Percolation phase transition in weight-dependent random connection models

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joint work with

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Percolation

Suppose $\mathscr{G}$ is a random graph with infinite vertex set and finite vertex degrees. *Percolation* is the event that there is an infinite connected component in $\mathscr{G}$. Our interest is in families of graphs $\mathscr{G}(\beta)$ where edge densities increase in β. Does percolation become impossible when β is decreased and possible if it is increased?

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The classical example is the lattice $\mathbb{Z}^d$ with $d \geq 2$, in which edges are removed independently with probability $p = 1 - \beta$. Broadbent and Hammersley (1957) introduced this model and showed that there is a percolation phase transition, i.e. there exists a critical edge density

$$0 < \beta_c < 1,$$

such that

- $\mathscr{G}(\beta)$ does *not percolate* almost surely if $\beta < \beta_c$ (the subcritical phase),
- $\mathscr{G}(\beta)$ *percolates* almost surely if $\beta > \beta_c$ (the supercritical phase).

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Problem: When does this hold for graph families $\mathscr{G}(\beta)$ with long-range dependencies and heavy-tailed degree distribution?

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$$B(x, \beta) \cap B(y, \beta) \neq \emptyset.$$

  Gilbert (1961) showed that there is a critical $0 < \beta_c < \infty$ such that the graph percolates if $\beta > \beta_c$ but does not percolate for $\beta < \beta_c$.

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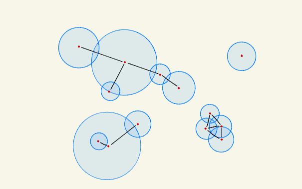
- Poisson-Boolean model: Take $(R_x : x \in \mathscr{X})$ positive iid random variables with $\mathbb{E}[R_x^d] < \infty$. Connect $x, y \in \mathscr{X}$ if

$$B(x, \beta R_x) \cap B(y, \beta R_y) \neq \emptyset.$$

  Then Gouéré (2008) showed that there is a percolation phase transition. This model includes graphs with heavy-tailed degree distribution.



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- Long-range percolation: Connect $x, y \in \mathscr{X}$ independently with probability

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- Summary: Neither the powerful vertices of graphs with heavy-tailed degree distribution, nor the long edges in long-range percolation models can remove the subcritical phase and ensure $\beta_c = 0$. Is this possible at all?

Percolation

- Scale-free percolation: In the scale-free percolation model of Deijfen, van der Hofstad, Hooghiemstra (2018) vertices have independent weights $W_x, x \in \mathscr{X}$. We connect two points $x, y \in \mathscr{X}$ independently with probability

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This is analogous to the behaviour of classical (non-spatial) scale-free networks. The behaviour depends only on the variance of the degree distribution and *not* on the geometry of the underlying space.

Our model: the weight-dependent random connection model

The vertex set of $\mathscr{G}(\beta)$ is a Poisson point process of unit intensity on

$$\mathbb{R}^d \times (0,1] \text{ for } d \geq 2.$$

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- a non-decreasing, symmetric *kernel* $g\colon (0,1] \times (0,1] \to \mathbb{R}_+$.

We assume (without loss of generality) that

$$\int_{\mathbb{R}^d} \rho(|x|^d)\, dx = 1. \tag{1}$$

Then, the degree distribution only depends on the kernel g and not on ρ.

Interesting kernels

Recall

$$\varphi(\mathbf{x}, \mathbf{y}) = \rho(g(t, s)|x - y|^d).$$

Our kernels g are defined in terms of a parameter $\gamma \in (0, 1)$ and have heavy tailed degree distributions with exponent

$$\tau = 1 + \tfrac{1}{\gamma}.$$

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To study the influence of long-range effects on the percolation problem, we focus primarily on regulary varying profile functions with index $\delta > 1$, that is

$$\lim_{r \uparrow \infty} \frac{\rho(cr)}{\rho(r)} = c^{-\delta} \quad \text{for all } c \geq 1.$$

Our result

Theorem 1

For the weight-dependent random connection model with preferential attachment kernel, sum kernel or min kernel and parameters $\delta > 1$ and $0 < \gamma < 1$, we have

(a) *if $\gamma < \frac{\delta}{\delta+1}$, then $\beta_c > 0$.*

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Theorem 2

For the product kernel we have

- (c) *if $\gamma \leq \frac{1}{2}$, then $\beta_c > 0$.*
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For the max kernel we always have $\beta_c = 0$ (Yukich (2006)).

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- Given the graph $\mathscr{G}_{t-}$, a vertex born at time t and placed at x is connected by an edge to each existing vertex at y born at time s independently with conditional probability

$$\rho\left(\frac{t\, d(x,y)^d}{\beta\left(\frac{t}{s}\right)^{\gamma}}\right).$$

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$(\mathscr{G}_t)_{t\geq 0}$ has a *giant component* if the largest connected component of $\mathscr{G}_t$ is of asymptotically linear size, it is *robust* if the percolated sequence $(\mathscr{G}_t^p)_{t\geq 0}$ has a giant component for every retention parameter $p > 0$.

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Theorem 3

The network $(\mathscr{G}_t)_{t\geq 0}$ is robust if $\gamma > \frac{\delta}{\delta+1}$, but non-robust if $\gamma < \frac{\delta}{\delta+1}$.

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- As $s \downarrow 0$ the probability that for a fixed vertex (x_0, s_0) with $s_0 < s$ there exists an infinite sequence of vertices

$$(x_0, s_0), (x_1, s_1), (x_2, s_2), \ldots$$

such that

  - $s_{k+1} < s_k^{\alpha_1}$ and $|x_{k+1} - x_k|^d < \frac{\beta}{2} s_k^{-\alpha_2}$, and
  - (x_k, s_k) is connected to (x_{k+1}, s_{k+1}) by *a path of length two*;

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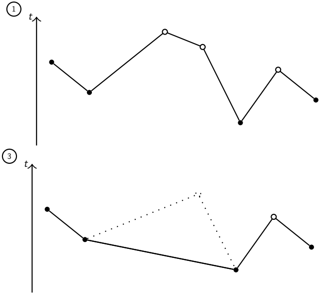
- *If* $\frac{1}{2} \leq \gamma < \frac{\delta}{\delta+1}$ we look at a path of length n and identify its *skeleton*.



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$$\mathbb{P}_{\mathbf{x},\mathbf{y}}\{\mathbf{x} \overset{k}{\leftrightarrow} \mathbf{y}\} \leq (C\beta)^{k-1}\mathbb{P}_{\mathbf{x},\mathbf{y}}\{\mathbf{x} \sim \mathbf{y}\}.$$

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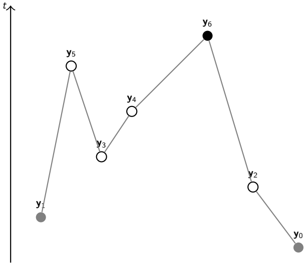

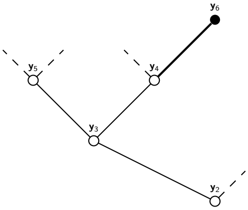

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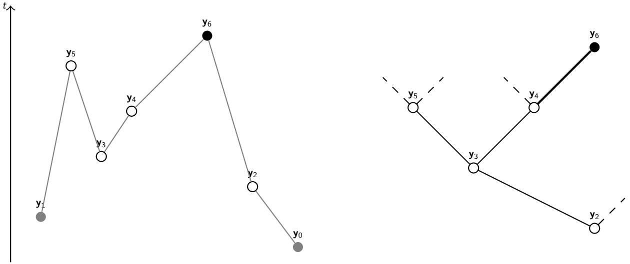

- Add local maxima successively according to the tree using

$$\int_{t_1}^{1} dt_2 \int_{\mathbb{R}^d} dy_2\ \varphi((y_0,t_0),(y_2,t_2))\,\varphi((y_2,t_2),(y_1,t_1)) \leq (C\beta)\,\mathbb{P}_{\mathbf{y}_0,\mathbf{y}_1}\{\mathbf{y}_0 \sim \mathbf{y}_1\}.$$

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- By stopping a path when it goes below the threshold and using our tool, we show that if β is below some positive constant depending only on ρ, γ and d, almost surely every infinite path is regular.

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- By stopping a path when it goes below the threshold and using our tool, we show that if β is below some positive constant depending only on ρ, γ and d, almost surely every infinite path is regular.
- The probability that there exists a regular path of length n can be bounded by $(C\beta)^n$ combining our tool with the first moment method. Hence for $\beta < \frac{1}{C}$ there are no infinite regular paths.

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- Study the weight-dependent random connection graphs with moving points, resp. time varying networks.
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Thank you very much for your attention!