Hydrodynamic limits for (2+1)-dimensional interface growth models in the AKPZ class

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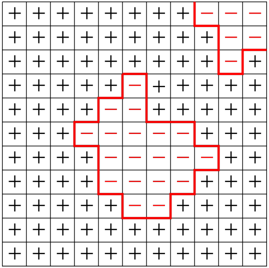
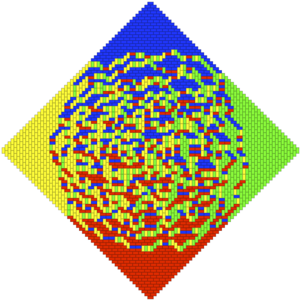
Outline

1. **Global picture**
2. Hydrodynamic limits for the PNG and its generalisations
   - $d = 1$: Polynuclear Growth model
   - $d \geq 2$: Isotropic case
   - $d = 2$: Anisotropic Gates-Westcott model
3. Hydrodynamic limit for the Borodin-Ferrari dynamic

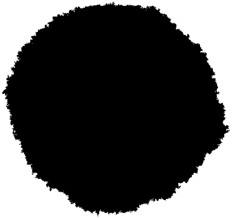
Interfaces in Statistical physics



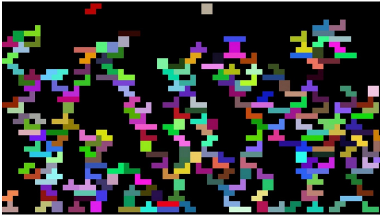
Spin dynamics (Ising model)



Dimer dynamics (Aztec diamond)



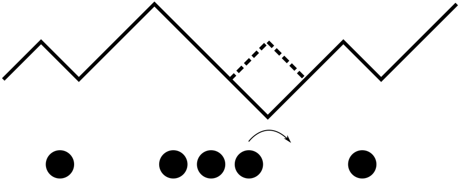
Eden model (First Passage Percolation)

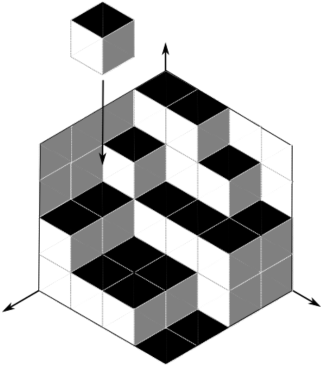


Random deposition (random tetris)

Discrete height functions

e.g Corner Growth Model
(particle systems, dimer model, LPP)





Height function $h : \mathbb{Z}^d \times \mathbb{R}_+ \to \mathbb{Z}$. Irreversible Markovian dynamics

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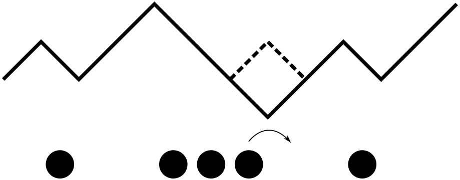
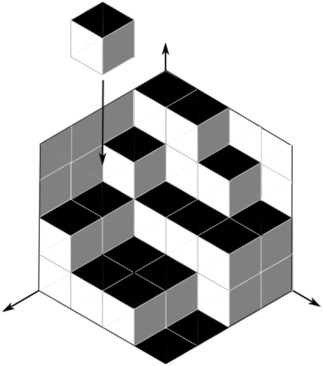




Height function $h : \mathbb{Z}^d \times \mathbb{R}_+ \to \mathbb{Z}$. Irreversible Markovian dynamics

Q

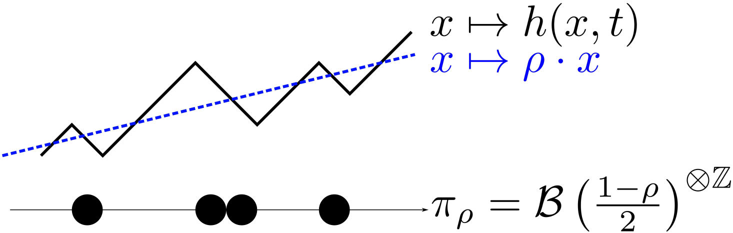
- Invariant measures of gradients?
- Law of large numbers / Hydrodynamic limits? $\hookrightarrow$ Non-linear PDEs
- Fluctuations? Universality? $\hookrightarrow$ Non-linear SPDEs

Invariant measures

When $t \to +\infty$, we expect that

$$(h(x+e_i,t)-h(x,t))_{x\in\mathbb{Z}^d,\, i\in\{1,\cdots d\}} \xrightarrow{\text{Law}} \pi_\rho$$

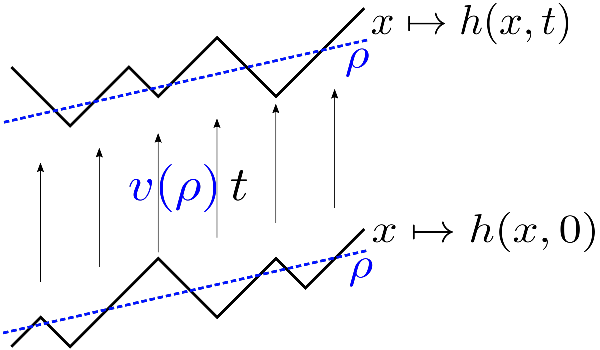
where π_ρ is an irreversible invariant measure and $\rho \in \mathbb{R}^d$ is a slope parameter: $\mathbb{E}_{\pi_\rho}[h(x)-h(0)] = \rho \cdot x$.



Speed of Growth

$$v(\rho) := \partial_t \mathbb{E}_{\pi_\rho}\left[h(x,t)\right]\Big|_{t=0}$$

$$\hookrightarrow \mathbb{E}_{\pi_\rho}\left[h(x,t) - h(0,0)\right] = \rho \cdot x + v(\rho)\, t$$



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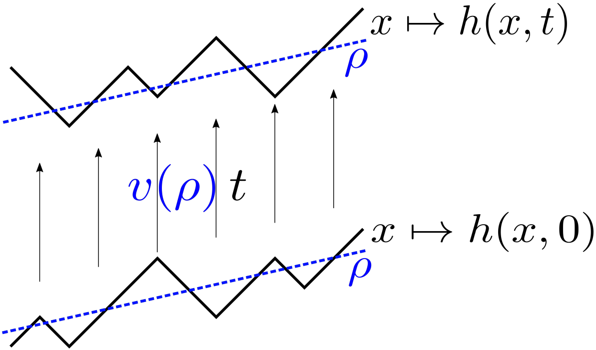


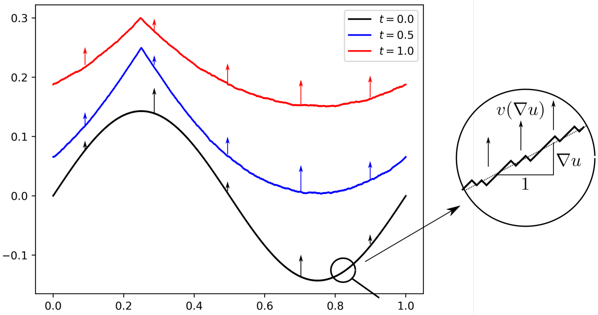
e.g $v(\rho) = 2\pi_\rho(\vee) = 2\pi_\rho(\bullet\circ) = 2\frac{1-\rho}{2}\left(1 - \frac{1-\rho}{2}\right) = \frac{1-\rho^2}{2}$

Hydrodynamic Limits

$$\text{If} \quad \frac{1}{L} h(\lfloor Lx \rfloor, 0) \underset{L\to\infty}{\longrightarrow} u_0(x) \quad \text{then} \quad \frac{1}{L} h(\lfloor Lx \rfloor, Lt) \underset{L\to\infty}{\longrightarrow} u(x,t)$$

with u the unique viscosity solution of the Hamilton-Jacobi non-linear PDE

$$\begin{cases} \partial_t u(x,t) = v(\nabla u(x,t)) \\ u(x,0) = u_0(x) \end{cases}$$



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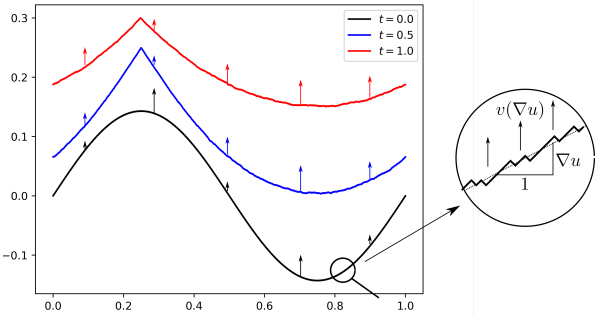

↪ Formation of shocks ↪ Variational formula when v is convex

Fluctuations and characteristic exponents

Universal characteristic exponents:

Roughness exponent α:

$$\mathrm{Var}_{\pi_\rho}(h(x) - h(y)) \underset{|x-y|\to\infty}{\sim} c_1 \, |x-y|^{2\alpha} + c_2$$

Growth exponent β:

$$\mathrm{Var}(h(x,t) - h(x,0)) \underset{t\to\infty}{\sim} c_1' \, t^{2\beta} + c_2'$$

Dynamical scaling exponent $z = \frac{\alpha}{\beta}$:

at time t, correlation length $= t^{1/z}$.

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Conjectured to only depend only on

- the dimension
- the symmetries of the model

Fluctuations and KPZ equation

Large-scale fluctuations along the characteristic lines of $\partial_t u = v(\nabla u)$ are expected to behave like the Kardar-Parisi-Zhang equation ('86):

$$\partial_t h = \nu \, \Delta h + \lambda \, \langle \nabla h, H \, \nabla h \rangle + \sqrt{D} \, \xi$$

with $H = \mathrm{D}^2_\rho(v)$ and ξ space-time white noise (regularised)

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Q: Behaviour of the solution on large scales?

Linear case $\lambda = 0$: Edwards-Wilkinson equation

- Stationary states π_ρ: massless Gaussian Free Field
- Characteristic exponent:

$$\alpha_{EW} = \frac{2-d}{2}, \ \beta_{EW} = \frac{2-d}{4}, \ z_{EW} = 2 \text{ (diffusive scaling).}$$

Rk: in dimension 2

$$\mathrm{Var}_{\pi_\rho}(h(x) - h(y)) \underset{|x-y| \to \infty}{\sim} c \log |x-y|, \ \mathrm{Var}(h(x,t)) \underset{t \to \infty}{\sim} c' \log t.$$

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Q: Relevance/Irrelevance of the non-linearity ($\lambda > 0$) on large scales?

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Case $\lambda > 0$:

- $d = 1$, the non-linearity is relevant:

$$(\alpha_{KPZ}, \beta_{KPZ}, z_{KPZ}) = \left(\frac{1}{2}, \frac{1}{3}, \frac{3}{2}\right) \neq \left(\frac{1}{2}, \frac{1}{4}, 2\right) = (\alpha_{EW}, \beta_{EW}, z_{EW}),$$

with Tracy-Widow universal limiting distribution Baik-Deift-Johansson '99, Johansson '00, convergence of a weakly asymetric limit of Corner Growth model to solution of the KPZ equation Bertini-Giacomin '97

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- $d \geq 3$, $H = Id$, $\lambda < \lambda_c(d)$, the non-linearity is irrelevant:

$$\varepsilon^{\alpha_{EW}} (h(x/\varepsilon, t/\varepsilon^2) - \mathbf{E}[h(x/\varepsilon, t/\varepsilon^2)]) \underset{\varepsilon \to 0}{\longrightarrow} \text{solution of EW equation}$$

  Magnen-Unterberger'17, Gu-Ryzhik-Zeituni'17, Comets-Cosco-Mukherjee'19

Wolf's conjecture

$$\partial_t h = \nu \Delta h + \lambda \langle \nabla h, H \nabla h \rangle + \sqrt{D}\, \xi$$

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- $d = 2$: **Wolf's conjecture** '91: (renormalisation group analysis)

$\det(H) > 0$ (Isotropic KPZ):

$$\alpha_{KPZ} \simeq 0.39,\ \beta_{KPZ} \simeq 0.24$$

relevance

$\det(H) \leq 0$ (Anisotropic KPZ):

$$\alpha_{KPZ} = 0,\ \beta_{KPZ} = 0$$

$$\mathrm{Var}(h(x) - h(y)) \underset{|x-y| \to \infty}{\sim} c \log |x - y|$$

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irrelevance ?

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irrelevance ?

Cannizzaro-Erhard-Toninelli '20: for $H = \mathrm{diag}(+1, -1)$, the correlation length is of order $t^{1/2} \, (\log t)^{\delta/2}$ with conjectural: $\delta = 1/2$.

$\hookrightarrow$ Relevance of the non-linearity !

Outline

1. Global picture

2. Hydrodynamic limits for the PNG and its generalisations
    - $d = 1$: Polynuclear Growth model
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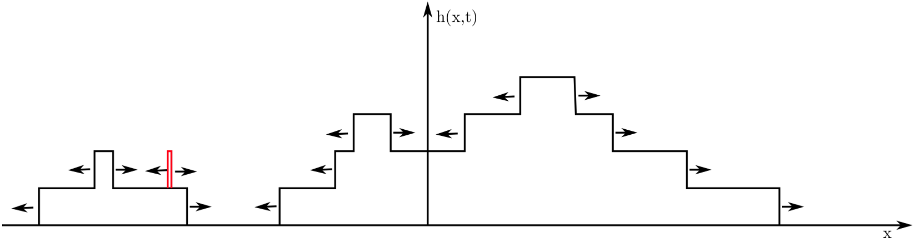
Outline

PolyNuclear Growth Model and dynamic

Layer by layer crystal growth model



$h : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{Z}$

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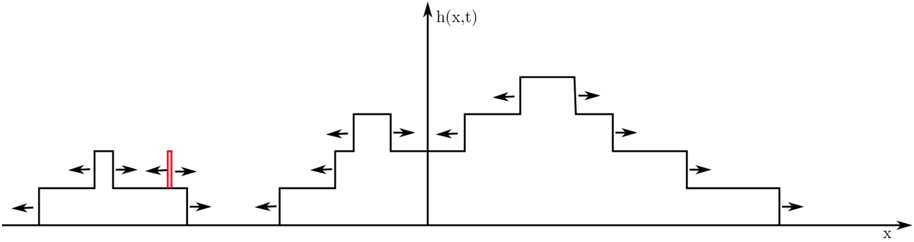

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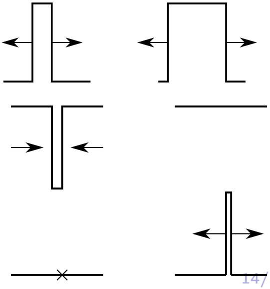
- **Lateral expansion** at speed 1
- **Annihilation**
- **Nucleations** given by Poisson Point Process



Envelope property

- **Monotonicity** $h^1(0) \leq h^2(0) \Longrightarrow h^1(t) \leq h^2(t)$

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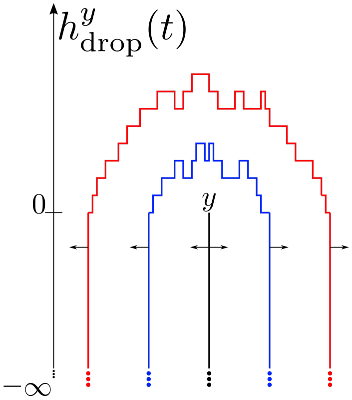
$\hookrightarrow$ Variational formula:

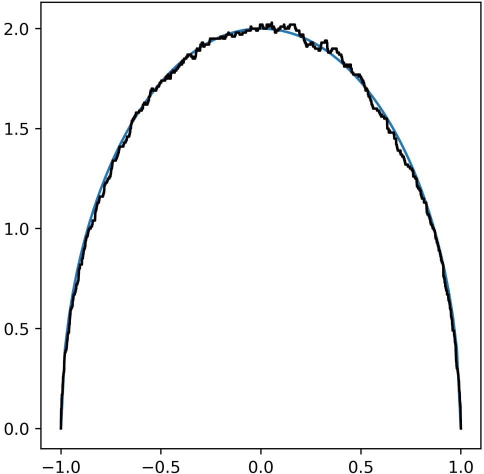
$$h(x,t) = \sup_{y \in \mathbb{R}}\{h(y,0) + h^y_{\text{drop}}(x-y,t)\}$$



Super-additivity

Super-additive ergodic argument (Seppäläinen, Rezakhanlou)

$$\frac{1}{L} h_{\mathrm{drop}}(Lx, Lt) \underset{L\to\infty}{\longrightarrow} t\, g(x/t) \qquad g \text{ concave}$$



Hydrodynamic limit

- If $\frac{1}{L} h(Lx, 0) \underset{L\to\infty}{\longrightarrow} u_0(x)$ for all x, then

$$\frac{1}{L} h(Lx, Lt) \underset{L\to\infty}{\longrightarrow} u(x,t) := \sup_{y\in\mathbb{R}} \left\{ u_0(y) + t\, g\left(\frac{x-y}{t}\right)\right\}$$

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Moreover, $u(x, t)$ is the Hopf-Lax formula for the viscosity solution of

$$\begin{cases} \partial_t u(x,t) = -g^*(\nabla u(x,t)) \\ u(x,0) = u_0(x) \end{cases} \qquad g^* \text{ concave conjugate of } g$$

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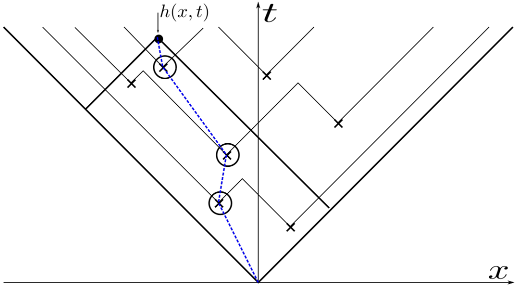
- Compatibility with affine profiles and stationary growth

$$\Longrightarrow -g^*(\rho) = v(\rho) = \sqrt{4+\rho^2}$$
$$\Longrightarrow g(x) = (g^*)^*(x) = 2\sqrt{1-x^2}$$

Comments about the PNG:

- Link with the longest-increasing subsequence of a random permutation (Ulam's problem '61), Hammersley process '72 and Random polymers



- Determinantal structure with Bessel Kernel
- Fluctuations scales like $t^{1/3}$ and converge to a Tracy Widom distribution (different geometries: Droplet, Flat, Equilibrium) (Baik-Deift-Johansson '99, Baik-Rains '00, 01')
- Convergence of multi-point fluctuations ($x \mapsto t^{-1/3}h(t, xt^{2/3})$) to Airy processes (Prähofer-Spohn '02, Ferrari '04)

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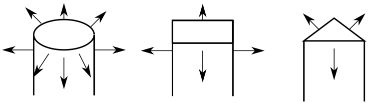
B-shaped PNG model

Terraces of shape B, unit ball of a norm in $\mathbb{R}^d$ (Prähofer '03)



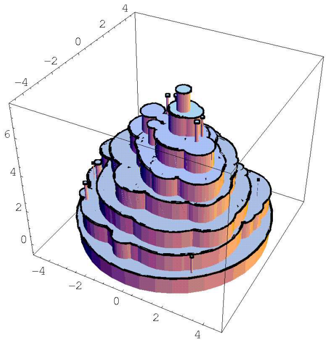


Image of Michael Prähofer

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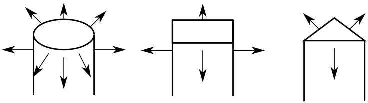

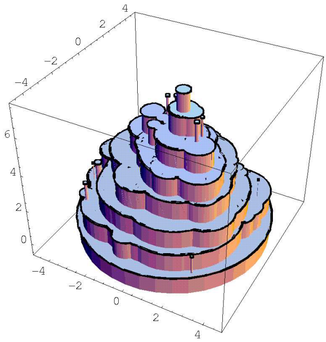

- Envelope property + Super-additivity
  $\hookrightarrow$ existence of Hydrodynamic Limits
  Hopf-Lax formula and convex speed
  $\hookrightarrow$ Isotropic KPZ
- $v(\rho)$ not explicit but for B euclidian ball or simplex (Seppäläinen '07), $v(\rho)$ explicit up to a multiplicative constant

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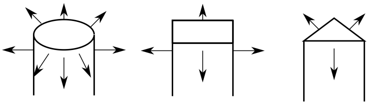

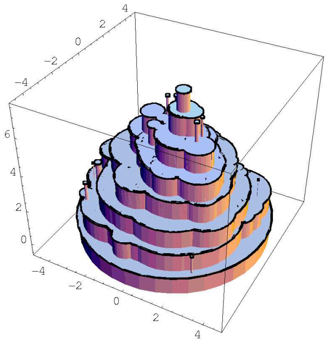

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- Other Isotropic examples: Ballistic deposition, Corner Growth Model and generalisations (Seppäläinen, Rezakhanlou)

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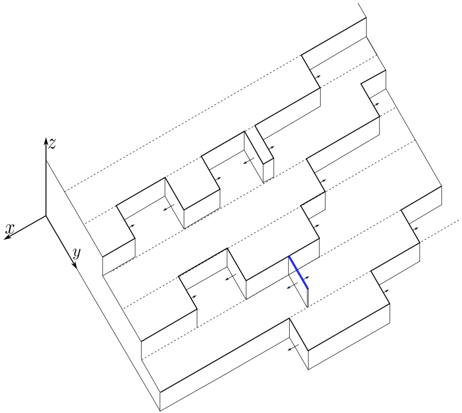
The Gates-Westcott model

Layer by layer Crystal Growth (Gates-Westcott '95)



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- Infinite collection of non-intersecting level lines that follow the PNG dynamic with nucleation deleted if two lines intersect
- $\hookrightarrow$ Non-trivial interactions

Stationary states and previous results

Prähofer-Spohn '97 found invariant measures π_ρ with slope $\rho = (\rho_1, \rho_2) \in \mathbb{R} \times (-1, 0)$ (case $\rho_1 = 0$ treated by Gates-Westcott '95).

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Using fermionic Fock space tools:

- computed the speed

$$v(\rho) = \frac{1}{\pi}\sqrt{\pi^2\rho_1^2 + 4\sin^2(\pi\rho_2)}$$

-

$$\det(\mathrm{D}^2_\rho(v)) < 0 \quad \text{for every } \rho$$

  $\hookrightarrow$ Anisotropic class

-

$$\mathrm{Var}_{\pi_\rho}(h(x) - h(y)) \sim c \log|x - y|$$

  $\hookrightarrow$ typical from Gaussian Free Field and Edward-Wilkinson universality class

Hydrodynamic limit and upper bound on fluctuations

Theorem 1 (L. '19)

If for all $R>0$, $\sup_{\|(x,y)\|\leq R}\left|\frac{1}{L}h(Lx,\lfloor Ly\rfloor,0)-u_0(x,y)\right|\underset{L\to\infty}{\longrightarrow}0$ *with* $u_0\in\mathcal{C}(\mathbb{R}^2)$, *then, for all* $T,R>0$,

$$\sup_{\|(x,y)\|\leq R,t\in[0,T]}\left|\frac{1}{L}h(Lx,\lfloor Ly\rfloor,Lt)-u(x,y,t)\right|\xrightarrow[L\to\infty]{\text{a.s}}0$$

with u *unique* *viscosity solution* *of* $\begin{cases}\partial_t u=v(\nabla u)\\ u(\cdot,\cdot,0)=u_0\ ,\end{cases}$ *and*

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Theorem 2 (L. '19)

$$\forall \rho \in \mathbb{R} \times (-1, 0), \quad \mathrm{Var}_{\pi_\rho}(h(x,y,t) - h(x,y,0)) = \underset{t\to\infty}{O}(\log t)$$

Outline

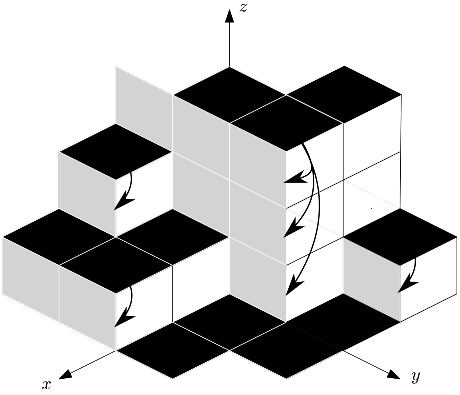
1. Global picture

2. Hydrodynamic limits for the PNG and its generalisations
   - $d = 1$: Polynuclear Growth model
   - $d \geq 2$: Isotropic case
   - $d = 2$: Anisotropic Gates-Westcott model

3. Hydrodynamic limit for the Borodin-Ferrari dynamic

The Borodin-Ferrari dynamic

Long-jump version of the Corner Growth model



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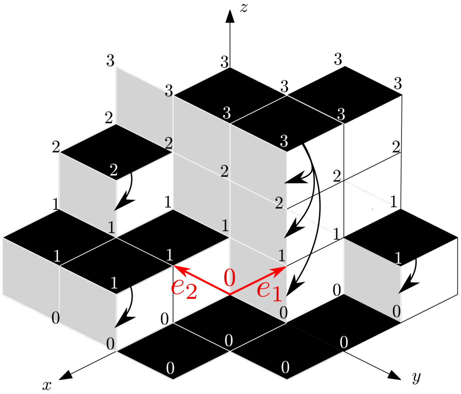

Height function $h : \mathbb{Z}^2 \times \mathbb{R}_+ \to \mathbb{Z}$

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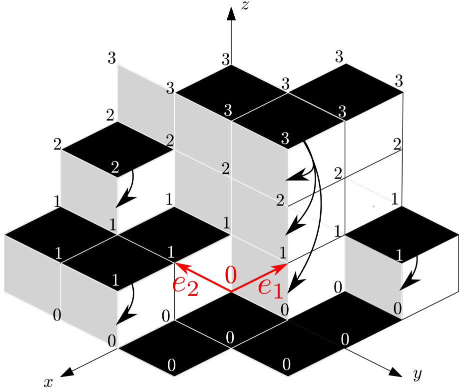

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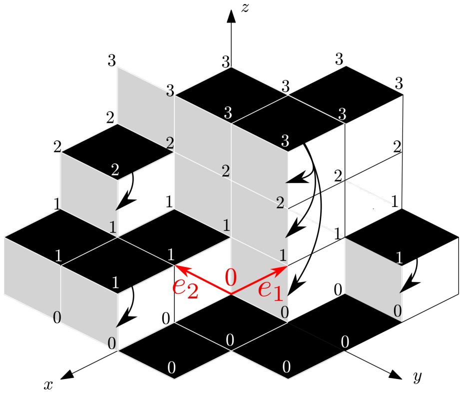

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  (partial) determinental correlations away from characteristic lines...

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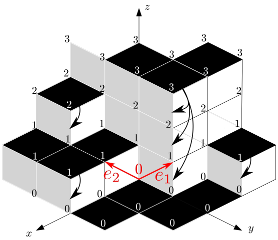

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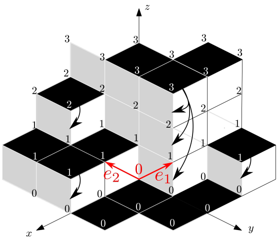

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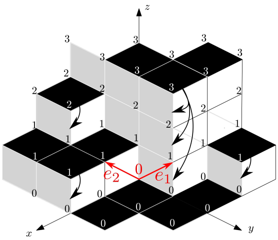

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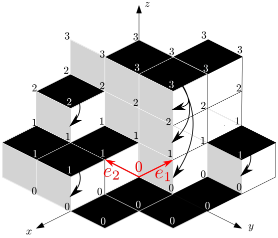

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- Hydrodynamic limit for smooth initial profile up to the time of shocks or for convex initial profile Legras-Toninelli '17.

Hydrodynamic limit

Theorem 3 (L.-Toninelli '20)

Technical condition: initial microscopic slopes $\rho_1 + \rho_2$ *stay uniformly away from* 1.

If for all $R > 0$, $\sup_{\|x\| \le R} \left| \frac{1}{L} h(\lfloor Lx \rfloor, 0) - u_0(x) \right| \underset{L\to\infty}{\longrightarrow} 0$, *with* $u_0 \in \mathcal{C}(\mathbb{R}^2)$,

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- Semi-group approach by Rezakhanlou '01
  $\hookrightarrow$ potentially robust but couldn't be applied for the models considered
- Domino shuffling dynamic: first full hydrodynamic limit X.Zhang '18

1) Properties of the semi-group associated to $\partial_t u = v(\nabla u)$.

$$S(s,t) : \begin{cases} \Gamma \to \Gamma & (\Gamma = \mathcal{C}(\mathbb{R}^2) \text{ with slope constraints}) \\ u_0 \mapsto u(\cdot, t-s) & u = (\text{viscosity solution started from } u_0) \end{cases}$$

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- *Translation Invariance*: $S(s,t)(f+c) = S(s,t)(f) + c,\ c \in \mathbb{R}$
- *Monotonicity*: $f \leq g \Longrightarrow S(s,t)(f) \leq S(s,t)(g)$
- *Finite speed of propagation*:
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Conversely, these are sufficient conditions

Idea of the proof of the hydrodynamic limits

2) Rescaled microscopic semi-group.

For a fix realisation of Poison Point Process ω

$$S_L(s,t,\omega) : \begin{cases} \Gamma \to \Gamma \\ f \mapsto \dfrac{1}{L} h\left(\lfloor L\cdot \rfloor, L(t-s); \varphi_L^f, \theta_{Ls}\omega\right) \end{cases}$$

with θ_s time-translation, φ_L^f discrete height function approaching f

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4) A.s identification of the limit. Any limit point $S_\infty(\cdot,\cdot,\omega)$ satisfies sufficient conditions (uses microscopic properties and stationary measures).

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Compared to Rezakhanlou '01, Zhang '18, our results

- hold in the strong almost sure sense of convergence
- presents additional non-trivial difficulties from unbounded spatial gradients (GW model) and divergence of $v(\rho)$ when $\rho_1 + \rho_2 \simeq 1$ with lack of a priori bound on microscopic slopes (BF dynamic).

Thank You for your attention!