{3\overline{v}t+i}{3\overline{v}t}e^{-\epsilon 3\overline{v}t}\right)$$

This is just an iteration of the result of Theorem B.4.4. Thanks to the hypothesis $H(L_0,M,t)$, we can apply case 1 or 2 of Theorem B.4.4 successively to $f\left(\theta_L\omega(t)\right)$, then to $\mu_{\{1\}}(f)\left(\theta_L\omega(t)\right)$ (which is a function of $\theta_{L+1}\omega(t)$), and so on up to $\mu_{[1,3\overline{v}t-L_0-1]}(f)\theta_L\omega(t)$ (which is a function of $\theta_{3\overline{v}t-1}\omega(t)$). Then, thanks again to the hypothesis $H(L_0,M,t)$, we apply case 3 of Theorem B.4.4 successively to $\mu_{[1,3\overline{v}t-L_0]}(f)\theta_L\omega(t)$, $\mu_{[1,3\overline{v}t-L_0+1]}(f)\theta_L\omega(t)$..., $\mu_{[1,M]}(f)\theta_L\omega(t)$ (which are functions respectively of $\theta_{3\overline{v}t}\omega(t)$, $\theta_{3\overline{v}t+1}\omega(t)$..., $\theta_M\omega(t)$). The result follows since

$$\sum_{i\geq 0}e^{-\epsilon(L_0+i)}=e^{-\epsilon L_0}\sum_{i\geq 0}e^{-\epsilon i}$$

and the sum converges.

2. We want to show that for any f on LO_0 such that $\|f\|_\infty \leqslant 1$,

$$\mathbb{E}_{\tilde{\mu}}\left[f\left(\theta_{L_0}\omega(t)\right)\right]=\tilde{\mu}(f)+O\left(e^{-\epsilon L_0}\right).$$

Assume $L_0 \leqslant 3\overline{v}t$. Define the event:

$$\mathcal{H}=\left\{\omega\in LO_0 \mid \omega \text{ satisfies } H(L_0,3\overline{v}t-L_0,t)\right\}.$$

Then

$$\mathbb{E}_{\tilde{\mu}}\left[f\left(\theta_{L_0}\omega(t)\right)\right] = \tilde{\mu}\left(\mathbf{1}_{\mathcal{H}}\mathbb{E}_\omega\left[f\left(\theta_{L_0}\omega(t)\right)\right]\right)+\tilde{\mu}\left(\mathbf{1}_{\mathcal{H}^c}\mathbb{E}_\omega\left[f\left(\theta_{L_0}\omega(t)\right)\right]\right)$$

But

$$\begin{aligned}\left|\tilde{\mu}\left(\mathbf{1}_{\mathcal{H}^c}\mathbb{E}_\omega\left[f\left(\theta_{L_0}\omega(t)\right)\right]\right)\right| &\leqslant \tilde{\mu}\left(\mathcal{H}^c\right)\\ &\leqslant \sum_{L=L_0}^{3\overline{v}t}\sum_{i=1}^{k(L)-\left\lfloor\frac{s(L)}{\alpha(L)}\right\rfloor}p^{(\overline{v}-\underline{v})\alpha(L)}\\ &\leqslant \sum_{L=L_0}^{3\overline{v}t}k(L)p^{(\overline{v}-\underline{v})\alpha(L)}\\ &= O\left(\sum_{L=L_0}^{3\overline{v}t}e^{-\epsilon' L}\right)\\ &= O\left(e^{-\epsilon L_0}\right)\end{aligned}$$

for some $\epsilon,\epsilon'>0$ (notice that for $L\leqslant 3\overline{v}t$, $k(q,L,t)$ is bounded by a constant depending only on q). By application of (B.29) (taking $M=3\overline{v}t-L_0$),

$$\left|\tilde{\mu}\left(\mathbf{1}_{\mathcal{H}}\left(\mathbb{E}_\omega\left[f\left(\theta_{L_0}\omega(t)\right)\right]-\mathbb{E}_\omega\left[\tilde{\mu}_{[\![1,3\overline{v}t-L_0]\!]}\left(f\left(\theta_{L_0}\omega(t)\right)\right)\right]\right)\right)\right| = O\left(e^{-\epsilon L_0}\right)$$

So that:

$$\mathbb{E}_{\tilde{\mu}}\left[f\left(\theta_{L_0}\omega(t)\right)\right]-\mathbb{E}_{\tilde{\mu}}\left[\tilde{\mu}_{[\![1,3\overline{v}t-L_0]\!]}\left(f\left(\theta_{L_0}\omega(t)\right)\right)\right]=O\left(e^{-\epsilon L_0}\right)$$

for some $\epsilon > 0$.

Since $\tilde{\mu}_{[\![1,3\overline{v}t-L_0]\!]}\left(f\left(\theta_{L_0}\omega(t)\right)\right)$ is a function of $\theta_{3\overline{v}t}\omega(t)$ bounded by 1, all that remains now is justify that we can choose $\epsilon > 0$ such that for all f with $\|f\|_\infty \leqslant 1$ and for $L_0 \geq 3\overline{v}t$:

$$\mathbb{E}_{\tilde{\mu}}\left[f\left(\theta_{L_0}\omega(t)\right)\right] - \tilde{\mu}(f) = O\left(e^{-\epsilon L_0}\right)$$

But for such L_0, with high probability, $X(\omega(t)) + L_0 > 0$ and f looks essentially at the positive half-line, where everything is at equilibrium, thanks to the orientation of the East model. Let us write this more precisely.

Call $R = F(0, -L_0/3, t)^c \cap F(0, L_0/3, t)^c$ (recall (B.5)). In particular, on this event, $|X(\omega(t))| \leqslant L_0/3$.

$$\begin{aligned}
\mathbb{E}_{\tilde{\mu}}\left[f\left(\theta_{L_0}\omega(t)\right)\right] &= \mathbb{E}_{\tilde{\mu}}\left[f\left(\theta_{L_0}\omega(t)\right)\mathbf{1}_R\right] + O\left(e^{-L_0/3}\right) && \text{(B.31)}\\
&= \sum_{x=-L_0/3}^{L_0/3} \mathbb{E}_\mu\left[f\left(\omega_{x+L_0+\cdot}(t)\right)\mathbf{1}_{\tilde{X}(t)=x}\mathbf{1}_R\right] + O\left(e^{-L_0/3}\right) && \text{(B.32)}
\end{aligned}$$

where $\tilde{X}(s), s \leqslant t$ is defined in the following way. Starting from a configuration $\omega \in \Omega$ (not necessarily in LO_0), couple the trajectories started from ω and $\tilde{\omega}$ using the same clocks and coin flips, where $\tilde{\omega}_x = 1$ if $x < 0$, $\tilde{\omega}_0 = 0$ and $\tilde{\omega}_x = \omega_x$ if $x > 0$. Then $\tilde{X}(t) = X(\tilde{\omega}(t))$ depends only on the clock rings, coin flips and $\omega_{|\mathbb{N}^*}$. We can go from (B.31) to (B.32) because on the event R, $X(\omega(t)) + L_0 > 2L_0/3$, so that f looks at sites that are included in $[2L_0/3, +\infty)$ and thus, thanks to the orientation of the East model, is uninfluenced by the choice of the initial configuration on $\mathbb{Z}\backslash\mathbb{N}^*$; also, R is an event that depends only on the Poisson processes, which means in particular that it is unchanged by a change in the initial configuration.

Now notice that, in the same way as in the proof of Theorem B.4.4, for $x \in [-L_0/3, L_0/3]$, the variables $\mathbf{1}_{-L_0/3 \leqslant \tilde{X}(t) \leqslant L_0/3}\mathbf{1}_{\tilde{X}(t)=x}$ and $f\left(\omega_{x+L_0+\cdot}(t)\right)$ are independent on an event of probability greater than $1 - O\left(e^{-L_0/3}\right)$.

So that:

$$\begin{aligned}
\mathbb{E}_{\tilde{\mu}}\left[f\left(\theta_{L_0}\omega(t)\right)\right] &= \sum_{x=-L_0/3}^{L_0/3} \mathbb{E}_\mu\left[f\left(\omega_{x+L_0+\cdot}(t)\right)\right]\mathbb{P}_\mu\left(\{\tilde{X}(t) = x\} \cap R\right) + O\left(e^{-L_0/3}\right)\\
&= \mu(f) + O\left(e^{-L_0/3}\right),
\end{aligned}$$

since μ is the equilibrium measure for the East dynamics on $\mathbb{Z}$. To conclude, since f is a function on LO_0, $\mu(f) = \tilde{\mu}(f)$.

✓

B.5 Invariant measure behind the front

In this section, we show the ergodicity of the process seen from the front. It is a process on LO_0. To write its generator, define the shift ϑ^+ (resp. ϑ^-) from LO_0 (resp. $\{\omega \in LO_0 \mid \omega_1 = 0\}$) into LO_0 such that:

$$\left(\vartheta^+\omega\right)_x = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x < 0 \\ \omega_{x-1} & \text{if } x > 0 \end{cases} \qquad \text{(B.33)}$$

and

$$\left(\vartheta^{-}\omega\right)_x = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x < 0 \\ \omega_{x+1} & \text{if } x > 0 \end{cases} \tag{B.34}$$

Now the generator of the process behind the front can be written:

$$\begin{aligned} \mathcal{L}^F f(\omega) &= q\left[f\left(\vartheta^{+}\omega\right) - f(\omega)\right] + p(1-\omega_1)\left[f\left(\vartheta^{-}\omega\right) - f(\omega)\right] \\ &\quad + \sum_{x\in\mathbb{N}^*} (1-\omega_{x+1})\left(p(1-\omega_x) + q\omega_x\right)\left[f(\omega^x) - f(\omega)\right] \end{aligned} \tag{B.35}$$

This is a combination of two processes: a shift process that comes from the jumps of the front (the first term corresponds to the front jumping to the left, the second to a jump to the right), and the East dynamics on the positive half-line.

Theorem B.5.1 *The process seen from the front has a unique invariant measure ν. For any distribution π on LO_0, recall that $\nu^{\pi}_{t,0,\infty}$ denotes the law of the configuration on the right of the front at time t starting from the distribution π. Then we also have:*

$$\nu^{\pi}_{t,0,\infty} \underset{t\to+\infty}{\Longrightarrow} \nu \tag{B.36}$$

We are going to use the following coupling argument, so we postpone the proof until after this result.

Theorem B.5.2 *Let $\omega, \sigma \in LO_0$. For any $t > 0$, there exist $L_0 = L_0(t) \in \mathbb{N}^*$, and a coupling $\left(\omega^{[t]}, \sigma^{[t]}\right)$ with law $\mathcal{P}$ between $\theta\left(\delta_\omega P_t\right)$ and $\theta\left(\delta_\sigma P_t\right)$ (the configurations seen from the front at time t started from ω and σ), such that $L_0(t) \underset{t\to\infty}{\longrightarrow} +\infty$ and the convergence*

$$\mathcal{P}\left(\left(\omega^{[t]}\right)_{[\![1,L_0]\!]} = \left(\sigma^{[t]}\right)_{[\![1,L_0]\!]}\right) \underset{t\to\infty}{\longrightarrow} 1 \tag{B.37}$$

occurs uniformly in ω, σ.

Proof of Theorem B.5.2

Let us introduce some notations. Fix $L_0, N \in \mathbb{N}^*$, $t_0, t_c, t'_c > 0$ to be chosen later so that

$$t = t_0 + N\left(t_c + t'_c\right)$$

(in particular, these quantities will grow with t). For $n = 0, ..., N$, define

$$\begin{aligned} t_n &= t_0 + n(t_c + t'_c) \\ \tilde{t}_n &= t_n - t'_c \\ L_n^+ &= L_0 + \overline{v}(t_N - t_n) \end{aligned}$$

For $n \in \mathbb{N}$, we define a coupling $\left(\omega^{(n)}, \sigma^{(n)}\right)$ between the configurations seen from the front at time t_n (resp. $\left(\tilde{\omega}^{(n)}, \tilde{\sigma}^{(n)}\right)$ between the configurations seen from the front at time $\tilde{t}_n$). Namely, $\omega^{(n)} \sim \theta\omega(t_n)$, $\sigma^{(n)} \sim \theta\sigma(t_n)$, $\tilde{\sigma}^{(n)} \sim \theta\sigma(\tilde{t}_n)$ and $\tilde{\omega}^{(n)} \sim \theta\omega(\tilde{t}_n)$. We want this coupling to be such

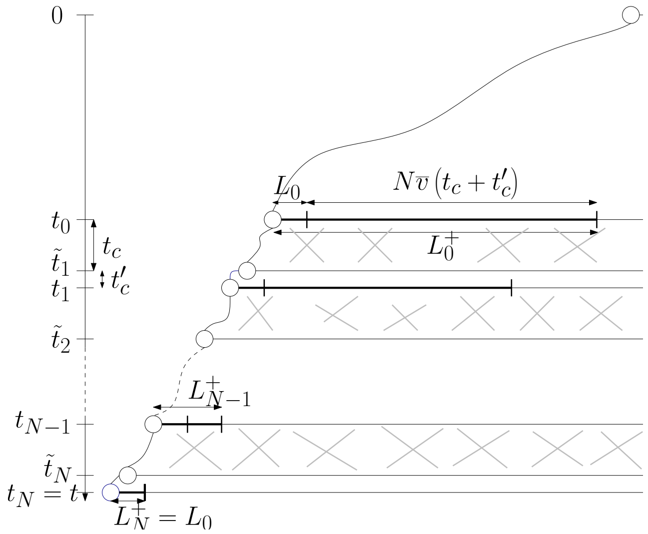

Figure B.7: We construct a coupling between the configurations behind the front started from ω, σ at the times $t_0, \tilde{t}_1, t_1, ...\tilde{t}_N, t_N = t$. The grey crosses, for instance on the interval $\left[t_0, \tilde{t}_1\right]$ are meant to emphasize the fact that the realization of the coupling between $\theta\omega(t_0+t_c)$ and $\theta\sigma(t_0+t_c)$ knowing $\theta\omega(t_0), \theta\sigma(t_0)$ cannot be interpreted as the outcome of an explicit dynamical coupling between times t_0 and $t_0 + t_c$. This part of the construction of the coupling is quite abstract (maximal coupling), and one should be careful that the configurations cannot be defined jointly during those crossed intervals. Also note that this remark holds only as long as the coupling is not successful –on this picture, we represented a case where the coupling is not successful before the last step.

that $\omega^{[t]} := \omega^{(N)}$ and $\sigma^{[t]} := \sigma^{(N)}$ agree on $[\![1, L_0]\!]$ with probability that goes to 1 when we choose the parameters in an appropriate way.

We are going to use the standard (or basic, or grand) coupling between the East dynamics starting from ω and σ, constructed via the graphical construction, using the same set of Poisson clocks and coin tosses. We denote by $(P_t^2)_{t\geq 0}$ the associated semigroup. We also use the maximal coupling (see [LPW09], in which it is called the optimal coupling, or [Kuk06]) between two probability measures π and π': it allows to construct a couple of random variables (Y, Y') such that $Y \sim \pi$, $Y' \sim \pi'$ and $\|\pi - \pi'\|_{TV} = P(Y \neq Y')$.

Let us now define our coupling:

- To sample $\left(\omega^{(0)}, \sigma^{(0)}\right)$, we run the dynamics started from ω and σ using the standard coupling, and take the configurations seen from the front at time t_0.
- For any $n = 1, ..., N$, let us assume the random variable $(\omega^{(n-1)}, \sigma^{(n-1)})$ has been constructed. Conditional on $\left(\omega^{(n-1)}, \sigma^{(n-1)}\right)$, we construct $\left(\tilde{\omega}^{(n)}, \tilde{\sigma}^{(n)}\right)$ in the following way:
  - If $\omega^{(n-1)}$ and $\sigma^{(n-1)}$ are not equal on $[\![1, L_{n-1}^+]\!]$ (i.e. the coupling has not been successful so far), we choose first the restriction of $\left(\tilde{\omega}^{(n)}, \tilde{\sigma}^{(n)}\right)$ to $[\![L_0+1, L_n^+]\!]$ using the maximal coupling between the laws of the configurations seen from the front at time t_c starting from $\omega^{(n-1)}$ and $\sigma^{(n-1)}$, restricted to $[\![L_0+1, L_n^+]\!]$. I.e. $\left(\tilde{\omega}^{(n)}_{[\![L_0+1,L_n^+]\!]}, \tilde{\sigma}^{(n)}_{[\![L_0+1,L_n^+]\!]}\right)$ is given by the maximal coupling between $\left(\theta\omega^{(n-1)}(t_c)\right)_{[\![L_0+1,L_n^+]\!]}$ and $\left(\theta\sigma^{(n-1)}(t_c)\right)_{[\![L_0+1,L_n^+]\!]}$. Conditional on the outcome, the rests of the configurations $\tilde{\omega}^{(n)}$ and $\tilde{\sigma}^{(n)}$ on $\mathbb{N}^* \setminus [\![L_0 + 1, L_n^+]\!]$ are then chosen independently so that $\tilde{\omega}^{(n)}$ and $\tilde{\sigma}^{(n)}$ have the law of the configurations seen from the front at time $\tilde{t}_n$ starting from ω and σ.
  - If $\omega^{(n-1)}$ and $\sigma^{(n-1)}$ are equal on $[\![1, L_{n-1}^+]\!]$ (i.e. the coupling has already been successful), we choose $\left(\tilde{\omega}^{(n)}, \tilde{\sigma}^{(n)}\right)$ as the configurations seen from the front at time t_c using the standard coupling starting from $\omega^{(n-1)}$ and $\sigma^{(n-1)}$.
- For $n = 1, ..., N$, assume $\left(\tilde{\omega}^{(n)}, \tilde{\sigma}^{(n)}\right)$ has been constructed. Conditional on $\left(\tilde{\omega}^{(n)}, \tilde{\sigma}^{(n)}\right)$, we choose $\left(\omega^{(n)}, \sigma^{(n)}\right)$ as the configurations seen from the front when we run the standard coupling started from $\left(\tilde{\omega}^{(n)}, \tilde{\sigma}^{(n)}\right)$ during time t_c'.

Denote by $\mathcal{P}$ the joint law of these couplings, and $\mathcal{E}$ the associated expectancy. $\mathcal{P}_{t_n}, \mathcal{E}_{t_n}$ refer to the law and expectancy of the couplings after time t_n.

The idea is the following: for any $n = 1, ..., N$, provided we manage to keep track of enough zeros, there is a high probability that the configurations obtained from the maximal coupling at step n will be equal on $[\![L_0 + 1, L_n^+]\!]$ (i.e. $\left(\left(\tilde{\omega}^{(n)}\right)_{[\![L_0+1,L_n^+]\!]} = \left(\tilde{\sigma}^{(n)}\right)_{[\![L_0+1,L_n^+]\!]}\right)$). Now, once the configurations at distance L from the front are coupled, there is a small but strictly positive probability that equality will propagate up to the front (see (B.40)), and thus to have $\left(\omega^{(n)}\right)_{[\![1,L_n^+]\!]} = \left(\sigma^{(n)}\right)_{[\![1,L_n^+]\!]}$. We just keep trying to couple the configurations close to the front until this works. Once the coupling has been successful, thanks to finite speed propagation, the two configurations will remain equal near the front. The difficulty that remains is to guarantee that the chance to couple the configurations at distance L at step n is not much lessened by the fact that previous attempts failed.

Let us introduce some useful events.

Definition B.5.3 *1. For $n = 0, ..., N-1$, we define the event that there are enough zeros at step n:*

$$\mathcal{H}_n = \left\{\omega^{(n)} \text{ and } \sigma^{(n)} \text{ satisfy } H\left(L_0, L_n^+ - L_0, t_c\right)\right\} \tag{B.38}$$

On these events, Theorem B.4.7 applies for any $M \leqslant L_n^+ - L_0$ with initial configuration $\omega^{(n)}$ or $\sigma^{(n)}$ and time t_c, so that on $\mathcal{H}_n$

$$\mathcal{P}\left(\left(\tilde{\omega}^{(n+1)}\right)_{|[\![L_0+1,L_n^+]\!]} \neq \left(\tilde{\sigma}^{(n+1)}\right)_{|[\![L_0+1,L_n^+]\!]}\right) \leqslant 2K\left(e^{-\epsilon L_0} + \sum_{i=(L_0-3\overline{v}t_c)\vee 1}^{(L_n^+-3\overline{v}t_c)\vee 0} \frac{3\overline{v}t_c + i}{3\overline{v}t_c} e^{-\epsilon t_c}\right) \tag{B.39}$$

2. Knowing $\left(\tilde{\omega}^{(n)}, \tilde{\sigma}^{(n)}\right)$, then $\left(\omega^{(n)}, \sigma^{(n)}\right)$ can be constructed using clock rings and coin tosses. We define particular events on which, if the configurations $\tilde{\omega}^{(n)}, \tilde{\sigma}^{(n)}$ are equal on $[\![L_0+1, L_n^+]\!]$ and they both have a zero on L_0+1, the clocks on $L_0, ..., 1$ ring in that order before t_c', and the associated coin flips are 0. In particular, all the clock rings in this chain are legal, so they result in the same value for both configurations. Moreover, we ask that no clock rings before t_c' on the sites -1, 0 and $L_n^+ + 1$. This is enough to guarantee propagation of the equality. More formally, we define T_1^+ (resp. T_1^-) the first clock ring on L_0+1 (resp. on L_0), T_2^+ (resp. T_2^-) the time of the first clock ring on L_0 (resp. L_0-1) after T_1^-, and so on up to $T_{L_0}^+, T_{L_0}^-$. Call $B_1, B_2, \dots B_{L_0}$ the outcomes of the coin tosses associated to $T_1^-, T_2^-, \dots T_{L_0}^-$. Finally, call $\tau_{L_n^++1}$ (resp. τ_{-1}, resp. τ_0) the time of the first clock ring in $L_n^+ + 1$ (resp. -1, resp. 0). One event on which equality could propagate at step n is:

$$\mathcal{D}_n = \left\{\forall i = 1, ..., L_0 \quad T_i^- < T_i^+, \quad B_i = 0, \text{ and } \tau_{L_n^++1} \wedge \tau_{-1} \wedge \tau_0 \geq t_c' \geq T_{L_0}^-\right\} \tag{B.40}$$

One important thing about $\mathcal{D}_n$ is that it doesn't depend on the configurations at time $\tilde{t}_n$, but is expressed only in terms of clock rings and coin flips after that time. In particular, it is independent of everything that happened up to time $\tilde{t}_n$.

3. The event "step n is good" (or the coupling is successful at step n) is:

$$G_n = \left\{\left(\tilde{\omega}^{(n)}\right)_{|[\![L_0+1,L_n^+]\!]} = \left(\tilde{\sigma}^{(n)}\right)_{|[\![L_0+1,L_n^+]\!]}\right\} \cap \left\{\tilde{\omega}_{L_0+1}^{(n)} = 0\right\} \cap \mathcal{D}_n \tag{B.41}$$

On G_n, the two configurations are equal on $[\![1, L_n^+]\!]$ at time t_n.

Let us now get to the proof. Once again, $K < \infty$ and $\epsilon > 0$ are constants depending only on q that may change from line to line.

First of all, we note that if we are in the event G_n for some n (i.e. the configurations are equal on $[\![1, L_n^+]\!]$, the lengths L_n^+ have been chosen so that at time t, thanks to the finite speed of propagation property, we still have $\left(\tilde{\omega}^{(N)}\right)_{|[\![1,L_0]\!]} = \left(\tilde{\sigma}^{(N)}\right)_{|[\![1,L_0]\!]}$ with probability larger than $1-(N-n)e^{-(t_c+t_c')}$:

$$\mathcal{P}\left(\left(\omega^{(N)}\right)_{[\![1,L_0]\!]} = \left(\sigma^{(N)}\right)_{[\![1,L_0]\!]}\right) \geq \mathcal{P}\left(\bigcup_{n=1}^{N} G_n\right) - Ne^{-(t_c+t_c')}$$

Then, for $n = 1, ..., N$ we evaluate $\mathcal{P}\left(G_n\right)$ on the event $\mathcal{H}_{n-1}$. Thanks to Theorem B.4.7 ($H\left(L_0, L_n^+ - L_0, t_c\right)$ is satisfied by $\omega^{(n-1)}, \sigma^{(n-1)}$ on $\mathcal{H}_{n-1}$), and the definition of the maximal coupling, on the event $\mathcal{H}_{n-1}$, we have:

$$\begin{aligned}
\mathcal{P}\left(G_n\right) &\geq \mathcal{P}(\mathcal{D}_n)\left(\mathcal{P}\left(\tilde{\omega}^{(n)}_{L_0+1}=0\right)-\mathcal{P}\left(\left(\tilde{\omega}^{(n)}\right)_{|[\![L_0+1,L_n^+]\!]}\neq\left(\tilde{\sigma}^{(n)}\right)_{|[\![L_0+1,L_n^+]\!]}\right)\right)\\
&\geq e^{-3t'_c}q^{L_0}2^{-L_0}e^{-2t'_c}\frac{(2t'_c)^{L_0}}{L_0!}\times\left(q-K\left(e^{-\epsilon L_0}+\sum_{i=(L_0-3\overline{v}t_c)\vee 1}^{(L_n^+-3\overline{v}t_c)\vee 0}\frac{3\overline{v}t_c+i}{3\overline{v}t_c}e^{-\epsilon t_c}\right)\right)\\
&\geq e^{-3t'_c}q^{L_0}2^{-L_0}e^{-2t'_c}\frac{(2t'_c)^{L_0}}{L_0!}\times\left(q-K\left(e^{-\epsilon L_0}+\left(L_n^+\right)^2e^{-\epsilon t_c}\right)\right),
\end{aligned}$$

where the second inequality comes from an estimate of $\mathcal{P}(\mathcal{D}_n)$, (B.39) and from the application of Theorem B.4.7 to $\mathcal{P}\left(\tilde{\omega}^{(n)}_{L_0+1}=0\right)$. The third inequality is a rough estimate of the sum appearing in the line above. Thus, there is $\beta>0$ a constant such that for $t'_c=\beta L_0$ and if

$$\left(L_n^+\right)^2e^{-\epsilon t_c}\ll 1 \tag{B.42}$$

we have for some constant $\Delta<\infty$:

$$\mathcal{P}\left(G_n \mid \mathcal{H}_{n-1}\right)\geq e^{-\Delta L_0} \tag{B.43}$$

for L_0 large enough.

Then we need to control the probability of keeping enough zeros throughout our coupling.

Lemma B.5.4 *There are constants $K<\infty$, $\epsilon>0$ such that if $t_0\geq K(L_0^+)^2$*

$$\mathcal{P}\left(\bigcap_{n=0}^{N-1}\mathcal{H}_n\right)\geq 1-KN(L_0^+)^2e^{-\epsilon t_c}$$

Proof of Lemma B.5.4
Thanks to the remark in Definition B.4.6, we have:

$$\begin{aligned}
\mathcal{P}\left(\bigcap_{n=0}^{N-1}\mathcal{H}_n\right)&\geq 1-\sum_{n=0}^{N-1}\mathcal{P}(\mathcal{H}_n^c)\\
&\geq 1-\sum_{n=0}^{N-1}\mathbb{P}_\omega\left(\sum_{L=\lfloor\frac{v}{1+2\underline{v}c_1}t_c\rfloor}^{L_n^+}\sum_{i=1}^{k(L,t_c)-\left\lfloor\frac{s(L,t_c)}{\alpha(L,t_c)}\right\rfloor}\mathbf{1}_{Z_i^{\underline{v}\alpha(L,t_c)}(\omega(t_n))-Z_{i-1}^{\underline{v}\alpha(L,t_c)}(\omega(t_n))\geq\overline{v}\alpha(L,t_c)}\neq 0\right)\\
&\quad-\sum_{n=0}^{N-1}\mathbb{P}_\sigma\left(\sum_{L=\lfloor\frac{v}{1+2\underline{v}c_1}t_c\rfloor}^{L_n^+}\sum_{i=1}^{k(L,t_c)-\left\lfloor\frac{s(L,t_c)}{\alpha(L,t_c)}\right\rfloor}\mathbf{1}_{Z_i^{\underline{v}\alpha(L,t_c)}(\sigma(t_n))-Z_{i-1}^{\underline{v}\alpha(L,t_c)}(\sigma(t_n))\geq\overline{v}\alpha(L,t_c)}\neq 0\right)
\end{aligned}$$

Now look carefully at the event

$$\left\{\sum_{L=\lfloor\frac{v}{1+2\underline{v}c_1}t_c\rfloor}^{L_n^+}\sum_{i=1}^{k(L,t_c)-\left\lfloor\frac{s(L,t_c)}{\alpha(L,t_c)}\right\rfloor}\mathbf{1}_{Z_i^{\underline{v}\alpha(L,t_c)}(\omega(t_n))-Z_{i-1}^{\underline{v}\alpha(L,t_c)}(\omega(t_n))\geq\overline{v}\alpha(L,t_c)}\neq 0\right\} \tag{B.44}$$

It depends only on $\theta_{\underline{v}\alpha\left(\left\lfloor\frac{v}{1+2c_1\underline{v}}t_c\right\rfloor-1\right)}\omega(t_n)$ restricted to $[\![1,k(L_n^+,t_c)\overline{v}\alpha(L_n^+,t_c)]\!]$. So, if

$$t_0 \geq c\left(\underline{v}\alpha\left(\left\lfloor\frac{\underline{v}}{1+2c_1\underline{v}}t_c\right\rfloor-1,t_c\right)+k(L_n^+,t_c)\overline{v}\alpha(L_n^+,t_c)\right), \tag{B.45}$$

thanks to Remark B.4.8, since $t_n \geq t_0$, ω and σ automatically satisfy the hypotheses

$$H\left(\underline{v}\alpha\left(\left\lfloor\frac{\underline{v}}{1+2c_1\underline{v}}t_c\right\rfloor-1\right),k(L_n^+,t_c)\overline{v}\alpha\left(\left\lfloor\frac{\underline{v}}{1+2c_1\underline{v}}t_c\right\rfloor-1\right),t_n\right).$$

Thanks to this remark, we can apply Theorem B.4.7 to the indicator function of the event (B.44) with t_0 such that (B.45) is verified to get:

$$\begin{aligned}
&\mathbb{P}_\omega\left(\sum_{L=\lfloor\frac{\underline{v}}{1+2\underline{v}c_1}t_c\rfloor}^{L_n^+}\sum_{i=1}^{k(L,t_c)-\lfloor\frac{s(L,t_c)}{\alpha(L,t_c)}\rfloor}\mathbf{1}_{Z_i^{\underline{v}\alpha(L,t_c)}(\omega(t_n))-Z_{i-1}^{\underline{v}\alpha(L,t_c)}(\omega(t_n))\geq\overline{v}\alpha(L,t_c)}\neq 0\right)\\
&\quad\leqslant \mu\left(\sum_{L=\lfloor\frac{\underline{v}}{1+2\underline{v}c_1}t_c\rfloor}^{L_n^+}\sum_{i=1}^{k(L,t_c)-\lfloor\frac{s(L,t_c)}{\alpha(L,t_c)}\rfloor}\mathbf{1}_{Z_i^{\underline{v}\alpha(L,t_c)}(\omega(t_n))-Z_{i-1}^{\underline{v}\alpha(L,t_c)}(\omega(t_n))\geq\overline{v}\alpha(L,t_c)}\neq 0\right)\\
&\qquad+Ke^{-\epsilon\underline{v}\alpha\left(\left\lfloor\frac{\underline{v}}{1+2c_1\underline{v}}t_c\right\rfloor-1\right)}\\
&\quad\leqslant \sum_{L=\lfloor\frac{\underline{v}}{1+2\underline{v}c_1}t_c\rfloor}^{L_n^+}\left(k(L,t_c)-\left\lfloor\frac{s(L,t_c)}{\alpha(L,t_c)}\right\rfloor\right)p^{(\overline{v}-\underline{v})\alpha(L,t_c)}+Ke^{-\epsilon\underline{v}\alpha\left(\left\lfloor\frac{\underline{v}}{1+2c_1\underline{v}}t_c\right\rfloor-1\right)}\\
&\quad= O\left((L_n^+)^2e^{-\epsilon t_c}\right),
\end{aligned}$$

where the second inequality uses two union bounds and the last equality is a rough estimate of the above line.

✓

Now we can finish proving the theorem, assuming L_0 large enough, $t_0 \geq K(L_0^+)^2$ and $N(L_0^+)^2e^{-\epsilon t_c} \ll 1$. The trick is to notice that the probability of success at step n, $\mathcal{P}(G_n)$, is greater than a positive constant as soon as we have enough zeros at time t_{n-1}, i.e. $\mathcal{H}_n$ is realised. At every step n, either G_n happens, or G_n^c, in which case we request that we be on $\mathcal{H}_n$. For the first step, we write, conditioning by $\left(\omega^{(0)},\sigma^{(0)}\right)$ and then by $\left(\omega^{(1)},\sigma^{(1)}\right)$:

$$\begin{aligned}
\mathcal{P}\left(\bigcup_{n=1}^N G_n\right) &\geq \mathcal{E}\left[\mathbf{1}_{\mathcal{H}_0}\mathbf{1}_{\bigcup_{n=1}^N G_n}\right]=\mathcal{E}\left[\mathbf{1}_{\mathcal{H}_0}\mathcal{P}_{t_0}\left(\bigcup_{n=1}^N G_n\right)\right]\\
&\geq \mathcal{E}\left[\mathbf{1}_{\mathcal{H}_0}\mathcal{E}_{t_0}\left[\mathbf{1}_{G_1}+\mathbf{1}_{G_1^c}\mathcal{P}_{t_1}\left(\bigcup_{n=2}^N G_n\right)\right]\right]\\
&\geq \mathcal{E}\left[\mathbf{1}_{\mathcal{H}_0}\mathcal{E}_{t_0}\left[\mathbf{1}_{G_1}+\mathbf{1}_{G_1^c}\mathbf{1}_{\mathcal{H}_1}\mathcal{P}_{t_1}\left(\bigcup_{n=2}^N G_n\right)\right]\right]
\end{aligned}$$

Then we iterate inside $\mathcal{P}_{t_1}$ with the same strategy, and so on until the last step:

$$\begin{aligned}
\mathcal{P}\left(\bigcup_{n=1}^N G_n\right) &\geq \mathcal{E}\left[\mathbf{1}_{\mathcal{H}_0}\mathcal{E}_{t_0}\left[\mathbf{1}_{G_1}+\mathbf{1}_{G_1^c}\mathbf{1}_{\mathcal{H}_1}\mathcal{E}_{t_1}\left[\mathbf{1}_{G_2}+\mathbf{1}_{G_2^c}\mathbf{1}_{\mathcal{H}_2}\mathcal{P}_{t_2}\left(\cup_{n=3}^N G_n\right)\right]\right]\right]\\
&\geq \mathcal{E}\left[\mathbf{1}_{\mathcal{H}_0}\mathcal{E}_{t_0}\left[\dots\mathcal{E}_{t_{N-2}}\left[\mathbf{1}_{G_{N-1}}+\mathbf{1}_{G_{N-1}^c}\mathbf{1}_{\mathcal{H}_{N-1}}\mathcal{P}_{t_{N-1}}(G_N)\right]\dots\right]\dots\right]
\end{aligned}$$

Now to exploit the bounds (B.43), we start by the last step:

$$\mathbf{1}_{\mathcal{H}_{N-1}}\mathcal{P}_{t_{N-1}}\left(G_N\right) \geq \mathbf{1}_{\mathcal{H}_{N-1}}e^{-\Delta L_0},$$

so that writing $\mathbf{1}_{G^c_{N-1}} = 1 - \mathbf{1}_{G_{N-1}}$, we get:

$$\mathcal{P}\left(\bigcup_{n=1}^{N} G_n\right) \;\; \geq \;\; \mathcal{E}\left[\mathbf{1}_{\mathcal{H}_0}\mathcal{E}_{t_0}\ldots\mathcal{E}_{t_{N-2}}\left[\mathbf{1}_{G_{N-1}}\left(1-e^{-\Delta L_0}\right)+\mathbf{1}_{\mathcal{H}_{N-1}}e^{-\Delta L_0}\right]\ldots\right]$$

Now that we have taken care of what happens at step N, let us look at the term inside $\mathcal{E}_{t_{N-3}}$, and do the same with step $N-1$:

$$\mathcal{E}_{t_{N-3}}\left[\mathbf{1}_{G_{N-2}}+\mathbf{1}_{G^c_{N-2}}\mathbf{1}_{\mathcal{H}_{N-2}}\left(\left(1-e^{-\Delta L_0}\right)\mathcal{P}_{t_{N-2}}(G_{N-1})+e^{-\Delta L_0}\mathcal{P}_{t_{N-2}}(\mathcal{H}_{N-1})\right)\right]$$

Again, thanks to (B.43), this is greater than:

$$\begin{aligned}
&\mathcal{E}_{t_{N-3}}\left[\mathbf{1}_{G_{N-2}}+\mathbf{1}_{G^c_{N-2}}\mathbf{1}_{\mathcal{H}_{N-2}}\left(\left(1-e^{-\Delta L_0}\right)e^{-\Delta L_0}+e^{-\Delta L_0}\mathcal{P}_{t_{N-2}}(\mathcal{H}_{N-1})\right)\right]\\
&\geq \mathcal{E}_{t_{N-3}}\left[\mathbf{1}_{G_{N-2}}\left(1-\cancel{\mathbf{1}_{\mathcal{H}_{N-2}}}((1-e^{-\Delta L_0})e^{-\Delta L_0}+e^{-\Delta L_0}\cancel{\mathcal{P}_{t_{N-2}}(\mathcal{H}_{N-1})})\right)\right.\\
&\qquad\qquad\qquad\left. +\ \mathbf{1}_{\mathcal{H}_{N-2}}\left(e^{-\Delta L_0}(1-e^{-\Delta L_0})+e^{-\Delta L_0}\mathcal{P}_{t_{N-2}}(\mathcal{H}_{N-1})\right)\right]\\
&\geq \mathcal{E}_{t_{N-3}}\left[\mathbf{1}_{G_{N-2}}(1-e^{-\Delta L_0})^2+\mathbf{1}_{\mathcal{H}_{N-2}}e^{-\Delta L_0}(1-e^{-\Delta L_0})+\mathbf{1}_{\mathcal{H}_{N-2}\cap\mathcal{H}_{N-1}}e^{-\Delta L_0}\right],
\end{aligned}$$

where we have put to one the crossed terms because the inequalities go in the right way. Iterating for steps $N-2, N-3..., 1$, we get:

$$\begin{aligned}
\mathcal{P}\left(\bigcup_{n=1}^{N} G_n\right) \;\; &\geq \;\; \sum_{n=0}^{N-1} e^{-\Delta L_0}\left(1-e^{-\Delta L_0}\right)^n\mathcal{P}\left(\bigcap_{i=0}^{N-n-1}\mathcal{H}_i\right)\\
&\geq \;\; \left(1-KN\left(L_0+N(t_c+t'_c)\right)^2e^{-\epsilon t_c}\right)\left(1-\left(1-e^{-\Delta L_0}\right)^N\right)
\end{aligned}$$

So let us choose:

$$\begin{aligned}
N &= \left\lfloor L_0e^{\Delta L_0}\right\rfloor\\
t_c &= L_0^2\\
t'_c &= \beta L_0\\
t_0 &= \left(L_0+L_0e^{\Delta L_0}\overline{v}\left(L_0^2+\beta L_0\right)\right)^2
\end{aligned}$$

and L_0 such that

$$t = t_0 + N\left(t_c + t'_c\right)$$

This respects condition (B.42), the hypothesis of Lemma B.5.4, and provides

$$\mathcal{P}\left(\left(\omega^{(N)}\right)_{[\![1,L_0]\!]}=\left(\sigma^{(N)}\right)_{[\![1,L_0]\!]}\right)\underset{t\to\infty}{\longrightarrow}1$$

✓

Proof of Theorem B.5.1

The existence of an invariant measure is just given by the compacity of the set of probability measures on the compact set Ω (see for instance [Lig85]): any limit along a subsequence of the distributions of the process seen from the front is invariant. For the uniqueness and the convergence property, let π be any probability measure on LO_0 and ν an invariant measure for the process seen from the front. It is enough to show that $\nu^{\pi}_{t,0,\infty}$ (recall the statement of Theorem B.5.1) converges to ν in distribution. Let f be a local function on LO_0. Since ν is invariant:

$$\begin{aligned}
\nu^{\pi}_{t,0,\infty}(f) - \nu(f) &= \mathbb{E}_{\pi}\left[f(\theta(\omega(t))\right] - \mathbb{E}_{\nu}\left[f(\theta(\sigma(t))\right] \\
&= \pi\nu\left(\mathbb{E}_{\omega}\left[f(\theta(\omega(t))\right] - \mathbb{E}_{\sigma}\left[f(\theta(\sigma(t))\right]\right)
\end{aligned}$$

Now for any $\omega, \sigma \in LO_0$, we use the coupling constructed in Theorem B.5.2:

$$\begin{aligned}
\left|\mathbb{E}_{\omega}\left[f(\theta(\omega(t))\right] - \mathbb{E}_{\sigma}\left[f(\theta(\sigma(t))\right]\right| &\leqslant \mathcal{E}\left[|f(\omega_t) - f(\sigma_t)|\right] \\
&\leqslant \|f\|_{\infty}\mathcal{P}\left((\omega_t)_{Supp(f)} \neq (\sigma_t)_{Supp(f)}\right) \\
&\underset{t\to+\infty}{\longrightarrow} 0
\end{aligned}$$

uniformly in ω, σ since $Supp(f)$ is finite. So ν is the only possible accumulation point for $\left(\nu^{\pi}_{t,0,\infty}\right)_{t\geq 0}$. Hence the convergence. ✓

Let us now give a few properties of the invariant measure ν.

Proposition B.5.5 *1. There exist constants $\epsilon > 0$, $K < \infty$ such that for any $L, M \in \mathbb{N}$, for any event A on LO_0 with support in $[\![L, L+M]\!]$*

$$|\nu(A) - \mu(A)| \leqslant Ke^{-\epsilon L} \tag{B.46}$$

2.

$$\nu \ll \tilde{\mu} \qquad \textit{(i.e. every property true } \tilde{\mu}\textit{-a.s. is also true } \nu\textit{-a.s.).} \tag{B.47}$$

Proof

1. Take such an event A. Define $\theta_L A = \{\theta_L\omega \mid \omega \in A\}$. By point 2 of Theorem B.4.7

$$\begin{aligned}
\left|\mathbb{E}_{\tilde{\mu}}\left[\mathbf{1}_A\left(\theta\omega(t)\right)\right] - \tilde{\mu}(A)\right| &= \left|\mathbb{E}_{\tilde{\mu}}\left[\mathbf{1}_{\theta_L A}\left(\theta_L\omega(t)\right)\right] - \tilde{\mu}(A)\right| \\
&\leqslant Ke^{-\epsilon L}
\end{aligned}$$

Moreover, we know by Theorem B.5.1 that

$$\mathbb{E}_{\tilde{\mu}}\left[\mathbf{1}_A\left(\theta\omega(t)\right)\right] \underset{t\to+\infty}{\longrightarrow} \nu(A)$$

2. First of all, let us extend the previous property to events A closed (for the topology of LO_0) depending only of the coordinates after L (but possibly with infinite support). For A such an event, for any $M \in \mathbb{N}$, define

$$A_M = \left\{\omega \in LO_0 \mid \exists \sigma \in \{0,1\}^{\{L+M+1,\dots\}}\ \mathbf{1}\cdot\omega\cdot\sigma \in A\right\}$$

where $\mathbf{1}\cdot\omega\cdot\sigma$ is the configuration in LO_0 equal to 1 on $\{1,...L-1\}$, to ω on $\{L,...,L+M\}$ and to σ on $\{L+M+1,...\}$.

Let us show that for any $\omega \in LO_0$, $\mathbf{1}_{A_M}(\omega) \underset{M\to+\infty}{\longrightarrow} \mathbf{1}_A(\omega)$.

Fix $\omega \in LO_0$. If $\omega \in A$, for any $M \in \mathbb{N}$, $\omega \in A_M$. Suppose $\omega \notin A$ and there exists a sequence $M_k \to +\infty$ such that $\forall k$, it holds $\omega \in A_{M_k}$. For every k, take $\sigma_{(k)} \in \{0,1\}^{\{L+M_k+1,...\}}$ such that $\mathbf{1}\cdot\omega\cdot\sigma_{(k)} \in A$. Then $\omega\cdot\sigma_{(k)} \in A$ for every k. Moreover, $\omega\cdot\sigma_{(k)} \underset{k\to+\infty}{\longrightarrow} \omega$. But that would imply $\omega \in A$ since A is closed, which is a contradiction.

This being established, by dominated convergence,

$$\nu(A_M) \underset{M\to+\infty}{\longrightarrow} \nu(A) \quad \text{and} \quad \tilde{\mu}(A_M) \underset{M\to+\infty}{\longrightarrow} \tilde{\mu}(A).$$

The previous result tells us that $|\nu(A_M) - \tilde{\mu}(A_M)| \leqslant Ke^{-\epsilon L}$, so that we have after taking the limit $|\nu(A) - \tilde{\mu}(A)| \leqslant Ke^{-\epsilon L}$.

Now let A be any event depending only of the coordinates after L. $\tilde{\mu}$ and ν are regular (Theorem 1.1 in [Bil09]): for any $\delta > 0$, there exist $O_{\tilde{\mu}}, O_\nu$ open sets and $F_{\tilde{\mu}}, F_\nu$ closed sets depending only on the coordinates after L such that:

$$\begin{array}{lll} F_\nu \subset A \subset O_\nu & \text{and} & \nu\left(O_\nu \backslash F_\nu\right) < \delta \\ F_{\tilde{\mu}} \subset A \subset O_{\tilde{\mu}} & \text{and} & \tilde{\mu}\left(O_{\tilde{\mu}} \backslash F_{\tilde{\mu}}\right) < \delta \end{array}$$

Thanks to the property we just established for closed events (and so immediately also for open events):

$$\begin{array}{rcl} \tilde{\mu}(A) - \nu(A) & \leqslant & \tilde{\mu}\left(O_{\tilde{\mu}} \cap O_\nu\right) - \nu\left(F_{\tilde{\mu}} \cup F_\nu\right) \\ & \leqslant & \mu(O_\nu) - \nu(F_\nu) \\ & \leqslant & \delta + Ke^{-\epsilon L} \end{array}$$

So that –using a similar reasoning for the other inequality– $|\tilde{\mu}(A) - \nu(A)| \leqslant Ke^{-\epsilon L}$.

Take A an event such that $\tilde{\mu}(A) = 0$. Essentially, all that remains to show is that the fact that A has probability zero doesn't depend on any finite set of coordinates.

Let $A^L = \left\{\omega \in \{0,1\}^{\{L+1,...\}} \mid \exists\sigma \in \{0,1\}^{\{1,...,L\}} \ \sigma\cdot\omega \in A\right\}$, where $\sigma\cdot\omega$ is the configuration in $\{0,1\}^{\mathbb{N}^*}$ equal to σ on $\{0,1\}^{\{1,...,L\}}$ and to ω on $\{0,1\}^{\{L+1,...\}}$. Also let $\tilde{A}^L = \{0,1\}^{\{1,...,L\}} \times A^L$.

For any $\sigma \in \{0,1\}^{\{1,...,L\}}$, $\tilde{\mu}\left(\{\sigma\} \times A^L\right) \leqslant \left(\frac{1}{p\wedge q}\right)^L \tilde{\mu}(A)$, so that

$$\tilde{\mu}\left(\tilde{A}^L\right) \leqslant \sum_{\sigma\in\{0,1\}^{\{1,...,L\}}} \tilde{\mu}\left(\{\sigma\} \times A^L\right) = 0$$

But $\tilde{A}^L$ depends only on the coordinates after L, so $\nu(\tilde{A}^L) \leqslant Ke^{-\epsilon L}$. Since $A \subset \tilde{A}^L$ for all $L \in \mathbb{N}$, $\nu(A) = 0$.

✓

B.6 Front speed

The ergodicity proven in Theorem B.5.1 is enough to say that

$$\frac{X(\omega(t))}{t} \underset{t\to+\infty}{\longrightarrow} p\nu(1-\omega_1)-q \qquad \mathbb{P}_\nu - \text{a.s.} \tag{B.48}$$

However, since we know very little about the measure ν, this is not a very practical property. In fact, the law of large numbers for the front is true more generally than ν-a.s.: we are able to show it in $\mathbb{P}_\omega$-probability for any initial configuration $\omega \in LO$ (i.e. requesting only that there be one zero in the initial configuration, which is obviously the minimal requirement one has to make in order to prove a law of large numbers for the front).

Theorem B.6.1 *For any $\omega \in LO_0$*

$$\frac{X(\omega(t))}{t} \xrightarrow[t\to+\infty]{\mathbb{P}_\omega} p\nu(1-\omega_1)-q \tag{B.49}$$

Proof of Theorem B.6.1

Define $v = p\nu(1-\omega_1)-q$.

Step 1: Convergence of the mean value

Let us first establish that:

$$\frac{1}{t}\mathbb{E}_\omega\left[X(\omega(t))\right] \underset{t\to+\infty}{\longrightarrow} v \tag{B.50}$$

In the same way as in [Lig99], III.4, for any $\omega \in LO_0$, we can write

$$X(\omega(t)) = \int_0^t \left(p\left(1-(\theta\omega(s))_1\right)-q\right)ds + M_t$$

where $(M_t)_{t\geq 0}$ is a martingale (so it converges nicely when divided by t). Thanks to Theorem B.5.1, we can apply Birkhoff Ergodic Theorem to the integral term and get the other convergence we need to have (B.50).

Step 2: Upper bound on the velocity.

The essential work of the proof will be to prove that

$$\overline{\lim_{t\to\infty}}\frac{1}{t}X(\omega(t)) \leqslant v \tag{B.51}$$

Classic arguments for this kind of result use subadditivity (see for instance [Lig85], chap. 2, section 2). Here we do not strictly have subadditivity (mainly due to the lack of attractiveness), but we can derive a quantitative version of this argument.

Fix $t>0$, $n\in\mathbb{N}$, $s=t/n$, $L\in\mathbb{N}$, such that $n=\lfloor\sqrt{t}\rfloor$. Note that in the end, we want to take the limit $t\to\infty$. For that purpose, from now on we assume that $L=o(s)$. Let us define the process in $\mathbb{Z}$, for $\omega\in LO$ (see Figure B.8):

$$\begin{aligned} D_L(\omega) &= \inf\{x\geq 0 \mid \omega_{X(\omega)+L+x}=0\} \qquad (\inf(\emptyset)=+\infty)\\ X^L_\omega(0) &= X(\omega)+L+D_L(\omega)\\ X^L_\omega(u)-X^L_\omega(0) &= X\left(\left(\theta_{L+D_L(\omega)}\omega\right)(u)\right) \qquad (=0 \text{ if } D_L(\omega)=+\infty) \end{aligned} \tag{B.52}$$

In words, given a configuration ω, we take the first zero at distance at least L from the front: $X^L_\omega(0)$. Then, using the graphical representation of the process, we follow this zero as if it were a front, i.e. as if we started from a configuration filled with ones on its left. The reader can check, thanks to the orientation of the East model, that this does give a process defined only in terms of the underlying graphical representation and $\theta_{L+D_L(\omega)}\omega$. Note that for any $\omega \in LO$, $L \in \mathbb{N}^*$, $u \geq 0$, $X^L_\omega(u) \geq X(\omega(u))$ by definition, since we used the same variables for the graphical representation. We can then write $\mathbb{P}_\omega$-a.s. for any $\omega \in LO_0$:

$$\begin{aligned} X(\omega(t+s)) &= X(\omega(t)) + X(\omega(t+s)) - X^L_{\omega(t)}(s) + X^L_{\omega(t)}(s) - X^L_{\omega(t)}(0) + X^L_{\omega(t)}(0) - X(\omega(t)) \\ &\leqslant X(\omega(t)) + \left(X^L_{\omega(t)}(s) - X^L_{\omega(t)}(0)\right) + (L + D_L(\omega(t))) \end{aligned}$$

Iterating the previous inequality, thanks to Lemma B.3.4 that implies $\frac{1}{t}X(\omega(s)) \underset{t\to\infty}{\longrightarrow} 0$, we can write:

$$\begin{aligned} \limsup \frac{1}{t} X(\omega(t)) &= \limsup \frac{1}{t}\sum_{j=1}^{n-1} \left[X(\omega((j+1)s)) - X(\omega(js))\right] \\ &\leqslant \limsup \frac{1}{t}\sum_{j=1}^{n-1} \left[X^L_{\omega(js)}(s) - X^L_{\omega(js)}(0) + L + D_L\left(\omega(js)\right)\right] \\ &\leqslant \limsup \frac{1}{t}\sum_{j=1}^{n-1} \left[X^L_{\omega(js)}(s) - X^L_{\omega(js)}(0)\right] + \frac{L}{s} + \frac{1}{t}\sum_{j=1}^{n-1} D_L\left(\omega(js)\right) \end{aligned} \tag{B.53}$$

Let us deal with the most problematic term first:

$$\frac{1}{t}\sum_{j=1}^{n-1} \left[X^L_{\omega(js)}(s) - X^L_{\omega(js)}(0)\right]$$

We want to say that the different terms in the sum are essentially i.i.d. This is of course not true, but we have showed that up to a reasonable distance, $\theta_L\omega(js)$ has almost law $\tilde{\mu}$. Since this coupling doesn't extend to infinity (see the discussion before Theorem B.4.7), we need to use the finite speed of propagation again. So we define the following process, for any $\omega \in LO$, $L, M \in \mathbb{N}$. It is pretty much the same as X^L_ω, except we put a zero boundary condition at $X^L_\omega(0) + M + 1$ in order to be restricted to a process in finite volume.

- $X^{L,M,\circ}_\omega(0) = X^L_\omega(0) \wedge (X(\omega) + L + M + 1)$
- For the rest of the definition, run the East dynamics on $(-\infty, X^L_\omega(0) + M]\!]$ with empty boundary condition at $X^L_\omega(0) + M + 1$. This dynamics can easily be coupled with the East dynamics on $\mathbb{Z}$ via the graphical representation.
- $X^{L,M,\circ}_\omega(u) - X^{L,M,\circ}_\omega(0) = X\left(\left(\theta_{L+D_L(\omega)}\omega\right)^{X^L_\omega(0)+M+1,\circ}(u)\right)$, where $\sigma^{X^L_\omega(0)+M+1,\circ}(u)$ denotes the configuration obtained at time u starting from σ and running the dynamics with zero boundary condition at $X^L_\omega(0) + M + 1$.

The replacement of X^L_ω by $X^{L,M,\circ}_\omega$ is a very mild modification: for the remaining of the proof, we take $M = 3\overline{v}s$, so that with high probability, the front won't notice the change. Namely, for any $j = 0, ..., n-1$, call R_j the event "there is no sequence of rings linking $X^L_{\omega(js)}(0) + 3\overline{v}s$ to the modified front during the time interval $[0, s]$".

$$\begin{aligned} X^L_{\omega(js)}(s) - X^L_{\omega(js)}(0) &= X^{L,3\overline{v}s,\circ}_{\omega(js)}(s) - X^{L,3\overline{v}s,\circ}_{\omega(js)}(0) \\ &\quad + \mathbf{1}_{R_j^c}\left(X^L_{\omega(js)}(s) - X^L_{\omega(js)}(0) - X^{L,3\overline{v}s,\circ}_{\omega(js)}(s) + X^{L,3\overline{v}s,\circ}_{\omega(js)}(0)\right) \end{aligned} \tag{B.54}$$

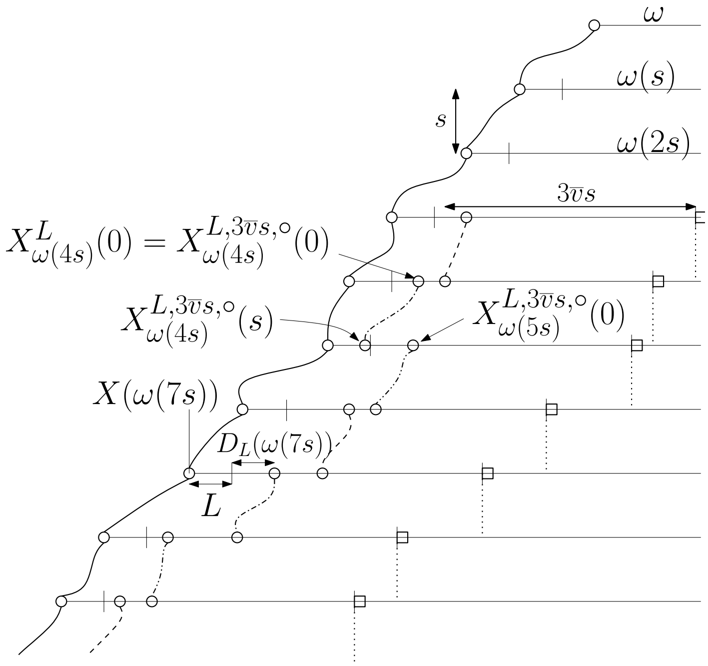

Figure B.8: The original trajectory of the front is on the left, cut into sections of length s. The trajectories of the modified fronts are in different dash styles : "- -","$- \cdot -$" or "$- \cdot\cdot -$". They necessarily stay on the right of the original one. The squares represent the zero boundary condition we use to define $X^{L,3\overline{v}s,\circ}_{\omega(ks)}(u)$. Here we have taken $j_0 = 3$: we consider separately the terms in the sum corresponding to the "- -","$- \cdot -$" and "$- \cdot\cdot -$" trajectories. Notice that after following the modified front for time s, we may either start again closer to the real front (e.g. in the picture $X^{L,3\overline{v}s,\circ}_{\omega(4s)}(0) \leqslant X^{L,3\overline{v}s,\circ}_{\omega(3s)}(s)$), or further from the real front (e.g. $X^{L,3\overline{v}s,\circ}_{\omega(5s)}(0) \geq X^{L,3\overline{v}s,\circ}_{\omega(4s)}(s)$).

We will deal later with the second, exceptional term. For now, let us focus on the first one, to which we substract its mean value:

$$\frac{1}{n}\sum_{j=1}^{n-1}\frac{1}{s}\left(X_{\omega(js)}^{L,3\overline{v}s,\circ}(s)-X_{\omega(js)}^{L,3\overline{v}s,\circ}(0)-\mathbb{E}_{\tilde{\mu}}\left[X_{\eta}^{L,3\overline{v}s,\circ}(s)\right]\right)$$

In order not to carry heavy notations through heavy computations, let us define:

$$\Delta_j=\frac{1}{s}\left(X_{\omega(js)}^{L,3\overline{v}s,\circ}(s)-X_{\omega(js)}^{L,3\overline{v}s,\circ}(0)-\mathbb{E}_{\tilde{\mu}}\left[X_{\eta}^{L,3\overline{v}s,\circ}(s)\right]\right) \tag{B.55}$$

As a preliminary, notice that Lemma B.3.4 can be easily generalized to: the Δ_j have moments of any order bounded by universal constants independent of j. Also let $j_0\geq 2$ be such that $(j_0-1)s\geq c(L+3\overline{v}s)$ for any s (see Remark B.4.8 and recall $L=o(s)$). The idea is that for $|j-j'|>j_0$, the terms of indices j and j' are almost independent of mean zero.

Let us forget the j_0-1 first terms and decompose $\frac{1}{n}\sum_{j=j_0}^{n-1}\Delta_j$ into (see Figure B.8; the different terms in the sum below correspond to different dash styles in the picture):

$$\frac{1}{n}\left(\sum_{\substack{k\text{ s.t.}\\ j_0\leqslant kj_0\leqslant n-1}}\Delta_{kj_0}+\sum_{\substack{k\text{ s.t.}\\ j_0\leqslant kj_0+1\leqslant n-1}}\Delta_{kj_0+1}+\cdots+\sum_{\substack{k\text{ s.t.}\\ j_0\leqslant kj_0+j_0-1\leqslant n-1}}\Delta_{kj_0+j_0-1}\right)$$

Remember that j_0 is fixed (in particular it doesn't depend on s), so that the following lemma is enough to conclude that

$$\frac{1}{n}\sum_{j=1}^{n-1}\Delta_j\underset{t\to+\infty}{\longrightarrow}0\quad\mathbb{P}_{\omega}\text{-a.s.} \tag{B.56}$$

Lemma B.6.2 *For any $i=0,...,j_0-1$, taking $L=\lfloor\sqrt{s}\rfloor$, we have*

$$\frac{1}{n}\sum_{\substack{k\text{ s.t.}\\ j_0\leqslant kj_0+i\leqslant n-1}}\Delta_{kj_0+i}\underset{t\to+\infty}{\longrightarrow}0\quad\mathbb{P}_{\omega}\text{-a.s.},$$

We postpone the proof of the lemma to see how we can deduce the upper bound (B.51). Putting together (B.53), (B.54) and (B.55), we have obtained:

$$\begin{aligned}\frac{1}{t}X(\omega(t))\quad&\leqslant\quad\frac{1}{n}\sum_{j=1}^{n-1}\Delta_j+\frac{L}{s}+\frac{1}{t}\sum_{j=1}^{n-1}D_L\left(\omega(js)\right)+\frac{1}{s}\mathbb{E}_{\tilde{\mu}}\left[X_{\eta}^{L,3\overline{v}s,\circ}(s)\right]\\&\quad+\frac{1}{t}\sum_{j=1}^{n-1}\mathbf{1}_{R_j^c}\left(X_{\omega(js)}^{L}(s)-X_{\omega(js)}^{L}(0)-X_{\omega(js)}^{L,3\overline{v}s,\circ}(s)+X_{\omega(js)}^{L,3\overline{v}s,\circ}(0)\right)\end{aligned}$$

and by (B.56), we know that the first term goes to zero. For the other terms:

1. Thanks to finite speed propagation and Lemma B.3.4

$$\mathbb{P}_{\omega}\left(\left|\frac{1}{t}\sum_{j=0}^{n-1}\mathbf{1}_{R_j^c}\left(X_{\eta_j}^{0,\overline{v}s,\circ}(s)-X_{\eta_j}^{0,\overline{v}s,\circ}(0)-X(\eta_j(s))+X(\eta_j)\right)\right|\geq\delta\right)$$
$$\leqslant\frac{K}{\delta}\left(e^{-\delta\epsilon' s}+e^{-\epsilon L}\right) \tag{B.57}$$

for some $\epsilon'>0$, so that this term goes to 0 almost surely.

2. By Borel-Cantelli lemma, $\frac{1}{t}\sum_{j=1}^{n-1} D_L\left(\omega(js)\right) \underset{t\to\infty}{\longrightarrow} 0$ since for $u \geq j_0 s$, for $\delta \leqslant 3\overline{v}s$, by Theorem B.4.7,
$$\mathbb{P}_\omega\left(D_L(\omega(u)) > \delta s\right) \leqslant p^{\delta s} + Ke^{-\epsilon L}$$

3. By an argument of finite speed propagation similar to (B.54) and (B.57) put together, we get
$$\left|\mathbb{E}_{\tilde{\mu}}\left[X_\eta^{L,3\overline{v}s,\circ}(s)\right] - \mathbb{E}_{\tilde{\mu}}\left[X(\omega(s))\right]\right| \underset{s\to\infty}{\longrightarrow} 0,$$
so that using step 1:
$$\frac{1}{s}\mathbb{E}_{\tilde{\mu}}\left[X_\eta^{L,3\overline{v}s,\circ}(s)\right] \underset{s\to\infty}{\longrightarrow} v,$$
which concludes the proof that
$$\overline{\lim_{t\to\infty}}\frac{1}{t}X(\omega(t)) \leqslant v.$$

Proof of Lemma B.6.2

Fix $\epsilon > 0$. We want to show that
$$\mathbb{P}_\omega\left(\left|\frac{1}{n}\sum_{\substack{k \text{ s.t.}\\ j_0\leqslant kj_0+i\leqslant n-1}} \Delta_{kj_0+i}\right| \geq \epsilon\right)$$
are summable to use the Borel-Cantelli lemma. Take for instance $i=0$. As the variables are weakly dependent, one can derive the law of large numbers by computing the fourth moment and evaluating the correlations:

$$\begin{aligned}
\epsilon^4 n^4 \mathbb{P}_\omega\left(\left|\frac{1}{n}\sum_{k=1}^{\left\lfloor\frac{n-1}{j_0}\right\rfloor}\Delta_{kj_0}\right| \geq \epsilon\right) &\leqslant \mathbb{E}_\omega\left[\left(\sum_{k=1}^{\left\lfloor\frac{n-1}{j_0}\right\rfloor}\Delta_{kj_0}\right)^4\right]\\
&\leqslant \sum_{k=1}^{\left\lfloor\frac{n-1}{j_0}\right\rfloor}\mathbb{E}_\omega\left[\Delta_{kj_0}^4\right]\\
&\quad + 6\sum_{i=1}^{\left\lfloor\frac{n-1}{j_0}\right\rfloor-1}\sum_{k=i+1}^{\left\lfloor\frac{n-1}{j_0}\right\rfloor}\mathbb{E}_\omega\left[\Delta_{ij_0}^2\Delta_{kj_0}^2\right]\\
&\quad + 4\sum_{i=1}^{\left\lfloor\frac{n-1}{j_0}\right\rfloor}\sum_{\substack{k=1\\k\neq i}}^{\left\lfloor\frac{n-1}{j_0}\right\rfloor}\mathbb{E}_\omega\left[\Delta_{ij_0}^3\Delta_{kj_0}\right]\\
&\quad + 12\sum_{i=1}^{\left\lfloor\frac{n-1}{j_0}\right\rfloor}\sum_{\substack{j=1\\j\neq i}}^{\left\lfloor\frac{n-1}{j_0}\right\rfloor-1}\sum_{\substack{k=j+1\\k\neq i}}^{n-1}\mathbb{E}_\omega\left[\Delta_{ij_0}^2\Delta_{jj_0}\Delta_{kj_0}\right]\\
&\quad + 24\sum_{i=1}^{\left\lfloor\frac{n-1}{j_0}\right\rfloor-3}\sum_{j=i+1}^{\left\lfloor\frac{n-1}{j_0}\right\rfloor-2}\sum_{k=j+1}^{\left\lfloor\frac{n-1}{j_0}\right\rfloor-1}\sum_{l=k+1}^{\left\lfloor\frac{n-1}{j_0}\right\rfloor}\mathbb{E}_\omega\left[\Delta_{ij_0}\Delta_{jj_0}\Delta_{kj_0}\Delta_{lj_0}\right]
\end{aligned}$$

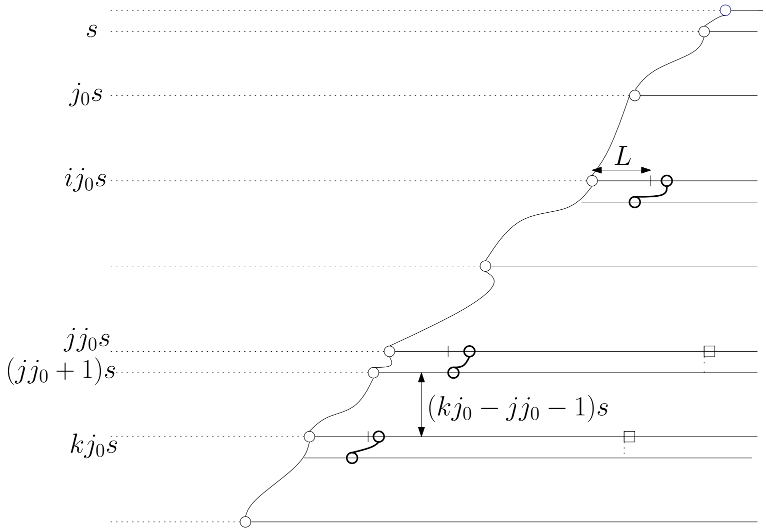

Figure B.9: To bound $\mathbb{E}_\omega\left[\Delta^2_{ij_0}\Delta_{jj_0}\Delta_{kj_0}\right]$, we apply the Markov property at time $(jj_0+1)s$.

Let us evaluate separately the different terms above.

1. The first three terms are of order $O(n^2)$ thanks to an easy generalization of Lemma B.3.4.

2. Let us now deal with the terms $\mathbb{E}_\omega\left[\Delta^2_{ij_0}\Delta_{jj_0}\Delta_{kj_0}\right]$ in the case $i<j<k$ (see Figure B.9).

   $$\mathbb{E}_\omega\left[\Delta^2_{ij_0}\Delta_{jj_0}\Delta_{kj_0}\right] = \frac{1}{s}\mathbb{E}_\omega\left[\Delta^2_{ij_0}\Delta_{jj_0}\mathbb{E}_{\omega((jj_0+1)s)}\left[X^{L,3\overline{v}s,\circ}_{\eta((kj_0-jj_0-1)s)}(s)-X^{L,3\overline{v}s,\circ}_{\eta((kj_0-jj_0-1)s)}(0)-\mathbb{E}_{\tilde{\mu}}\left[X^{L,3\overline{v}s,\circ}_{\eta}(s)\right]\right]\right],$$

   where $\eta(0)=\omega((jj_0+1)s)$.

   For any $\eta\in LO$, given our choice of j_0, we can apply Theorem B.4.7 to

   $$\mathbb{E}_\eta\left[X^{L,3\overline{v}s,\circ}_{\eta((kj_0-jj_0-1)s)}(s)-X^{L,3\overline{v}s,\circ}_{\eta((kj_0-jj_0-1)s)}(0)\right]$$

   since $H\left(L,3\overline{v}s,(kj_0-jj_0-1)s\right)$ is satisfied by any configuration, getting that

   $$\mathbb{E}_\eta\left[X^{L,3\overline{v}s,\circ}_{\eta((kj_0-jj_0-1)s)}(s)-X^{L,3\overline{v}s,\circ}_{\eta((kj_0-jj_0-1)s)}(0)\right]=\mathbb{E}_{\tilde{\mu}}\left[X^{L,3\overline{v}s,\circ}_{\eta}(s)\right]+O\left(e^{-\epsilon L}\right)$$

   So that:

   $$\mathbb{E}_\omega\left[\Delta^2_{ij_0}\Delta_{jj_0}\Delta_{kj_0}\right]=O\left(e^{-\epsilon L}\right)$$

   The cases of $\mathbb{E}_\omega\left[\Delta^2_{ij_0}\Delta_{jj_0}\Delta_{kj_0}\right]$ with $j<i<k$ and $\mathbb{E}_\omega\left[\Delta_{ij_0}\Delta_{jj_0}\Delta_{kj_0}\Delta_{lj_0}\right]$ with $i<j<k<l$ can be treated in the same way.

3. The only terms remaining are the $\mathbb{E}_\omega\left[\Delta^2_{ij_0}\Delta_{jj_0}\Delta_{kj_0}\right]$ with $j<k<i$.

   $$s^2\mathbb{E}_\omega\left[\Delta^2_{ij_0}\Delta_{jj_0}\Delta_{kj_0}\right] = \mathbb{E}_\omega\left[\Delta_{jj_0}\Delta_{kj_0}\mathbb{E}_{\omega((kj_0+1)s)}\left[\left(X^{L,3\overline{v}s,\circ}_{\eta(((i-k)j_0-1)s)}(s)-X^{L,3\overline{v}s,\circ}_{\eta(((i-k)j_0-1)s)}(0)-\mathbb{E}_{\tilde{\mu}}\left[X^{L,3\overline{v}s,\circ}_{\eta}(s)\right]\right)^2\right]\right]$$

For any $\eta \in LO$, applying Theorem B.4.7 to

$$\mathbb{E}_\eta\left[\left(X^{L,3\overline{v}s,\circ}_{\eta((ij_0-kj_0-1)s)}(s) - X^{L,3\overline{v}s,\circ}_{\eta((ij_0-kj_0-1)s)}(0) - \mathbb{E}_{\tilde{\mu}}\left[X^{L,3\overline{v}s,\circ}_{\eta}(s)\right]\right)^2\right]$$

with $L_0 = L$, $M = 3\overline{v}s$ and $t = (ij_0 - kj_0 - 1)s$ yields

$$\mathbb{E}_\omega\left[\Delta^2_{ij_0}\Delta_{jj_0}\Delta_{kj_0}\right] = \frac{1}{s^2}\mathbb{E}_\omega\left[\Delta_{jj_0}\Delta_{kj_0}\right]\left(\mathbb{E}_{\tilde{\mu}}\left[\left(X^{L,3\overline{v}s,\circ}_{\eta}(s) - X^{L,3\overline{v}s,\circ}_{\eta}(0) - \mathbb{E}_{\tilde{\mu}}\left[X^{L,3\overline{v}s,\circ}_{\eta}(s)\right]\right)^2\right] + O\left(e^{-\epsilon L}\right)\right)$$

In the same way as above, we can now say that $\mathbb{E}_\omega\left[\Delta_{jj_0}\Delta_{kj_0}\right] = O\left(e^{-\epsilon L}\right)$.

In conclusion, we have shown that:

$$\mathbb{P}_\omega\left(\left|\frac{1}{n}\sum_{j=1}^{\left\lfloor\frac{n-1}{j_0}\right\rfloor}\Delta_{jj_0}\right| \geq \epsilon\right) = O\left(\frac{1}{n^2} + e^{-\epsilon L}\right),$$

so that since $L = \lfloor\sqrt{s}\rfloor$, by the Borel-Cantelli lemma:

$$\frac{1}{n}\sum_{j=1}^{\left\lfloor\frac{n-1}{j_0}\right\rfloor}\Delta_{jj_0} \underset{n\to\infty}{\longrightarrow} 0 \quad \mathbb{P}_\omega\text{-a.s.}$$

✓

Step 3: Lower bound

Now we just have to show that for any $\epsilon > 0$, t big enough

$$\mathbb{P}_\omega\left(\frac{1}{t}X(\omega(t)) - v < -\epsilon\right) \leqslant \epsilon \tag{B.58}$$

Indeed,

$$\mathbb{P}_\omega\left(\left|\frac{1}{t}X(\omega(t)) - v\right| > \epsilon\right) \leqslant \mathbb{P}_\omega\left(\frac{1}{t}X(\omega(t)) - v < -\epsilon\right) + \mathbb{P}_\omega\left(\frac{1}{t}X(\omega(t)) - v > \epsilon\right)$$

and we have just proven that the second term goes to zero as $t \to \infty$.

For simplicity, let us call $Y_t = \frac{1}{t}X(\omega(t)) - v$. Fix $\epsilon > 0$ and define $\epsilon' = \epsilon^2/3$. We have:

$$\begin{aligned}\mathbb{E}_\omega\left[Y_t\right] &= \mathbb{E}_\omega\left[Y_t\mathbf{1}_{Y_t<-\epsilon}\right] + \mathbb{E}_\omega\left[Y_t\mathbf{1}_{-\epsilon\leqslant Y_t\leqslant\epsilon'}\right] + \mathbb{E}_\omega\left[Y_t\mathbf{1}_{Y_t>\epsilon'}\right]\\ &< -\epsilon\mathbb{P}_\omega\left(Y_t < -\epsilon\right) + \mathbb{E}_\omega\left[Y_t\mathbf{1}_{-\epsilon\leqslant Y_t\leqslant\epsilon'}\right] + \mathbb{E}_\omega\left[Y_t\mathbf{1}_{Y_t>\epsilon'}\right]\end{aligned}$$

So that:

$$\mathbb{P}_{\tilde{\mu}}\left(Y_t < -\epsilon\right) < \frac{1}{\epsilon}\left(\epsilon' + \mathbb{E}_{\tilde{\mu}}\left[Y_t\mathbf{1}_{Y_t>\epsilon'}\right] - \mathbb{E}_{\tilde{\mu}}\left[Y_t\right]\right)$$

Now, thanks to the dominated convergence theorem, (B.51) and Lemma B.3.4 (for the second term), and (B.50) (for the third term), for t big enough:

$$\mathbb{P}_{\tilde{\mu}}\left(Y_t < -\epsilon\right) < \frac{1}{\epsilon}3\epsilon' \leqslant \epsilon$$

✓

Appendix C

Tracer diffusion in low temperature KCSM

This article has been submitted, excluding the section about the windmill model.

Contents

We describe the motion of a tracer in an environment given by a kinetically constrained spin model (KCSM) at equilibrium. We check convergence of its trajectory properly rescaled to a Brownian motion and positivity of the diffusion coefficient D as soon as the spectral gap of the environment is positive (which coincides with the ergodicity region under general conditions). Then we study the asymptotic behaviour of D when the density $1-q$ of the environment goes to 1 in two classes of KCSM. For non-cooperative models, the diffusion coefficient D scales like a power of q, with an exponent that we compute explicitly. In the case of the Fredrickson-Andersen one-spin facilitated model, this proves a prediction made in [JGC04]. For the East model, instead we prove that the diffusion coefficient is comparable to the spectral gap, which goes to zero faster than any power of q. This result contradicts the prediction of physicists ([JGC04]), based on numerical simulations, that suggested $D \sim \text{gap}^\xi$ with $\xi < 1$.

C.1 Introduction

Kinetically constrained models (KCSM) have been introduced in the physics literature to model glassy dynamics. They are Markov processes on $\{0,1\}^{\mathbb{Z}^d}$ (or more generally on the set of configurations on a graph), where zeros mark empty sites, and ones mark sites occupied by a particle. The dynamics is of Glauber type: with rate one, each site refreshes its occupation variable: to a zero with probability q, and to a one with probability $1-q$, on the condition that a specific constraint be satisfied by the configuration around the to-be-updated site. This constraint takes the form “a certain set of zeros should be present in a fixed neighbourhood", but does not involve the configuration *at* the to-be-updated site, so that the product Bernoulli measure on $\mathbb{Z}^d$ with parameter $1-q$ is reversible for the dynamics.

A tracer particle evolves in an environment given by a KCSM. The environment is not influenced by the tracer, which performs a simple random walk constrained to jumping only between two empty sites. Properly rescaled, the tracer trajectory is expected to converge to a Brownian motion with a diffusion coefficient depending on the environment. Standard results and strategy ([KV86],[DMFGW89], [Spo90]) allow us to show that in the ergodic regime for the environment there is indeed convergence to a Brownian motion, and to give a variational formula for the diffusion coefficient (see Proposition C.3.1 and Lemma C.3.2). A general argument then implies that, as soon as the environment has a positive spectral gap, the diffusion coefficient is also positive, so that the convergence result is non-degenerate (Proposition C.3.4). Note that the ergodicity regime of KCSM has been identified in [CMRT08], and has been shown to coincide with the region of positivity of the spectral gap in great generality, including all the models we consider. Thus we prove in fact positivity of the diffusion coefficient in the ergodic regime of the dynamical environment. The variational formula also yields an immediate upper bound on the diffusion coefficient. A similar study was carried in [BT04] with environments given by some non-cooperative constrained models with Kawasaki dynamics.

The main focus of this paper is to compute the asymptotics of the diffusion coefficient when $q \to 0$. This study is inspired by the papers [JGC04] and [JGC05], which in turn have the following physical motivation. In homogeneous liquid systems, physicists argue that the relaxation time τ (measured as the viscosity of the liquid), the temperature T and the diffusion coefficient D of a particle moving inside the system satisfy the following relation, called the Stokes-Einstein relation

$$D \propto T\tau^{-1}, \tag{C.1}$$

This relation is well obeyed in liquids at high enough temperature. Instead, in supercooled liquids it is experimentally observed (see for instance [EEH+12], [CE96], [CS97], [SBME03]) that $D\tau/T$ increases by 2-3 orders of magnitude when decreasing T towards the glass transition temperature. In particular both D and τ^{-1} decrease faster than any power law when the temperature is lowered and for many supercooled liquids a good fit of data is

$$D \propto \tau^{-\xi} \text{ with } \xi < 1. \tag{C.2}$$

In other words, the self-diffusion of particles becomes much faster than structural relaxation and the Stokes Einstein relation is violated. This decoupling between translational diffusion and global relaxation is interpreted as a landmark of dynamical heterogeneities in glassy systems, namely the existence of spatially correlated regions of relatively high or low mobility that persist for a finite lifetime in the liquid, and that grow in size as one approaches the glass transition. More precisely, the decoupling should be due to the fact that diffusion is dominated by the fastest regions whereas structural relaxation is dominated by the slowest regions.

In order to investigate the possible violation of the Stokes Einstein relation in KCM, which are used as simplified models of glassy dynamics, in [JGC04] and [JGC05] the authors run simulations of a tracer in two systems with constrained dynamics in one dimension: the FA-1f model (in which the constraint requests that at least one neighbour be empty), and the East model (in which the constraint is satisfied if the neighbour in the East direction is empty). They predict in both cases a breakdown of the Stokes-Einstein relation. More precisely, they predict that in the FA-1f model in one dimension

$$D \sim q^2 \sim \text{gap}^{2/3} \tag{C.3}$$

and in the East model

$$D \approx \mathrm{gap}^{\xi} \quad \text{with} \quad \xi \approx 0.73. \tag{C.4}$$

Our results confirm (C.3) but invalidate (C.4). Indeed we prove that for the East model $D \approx$ gap up to polynomial corrections (Theorem C.4.1). For this model simulations are much harder to run than for FA model due to the very fast divergence of the relaxation time when $q \to 0$ (faster than any power of $1/q$; see (C.64)), thus accounting for the wrong numerical prediction.

More generally we show that, in any dimension, if the model is defined by the constraint "there should be at least k zeros in a ball of radius k around the to-be-updated site" the diffusion coefficient is of order q^{k+1} ($k = 1$ corresponds to the FA-1f model, so the result confirms the conjecture in [JGC04]; see Theorem C.4.1). The proof of this result relies on the introduction of an auxiliary dynamics whose diffusion coefficient gives a lower bound for D. This dynamics is similar to that in [Spo90], though it is less immediate to derive because it does not appear by just suppressing terms in the variational formula. The very construction of this auxiliary dynamics is in fact quite informative about the effective dynamics of the tracer, and can be generalized to other non-cooperative models (see Definition C.2.1). Back to the FA-1f model, in dimension 2, our result and the estimate of the spectral gap in [CMRT08, Theorem 6.4] show that $D \propto$ gap. When $d \geq 3$, whereas we do know the asymptotic behaviour of D, for want of precision in the estimates on the spectral gap of the FA-1f model, we cannot decide whether $D \propto \mathrm{gap}^{\xi}$ for some exponent ξ, but our results do imply that ξ cannot be strictly smaller than one.

We also study the diffusion coefficient when the environment is given by the East model, which does not belong to the non-cooperative class. As mentioned above we prove in this case $D \approx$ gap up to polynomial corrections (Theorem C.5.2), contradicting (C.4). The strategy used in that context is very different from the one we designed for the "k-zeros" model, because the dynamics of the East model is cooperative, so that restricting the dynamics only to a neighbourhood of the tracer is not relevant. The proof relies instead on precise estimates of the energy barriers that have to be overcome in order for the tracer to cross the typical distance between two zeros at equilibrium, $1/q$. These estimates have been established mostly in [CMRT08] and [CFM12]. As an extension of results in these two papers, we provide in particular a better estimate on the spectral gap in infinite volume (Lemma C.5.5).

The paper is organized as follows. In section C.2, we define the processes of the environment, the tracer dynamics and the environment seen from the tracer. In section C.3, we prove convergence of the tracer trajectory to a Brownian motion with positive diffusion coefficient in the ergodic regime. Section C.4 is devoted to retrieve the right asymptotics for the diffusion coefficient when the density goes to 1 in non-cooperative models. In section C.5, we show that asymptotically the diffusion coefficient in the East model is of the same order as the spectral gap, up to polynomial corrections. The last two sections give some more insight on the non-cooperative case. Section C.6 provides an alternative proof of C.3, and Section C.7 shows in an example how to deal with non-cooperative models more general that the "k-zeros".

C.2 Models and notations

Let $\Omega = \{0,1\}^{\mathbb{Z}^d}$. For $\omega \in \Omega$, $x \in \mathbb{Z}^d$ we define ω^x the configuration such that

$$\omega^x_y = \begin{cases} \omega_y & \text{if } y \neq x \\ 1 - \omega_x & \text{if } y = x. \end{cases} \tag{C.5}$$

A KCSM is defined by its equilibrium density $p = 1 - q$ and constraints $(c_x(\omega))_{x \in \mathbb{Z}, \omega \in \Omega}$, taking values 0 and 1. We require that the constraints be translation invariant, that c_x depend on a fixed finite neighborhood of x and not on ω_x (i.e. $c_x(\omega) = 1$ iff $c_x(\omega^x) = 1$). We also want the constraints to be monotone (if $\forall x \in \mathbb{Z}^d$, $\omega_x \leqslant \omega'_x$, then $\forall x \in \mathbb{Z}^d$, $c_x(\omega) \geq c_x(\omega')$). We will denote by $\mathcal{L}_E$ the generator of the environment process: for f a local function on $\{0,1\}^{\mathbb{Z}}$

$$\mathcal{L}_E f(\omega) = \sum_{y\in\mathbb{Z}} c_y(\omega)((1-q)(1-\omega_y) + q\omega_y)\left[f(\omega^y) - f(\omega)\right]. \tag{C.6}$$

In words, a zero (resp. each one) at site x in configuration η turns into a one (resp. a zero) at rate $(1-q)$ (resp. q), provided the constraint is satisfied at x, i.e. $c_x(\eta) = 1$. This process satisfies the detailed balance property w.r.t. μ the product Bernoulli measure on $\{0,1\}^{\mathbb{Z}^d}$ of parameter $1-q$, so it is reversible.

A transition $\omega \to \omega^x$ is *legal* if $c_x(\omega) = 1$. Note that $\omega \to \omega^x$ is legal iff $\omega^x \to \omega$ is. A KCSM is *non-cooperative* if a finite empty set is enough to empty the whole configuration through legal transitions. More precisely

Definition C.2.1 *A KCSM is* non-cooperative *if the following holds:*

There exists a finite set $A \subset \mathbb{Z}^d$ such that for every $\omega \in \Omega$, if $\omega_{|A} \equiv 0$, for every $x \in \mathbb{Z}^d$ such that $\omega_x = 1$, there is a finite sequence $\omega^{(0)}, ..., \omega^{(n)}$ such that $\omega^{(0)} = \omega$, $\left(\omega^{(n)}\right)_x = 0$, and for all $i = 1, ..., n$, $\omega^{(i)} = \left(\omega^{(i-1)}\right)^{x_i}$ where $x_i \in \mathbb{Z}^d$ such that $c_{x_i}\left(\omega^{(i-1)}\right) = 1$.

The ergodic regime for KCSM was identified in [CMRT08]. In general, there is a critical parameter $q_c \in [0,1]$ such that the process is ergodic for $q > q_c$ and non-ergodic for $q < q_c$. $p_c = q_c$ is characterized as the critical density of an appropriate bootstrap percolation model; basically, it is the density above which blocked clusters (i.e. clusters of occupied sites that cannot be emptied through legal transitions) appear with positive probability. A non-cooperative model is ergodic at every density $p = 1-q \in (0,1)$ ($q_c = 0$).

We consider an environment given by a KCSM, and we inject a tracer at the origin. The tracer jumps at rate one to one of its nearest neighbours, provided that both the site where it sits and the site where it wants to jump are empty (for the environment). More formally, let $(\omega(t), X_t)$ be the joint evolution of the KCSM and the tracer. It is a Markov process on $\{0,1\}^{\mathbb{Z}^d} \times \mathbb{Z}^d$ given by the generator

$$\begin{aligned}\mathcal{L}_0 f(\omega, x) &= \sum_{y\in\mathbb{Z}^d} c_y(\omega)((1-q)(1-\omega_y) + q\omega_y)\left[f(\omega^y, x) - f(\omega, x)\right] \\ &\quad + \sum_{i=1}^{d}\sum_{\alpha=\pm 1}(1-\omega_x)(1-\omega_{x+\alpha e_i})\left[f(\omega, x+\alpha e_i) - f(\omega, x)\right].\end{aligned} \tag{C.7}$$

We consider the process $\eta(t)$ of the environment seen from the tracer, whose generator is given by

$$\begin{aligned}\mathcal{L} f(\eta) &= \sum_{y\in\mathbb{Z}^d} c_y(\eta)((1-q)(1-\eta_y) + q\eta_y)\left[f(\eta^y) - f(\eta)\right] \\ &\quad + \sum_{i=1}^{d}\sum_{\alpha=\pm 1}(1-\eta_0)(1-\eta_{\alpha e_i})\left[f(\eta_{\alpha e_i + \cdot}) - f(\eta)\right],\end{aligned} \tag{C.8}$$

where $\eta_{y+\cdot}$ denotes the configuration such that $\left(\eta_{y+\cdot}\right)_x = \eta_{y+x}$. This is again a reversible process w.r.t. μ the product Bernoulli measure on $\{0,1\}^{\mathbb{Z}^d}$ of parameter $1-q$ (it satisfies detailed balance).

C.3 Convergence to a non-degenerate Brownian motion

We follow the strategy of [KV86], [DMFGW89], [Spo90] to establish

Proposition C.3.1 *If the environment process is ergodic ($q > q_c$), we have*

$$\lim_{\epsilon\to 0} \epsilon X_{\epsilon^{-2}t} = \sqrt{2D}B_t, \tag{C.9}$$

where B_t is the standard Brownian motion, the convergence holds in the sense of weak convergence of path measures on $D([0,\infty),\mathbb{R}^d)$ and the diffusion matrix D is given by

$$u.Du = q^2\|u\|_2^2 - \int_0^\infty \mu\left(j_u e^{\mathcal{L}t} j_u\right) dt, \tag{C.10}$$

where for any $u = (u_1, ..., u_d) \in \mathbb{Z}^d$

$$j_u(\eta) = (1-\eta_0)\sum_{i=1}^{d}\sum_{\alpha=\pm 1}(1-\eta_{\alpha e_i})\alpha u_i. \tag{C.11}$$

Proof

Considering the martingale

$$M_t^u = u.X_t - \int_0^t j_u(\eta(s))ds \tag{C.12}$$

and following the steps of [DMFGW89], [Spo90], using reversibility, we get

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\left[(u.X_t)^2\right] \;=\; \sum_{i=1}^{d}\sum_{\alpha=\pm 1} u_i^2\mu\left((1-\eta_0)(1-\eta_{\alpha e_i})\right) - 2\int_0^\infty \mu\left(j_u e^{t\mathcal{L}} j_u\right) dt \tag{C.13}$$

In particular, $\int_0^\infty \mu\left(j_u e^{t\mathcal{L}} j_u\right) dt < \infty$, so that, since the process of generator $\mathcal{L}$ is ergodic, Theorem 1.8 of [KV86] applies to $\int_0^t j_u(\eta_s)ds$, yielding

$$\epsilon u.X_{\epsilon^{-2}t} = \epsilon\left(M^u_{\epsilon^{-2}t} + N_{\epsilon^{-2}t}\right) + Q^\epsilon(t), \tag{C.14}$$

where $M_t + N_t$ is a martingale in $L^2(\mathbb{P})$ with stationary increments and $Q^\epsilon(t)$ is an error term that vanishes when ϵ goes to 0. This implies the convergence of $\epsilon X_{\epsilon^{-2}t}$ to $\sqrt{2D}B_t$ with D given by (C.10).

✓

For the previous result to be meaningful, we need to prove $D > 0$. We show that this is true as soon as the KCSM has a positive spectral gap. In [CMRT08], it has been proved for a large class of KCSM that the spectral gap is positive in the whole ergodic regime, so that this requirement is not a big restriction. In particular, the spectral gap is positive at every density $p = 1-q \in (0,1)$ for the East model and non-cooperative models. A first step in the direction of proving $D > 0$ is to give a variational formula for D, which is the adaptation to our context of Prop. 2 in [Spo90].

Lemma C.3.2

$$
\begin{aligned}
u.Du &= \frac{1}{2}\inf_f \left\{ \sum_{y\in\mathbb{Z}^d} \mu\left(c_y(\eta)((1-q)(1-\eta_y)+q\eta_y)\left[f(\eta^y)-f(\eta)\right]^2\right)\right. \\
&\left. + \sum_{i=1}^{d}\sum_{\alpha=\pm1} \mu\left((1-\eta_0)(1-\eta_{\alpha e_i})\left[\alpha u_i + f(\eta_{\alpha e_i+\cdot}) - f(\eta)\right]^2\right)\right\}, \quad \text{(C.15)}
\end{aligned}
$$

where the infimum is taken over local functions f on Ω.

Proof

We notice, as in [Spo90], that

$$
\int_0^\infty \mu\left(j_u e^{t\mathcal{L}} j_u\right) dt = -\inf\left\{-2\mu(j_u f) - \mu(f\mathcal{L}f)\right\}, \quad \text{(C.16)}
$$

where the infimum is taken over local functions on Ω. Then, using detailed balance, notice that we can write

$$
-4\mu(j_u f) = 2\sum_{i=1}^{d}\sum_{\alpha=\pm1} \alpha u_i \mu\left((1-\eta_0)(1-\eta_{\alpha e_i})\left[f(\eta_{\alpha e_i+\cdot}) - f(\eta)\right]\right). \quad \text{(C.17)}
$$

Moreover,

$$
\begin{aligned}
-2\mu(f\mathcal{L}f) &= \sum_{y\in\mathbb{Z}^d} \mu\left(c_y(\eta)(p(1-\eta_y)+(1-p)\eta_y)\left[f(\eta^y)-f(\eta)\right]^2\right) \\
&+ \sum_{i=1}^{d}\sum_{\alpha=\pm1} \mu\left((1-\eta_0)(1-\eta_{\alpha e_i})\left[f(\eta_{\alpha e_i+\cdot}) - f(\eta)\right]^2\right) \quad \text{(C.18)}
\end{aligned}
$$

Inserting (C.17) and (C.18) into (C.13) and rearranging the terms, we get (C.15). ✓

Now we can prove $D > 0$ when the spectral gap of the environment is positive. Recall its definition

Definition C.3.3

$$
\text{gap}(\mathcal{L}_E) = \inf \frac{-\mu(f\mathcal{L}_E f)}{Var_\mu(f)}, \quad \text{(C.19)}
$$

where the infimum is taken over all functions in $L^2(\mu)$ with $Var_\mu(f) \neq 0$.

Proposition C.3.4

$$
q^2\|u\|_2^2 \geq u.Du \geq \frac{\text{gap}(\mathcal{L}_E)}{4d + \text{gap}(\mathcal{L}_E)} q^2\|u\|_2^2 \quad \text{(C.20)}
$$

Proof

The upper bound follows directly from (C.10), since the second term is non-negative.

For the lower bound, consider the expression of D given in (C.15). The first sum in the infimum is $-2\mu(f\mathcal{L}_E f)$, so that by definition of the spectral gap

$$u.2Du \geq \inf\left\{2\,\mathrm{gap}(\mathcal{L}_E)Var_\mu(f) + \sum_{i=1}^{d}\sum_{\alpha=\pm 1}\mu\left((1-\eta_0)(1-\eta_{\alpha e_i})\left[\alpha u_i + f(\eta_{\alpha e_i+\cdot}) - f(\eta)\right]^2\right)\right\}. \tag{C.21}$$

To bound the double sum, we use the inequality $(a+b)^2 \geq \gamma a^2 - \frac{\gamma}{1-\gamma}b^2$ for $\gamma < 1$. This yields

$$\begin{aligned}
&\mu\left((1-\eta_0)(1-\eta_{\alpha e_i})\left[\alpha u_i + f(\eta_{\alpha e_i}) - f(\eta)\right]^2\right)\\
&\qquad \geq \gamma q^2 u_i^2 - \frac{\gamma}{1-\gamma}\mu\left((1-\eta_0)(1-\eta_{\alpha e_i})\left[f(\eta_{\alpha e_i+\cdot}) - f(\eta)\right]^2\right)\\
&\qquad \geq \gamma q^2 u_i^2 - 4\frac{\gamma}{1-\gamma}Var_\mu(f)
\end{aligned}$$

So that, injecting this in (C.21), we get

$$u.Du \geq \inf\left\{\left(\mathrm{gap}(\mathcal{L}_E) - 4d\frac{\gamma}{1-\gamma}\right)Var_\mu(f) + \gamma q^2\|u\|_2^2\right\}. \tag{C.22}$$

Choosing $\gamma = \frac{\mathrm{gap}(\mathcal{L}_E)}{4d+\mathrm{gap}(\mathcal{L}_E)} < 1$, we get the desired lower bound. ✓

Note that at high density ($q \to 0$), the spectral gap of the East model is of order higher than any polynom in q, so that the term q^2 is negligible. In fact, for the East model, the lower bound here is quite accurate (Theorem C.5.2). For non-cooperative models however, we are able to do much better. In particular, for FA-1f in one dimension, this gives $D \geq Cq^5$, which is pretty poor, given that D is in fact of order q^2, as predicted in [JGC04]. Except in the FA-1f model, the upper bound also needs refinement. Designing more precise bounds on D when $q \to 0$ is the object of the next sections.

C.4 Correct order of D for small q in non-cooperative models

For the sake of simplicity, we give the following result only in the case of a particular class of non-cooperative models. However, we expect our method to work more generally for non-cooperative models, and give the right order in q at high density.

Let us define a class of non-cooperative KCSM, which we will call "k-zeros" for a positive integer k. Let $\|.\|_1$ denote the 1-norm on $\mathbb{Z}^d$, i.e. the norm induced by the graph distance. Let

$$\mathcal{N}_k(x) = \left\{y \in \mathbb{Z}^d \mid 0 < \|y-x\|_1 \leqslant k\right\} \tag{C.23}$$

be the k-neighbourhood of x (see Figure C.1).

The model "k-zeros" in $\mathbb{Z}^d$ is defined by the following constraints (recall (C.6))

$$c_x(\omega) = \left\{\begin{array}{ll} 1 & \text{if} \quad \sum_{y\in\mathcal{N}_k(x)}(1-\omega_y) \geq k \\ 0 & \text{else}\end{array}\right. , \tag{C.24}$$

i.e. the constraint is satisfied if there are at least k zeros within distance k. It is non-cooperative since it is enough to empty $0, e_1, 2e_1, ..., (k-1)e_1$ to empty the whole lattice through legal transitions. For $k = 1$, the "1-zero" model is better known as the one-flip Fredrickson-Andersen (or FA-1f) model.

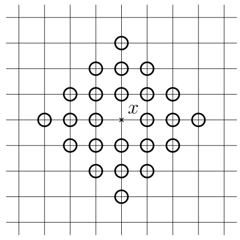

Figure C.1: $\mathcal{N}_3(x)$, the 3-neighbourhood of x in $\mathbb{Z}^2$.

Theorem C.4.1 *For the tracer diffusion in the model "k-zeros", there exist constants $0 < c \leqslant C < \infty$ depending only on d such that for all $u \in \mathbb{Z}^d$*

$$cq^{k+1}\|u\|_2^2 \leqslant u.Du \leqslant Cq^{k+1}\|u\|_2^2. \tag{C.25}$$

Remark C.4.2 *We believe that the techniques developed below can be adapted to show the equivalent of Theorem C.4.1 for any non-cooperative model, k being the minimal number of zeros needed to empty the whole lattice (see Definition C.2.1), and 1 being replaced by m the minimal number of extra zeros needed to move a minimal cluster around. We propose a heuristic for the order q^{k+m}. Consider for a moment a simple symmetric random walk on the interval $\{-1/(2q), ..., 1/(2q)\}$ of length $1/q$. For large times T, the time spent in 0 by the random walk is approximately Tq. Since $1/q$ is the typical distance between two zeros under the product Bernoulli measure μ, the fraction of time during which there is a zero at 0 before time T is approximately Tq. When that happens, a tracer sitting in 0 has a probability of order q to jump, which gives a diffusion coefficient for the tracer in the FA-1f model of order $Tq \times q/T = q^2$. How does this adapt to another non-cooperative environment, where $k \geq 1$, $m \geq 1$ (for instance the "k-zeros" model, $k > 1$, in which case $m = 1$)? A single zero cannot move on its own in such a model, but a group of k zeros can, and since the number of extra zeros it needs to move is m, the diffusion coefficient of such a group is of order q^m. So we have to consider the fraction of time spent in 0 by a group of k zeros performing a random walk on $\{-1/(2q^k), ..., 1/(2q^k)\}$ before time T ($1/q^k$ being the typical distance between two such groups under μ), that is Tq^k. During the time the group of k zeros is in contact with the tracer (i.e. at site 0), the tracer diffuses with it, which means with rate q^m. In the end, the diffusion coefficient of the tracer should therefore be of order $Tq^k \times q^m/T$.*

C.4.1 Lower bound in Theorem C.4.1

The key to the proof of the lower bound we give below is that we are able to come down to studying a *local* dynamics (see Lemma C.4.3 and the description of the dynamics in the proof of Lemma C.4.4). The possibility of doing this simplification is strongly related to the fact that we are working with non-cooperative models.

For the sake of simplicity, this proof is written for $k = 3$, but it generalizes without difficulty to any $k \geq 1$. It is widely inspired by the fourth section in [Spo90].

The first step is to give a lower bound on D in terms of the diffusion coefficient $\overline{D}$ of another dynamics (Lemma C.4.3), for which we can prove positivity (Lemma C.4.4). In the auxiliary dynamics, the only allowed transitions are jumps of the tracer between empty sites and swaps of its left and right neighbourhood, which can be reconstructed using only transitions that are allowed in the initial dynamics (see Figures C.2 and C.3). We need some notations to be more specific.

Let $\mu^{(3)}$ be the product Bernoulli measure *on* $\mathbb{Z}$ conditioned to having at least three consecutive zeros, one of which at the origin, i.e. let $A \subset \Omega$ be defined as

$$A = \Big\{\eta \in \Omega \mid \eta_0 = 0 \text{ and } (1-\eta_1)(1-\eta_2) + (1-\eta_{-1})(1-\eta_1) + (1-\eta_{-2})(1-\eta_{-1}) \geq 1\Big\} \tag{C.26}$$

and

$$\mu^{(3)} = \mu\,(.\;|A)\,. \tag{C.27}$$

Also, if $\eta \in \Omega$, denote by $\eta^{\leftrightarrow}$ the configuration obtained by exchanging the occupation numbers in sites -1 and $+1$, and -2 and $+2$

$$\eta_y^{\leftrightarrow} = \begin{cases} \eta_1 & \text{if } y = -1 \\ \eta_{-1} & \text{if } y = 1 \\ \eta_2 & \text{if } y = -2 \\ \eta_{-2} & \text{if } y = 2 \\ \eta_y & \text{else.} \end{cases} \tag{C.28}$$

We also generalize the notation η^x by defining $\eta^{x_1,\dots,x_n}$ as the configuration η flipped at sites $x_1, \dots, x_n$ (the x_i being distinct).

We can now state

Lemma C.4.3 *If $\overline{D}$ is defined by*

$$\begin{aligned} \overline{D} \;=\; & \frac{1}{2}\inf_f \Big\{\mu^{(3)}\left((1-(1-\eta_1)(1-\eta_{-1}))\left[f\left(\eta^{\leftrightarrow}\right) - f(\eta)\right]^2\right) \qquad \text{(C.29)} \\ & + \mu^{(3)}\left((1-\eta_1)\left[1 + f(\eta_{1+\cdot}) - f(\eta)\right]^2\right) + \mu^{(3)}\left((1-\eta_{-1})\left[-1 + f(\eta_{-1+\cdot}) - f(\eta)\right]^2\right)\Big\}\,, \end{aligned}$$

where the infimum is taken over local functions on Ω, then we have

$$e_1.De_1 \geq \frac{1+2p}{4} q^4 \overline{D}. \tag{C.30}$$

Proof of Lemma C.4.3

For briefness, we define

$$r_x(\eta) = (1-q)(1-\eta_x) + q\eta_x. \tag{C.31}$$

Then we have, given the definition of $\mu^{(3)}$ (C.27), for every local function f

$$\begin{aligned} & \mu^{(3)}\left((1-(1-\eta_1)(1-\eta_{-1}))\left[f\left(\eta^{\leftrightarrow}\right) - f(\eta)\right]^2\right) \\ & \quad = \mu^{(3)}\left((1-\eta_1)(1-\eta_2)\eta_{-1}\left[f\left(\eta^{\leftrightarrow}\right) - f(\eta)\right]^2\right) \\ & \qquad + \mu^{(3)}\left((1-\eta_{-1})(1-\eta_{-2})\eta_1\left[f\left(\eta^{\leftrightarrow}\right) - f(\eta)\right]^2\right) \end{aligned} \tag{C.32}$$

Our aim is to reconstruct the swap changing η into $\eta^{\leftrightarrow}$, using only legal (for the “3-zeros" model dynamics) flips. The first term of the r.h.s. in (C.32) can be rewritten as

$$\begin{aligned} & \mu^{(3)}\left((1-\eta_1)(1-\eta_2)\eta_{-1}\eta_{-2}\left[f\left(\eta^{\leftrightarrow}\right) - f(\eta)\right]^2\right) \\ & \quad + \mu^{(3)}\left((1-\eta_1)(1-\eta_2)\eta_{-1}(1-\eta_{-2})\left[f\left(\eta^{\leftrightarrow}\right) - f(\eta)\right]^2\right). \end{aligned} \tag{C.33}$$

Let us focus on the first term. See in Figure C.2 a representation of the successive flips used to reconstruct the swap. Writing that, when $\eta_{-1} = \eta_{-2} = (1-\eta_1) = (1-\eta_2) = 1$

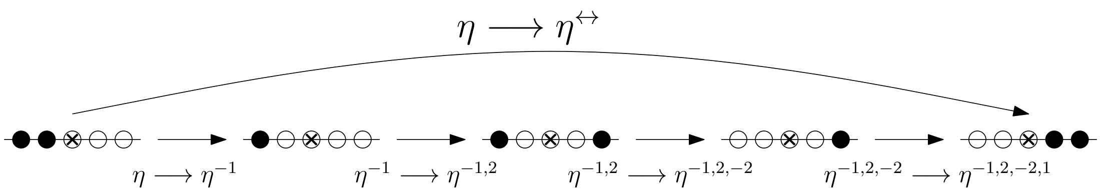

Figure C.2: The four legal flips used to reconstruct the swap $\eta \longrightarrow \eta^{\leftrightarrow}$ when $\eta_{-1} = \eta_{-2} = (1-\eta_1) = (1-\eta_2) = 1$. The cross recalls that the tracer is sitting at the origin.

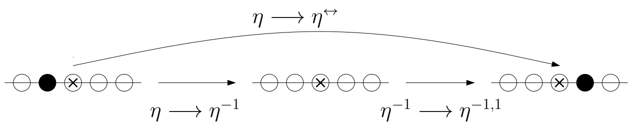

Figure C.3: The two legal flips used to reconstruct the swap $\eta \longrightarrow \eta^{\leftrightarrow}$ when $\eta_{-1} = \eta_{-2} = (1-\eta_1) = \eta_2 = 1$.

$$
\begin{aligned}
f\left(\eta^{\leftrightarrow}\right) - f(\eta) \quad = \quad & f\left(\eta^{-1,2,-2,1}\right) - f\left(\eta^{-1,2,-2}\right) + f\left(\eta^{-1,2,-2}\right) - f\left(\eta^{-1,2}\right) \\
& + f\left(\eta^{-1,2}\right) - f\left(\eta^{-1}\right) + f\left(\eta^{-1}\right) - f(\eta) \qquad \text{(C.34)}
\end{aligned}
$$

and using the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
& \mu^{(3)}\left((1-\eta_1)(1-\eta_2)\eta_{-1}\eta_{-2}\left[f\left(\eta^{\leftrightarrow}\right) - f(\eta)\right]^2\right) \qquad \text{(C.35)} \\
& \leqslant \quad 4\mu^{(3)}\left((1-\eta_1)(1-\eta_2)\eta_{-1}\eta_{-2}\left[f\left(\eta^{-1,2,-2,1}\right) - f\left(\eta^{-1,2,-2}\right)\right]^2\right) \\
& \quad + 4\mu^{(3)}\left((1-\eta_1)(1-\eta_2)\eta_{-1}\eta_{-2}\left[f\left(\eta^{-1,2,-2}\right) - f\left(\eta^{-1,2}\right)\right]^2\right) \\
& \quad + 4\mu^{(3)}\left((1-\eta_1)(1-\eta_2)\eta_{-1}\eta_{-2}\left[f\left(\eta^{-1,2}\right) - f\left(\eta^{-1}\right)\right]^2\right) \\
& \quad + 4\mu^{(3)}\left((1-\eta_1)(1-\eta_2)\eta_{-1}\eta_{-2}\left[f\left(\eta^{-1}\right) - f(\eta)\right]^2\right) \qquad \text{(C.36)}
\end{aligned}
$$

Note that all the flips involved are legal for the dynamics "3-zeros": there are always at least three zeros in the 3-neighbourhood of the site that is flipped. Then, we make a change of variables in the first three terms above to get

$$
\begin{aligned}
& \mu^{(3)}\left((1-\eta_1)(1-\eta_2)\eta_{-1}\eta_{-2}\left[f\left(\eta^{\leftrightarrow}\right) - f(\eta)\right]^2\right) \\
& \leqslant \quad 4\frac{1-q}{q}\mu^{(3)}\left((1-\eta_1)\eta_2(1-\eta_{-1})(1-\eta_{-2})\left[f\left(\eta^{1}\right) - f(\eta)\right]^2\right) \\
& \quad + 4\mu^{(3)}\left((1-\eta_1)\eta_2(1-\eta_{-1})\eta_{-2}\left[f\left(\eta^{-2}\right) - f(\eta)\right]^2\right) \\
& \quad + 4\frac{1-q}{q}\mu^{(3)}\left((1-\eta_1)(1-\eta_2)(1-\eta_{-1})\eta_{-2}\left[f\left(\eta^{2}\right) - f(\eta)\right]^2\right) \\
& \quad + 4\mu^{(3)}\left((1-\eta_1)(1-\eta_2)\eta_{-1}\eta_{-2}\left[f\left(\eta^{-1}\right) - f(\eta)\right]^2\right) \qquad \text{(C.37)}
\end{aligned}
$$

In the same way (following the strategy represented in Figure C.3), we get

$$
\begin{aligned}
& \mu^{(3)}\left((1-\eta_1)(1-\eta_2)\eta_{-1}(1-\eta_{-2})\left[f\left(\eta^{\leftrightarrow}\right) - f(\eta)\right]^2\right) \\
& \leqslant \quad \frac{2(1-q)}{q}\mu^{(3)}\left((1-\eta_{-1})(1-\eta_2)(1-\eta_{-2})(1-\eta_1)\left[f(\eta^{1}) - f(\eta)\right]^2\right) \\
& \quad + 2\mu^{(3)}\left((1-\eta_1)(1-\eta_2)(1-\eta_{-2})\eta_{-1}\left[f(\eta^{-1}) - f(\eta)\right]^2\right) \qquad \text{(C.38)}
\end{aligned}
$$

Combining (C.33), (C.37) and (C.38), and doing the same for the second term in (C.32), we get:

$$\begin{aligned}
&\mu^{(3)}\left((1-(1-\eta_1)(1-\eta_{-1}))\left[f\left(\eta^{\leftrightarrow}\right)-f(\eta)\right]^2\right)\\
&\qquad\leqslant \frac{4}{q}\mu^{(3)}\left((1-\eta_{-1})(1-\eta_{-2})r_1(\eta)\left[f\left(\eta^1\right)-f\left(\eta\right)\right]^2\right)\\
&\qquad\quad+\frac{4}{q}\mu^{(3)}\left((1-\eta_1)(1-\eta_{-1})r_{-2}(\eta)\left[f\left(\eta^{-2}\right)-f\left(\eta\right)\right]^2\right)\\
&\qquad\quad+\frac{4}{q}\mu^{(3)}\left((1-\eta_1)(1-\eta_{-1})r_2(\eta)\left[f\left(\eta^2\right)-f\left(\eta\right)\right]^2\right)\\
&\qquad\quad+\frac{4}{q}\mu^{(3)}\left((1-\eta_1)(1-\eta_2)r_{-1}(\eta)\left[f\left(\eta^{-1}\right)-f\left(\eta\right)\right]^2\right)
\end{aligned}\tag{C.39}$$

Now notice that we have:

$$\begin{aligned}
&\mu^{(3)}\left((1-\eta_{-1})(1-\eta_{-2})r_1(\eta)\left[f\left(\eta^1\right)-f\left(\eta\right)\right]^2\right)\\
&\qquad=\frac{1}{\mu(A)}\mu\left((1-\eta_0)(1-\eta_{-1})(1-\eta_{-2})r_1(\eta)\left[f\left(\eta^1\right)-f\left(\eta\right)\right]^2\right)
\end{aligned}$$

and similarly for the other terms in (C.39), so that we have proved the following inequality – recalling that $\mu(A)=q^3(1+2p)$:

$$\begin{aligned}
&\textstyle\sum_{y\in\mathbb{Z}^d}\mu\left(c_y(\eta)((1-q)(1-\eta_y)+q\eta_y)\left[f(\eta^y)-f(\eta)\right]^2\right)\\
&\qquad\geq q^4\frac{(1+2(1-q))}{4}\mu^{(3)}\left((1-(1-\eta_1)(1-\eta_{-1}))\left[f\left(\eta^{\leftrightarrow}\right)-f(\eta)\right]^2\right).
\end{aligned}\tag{C.40}$$

We are almost done: it remains to notice that

$$\mu^{(3)}\left((1-\eta_1)\left[1+f(\eta_{1+\cdot})-f(\eta)\right]^2\right)\quad\leqslant\quad\frac{1}{\mu(A)}\mu\left((1-\eta_0)(1-\eta_1)\left[1+f(\eta_{1+\cdot})-f(\eta)\right]^2\right)$$

and similarly with 1 replaced by -1, so that a fortiori:

$$\begin{aligned}
&\mu^{(3)}\left((1-\eta_1)\left[1+f(\eta_{1+\cdot})-f(\eta)\right]^2\right)+\mu^{(3)}\left((1-\eta_{-1})\left[-1+f(\eta_{-1+\cdot})-f(\eta)\right]^2\right)\\
&\qquad\leqslant\frac{4}{q^4(1+2(1-q))}\sum_{i=1}^{d}\sum_{\alpha=\pm1}\mu\left((1-\eta_0)(1-\eta_{\alpha e_i})\left[\alpha\delta_{1i}+f(\eta_{\alpha e_i+\cdot})-f(\eta)\right]^2\right).
\end{aligned}\tag{C.41}$$

Combining (C.40) and (C.41), and recalling (C.15), we get the lemma. ✓

Of course there is nothing special about the direction e_1, and the lemma is valid in all directions. Notice that it does not depend on the dimension. Now we need to see how to obtain a universal lower bound for $\overline{D}$.

Lemma C.4.4

$$\overline{D}\geq 4/9\tag{C.42}$$

Proof of Lemma C.4.4

Following the same lines as in the proof of Proposition C.3.1 and Lemma C.3.2, we see that $\overline{D}$ is the diffusion coefficient of the dynamics reversible w.r.t. $\mu^{(3)}$ described below

- with rate 1, if $\eta_1=0$, the tracer jumps to the right, i.e. we go from η to $\eta_{1+\cdot}$.

- with rate 1, if $\eta_{-1} = 0$, the tracer jumps to the left, i.e. we go from η to $\eta_{-1+\cdot}$.
- with rate 1, if either $\eta_1 = 1$ or $\eta_{-1} = 1$, $\{-2, -1\}$ and $\{2, 1\}$ are swapped, i.e. we go from η to $\eta^{\leftrightarrow}$.

As in [Spo90], starting from a configuration η chosen after $\mu^{(3)}$, we can index by $\mathbb{Z}$ all the configurations that can be reached by this dynamics in the following way. $\eta^{(0)} = \eta$ is the initial configuration, that is almost surely in A. Then we define inductively $\eta^{(n)}$, $n \in \mathbb{Z}$. If $\eta_1^{(n)} = 0$, $\eta^{(n+1)} = \eta_{1+\cdot}^{(n)}$. If $\eta_1^{(n)} = 1$, $\eta^{(n+1)} = \left(\eta^{(n)}\right)^{\leftrightarrow}$. Similarly, if $\eta_{-1}^{(n)} = 0$, $\eta^{(n-1)} = \eta_{-1+\cdot}^{(n)}$. If $\eta_{-1}^{(n)} = 1$, $\eta^{(n-1)} = \left(\eta^{(n)}\right)^{\leftrightarrow}$. Note that this definition is consistent ($\eta^{(n+1-1)} = \eta^{(n)}$).

Using this labelling with integers of all attainable configurations, the dynamics described above can be equivalently defined in the following way: if the system is in the configuration $\eta^{(n)}$, it goes to $\eta^{(n+1)}$ with rate one, and to $\eta^{(n-1)}$ also with rate one. So we can rewrite the process starting from η as $\eta(t) = \eta^{N_t}$ where $(N_t)_{t\geq 0}$ is a simple random walk on $\mathbb{Z}$.

Now to conclude, we just need to notice that if X_t is the position of the tracer at time t in this dynamics, we have

$$X_t \geq \left\lfloor \frac{2}{3} N_t \right\rfloor,$$

since two out of three times N moves to the right, X also jumps by one.

$$2\overline{D} = \lim_{t\to+\infty} \frac{1}{t}\mathbb{E}\left[X_t^2\right] \geq \frac{4}{9} \lim_{t\to+\infty} \frac{1}{t}\mathbb{E}\left[N_t^2\right] = 8/9.$$

✓

To deduce Theorem C.4.1 from Lemma C.4.3 and Lemma C.4.4, let $u \in \mathbb{R}^d$ be such that $\|u\|_2 = 1$ and notice that we can use comparisons with the auxiliary dynamics above in all directions to get

$$\begin{aligned}
2u.Du \quad \geq \quad & \sum_{i=1}^{d} \inf_{f_i} \left\{ \frac{1}{d} \sum_{x\in\mathbb{Z}^d} \mu\left(c_x(\eta) r_x(\eta) \left[f_i(\eta^x) - f_i(\eta)\right]^2\right) \right. && \text{(C.43)} \\
& \left. + \sum_{\alpha=\pm1} \mu\left((1-\eta_0)(1-\eta_{\alpha e_i})\left[\alpha u_i + f_i(\eta_{\alpha e_i + \cdot}) - f_i(\eta)\right]^2\right) \right\} && \text{(C.44)} \\
\geq \quad & \sum_{i=1}^{d} u_i^2 \inf_{f_i} \left\{ \frac{1}{d} \sum_{x\in\mathbb{Z}\cdot e_i} \mu\left(c_x^i(\eta) r_x(\eta) \left[f_i(\eta^x) - f_i(\eta)\right]^2\right) \right. && \text{(C.45)} \\
& \left. + \frac{1}{d} \sum_{\alpha=\pm1} \mu\left((1-\eta_0)(1-\eta_{\alpha e_i})\left[\alpha + f_i(\eta_{\alpha e_i + \cdot}) - f_i(\eta)\right]^2\right) \right\} && \text{(C.46)} \\
\geq \quad & \frac{2}{d} D_1, && \text{(C.47)}
\end{aligned}$$

where $c_x^i(\eta)$ is one iff the constraint is satisfied using only zeros in the direction i, D_1 is the diffusion coefficient in one dimension and we used $\sum_{i=1}^{d} u_i^2 = 1$. Theorem C.4.1 follows from this inequality and the two previous Lemmas.

Remark C.4.5 *This strategy can be applied to other non-cooperative models. However, the auxiliary dynamics (the one involving swaps around the origin and jumps of the tracer) will be model dependant and may not be strictly one-dimensional. It may be encoded by a random walks on graphs slightly more complex than $\mathbb{Z}$, but still with a uniformly positive diffusion coefficient. We believe that this technique allows to retrieve the correct exponent at low temperature for non-cooperative models.*

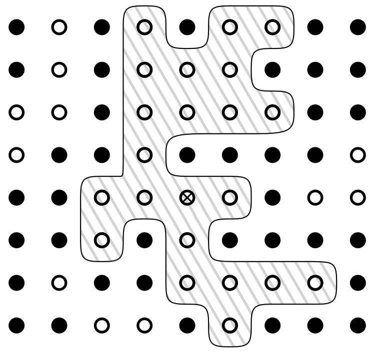

Figure C.4: An example of $C(\eta)$. Zeros are represented by empty circles, ones by filled disks and the origin is marked by a cross. The cluster of zeros containing the origin is circled by a line and tiled in gray. In this case, $f(\eta) = 4$.

C.4.2 Upper bound in Theorem C.4.1

In view of (C.15), to find an upper bound on D, we need to find an appropriate test function. As a warming, suppose that $d = 1$. Then, looking for a function that cancels the second line in (C.15), we find that a natural function to consider is

$$f(\eta) = \min\{x \in \mathbb{N} \mid \eta_x = 1\}. \tag{C.48}$$

Then it is not too difficult to check that if we plug this function in the first line of (C.15), we get an expression of order q^{k+1}: the factor q^k comes from the constraint, and the extra q comes from the extra empty site we need in order to evolve.

In higher dimension, we are going to find a good test function to evaluate $e_1.De_1$. Define $C(\eta)$ the connected cluster of zeros containing the origin in the configuration η ($C(\eta) = \varnothing$ if $\eta_0 = 1$). See Figure C.4 for an example.

Now we can define our test function.

$$f(\eta) = \min\left\{x \in \mathbb{N} \mid C(\eta) \subset (-\infty, x-1] \times \mathbb{Z}^{d-1}\right\}. \tag{C.49}$$

For instance, if $\eta_0 = 1$, $f(\eta) = 0$. In Figure C.4, $f(\eta) = 4$. Note that this function coincides with that in (C.48) when $d = 1$. This function cancels the second line in (C.15) when $u = e_1$. Indeed, when $(1-\eta_0)(1-\eta_{\alpha e_i} \neq 0$, 0 and αe_i belong to the same cluster of zeros. So what we need to do is show that

$$\sum_{y \in \mathbb{Z}^d} \mu\left(c_y(\eta)((1-q)(1-\eta_y) + q\eta_y)\left[f(\eta^y) - f(\eta)\right]^2\right) \leqslant Cq^{k+1} \tag{C.50}$$

for some finite C. Let us split the l.h.s. in two terms and treat them separately: we need to show that

$$S_0 = \sum_{y\in\mathbb{Z}^d} \mu\left(c_y(\eta)(1-\eta_y)\left[f(\eta^y) - f(\eta)\right]^2\right) \leqslant Cq^{k+1} \tag{C.51}$$

$$S_1 = \sum_{y\in\mathbb{Z}^d} \mu\left(c_y(\eta)\eta_y\left[f(\eta^y) - f(\eta)\right]^2\right) \leqslant Cq^{k} \tag{C.52}$$

Thanks to detailed balance, $(1-q)S_0 = qS_1$, so we only need to show (C.51).

Let us now study S_0. The mechanism involved here is the removal of part of the cluster of zeros around the origin. In particular, when $(1-\eta_y)\left[f(\eta^y) - f(\eta)\right]^2 \neq 0$, we certainly have

$[f(\eta^y) - f(\eta)]^2 \leqslant |C(\eta)|^2$, where $|C(\eta)|$ is the cardinal of $C(\eta)$. So that

$$S_0 \leqslant \mu\left(|C(\eta)|^2 \sum_{y\in C(\eta)} c_y(\eta)(1-\eta_y)\right) \tag{C.53}$$

$$\leqslant \sum_{n\geq 0}\mu\left(|C(\eta)|^2 \mathbf{1}_{0\leftrightarrow\partial B_n,\ 0\not\leftrightarrow\partial B_{n+1}} \sum_{y\in C(\eta)} c_y(\eta)(1-\eta_y)\right), \tag{C.54}$$

where ∂B_n denotes the set of points at distance n from 0, and $\{0 \leftrightarrow \partial B_n\}$ is the event that there is a site at distance n from 0 in $C(\eta)$. Since on the event $\{0 \leftrightarrow \partial B_n,\ 0 \not\leftrightarrow \partial B_{n+1}\}$, $C(\eta) \subset B_1(0,n)$, we have

$$S_0 \leqslant \sum_{n\geq 0}(2n+1)^{2d} \sum_{y\in B_1(0,n)} \mu\left(c_y(\eta)(1-\eta_y)\mathbf{1}_{0\leftrightarrow\partial B_n,\ 0\not\leftrightarrow\partial B_{n+1}}\right). \tag{C.55}$$

On the one hand, for any y, we have for some constant C depending only on d

$$\mu\left(c_y(\eta)(1-\eta_y)\right) \leqslant Cq^{k+1}, \tag{C.56}$$

since the constraint requires at least k zeros to be satisfied, and c_y is independent from η_y. On the other hand, if $0 \leftrightarrow \partial B_n$, there is a self-avoiding walk of length n starting at 0 which is empty. So a rough bound on the number of self-avoiding walks of length n yields

$$\mu\left(0 \leftrightarrow \partial B_n,\ 0 \not\leftrightarrow \partial B_{n+1}\right) \leqslant (2d)^n q^n. \tag{C.57}$$

Putting together (C.56) and (C.57), we get

$$S_0 \leqslant \sum_{n\geq 0}(2n+1)^{3d}\left[Cq^{k+1} \wedge (2dq)^n\right] \leqslant C'q^{k+1} \tag{C.58}$$

for q small enough. So we have proved (C.51).

A general argument allows to retrieve the upper bound in Theorem C.4.1 for any $u \in \mathbb{R}^d$ from the result for $e_1, ..., e_d$. Write $u = \sum_{i=1}^d u_i e_i$ and compute

$$u.Du = \sum_{i=1}^d u_i^2 e_i.De_i + \sum_{i\neq j} u_i u_j e_i.De_j \tag{C.59}$$

Notice that D is symmetric and positive (by Proposition C.3.4), so that the application $(u,v) \mapsto u.Dv$ is a scalar product. We can therefore apply Cauchy-Schwarz inequality to the terms $e_i.De_j$ and get

$$u.Du \leqslant Cq^{k+1}\left(\sum_{i=1}^d |u_i|\right)^2 \leqslant C'q^{k+1}, \tag{C.60}$$

where C' depends only on d by equivalence of the norms in finite dimension.

C.5 In the East model, $D \approx \text{gap}$

In this section, the environment process is given by the East model, a one-dimensional KCSM for which the constraint is that the East neighbour of the to-be-updated site be vacant. The corresponding generator is

$$\mathcal{L}_E f(\omega) = \sum_{y\in\mathbb{Z}} (1-\omega_{y+1})((1-q)(1-\omega_y) + q\omega_y)\,[f(\omega^y) - f(\omega)]\,. \tag{C.61}$$

Before getting into the results concerning the tracer, let us recall briefly the definition and basic property of the so-called distinguished zero, a very useful tool for the study of the East model, which was introduced in [AD02].

Definition C.5.1 *Consider $\omega \in \Omega$ a configuration with $\omega_x = 0$ for some $x \in \mathbb{Z}$. Define $\xi(0) = x$. Call $T_1 = \inf\{t \geq 0 \mid \text{the clock in } x \text{ rings and } \omega_{x+1}(t) = 0\}$, the time of the first legal ring at x. Let $\xi(s) = x$ for $s < T_1$, $\xi(T_1) = x+1$ and start again to define recursively $(\xi(s))_{s\geq 0}$.*

Notice that for any $s \geq 0$, $\omega_{\xi(s)}(s) = 0$, and that $\xi : \mathbb{R}^+ \to \mathbb{Z}$ is almost surely càdlàg and increasing by jumps of 1.

This distinguished zero has an important property: as it moves forward, it leaves equilibrium on its left (see [AD02, Lemma 4] or [CMST10, Lemma 3.5]). In particular, if ω is such that $\omega_x = 0$ and A an event depending only on the configuration restricted to $[x_-, x_+]$, with $x_+ < x$, letting $V = \{x_-, ..., x-1\}$, then we have the following estimate

$$\mathbb{P}_\omega\left(\omega(t) \in A\right) \leqslant \mu_V(\omega_{|V})^{-1}\mathbb{P}_{\mu_V\cdot\omega}\left(\omega(t) \in A\right) = \mu(\omega_{|V})^{-1}\mu(A), \tag{C.62}$$

where μ_V is the Bernoulli$(1-q)$ product measure on $\{0,1\}^V$, $\mu_V \cdot \omega$ denotes the law of a random configuration equal to ω on $\mathbb{Z}\backslash V$ and chosen with law μ_V on V. In the above estimate, the factor $\mu_V(\omega_{|V})$ comes from a change of measure to start from μ in V, and the last equality comes from the property of the distinguished zero mentioned above.

For briefness, in this section, we will denote the spectral gap of the East process by gap (see C.19).

Theorem C.5.2 *When the environment is given by the East model, there exist constants $C, c > 0$ and α such that*

$$cq^2\,\text{gap} \leqslant D \leqslant Cq^{-\alpha}\,\text{gap}\,. \tag{C.63}$$

Remark C.5.3 *In [AD02] and [CMRT08], it was established that*

$$\lim_{q\to 0} \frac{\log(1/\,\text{gap})}{(\log(1/q))^2} = (2\log 2)^{-1}. \tag{C.64}$$

In particular, this means that the powers of q appearing in (C.63) are merely corrections to the correct asymptotic for D, which is governed by the spectral gap of the East model. (C.63) is therefore incompatible with the prediction in [JGC04] that $D \approx \text{gap}^\xi$ for some $\xi < 1$.

Proof of Theorem C.5.2

The first inequality was already contained in Proposition C.3.4.

For the proof of the second inequality, fix $t > 0$ and $\tau \ll t$ to be chosen later, such that t/τ is an integer and $\tau \lesssim \mathrm{gap}^{-1}$ (more precisely, $\tau = q^\beta \,\mathrm{gap}^{-1}$).

Then we can write

$$\begin{aligned}
\mathbb{E}\left[X_t^2\right] &= \mathbb{E}\left[\left(\sum_{k=1}^{t/\tau} X_{k\tau} - X_{(k-1)\tau}\right)^2\right] \\
&= \sum_{k=1}^{t/\tau} \mathbb{E}\left[\left(X_{k\tau} - X_{(k-1)\tau}\right)^2\right] + \sum_{k\neq k'} \mathbb{E}\left[\left(X_{k\tau} - X_{(k-1)\tau}\right)\left(X_{k'\tau} - X_{(k'-1)\tau}\right)\right] \\
&= \frac{t}{\tau}\mathbb{E}\left[X_\tau^2\right] + \sum_{k\neq k'} \mathbb{E}\left[\left(X_{k\tau} - X_{(k-1)\tau}\right)\left(X_{k'\tau} - X_{(k'-1)\tau}\right)\right] \qquad \text{(C.65)}
\end{aligned}$$

We need to show that (C.65) is smaller that $tq^{-\alpha}\,\mathrm{gap}$ for some α when τ is well chosen. We are going to bound the first term using the fact that energy barriers make it very costly to cross a distance greater than $1/q$ in time $\tau \lesssim \mathrm{gap}^{-1}$. To bound the second term, we use the symmetry of the model and the fact that the process seen from the tracer has a positive spectral gap.

Proposition C.5.4 *There exists $\beta, C < \infty$ such that, if $\tau = q^\beta \,\mathrm{gap}^{-1}$,*

$$\mathbb{E}\left[X_\tau^2\right] \leqslant Cq^{-C}. \qquad \text{(C.66)}$$

First we need two lemmas, that rely on precise estimates on the spectral gap of the East model on lengths of order at most $1/q$, and related energy barriers, that have been established in [CFM12]. We start by showing a precise comparison between the relaxation time in infinite volume and the relaxation time in volume $1/q$. Recall that it was shown in [CMRT08] that for any $\delta > 0$

$$\mathrm{gap}^{-1} \leqslant C_\delta \left(\frac{1}{q}\right)^{\log_2(1/q)/(2-\delta)}. \qquad \text{(C.67)}$$

Lemma C.5.5 *Let $n = \lceil \log_2(1/q) \rceil$ and $T_{rel}(L)$ be the relaxation time of the East model on length L with empty boundary condition. Then there exist finite constants C, C' such that*

$$\mathrm{gap}^{-1} \leqslant Cq^{-C} T_{rel}(1/q) \leqslant C'q^{-C'} \frac{n!}{q^n 2^{\binom{n}{2}}}. \qquad \text{(C.68)}$$

Proof

The second inequality follows immediately from Theorem 2 in [CFM12]. To prove the first one, we refine the bisection technique used in [CMRT08] to prove (C.67). Let $\delta(q) = 10/\log(1/q)$, $l_k = 2^k$, $\delta_k = \left\lfloor l_k^{1-\delta/2} \right\rfloor$, $s_k = \left\lfloor l_k^{\delta/6} \right\rfloor$. These are the same definitions as in [CMRT08], except that instead of a fixed $\delta > 0$, we take δ to 0 with q. With these definitions, we have for every $k \geq k_\delta := 6/\delta$ the following estimate[1] (see [CMRT08, (6.3)])

$$\mathrm{gap}^{-1} \leqslant T_{\mathrm{rel}}(l_k + l_k^{1-\delta/6}) \prod_{j=k}^{\infty} \left(\frac{1}{1 - p^{\delta_j/2}}\right) \prod_{j=k}^{\infty} \left(1 + s_j^{-1}\right). \qquad \text{(C.69)}$$

[1] This condition is not necessary, but sufficient; it comes from the fact that Lemma 4.2 in [CMRT08] has to be satisfied in order to apply the bisection technique.

As in [CMRT08], let

$$j_* = \min\left\{j \mid p^{\delta_j/2} \leqslant e^{-1}\right\} \approx \log_2(1/q)/(1-\delta/2). \tag{C.70}$$

As long as $j_* \geq k_\delta$, which is true thanks to our choice of δ, we can replace k by j_* in (C.69). Now we have (see the computations in [CMRT08], top of page 484 for the first estimate)

$$\prod_{j=j_*}^{\infty}\left(\frac{1}{1-p^{\delta_j/2}}\right) \leqslant C \tag{C.71}$$

$$\prod_{j=j_*}^{\infty}\left(1+s_j^{-1}\right) \leqslant q^{-C}, \tag{C.72}$$

for C some constant not depending on q. Noticing that $l_{j_*} + l_{j_*}^{1-\delta/6} \leqslant d/q$ for some constant d, we get

$$\text{gap}^{-1} \leqslant Cq^{-C}T_{\text{rel}}(d/q). \tag{C.73}$$

Now it is enough to recall Theorem 4 in [CFM12], that states that there is no time scale separation on scale $1/q$

$$T_{\text{rel}}(d/q) \sim T_{\text{rel}}(1/q) \tag{C.74}$$

✓

Now we can use Lemma C.5.5 to prove the following estimate, which basically means that in times smaller than gap^{-1}, it will be extremely difficult for the system to erase a row of $1/q$ ones.

Lemma C.5.6 *Recall that $\tau = q^\beta \,\text{gap}^{-1}$. Let $l = 1/q$ and $\mathbb{P}_{\mathbf{10}}(\cdot)$ denote (abusively) the law of the East process starting from a configuration equal to one on $\{1,...,l\}$, with a zero in $l+1$. Let T_0 be the first time there is a zero at 1. Independently of the choice of the initial configuration outside $\{1,...,l,l+1\}$, we have, if β is large enough (independently of q)*

$$\mathbb{P}_{\mathbf{10}}(T_0 \leqslant \tau) \leqslant Cq. \tag{C.75}$$

Proof of Lemma C.5.6

In [CFM12][2], the authors define a certain set ∂A_* of configurations in $\{0,1\}^l$ that has two interesting properties (it is defined in paragraph 5.2.1 of [CFM12], the properties below are stated in Remark 5.8 and Corollary 5.10)

- Starting from a configuration equal to one on $\{1,...,l\}$, with a zero in 0, in order to put a one in 0 before time τ, the dynamics restricted to $\{1,...,l\}$ has to go through the set ∂A_* at some time $s \leqslant \tau$.
- For some $\alpha' < \infty$, if $n = \lceil \log_2 l \rceil$

$$\mu(\partial A_*) \leqslant \frac{q^n 2^{\binom{n}{2}}}{n!} q^{-\alpha'}. \tag{C.76}$$

[2] Note that the orientation convention is reversed in that paper: contrary to here, the constraint that has to be satisfied to update x is that $x-1$ should be empty.

Put another way, ∂A_* is a bottleneck separating the events $\{\eta_0 = \eta_{l+1} = 0,\ \eta_1 = ... = \eta_l = 1\}$ and $\{\eta_0 = 1\}$ in the East dynamics.

Call τ_0 the first time there is a one in 0. Denote (abusively) by $0\mathbf{1}0$ any configuration equal to zero in 0 and $l+1$, and to one on $\{1,...,l\}$, by T an exponential variable of parameter 2 independent of T_0, and by τ_0 the first time at which there is a one in position 0. Notice that, once there is a zero in 1, if the clock attached to site 0 rings before that attached to 1, and if the associated Bernoulli variable is a one, then the configuration at site 0 takes value one. So that

$$\frac{1-q}{2}\mathbb{P}_{\mathbf{1}0}\left(T_0 + T \leqslant \tau + 1/2\right) \leqslant \mathbb{P}_{0\mathbf{1}0}\left(\tau_0 \leqslant \tau + 1/2\right), \tag{C.77}$$

where $\mathbf{1}0$ and $0\mathbf{1}0$ are equal except maybe in 0. The constant $1/2$ appears to allow the following estimate

$$\mathbb{P}_{\mathbf{1}0}\left(T_0 + T \leqslant \tau + 1/2\right) \geq \mathbb{P}_{\mathbf{1}0}\left(T_0 \leqslant \tau\right)\mathbb{P}\left(T \leqslant 1/2\right) = (1-e^{-1})\mathbb{P}_{\mathbf{1}0}\left(T_0 \leqslant \tau\right). \tag{C.78}$$

(C.77) and (C.78) yield

$$\mathbb{P}_{\mathbf{1}0}\left(T_0 \leqslant \tau\right) \leqslant \frac{2}{(1-q)(1-e^{-1})}\mathbb{P}_{0\mathbf{1}0}\left(\tau_0 \leqslant \tau + 1/2\right). \tag{C.79}$$

Now we use the first property of ∂A_* to get

$$\mathbb{P}_{0\mathbf{1}0}\left(\tau_0 \leqslant \tau + 1/2\right) \quad \leqslant \quad \mathbb{P}_{\mathbf{1}0}\left(\exists\ s \leqslant \tau + 1/2 \quad \text{s.t.} \quad (\omega(s))_{[1,l]} \in \partial A_*\right) \tag{C.80}$$

To evaluate the r.h.s., we condition on $N_{\tau+1/2}$ the number of rings occurring in $[1,l]$ before time $\tau + 1/2$ in the graphical construction with a union bound to get

$$\begin{aligned}
\mathbb{P}_{\mathbf{1}0}\left(\exists\ s \leqslant \tau + 1/2 \quad \text{s.t.} \quad (\omega(s))_{[1,l]} \in \partial A_*\right) \quad &\leqslant \quad \mathbb{E}\left[N_{\tau+1/2}\right] \sup_{s\,\leqslant\,\tau+1/2} \mathbb{P}_{\mathbf{1}0}\left((\omega(s))_{[1,l]} \in \partial A_*\right) && \text{(C.81)}\\
&\leqslant \quad (\tau + 1/2)\, l \sum_{\sigma\in\partial A_*} \sup_{s\,\leqslant\,\tau+1/2} \mathbb{P}_{\mathbf{1}0}\left((\omega(s))_{[1,l]} = \sigma\right) && \text{(C.82)}\\
&\leqslant \quad (\tau + 1/2)\, l \sum_{\sigma\in\partial A_*} p^{-l}\mu(\sigma) && \text{(C.83)}\\
&\leqslant \quad (\tau + 1/2)\, l (1-q)^{-l}\mu(\partial A_*)\\
&\leqslant \quad (\tau + 1/2)\,(1-q)^{-l}\frac{q^n 2^{\binom{n}{2}}}{n!}q^{-(\alpha'+1)}, && \text{(C.84)}
\end{aligned}$$

where we used (C.62) with the distinguished zero starting at $l+1$ to get the third inequality, and the second property of ∂A_* (C.76) to get the last one. Now collect (C.79), (C.80) and (C.84) to get

$$\mathbb{P}_{\mathbf{1}0}\left(T_0 \leqslant \tau\right) \leqslant \frac{2}{(1-q)(1-e^{-1})}\,(\tau + 1/2)\,(1-q)^{-l}\frac{q^n 2^{\binom{n}{2}}}{n!}q^{-(\alpha'+1)}. \tag{C.85}$$

For q small enough and $\tau = q^{\beta}\,\mathrm{gap}^{-1}$, $\tau + 1/2 \leqslant q^{\beta-1}\,\mathrm{gap}^{-1}$, so that Lemma C.5.5 yields

$$\mathbb{P}_{\mathbf{1}0}\left(T_0 \leqslant \tau\right) \leqslant Cq^{-\alpha''}q^{\beta-1}, \tag{C.86}$$

for some C, α'' independent of q.

✓

Proof of Proposition C.5.4

First of all, let us reformulate what we want to show.

$$\begin{aligned}
\mathbb{E}\left[X_\tau^2\right] &= \sum_{x=1}^{\infty}(2x-1)\mathbb{P}\left(|X_\tau| \geq x\right) \\
&= 2\sum_{x=1}^{\infty}(2x-1)\mathbb{P}\left(X_\tau \geq x\right) \\
&\leqslant 4\sum_{m=1}^{\infty} q^{-m}\mathbb{P}\left(X_\tau \geq q^{-m}\right) \qquad \text{(C.87)}
\end{aligned}$$

In light of Lemma C.5.6, we can now notice that in order to have $X_\tau \geq q^{-m}$ for $m \geq 2$, the system will have to overcome a large number of energy barriers (i.e. rows of ones of length larger than $1/q$), so that the probability of this event will become very small.

Fix $m > 2$, and let us study $\mathbb{P}\left(X_\tau \geq q^{-m}\right)$. Throughout the proof, to simplify the notations, if $C(q)$ is a quantity going to infinity when $q \to 0$, we will not make the distinction between $C(q)$ and $\lfloor C(q) \rfloor$. We divide $\{0, ..., q^{-m}\}$ into $q^{-m+2}(3m)^{-1}$ groups of $3m$ blocks of length q^{-2}. Given a configuration, we say that a block of q^{-2} sites is well-behaved if we can find a row of consecutive ones of length at least $1/q$ that ends with a zero inside it. We can estimate the probability of a block having this property by

$$\mu(\text{a given block is not well-behaved}) \leqslant \left(1 - q(1-q)^{1/q}\right)^{1/q} \leqslant c < 1 \qquad \text{(C.88)}$$

for some constant c.

Let A be the event that in all of these $q^{-m+2}(3m)^{-1}$ groups of blocks, there is one of the $3m$ blocks that is not well-behaved. With this definition, on A^c, there is a group of $3m$ well-behaved blocks. Let us estimate the probability of A under μ using (C.88)

$$\begin{aligned}
\mu(A) &\leqslant \left(1 - \mu(\text{a given block is well-behaved})^{3m}\right)^{q^{-m+2}(3m)^{-1}} \\
&\leqslant \left(1 - (1-c)^{3m}\right)^{q^{-m+2}(3m)^{-1}} \qquad \text{(C.89)} \\
&\leqslant e^{-Cq^2 q^{-\gamma m}}, \qquad \text{(C.90)}
\end{aligned}$$

with $\gamma > 0$, $C < \infty$.

So we can write

$$\mathbb{P}\left(X_\tau \geq q^{-m}\right) \leqslant \mu(A) + \mu\left(\mathbf{1}_{A^c}(\eta)\mathbb{P}_\eta\left(X_\tau \geq q^{-m}\right)\right). \qquad \text{(C.91)}$$

Denote by B_1 the first block of length q^{-2}, $B_2 = B_1 + q^{-2}$,..., $B_{3m} = B_1 + (3m-1)\,q^{-2}$. We have the following estimate

$$\mu\left(\mathbf{1}_{A^c}(\eta)\mathbb{P}_\eta\left(X_\tau \geq q^{-m}\right)\right) \leqslant q^{-m+2}(3m)^{-1}\mu\left(\prod_{i=1}^{3m}\mathbf{1}_{B_i \text{ well-behaved}}(\eta)\mathbb{P}_\eta\left(X_\tau \geq q^{-m}\right)\right). \qquad \text{(C.92)}$$

Let η be a configuration in which all the B_i are well-behaved. Let x_i be the starting point of the first row of $1/q$ ones ended by a zero in B_i, and T_i the first time this site is empty. We denote by $\left(\xi_i(s)\right)_{s \leqslant \tau}$ the trajectory of the distinguished zero started from the position of the zero at the end of the row of ones starting at x_i, up to time τ.

$$\begin{aligned}
\mathbb{P}_\eta\left(X_\tau \geq q^{-m}\right) &\leqslant \mathbb{P}_\eta\left(\forall i = 1, ..., 3m \quad T_i \leqslant \tau\right) \\
&\leqslant \mathbb{P}_\eta\left(T_{3m} \leqslant \tau\right)\mathbb{P}_\eta\left(\forall i = 1, ..., 3m-1 \quad T_i \leqslant \tau \mid T_{3m} \leqslant \tau\right) \\
&\leqslant \mathbb{P}_\eta\left(T_{3m} \leqslant \tau\right)\mathbb{E}_\eta\left[\mathbb{P}_\eta\left(\forall i = 1, ...3m-1\ T_i \leqslant \tau \mid \left(\xi_{3m-1}(s)\right)_{s \leqslant \tau}\right) \mid T_{3m} \leqslant \tau\right] \qquad \text{(C.93)}
\end{aligned}$$

since the dynamics on the left of $x_{3m-1}+1/q$ knowing $(\xi_{3m-1}(s))_{s\leqslant\tau}$ does not depend on what happens on the right of $(\xi_{3m-1}(s))_{s\leqslant\tau}$.

Let us show iteratively that, uniformly in the trajectory $(\xi_k(s))_{s\leqslant\tau}$,

$$\mathbb{P}_\eta\left(\forall i=1,...k\ T_i\leqslant\tau \mid (\xi_k(s))_{s\leqslant\tau}\right)\leqslant (Cq)^k. \tag{C.94}$$

For $k=1$, *mutatis mutandis*, the proof of Lemma C.5.6 applies. Let $k>1$.

$$\mathbb{P}_\eta\left(\forall i=1,...k\ T_i\leqslant\tau \mid (\xi_k(s))_{s\leqslant\tau}\right)$$

is equal to

$$\mathbb{P}_\eta\left(T_k\leqslant\tau \mid (\xi_k(s))_{s\leqslant\tau}\right)\mathbb{P}_\eta\left(\forall i=1,...k-1\ T_i\leqslant\tau \mid (\xi_k(s))_{s\leqslant\tau}, T_k\leqslant\tau\right), \tag{C.95}$$

which can be rewritten

$$\mathbb{P}_\eta\left(T_k\leqslant\tau \mid (\xi_k(s))_{s\leqslant\tau}\right)\mathbb{E}_\eta\left[\mathbb{P}_\eta\left(\forall i=1,...k-1\ T_i\leqslant\tau \mid (\xi_{k-1}(s))_{s\leqslant\tau}\right)\mid (\xi_k(s))_{s\leqslant\tau}, T_k\leqslant\tau\right], \tag{C.96}$$

and the induction hypothesis applies.

Putting together (C.93), (C.94) and (C.92), we get for some constant C

$$\mu\left(\mathbf{1}_{A^c}(\eta)\mathbb{P}_\eta\left(X_\tau\geq q^{-m}\right)\right)\leqslant (Cq)^{2m}. \tag{C.97}$$

Recalling (C.91), (C.90) and (C.87), we get Proposition C.5.4. ✓

What now remains is to show there is enough decorrelation to bound the second sum in (C.65). This is not difficult, once we make the following remark.

Lemma C.5.7 *Denote by* gap_T *the spectral gap of the process seen from the tracer (recall* (C.8)*)*

$$\mathrm{gap}_T=\inf\frac{-\mu(f\mathcal{L}f)}{Var_\mu(f)}, \tag{C.98}$$

where the infimum is taken over non-constant functions $f\in L^2(\mu)$*. Then we have*

$$\mathrm{gap}_T\geq\mathrm{gap}\,. \tag{C.99}$$

Proof

This follows directly from (C.18) and the definition of gap and gap_T (recall (C.19)). ✓

Now we are armed to study the terms $\mathbb{E}\left[\left(X_{k\tau}-X_{(k-1)\tau}\right)\left(X_{k'\tau}-X_{(k'-1)\tau}\right)\right]$. First of all, by stationarity, this quantity depends only on τ and $|k-k'|$. So we only need to study $\mathbb{E}\left[\left(X_\tau\right)\left(X_{k\tau}-X_{(k-1)\tau}\right)\right]$ for $k\geq 2$. In fact, using Cauchy-Schwarz inequality and Proposition C.5.4, we only need to study this term for $k\geq 3$, which allows some decorrelation to take place between times τ and $(k-1)\tau$. Let us denote by $\left(P^T_s\right)_{s\geq 0}$ the semigroup associated to $\mathcal{L}$. $\mathbb{E}_{(\omega,x)}\left[\cdot\right]$ will denote the law of the process with generator $\mathcal{L}_0$ starting from the configuration ω with the tracer in position x ($\mathbb{E}\left[\cdot\right]$ is still the law of the process starting from μ and the tracer at the origin). Using successively the Markov property at time τ, we can write

$$\mathbb{E}\left[\left(X_\tau\right)\left(X_{k\tau}-X_{(k-1)\tau}\right)\right] \;=\; \mathbb{E}\left[X_\tau\mathbb{E}_{(\omega(\tau),X_\tau)}\left[X'_{(k-1)\tau}-X'_{(k-2)\tau}\right]\right], \tag{C.100}$$

where $(X'_s)_{s\geq 0}$ denotes the trajectory of the tracer under the law $\mathbb{E}_{(\omega(\tau),X_\tau)}[\cdot]$. Now we use successively Cauchy-Schwarz inequality and stationarity of the process seen from the tracer to get

$$\begin{aligned}
\mathbb{E}\left[(X_\tau)\left(X_{k\tau}-X_{(k-1)\tau}\right)\right]^2 &\leqslant \mathbb{E}\left[X_\tau^2\right]\mathbb{E}\left[\mathbb{E}_{(\omega(\tau),X_\tau)}\left[X_{(k-1)\tau}-X_{(k-2)\tau}\right]^2\right] && \text{(C.101)}\\
&\leqslant \mathbb{E}\left[X_\tau^2\right]\mathbb{E}\left[\mathbb{E}_{\left((\omega(\tau))_{X_\tau+\cdot},0\right)}\left[X_{(k-1)\tau}-X_{(k-2)\tau}\right]^2\right] && \text{(C.102)}\\
&\leqslant \mathbb{E}\left[X_\tau^2\right]\mu\left(\mathbb{E}_{(\omega,0)}\left[X_{(k-1)\tau}-X_{(k-2)\tau}\right]^2\right), && \text{(C.103)}
\end{aligned}$$

Let us focus on $\mathbb{E}_{(\omega,0)}\left[X_{(k-1)\tau}-X_{(k-2)\tau}\right]$. Using the Markov property at time $(k-2)\tau$, we get

$$\begin{aligned}
\mathbb{E}_{(\omega,0)}\left[X_{(k-1)\tau}-X_{(k-2)\tau}\right] &= \mathbb{E}_{(\omega,0)}\left[\mathbb{E}_{\left(\omega((k-2)\tau),X_{(k-2)\tau}\right)}\left[X'_\tau-X'_0\right]\right]\\
&= \mathbb{E}_{(\omega,0)}\left[\mathbb{E}_{\left((\omega((k-2)\tau))_{X_{(k-2)\tau+\cdot}},0\right)}\left[X'_\tau\right]\right]\\
&= P^T_{(k-2)\tau}g(\omega), && \text{(C.104)}
\end{aligned}$$

where $g(\omega)=\mathbb{E}_{(\omega,0)}[X_\tau]$, and the X'_s in the first and second line denote respectively the trajectory of the tracer under the laws $\mathbb{E}_{\left(\omega((k-2)\tau),X_{(k-2)\tau}\right)}[\cdot]$ and $\mathbb{E}_{\left((\omega((k-2)\tau))_{X_{(k-2)\tau+\cdot}},0\right)}[\cdot]$. Therefore, using the spectral gap inequality, and the fact that g is a mean-zero function in $L^2(\mu)$ thanks to stationarity and Proposition C.5.4, we get

$$\begin{aligned}
\mathbb{E}\left[(X_\tau)\left(X_{k\tau}-X_{(k-1)\tau}\right)\right]^2 &\leqslant \mathbb{E}\left[X_\tau^2\right]\mu\left(\left(P^T_{(k-2)\tau}g\right)^2\right)\\
&\leqslant \mathbb{E}\left[X_\tau^2\right]^2 e^{-2(k-2)\tau\,\mathrm{gap}_T}\\
&\leqslant \mathbb{E}\left[X_\tau^2\right]^2 e^{-2(k-2)q^\beta} && \text{(C.105)}
\end{aligned}$$

Since $\sum_{k\geq 1}e^{-kq^\beta}\lesssim q^{-\beta}$, the second term in (C.65) is

$$\sum_{k\neq k'}\mathbb{E}\left[\left(X_{k\tau}-X_{(k-1)\tau}\right)\left(X_{k'\tau}-X_{(k'-1)\tau}\right)\right]\leqslant C\lfloor t/\tau\rfloor\,\mathbb{E}\left[X_\tau^2\right]q^{-\beta}. \tag{C.106}$$

Putting this into (C.65) together with Proposition C.5.4, we get Theorem C.5.2. ✓

C.6 An alternative proof in the FA-1f model

When the environment is given by the one-spin Fredrickson-Andersen model (FA-1f), in which $c_x(\eta)=1-\prod_{i=1}^d\eta_{e_i}\eta_{-e_i}$ (the constraint requires at least one nearest neighbour to be empty), the diffusion coefficient at low density is of order q^2. This means that in this particular case, the correct order is already given by the first term in (C.10), which allows to design another strategy to find the lower bound in Theorem C.4.1 when $k=1$. Since the diffusion coefficient is of order lower than q^2 in the k-zeros model with $k>1$, this technique does not apply. For simplicity, we write the proof in dimension $d=1$.

We follow the strategy devised to prove Lemma 6.25 in [KLO12], i.e. we prove that

$$\sup\{2\mu(jf)-\mathcal{D}(f)\}\leqslant cq^2, \tag{C.107}$$

where $c < 1$ does not depend on q and $\mathcal{D}(f) = -\mu(f\mathcal{L}f)$. Seeing (C.10) and (C.16), this is sufficient to prove Theorem C.4.1 when $k=1$, $d=1$. To obtain that result, we define

$$\begin{aligned}\mathcal{D}_{jump}(f) &= \frac{1}{2}\mu\left((1-\eta_0)(1-\eta_{\alpha e_i})\left[f(\eta_{\alpha e_i+\cdot}) - f(\eta)\right]^2\right) && \text{(C.108)}\\ \mathcal{D}_{FA}(f) &= \frac{1}{2}\sum_{y\in\mathbb{Z}}\mu\left(c_y(\eta)((1-q)(1-\eta_y)+q\eta_y)\left[f(\eta^y)-f(\eta)\right]^2\right), && \text{(C.109)}\end{aligned}$$

so that $\mathcal{D}(f) = \mathcal{D}_{jump}(f) + \mathcal{D}_{FA}(f)$, and we show separately that for all f

$$\begin{aligned}2\mu(jf) - \mathcal{D}_{jump}(f) &\leqslant q^2 && \text{(C.110)}\\ 2\mu(jf) - \mathcal{D}_{FA}(f) &\leqslant Cq^2, && \text{(C.111)}\end{aligned}$$

where $C \geq 1$ is a constant that does not depend on q. To get the result from (C.110) and (C.111), we write that for any $\lambda > 0$, for any local function f

$$\lambda^{-1}\Big(2\mu(jf) - \mathcal{D}_{jump}(f) - \mathcal{D}_{FA}(f)\Big) = 2\mu\left(j\lambda^{-1}f\right) - \lambda\mathcal{D}_{jump}\left(\lambda^{-1}f\right) - \lambda\mathcal{D}_{FA}\left(\lambda^{-1}f\right)$$

So that

$$\lambda^{-1}\sup\Big\{2\mu(jf) - \mathcal{D}_{jump}(f) - \mathcal{D}_{FA}(f)\Big\} \leqslant \sup\Big\{2\mu\left(jg\right) - \lambda\mathcal{D}_{jump}\left(g\right) - \lambda\mathcal{D}_{FA}\left(g\right)\Big\}$$

Take for instance $\lambda = C/(C+1)$. We have $\lambda \geq 1-\lambda$, so that

$$\begin{aligned}\lambda^{-1}\sup\Big\{2\mu(jf) - \mathcal{D}_{jump}(f) - \mathcal{D}_{FA}(f)\Big\} &\leqslant \sup\Big\{2\mu\left(jg\right) - \lambda\mathcal{D}_{jump}\left(g\right) - (1-\lambda)\mathcal{D}_{FA}\left(g\right)\Big\}\\ &\leqslant \left[\lambda + (1-\lambda)C\right]q^2 = q^2,\end{aligned}$$

using (C.110) and (C.111), so that (C.107) is proven.

1. Proof of (C.110).

For any local function f, we can rewrite $\mu(jf)$ in terms of the "jumps" $\eta\to\eta_{1+\cdot}$ and $\eta\to\eta_{-1+\cdot}$

$$2\mu(jf) = -\mu\left((1-\eta_0)(1-\eta_1)\left[f(\eta_{1+\cdot}) - f(\eta)\right]\right) + \mu\left((1-\eta_0)(1-\eta_{-1})\left[f(\eta_{-1+\cdot}) - f(\eta)\right]\right)$$

Now using the inequality $ab \leqslant (a^2+b^2)/2$, the Dirichlet form $\mathcal{D}_{jump}(f)$ appears in the r.h.s.

$$\begin{aligned}2\mu(jf) &\leqslant q^2 + \frac{1}{2}\mu\left((1-\eta_0)(1-\eta_1)\left[f(\eta_{1+\cdot}) - f(\eta)\right]^2\right)\\ &\quad + \frac{1}{2}\mu\left((1-\eta_0)(1-\eta_{-1})\left[f(\eta_{-1+\cdot}) - f(\eta)\right]^2\right)\\ &\leqslant q^2 + \mathcal{D}_{jump}(f)\end{aligned}$$

2. Proof of (C.111).

We need only to prove it for small q. First we make a few computations to express $\mu(jf)$ in terms of allowed flips ($\eta\to\eta^1$ or $\eta\to\eta^{-1}$). Then we use the same optimization technique performed in the proof of Lemma 6.13 in [KLO12] to get the desired bound. We have the following equalities

$$\begin{aligned}\mu\left((1-\eta_0)(1-\eta_1)f(\eta)\right) &= \frac{q}{1-2q}\mu\left((1-\eta_0)\left[f(\eta^1) - f(\eta)\right]\right) + \frac{q}{1-q}\mu\left((1-\eta_0)\eta_1 f(\eta)\right),\\ \mu\left((1-\eta_0)(1-\eta_{-1})f(\eta)\right) &= \frac{q}{1-2q}\mu\left((1-\eta_0)\left[f(\eta^{-1}) - f(\eta)\right]\right) + \frac{q}{1-q}\mu\left((1-\eta_0)\eta_{-1} f(\eta)\right),\\ \mu\left((1-\eta_0)\eta_1 f(\eta)\right) &= (1-q)\mu\left((1-\eta_0)(1-\eta_1)\left[f(\eta^1) - f(\eta)\right]\right)\\ &\quad + (1-q)\mu\left((1-\eta_0)f(\eta)\right),\\ \mu\left((1-\eta_0)\eta_{-1} f(\eta)\right) &= (1-q)\mu\left((1-\eta_0)(1-\eta_{-1})\left[f(\eta^{-1}) - f(\eta)\right]\right)\\ &\quad + (1-q)\mu\left((1-\eta_0)f(\eta)\right)\end{aligned}$$

So that, computing differences, we get

$$\begin{aligned}\mu(jf) &= \frac{q}{p-q}\left[\mu\left((1-\eta_0)\left[f(\eta^1)-f(\eta)\right]\right)-\mu\left((1-\eta_0)\left[f(\eta^{-1})-f(\eta)\right]\right)\right] \\ &\quad + q\left[\mu\left((1-\eta_0)(1-\eta_1)\left[f(\eta^1)-f(\eta)\right]\right)-\mu\left((1-\eta_0)(1-\eta^{-1})\left[f(\eta^{-1})-f(\eta)\right]\right)\right]\end{aligned}$$

Assume $q < 1/2$. Using the inequality $ab \leqslant (a^2+b^2)/2$, we get for any $\alpha, \beta > 0$

$$\begin{aligned}&\frac{\mu(jf)}{q} \leqslant \\ &\quad \frac{1}{1-2q}\left\{\alpha q + \frac{1}{2\alpha}\left[\mu\left((1-\eta_0)\left[f(\eta^1)-f(\eta)\right]^2\right)+\mu\left((1-\eta_0)\left[f(\eta^{-1})-f(\eta)\right]^2\right)\right]\right\}+\beta \\ &\quad + \frac{1}{2\beta}\left[\mu\left((1-\eta_0)(1-\eta_1)\left[f(\eta^1)-f(\eta)\right]^2\right)+\mu\left((1-\eta_0)(1-\eta_{-1})\left[f(\eta^{-1})-f(\eta)\right]^2\right)\right]\end{aligned}$$

We insert the missing rates to recover terms appearing in $\mathcal{D}_{FA}(f)$. For instance, since we assumed $q < 1/2$

$$\begin{aligned}&\mu\left((1-\eta_0)\left[f(\eta^1)-f(\eta)\right]^2\right) \\ &\qquad \leqslant \frac{1}{q}\mu\left((1-\eta_0)((1-q)(1-\eta_1)+q\eta_1)\left[f(\eta^1)-f(\eta)\right]^2\right) \\ \text{and}\quad &\mu\left((1-\eta_0)(1-\eta_1)\left[f(\eta^1)-f(\eta)\right]^2\right) \\ &\qquad \leqslant \frac{1}{p}\mu\left((1-\eta_0)((1-q)(1-\eta_1)+q\eta_1)\left[f(\eta^1)-f(\eta)\right]^2\right)\end{aligned}$$

So that we get

$$\mu(jf) \leqslant \frac{q}{1-2q}\left\{\alpha q + \frac{1}{\alpha q}\mathcal{D}_{FA}(f)\right\} + q\left\{\beta + \frac{1}{\beta(1-q)}\mathcal{D}_{FA}(f)\right\} \tag{C.112}$$

Optimizing in α, β, this yields

$$\mu(jf) \leqslant \frac{2q}{1-2q}\sqrt{\mathcal{D}_{FA}(f)} + 2\sqrt{\mathcal{D}_{FA}(f)/(1-q)} \tag{C.113}$$

This is enough to prove (C.111) for small q (see [KLO12, Section 6.3]).

C.7 Lower bound for the windmill model

Let us define a non-cooperative model in two dimensions, to which we can apply the techniques of Section C.4.1, but with an auxiliary dynamics that is not strictly one-dimensional.

Definition C.7.1 *Define the sets (see Figure C.5):*

$$\begin{aligned} NE &= \{e_1+e_2, 2e_1+e_2\} && \text{(C.114)} \\ WN &= \{-e_1+e_2, -e_1+2e_2\} && \text{(C.115)} \\ SW &= \{-e_1-e_2, -2e_1-e_2\} && \text{(C.116)} \\ ES &= \{e_1-e_2, e_1-2e_2\} && \text{(C.117)}\end{aligned}$$

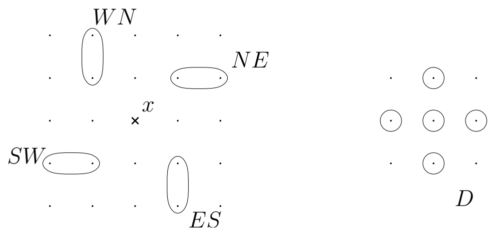

Figure C.5: The windmill constraint at x requires that NE, WN, SW or ES be empty. On the right is a minimal set of zeros that allow to empty to whole lattice (see Definition C.2.1).

Figure C.6: An example of transformation from η to $\eta^{\rightarrow}$ and $\eta^{\leftarrow}$ respectively. The origin is marked by a cross. Note the difference between the sets to swap.

The windmill constraint at the origin is

$$c_0(\eta) = \begin{cases} 1 & \textit{if } \eta_{|NE} \equiv 0 \textit{ or } \eta_{|WN} \equiv 0 \textit{ or } \eta_{|SW} \equiv 0 \textit{ or } \eta_{|ES} \equiv 0 \\ 0 & \textit{else} \end{cases} \tag{C.118}$$

The windmill model is then defined by $c_x(\eta) = c_0(\eta_{x+\cdot})$.

This is a non-cooperative model. See Figure C.5 for an example of set of zeros that can empty $\mathbb{Z}^2$ through legal flips.

Let us now describe the auxiliary dynamics for this model. Let the minimal set above be

$$D = \{0, e_1, e_2, -e_1, -e_2\}, \tag{C.119}$$

A the event

$$A = \Big\{\eta_{|D} \equiv 0 \text{ or } (\eta_{e_1+\cdot})_{|D} \equiv 0 \text{ or } (\eta_{-e_1+\cdot})_{|D} \equiv 0\Big\}, \tag{C.120}$$

the conditioned measure

$$\overline{\mu} = \mu(.|A). \tag{C.121}$$

and finally the swapped configurations $\eta^{\rightarrow}$ and $\eta^{\leftarrow}$ (see Figure C.6)

$$\eta^{\rightarrow}_y = \begin{cases} \eta_{e_1+e_2} & \text{if } y = e_2 \\ \eta_{e_2} & \text{if } y = e_1 + e_2 \\ \eta_{-e_1} & \text{if } y = 2e_1 \\ \eta_{2e_1} & \text{if } y = -e_1 \\ \eta_{e_1-e_2} & \text{if } y = -e_2 \\ \eta_{-e_2} & \text{if } y = e_1 - e_2 \\ \eta_y & \text{else} \end{cases}, \quad \eta^{\leftarrow}_y = \begin{cases} \eta_{-e_1+e_2} & \text{if } y = e_2 \\ \eta_{e_2} & \text{if } y = -e_1 + e_2 \\ \eta_{-2e_1} & \text{if } y = e_1 \\ \eta_{e_1} & \text{if } y = -2e_1 \\ \eta_{-e_1-e_2} & \text{if } y = -e_2 \\ \eta_{-e_2} & \text{if } y = -e_1 - e_2 \\ \eta_y & \text{else} \end{cases} \tag{C.122}$$

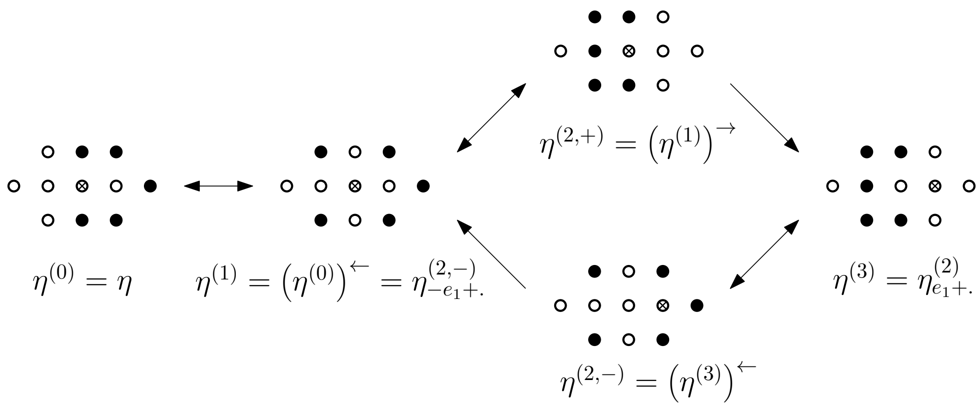

Figure C.7: The origin is marked by a cross. The configurations that can be reached starting from η can be indexed by the graph with oriented edges described by the arrows. This graph is $\mathbb{Z}$, except that now and then one site is doubled and a different route must be taken to go from left to right than from right to left. The dynamics started from η is just a simple random walk on this graph.

The auxiliary dynamics defined by the following generator is reversible w.r.t $\overline{\mu}$:

$$\begin{aligned}
\overline{\mathcal{L}}f(\eta) \;=\; & \left(1-\prod_{y\in D}(1-\eta_{e_1+y})\right)\prod_{y\in D}(1-\eta_y)\left[f(\eta^{\rightarrow})-f(\eta)\right] \\
& + \left(1-\prod_{y\in D}(1-\eta_{e_1+y})\right)\left(1-\prod_{y\in D}(1-\eta_y)\right)\prod_{y\in D}(1-\eta_{-e_1+y})\left[f(\eta^{\leftarrow})-f(\eta)\right] \\
& + \left(1-\prod_{y\in D}(1-\eta_{-e_1+y})\right)\prod_{y\in D}(1-\eta_y)\left[f(\eta^{\leftarrow})-f(\eta)\right] \\
& + \left(1-\prod_{y\in D}(1-\eta_{-e_1+y})\right)\left(1-\prod_{y\in D}(1-\eta_y)\right)\prod_{y\in D}(1-\eta_{e_1+y})\left[f(\eta^{\rightarrow})-f(\eta)\right] \\
& + \prod_{y\in D}(1-\eta_{e_1+y})\left[f(\eta_{e_1+\cdot})-f(\eta)\right]+\prod_{y\in D}(1-\eta_{-e_1+y})\left[f(\eta_{-e_1+\cdot})-f(\eta)\right] \quad \text{(C.123)}
\end{aligned}$$

In words, if the tracer finds an empty cross D on its right or left, it jumps there with rate 1. If the tracer is sitting on an empty cross, and does not have an empty cross on its right (resp. left), it does the first swap (resp. second swap) in Figure C.6 with rate 1. If the tracer is not sitting on an empty cross, and does not have an empty cross on its right (resp. left) –which means $\overline{\mu}$-a.s. that it has an empty cross on its left (resp. right)–, it does the second (resp. first) swap in Figure C.6 with rate 1.

To reproduce the proof of the lower bound in Theorem C.4.1, we need to describe the analogue of the SRW on $\mathbb{Z}$ taking part in the proof of Lemma C.4.4. That is, to describe how we can index by a graph much like $\mathbb{Z}$ all the configurations that can be attained by this dynamics starting from a given configuration in A. This is described informally in Figure C.7. The diffusion coefficient of a random walk on such a graph can be bounded from below by a positive constant uniformly in η. Moreover, in the auxiliary dynamics described by $\overline{\mathcal{L}}$, the tracer moves to the right (resp. left) at least every three steps takento the right (resp. left) on this graph.

Now to show the analogue of Lemma C.4.3, we need to see how to reconstruct the swaps $\eta \to \eta^{\rightarrow}$ (resp. $\eta \to \eta^{\leftarrow}$) using only legal flips and no more than an extra zero in the situations

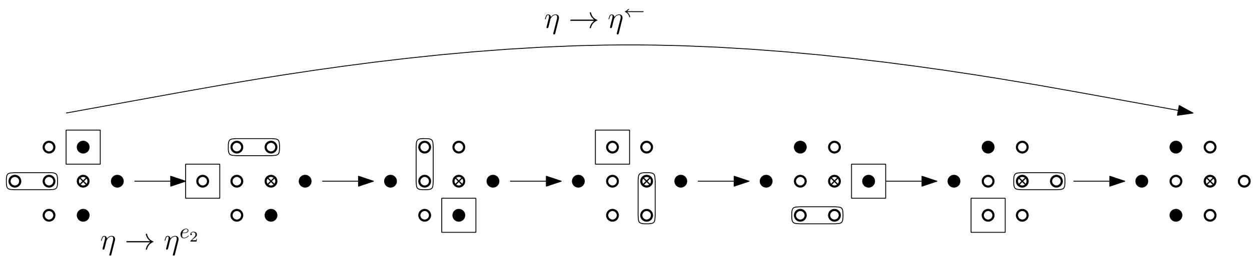

Figure C.8: An example of the reconstitution of the swap $\eta \to \eta^{\leftarrow}$ using only legal flips and an extra zero (in the zone concerned, at any step there are at most $5+1$ zeros). Before each step, the site to-be-flipped is inside a square, and the zeros that make the flip legal circled.

where they occur. This is the object of Figure C.8. Using this strategy, computations analogous to those in the proof of Lemma C.4.3 show that the diffusion coefficient in the windmill model satisfies

$$u.Du \geq c\|u\|^2 q^6. \tag{C.124}$$

Appendix D

Is there a breakdown of the Stokes-Einstein relation in Kinetically Constrained Models at low temperature?

In collaboration with

Cristina Toninelli

This is a letter written for the physical community exposing the results of [Blo13b] along with more heuristics and conjectures; submitted.

We study the motion of a tracer particle injected in facilitated models which are used to model supercooled liquids in the vicinity of the glass transition. We consider the East model, FA1f model and a more general class of non-cooperative models. For East previous works had identified a fractional violation of the Stokes-Einstein relation with a decoupling between diffusion and viscosity of the form $D \sim \tau^{-\xi}$ with $\xi \sim 0.73$. We present rigorous results proving that instead $D \sim \tau^{-1}$ at leading order for very large time-scales. Our results still suggest a violation but weaker, $D\tau \sim 1/q^{\alpha}$, where q is the density of excitations. We discuss the expected value of α. For FA1f we prove fractional Stokes Einstein in dimension 1, and $D \sim \tau^{-1}$ in dimension 2 and higher, confirming previous works. Our results extend to a larger class of non-cooperative models.

A microscopic understanding of the liquid/glass transition and of the glassy state of matter remains a challenge for condensed matter physicists (see [BB11, BG13] for recent surveys). In the last years many experimental and theoretical works have been devoted to understanding the spatially heterogeneous relaxation which occurs when temperature is lowered towards the glass transition [SE11, CE96, EEH+12, CS97, SBME03, MSKE06, Edi00, Sil99, Ric02, VI00, Glo00, HMCG07, Ber11b]. In this regime dynamics slows down and relaxation is characterized by the occurrence of correlated regions of high and low mobility whose typical size grows when temperature decreases. One of the most striking experimental consequences of dynamical heterogeneities is the violation of Stokes-Einstein relation, namely the decoupling of self-diffusion coefficient (D) and viscosity (η). In high temperature homogeneous liquids, self-diffusion and viscosity are related by the Stokes-Einstein relation $D\eta/T \sim const$ [HM06b]. Instead in supercooled fragile liquids the self-diffusion coefficient does not decrease as fast as the viscosity increases and $D\eta$ increases by 2-3 orders of magnitude approaching the glass transition

[SE11, CE96, EEH+12, CS97, SBME03, MSKE06]. A good fit of several experimental data is $D \sim \eta^{-\xi}$ with $\xi < 1$ an exponent depending on the specific liquid. Such a violation is instead absent or much weaker in strong liquids, consistently with the idea that the decoupling is related to heterogeneities which are indeed more important for more fragile liquids. A natural explanation of this effect is that different observables probe differently the underlying broad distribution of relaxation times [Edi00]: D is dominated by the more mobile particles, while η probes the time scale needed for every particle to move.

Different theories of the glass transition have been tested by measuring their capability to predict Stokes-Einstein breakdown. In particular, several works [JGC04,JGC05,CGJ+06,LDJ05] have analysed the self-diffusion coefficient of a probe particle injected in a facilitated (or kinetically constrained) model. In this setting supercooled liquids are modeled by a coarse-grained mobility field evolving with a Markovian stochastic dynamics with simple thermodynamic properties and non-trivial kinetic constraints. More precisely facilitated models are lattice models described by configurations $\{n_i\}$, $n_i = 0, 1$, with $n_i = 1$ if the lattice site i is active and $n_i = 0$ if i is inactive. Active and inactive sites essentially correspond to coarse grained unjammed and jammed regions, respectively. Active sites are also called defects. The dynamics is described by the following transition rates

$$n_i = 0 \quad \xrightarrow{qc_i} \quad n_i = 1 \tag{D.1}$$

$$n_i = 1 \quad \xrightarrow{pc_i} \quad n_i = 0, \tag{D.2}$$

where c_i encodes the model dependent constraints and is zero or one depending on the local configuration around i, $q = 1/(1+\exp(1/\tilde{T}))$, $p = 1-q$ and $\tilde{T}$ is a reduced temperature. Since c_i does not depend on the configuration on i, dynamics satisfies detailed balance w.r.t. the product measure that gives weight q to active sites and p to inactive sites, which is therefore an equilibrium distribution. Two very popular models are the one-spin facilitated model, FA1f [FA84], and the East model [JE91]. For FA1f $c_i = 1$ iff site i has at least an active nearest neighbour, while for East in one dimension $c_i = 1$ iff the right neighbour of i is active (namely $c_i = n_{i+1}$). The injection of a probe particle into these models is performed as follows [JGC04, JGC05]. Initially the lattice configuration is distributed with the equilibrium product measure and the probe particle is at the origin. Then one lets the lattice configuration (the environment) evolve according to the facilitated model dynamics while the probe is allowed to jump only between active sites, namely

$$X \longrightarrow X \pm e_\alpha \qquad \text{at rate} \qquad n_X n_{X \pm e_\alpha} \tag{D.3}$$

where X is the position of the probe, $\alpha = 1, ..., d$ is one of the d directions and e_α is the unit vector in this direction. Then the self diffusion matrix D is defined as usual by

$$e_\alpha.2De_\alpha = \lim_{t\to\infty} \frac{\left\langle (X_t \cdot e_\alpha)^2 \right\rangle}{t}.$$

A numerical analysis for the FA1f model lead in [JGC04,JGC05] to the conclusion that $D \sim q^2$ in any dimension. Previous numerical [BG13] and renormalisation group analysis [WBG04] suggested $\tau = 1/q^{2+\epsilon(d)}$ with $\epsilon(1) = 1$, $\epsilon(2) \simeq 0.3$, $\epsilon(3) \simeq 0.1$ and $\epsilon(d \geq 4) \simeq 0$. These estimates led [JGC04, JGC05] to the conclusion that Stokes-Einstein relation is violated with $\xi \simeq 2/3, 2/2.3, 2/2.1$ for FA1f in $d = 1, 2, 3$ and is not violated in higher dimensions. In [JMS06] the scaling of τ was deduced via an exact mapping into a diffusion limited aggregation model leading instead to $\epsilon(d \geq 2) = 0$. This finding is supported by the mathematical results in [CMRT07] which confirm $\epsilon(2) = 0$ and yield $\epsilon(3) \leqslant 0$. In consequence the result for the diffusion coefficient in [JGC04, JGC05] was reinterpreted [TGS] by saying that $\xi = 2/3$ in $d = 1$ while no violation occurs in $d \geq 2$. This is consistent with the idea that FA1f is a non cooperative model dominated by the diffusion of active sites and it is a model for strong rather than for fragile liquids. Instead for the East model the analysis in [JGC04, JGC05] leads to $D = \tau^{-\xi}$ with $\xi \simeq 0.73$, a result which is expected to hold also in higher dimensions. The exponent is consistent with the one observed experimentally and numerically in fragile glass-forming liquids [SBME03],[YO98],[Ber04].

Here we report recent rigorous mathematical results for East, for FA1f models and for more general non-cooperative models (details can be found in [Blo13b]). For the one dimensional East model we prove that there exists a constant $\alpha > 0$ such that

$$q^2\tau^{-1} \leqslant D \leqslant q^{-\alpha}\tau^{-1} \tag{D.4}$$

which yields at leading order

$$D \sim \tau^{-1} \tag{D.5}$$

since τ diverges faster than polynomial as $q \to 0$. Thus we establish that a fractional Stokes-Einstein relation cannot hold, in contrast with the predictions in [JGC04, JGC05]. Our result (D.4) does not exclude the possibility of a weaker violation of the form $D\tau \sim 1/q^{\alpha}$. Indeed, as we will explain, a natural conjecture is that this polynomial violation occurs with $\alpha = 2$. We provide a heuristic for our result, which is related to the estimate of the energy barriers that the probe has to overcome in order to cross the typical distance between two active sites at equilibrium. We also provide our understanding of which are the problems in the analysis performed in previous works. Then we consider non-cooperative models and we prove that in any dimension for FA1f it holds

$$cq^2 \leqslant D \leqslant c'q^2 \tag{D.6}$$

with c, c' constants independent on q. We also prove

$$cq^{k+1} \leqslant D \leqslant c'q^{k+1} \tag{D.7}$$

for a more general model in which k (instead of one) active sites are required in the vicinity of the to be updated site. We provide a heuristic both for the diffusion coefficient and the relaxation time which leads to a fractional Stokes-Einstein for $d = 1$ and to $D \sim \tau^{-1}$ for $d \geq 2$. In particular our heuristics clearly explains the scaling $\tau = 1/q^2$ in $d \geq 2$ for the FA1f model. Note that (D.6) together with the results in [CMRT07] imply that for FA1f in $d \geq 3$ it holds $D\tau \leqslant const$: any form of decoupling cannot hold in this case (while a logarithmic decoupling may occur in $d = 2$). Finally we obtain for any choice of the kinetic constraints a variational formula for the diffusion matrix, which we will present and discuss at the end in order to avoid technicalities at this stage. As a consequence we obtain for any facilitated model

$$q^2\tau^{-1} \leqslant e_\alpha.De_\alpha \leqslant q^2. \tag{D.8}$$

Let us start with the analysis of the East model. The relaxation time has in this case an exponential inverse temperature squared (EITS) form. Namely, up to polynomial corrections,

$$\tau \sim e^{\ln(1/q)^2/2\ln 2}. \tag{D.9}$$

The form $\tau \sim e^{cst/T^2}$ was first given in [SE99] with $cst = 1/\ln 2$, which was derived via energy barrier considerations. This value of the constant was proved to be wrong by a factor $1/2$ in [CMRT08]. Indeed, taking into account an entropy factor which was missing in the previous works (see also [FMRT12b] for a more extended explanation) and using the lower bound of

[AD02], in [CMRT08] it was proven instead that $cst = 1/2\ln 2$. This scaling can be explained through combinatorics arguments. Consider a configuration of only inactive sites on a typical equilibrium length $1/q$, with a fixed active site at the right boundary. Recall that, due to the orientation of the constraint, the left-most site can only become active if all sites on its right became active before it. It was proven in [SE99, CDG01] that before the leftmost site can become active, the system needs to visit configurations with at least $\ln(1/q)/\ln 2$ active sites. The equilibrium probability of such a configuration is less than $e^{-\ln(1/q)^2/\ln 2}$ when $q \to 0$, which accounts for the EITS form. Moreover, the set of configurations attainable using at most $n = \ln(1/q)/\ln 2$ active sites simultaneously has a cardinality of order $2^{\binom{n}{2}} n! \approx e^{\ln(1/q)^2/2\ln 2}$ [CDG01], so that the entropy factor changes the constant in the EITS form by a factor 2 and yields (D.9). This fast divergence of τ makes it very difficult to approach zero temperature through simulations and allows to neglect polynomial terms in q when an estimate involves τ. The above discussion actually explains the scale of the *persistence* time rather than the relaxation time. However, for the East model these characteristic times coincide [CFM13]. Let us provide the heuristics behind our result (D.4) which establishes that also diffusion occurs on this time scale at leading order. In the initial configuration, the first active site (i_a) on the right of the probe particle is typically at distance $\sim 1/q$. Before the tracer can move its first step to the right it needs at least to wait for its right neighbour to become active. This occurs thanks to the fact that sites are activated from right to left starting from i_a and thus requires a time proportional to the persistence time. Note that the arrival of the excitation sent from i_a does not influence the configuration on the right of i_a. In particular once the probe has arrived at i_a it has typically to face again the same energy barrier. In summary, for each distance of $1/q$ the probe covers towards the right we need a time at least τ and this, together with the symmetry of the motion of the probe and the fact that any polynomial in q is negligible with respect to τ, yields (D.5). Note that our result (D.4) allows a weak violation of the Stokes Einstein relation: $D\tau$ can diverge when $q \to 0$ as a polynomial in $1/q$. Based on the above energy barrier considerations and on the evaluation of the typical number of independent activation events coming from i_a, we indeed conjecture $D\tau \sim 1/q^2$.

We believe that the discrepancy between our result and the findings $D \sim \tau^{-\xi}$ with $\xi \sim 0.73$ in [JGC04, JGC05] is due the difficulty to approach zero temperature in simulations. In particular, among the diffusion coefficient data reported on Fig.3 of [JGC04], on all data except the last one the value of $1/T$ is such that $1/q^2 > e^{\ln(1/q)^2/2\ln 2}$. Thus these data, even though very accurate and asymptotic in time, are not sufficiently in the low temperature regime and do not allow to capture the asymptotic form of D vs τ^{-1} when $q \to 0$. The presumed fractional decoupling for East was considered (see e.g. [BG13],[CG10]) to be a consequence of the fluctuations in the dynamic. More precisely it was explained by the fact that, even if the first move is governed by the persistence time, then the probe is supposed to move faster since the typical time for the next events was considered to be the (shorter) mean time between changes of mobility for a given site (exchange time). To use the expression of [CG10], the probe should surf on excitation lines and thus move faster than the typical relaxation time. Due to the directed nature of the constraint, the excitation line cannot expand to the right of the site where it has originated, therefore the probe can perform this fast surfing only up to a distance $1/q$: the persistence time remains the leading order in the diffusion time scale while fluctuations should give rise to a polynomial violation of Stokes Einstein.

We turn now to non-cooperative models, and more specifically to the k-defects model which we define as follows: $c_i = 1$ if and only if there are at least k defects at distance at most k around i. Note that for $k = 1$, we recover the FA1f model and that any k-defects model is non-cooperative: if the initial system contains k active neighbours, any site can be activated through allowed transitions. Also, at low q we expect dynamics to be dominated by the diffusion of the group of k defects, which occurs at rate q because in order to shift of one step the group of vacancies we need to create an additional vacancy in the direction of the move (and then remove one vacancy of the group in the opposite direction). As stated in (D.7), we prove in all dimensions $D \sim q^{k+1}$, which agrees with the numerical results in [JGC04] for FA1f ($k = 1$).The heuristics behind (D.7) is the following. Consider a box of size q^{-k} centred on the probe particle. Typically at equilibrium there is one group of k active sites inside this box, so that the proportion of time during which the probe particle is on such a group is q^k. During that portion of time, the probe particle diffuses at the same rate as this group of k active sites which, as already explained, is q. In the end, the diffusion coefficient of the probe particle is of order $q^k \times q = q^{k+1}$. Concerning the relaxation time we expect $\tau \sim 1/q^{2k+1}$ in one dimension and $\tau \sim 1/q^{k+1}$ in $d \geq 2$. This, together with (D.7), implies that a fractional violation of the Stokes Einstein relation does not occur in $d \geq 2$ and occurs in $d = 1$. In $d = 1$ the result for τ should come from the fact that relaxation requires the group of k-vacancies to overcome the typical distance $1/q^k$ among two subsequent groups by diffusing at rate q. In $d \geq 2$ around each group of k-defects there is typically a ball of radius $r = 1/q^{k/d}$ without any such group. Relaxation requires that a fraction of the sites of the ball is covered by the active group which is essentially a walker at rate q. Classic results on random walks [Ald83, DPRZ04] imply that this requires a time (up to log corrections) r^d times the inverse of the diffusion rate of the walker, which indeed yields $\tau \sim 1/q^{k+1}$.

Before sketching the ideas that allow us to prove (D.7) rigorously, we wish to present our variational formula for the diffusion matrix, which is valid for any choice of the constraints and in particular yields (D.8). Denote by $\eta_i(t)$ the state of site $X_t + i$ at time t, *i.e.* $\eta(t)$ is the configuration *seen from the probe particle* at time t. In particular, the state of the system at the position of the tracer at time t is given by $\eta_0(t)$. We call j_α the current of the probe in the direction $\alpha = 1, ..., d$, namely

$$j_\alpha(\eta) = \eta_0 \left(\eta_{e_\alpha} - \eta_{-e_\alpha}\right). \tag{D.10}$$

Finally, we denote by $\mathcal{L}$ the Liouvillian operator associated to the master equation for the dynamics *i.e.* $\mathcal{L}$ is the operator such that $\partial_t \langle f(\eta(t))\rangle = -\langle \mathcal{L} f(\eta(t))\rangle$, where $\langle\cdot\rangle$ denotes the mean over trajectories and over the initial configuration distributed with the equilibrium measure. This is the adjoint of the operator $\mathbb{W}$ governing the master equation: $\partial_t |P\rangle = -\mathbb{W}|P\rangle$. We use this operator to express the typical value of f at time t as $\langle f(\eta(t))\rangle = \langle e^{-\mathcal{L}t} f\rangle$. Note that $\mathcal{L} = \mathcal{L}_{env} + \mathcal{L}_{jump}$, where $\mathcal{L}_{env}$ is the Liouvillian operator for the evolution of the environment (the facilitated model without the probe), and $\mathcal{L}_{jump}$ describes the evolution caused by the jumps of the probe particle. Using standard methods [Spo90] we compute the limit of the rescaled position of the probe particle in terms of the current and get the following result for $e_\alpha.2De_\alpha$ [Blo13b]

$$\sum_{y=\pm e_\beta} \langle \eta_0 \eta_{\pm e_\beta}\rangle \quad - \lim_{t\to\infty} \frac{1}{t} \left\langle \left(\int_0^t j_\alpha(\eta(s))ds\right)^2 \right\rangle,$$

where $\langle\cdot\rangle$ has the same meaning as above. In the r.h.s., the first term is just q^2 and the second one is $-\int_0^\infty \langle j_\alpha(\eta(0)) j_\alpha(\eta(s))\rangle$, which is $-\int_0^\infty \left\langle j_\alpha e^{-t\mathcal{L}} j_\alpha\right\rangle$ in the above formulation and can be rewritten as $\langle j_\alpha \mathcal{L}^{-1} j_\alpha\rangle = -\inf_f \{2\mu(j_\alpha f) - \langle f\mathcal{L} f\rangle\}$. Then some computations (see [Blo13b] for details) yield the following variational formula for $e_\alpha.2De_\alpha$:

$$\inf_f \left\{ \langle f \mathcal{L}_{env} f\rangle + \sum_{y=\pm e_\beta} \left\langle \eta_0 \eta_y \left[y_\alpha + f(\tau_y \eta) - f(\eta)\right]^2 \right\rangle \right\}, \tag{D.11}$$

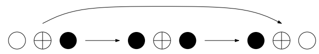

Figure D.1: How to reconstruct a swap using flips allowed by FA1f. Active (inactive) sites are in white (black) and the probe is marked by a cross.

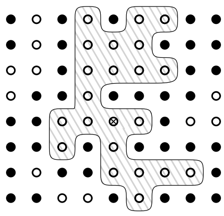

Figure D.2: Here the origin is crossed, the connected cluster of active sites is hatched and $f(\eta) = 4$.

where $\langle \cdot \rangle$ denotes the mean w.r.t. the equilibrium measure and $\tau_y \eta$ is η translated by the vector y.

We are now ready to sketch the ideas that allow us to prove (D.7). To establish $D \geq cq^{k+1}$, we show that $D \geq cq^{k+1}\overline{D}$, where $\overline{D}$ is the diffusion coefficient of a k-dependent auxiliary dynamics which we describe in the case $k = 1$ (FA1f) in dimension one. Take an initial configuration at equilibrium, with the probe at the origin, an active site at the origin and at least an active site among its neighbours. Then define the auxiliary dynamics as follows. The probe particle can jump to a neighbouring active site with rate 1, and the two neighbours of the probe particle can swap: if one of them is active and the other inactive, they exchange their activity state with rate 1. Note that with these rules the probe particle is always on an active site and has always an active neighbour. In particular, we can show that the diffusion coefficient for this auxiliary dynamics $\overline{D}$ is positive and does not depend on q. Then $D \geq cq^2\overline{D}$ can be established because it is possible to reconstruct any possible move in the auxiliary dynamics using a finite number of moves allowed by the original dynamics (see Fig. D.1). The term q^2 comes from the cost of imposing an active site at the origin and on one of its neighbours in the equilibrium configuration. The extension to other values of k and higher dimensions are detailed in [Blo13b]. In order to show $D \leqslant Cq^{k+1}$ we look for an observable f that captures the order of the diffusion when plugged in the variational formula (D.11). We treat the case $\alpha = 1$. In a configuration at equilibrium, consider the connected cluster of active sites containing the origin. This is the cluster that the probe could span if the environment remained frozen. We choose $f(\eta)$ to be the smallest non-negative coordinate z such that this cluster is contained in the half-space on the left of z, and we let $f(\eta) = 0$ if the origin is inactive. The calculations in [Blo13b] show that the test function f captures indeed the correct behaviour of the diffusion matrix.

In summary, we proved that for the East model in dimension one the self-diffusion coefficient of a probe particle scales as $D \sim \tau^{-1}$ in the low temperature regime ($q \to 0$), at variance with previous results claiming a fractional Stokes-Einstein relation of the form $D \sim \tau^{-\xi}$ with $\xi < 1$. Our results suggest a weaker violation of the form $D\tau \sim 1/q^2$. We also establish a variational formula for D which is valid for any kinetically constrained spin model in the ergodic regime. For FA1f model and more generally "k-defects" models, a detailed study of this variational formula allowed us to prove the exact order of the diffusion coefficient : $D \sim q^{k+1}$. This, together with the heuristics we provide for the scaling of the relaxation time, implies a fractional breakdown of the Stokes-Einstein relation only in dimension one.

In [JGC05] higher dimensional generalisations of the East model have been considered and a fractional Stokes-Einstein with $\xi \sim 0.7 - 0.8$ weakly dimensionally dependent has been observed. Since time is again larger than any polynomial in $1/q$ and the distance of the active sites is $1/q^{1/d}$, again a decoupling cannot occur as a consequence of the difference between persistence and exchange times and we expect $D \sim \tau^{-1}$. However to extend our mathematical proof to higher dimensions we need that persistence and relaxation times remain of the same order, a fact that is usually true for kinetically constrained models and should deserve further investigation. In the future, we also wish to investigate other cooperative models such as Fredrickson-Andersen two spin facilitated model (FA2f) [FA84] or the spiral model [TBF07]. In this case the event which triggers the moves of the probe could be more cooperative and it could modify the configuration up to a distance larger than a polynomial in $1/q$. Thus the fractional violation of Stokes Einstein observed in supercooled liquids could be reproduced by these kinetically constrained models.

We acknowledge very useful discussions with G.Biroli and T.Bodineau and thank V. Lecomte for comments on a draft version.

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