Critical Exponents for a 3D percolation model

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jt. with $\begin{cases} \text{A. Drewitz} \\ \text{A. Prévost.} \end{cases}$

Goal : understand (some) percolation model in "dimension"

- high enough s.t. no "planar tricks"

- low enough s.t. no "mean-field"

Setup

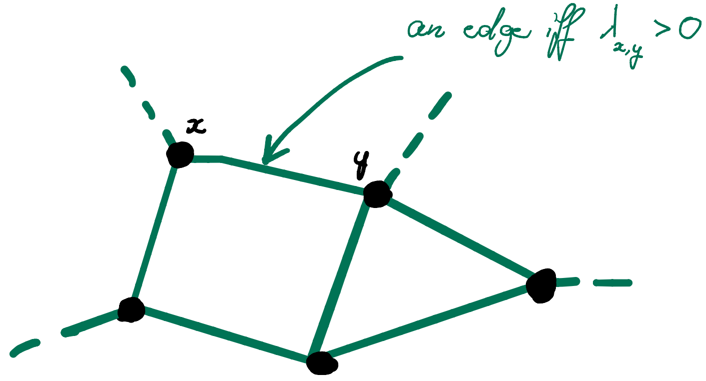
vertex set   weights

$$\mathcal{G} = (G, \lambda) : \text{weighted graph}$$

G : infinite (countable)

$\lambda : G \times G \longrightarrow [0,\infty)$ $\lambda_{x,y} = \lambda_{y,x} \ (\geq 0)$

$\lambda_{x,x} = 0$

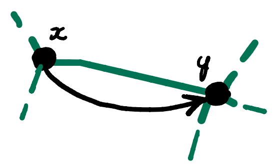


$\mathcal{G}$ conn., loc. finite

Example : $G = \mathbb{Z}^d$, $d \geq 3$, $\lambda = 1\{|x-y|=1\}$, ...

X : RW on $\mathcal{G}$

$$\mathcal{L}f(x) = \sum_y \lambda_{xy} \left(f(y) - f(x) \right)$$



$P_{x,y} = \lambda_{x,y} / \lambda_x$, $x,y \in G$

$\lambda_x = \sum_y \lambda_{x,y}$

law P_x, $x \in G$

Assume $\mathcal{G}$ transient.

$\varphi = (\varphi_x)_{x \in G}$ GFF on $\mathcal{G}$:

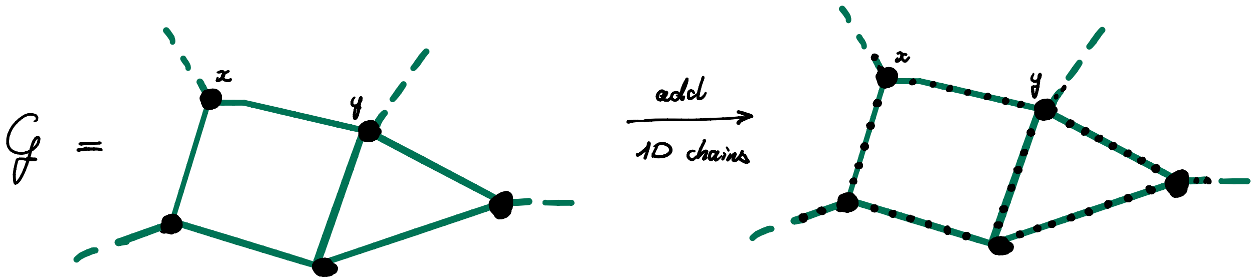
$$\mathbb{E}[\varphi_x] = 0, \quad \mathbb{E}[\varphi_x \varphi_y] = E_x\left[\int_0^\infty 1\{X_t = y\}\, dt \right], \quad x,y \in G$$

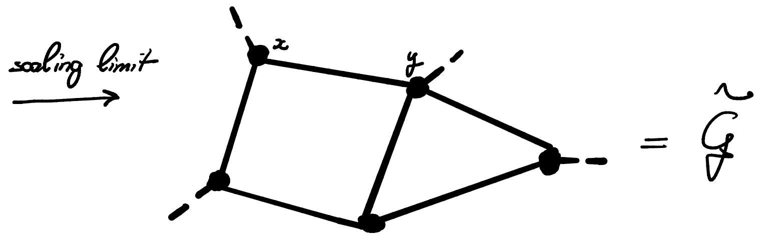
law $\mathbb{P}$

Metric graph

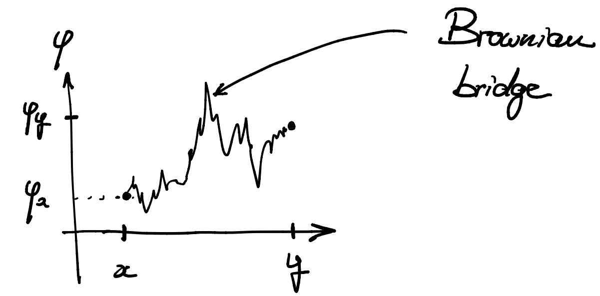
$\mathcal{G} = (G, \lambda)$: transient weighted graph $\rightsquigarrow$ $\tilde{\mathcal{G}}, \tilde{\varphi}$

$\varphi = (\varphi_x)_{x \in G}$: GFF on $\mathcal{G}$





$\tilde{\varphi} = (\tilde{\varphi}_x)_{x \in \tilde{\mathcal{G}}}$ s.t. $\tilde{\varphi}|_G = \varphi$



Why? $\tilde{\varphi}$ is continuous ($\rightsquigarrow$ 'Integrability')

Percolation

0 : a point in $\tilde{\mathcal{G}}$

$\tilde{K}^a$ = the connected comp. of 0 in $\{x \in \tilde{\mathcal{G}} : \tilde{\varphi}_x \geq a\}$, $a \in \mathbb{R}$

$\tilde{\theta}_0(a) \overset{\text{def.}}{=} \mathbb{P}[\tilde{K}^a \text{ is bounded}]$ ($\nearrow$ in a)   percolation function

$|\tilde{K}^a \cap G| < \infty$

$\tilde{a}_* \overset{\text{def.}}{=} \sup\{a \in \mathbb{R} : \tilde{\theta}_0(a) < 1 \text{ for some pt. } 0 \in \tilde{\mathcal{G}}\}$ critical parameter.

SUPERCRITICAL   SUBCRITICAL

$\tilde{\theta}_0(a) < 1$ $\tilde{a}_*$ $\tilde{\theta}_0(a) = 1$

Known: • $\boxed{\tilde{a}_* \geq 0}$ (Bricmont – Lebowitz – Maes '87 , interlacements !)

$\tilde{K}^a \supset \mathcal{I}^{a^2/2}$, $a < 0$

• For many $\mathcal{G}$'s (see below; e.g. transitive)

$\boxed{(\text{SIGN}) \quad \tilde{\theta}_0(a=0) = 1 , \quad 0 \in \tilde{\mathcal{G}}}$ ($\Longrightarrow \tilde{a}_* = 0$) $\rightsquigarrow$ loop soup clusters in $\mathcal{L}_{\frac{1}{2}}$

Q : behavior of $\tilde{\theta}_0(a)$ as $a \to 0^-$?

Differential formulas

K^a = conn. comp. of 0 in $\{\tilde\varphi \geq a\}$

$\mathbb{E}_U[\cdot] = \mathbb{E}[(\cdot)\, 1\{K^a \subset U\}]$, $U \subset \tilde{\mathcal{G}}$ conn., compact (e.g. $U = B(0,N)$, $N \gg 1$)

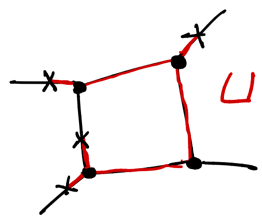
Observable $F(K^a)$; $F(\emptyset) = 0$

Lemma If $F(K^a)\,1\{K^a \subset U\} \in L^1(\mathbb{P})$, then

$$\boxed{-\frac{d}{da}\mathbb{E}_U[F(K^a)] = \mathbb{E}_U[M_U\, F(K^a)]}, \quad M_U \overset{\text{def.}}{=} \langle e_U, \tilde\varphi\rangle$$

$e_U(x) = P_x^X[\text{never return to } U]\, 1\{x \in U\}$

equilibrium measure of U



X: add vertex $\rightsquigarrow P_x^X$

$\mathrm{cap}(U) = \int e_U$

Pf: $\tilde\varphi = \tilde\psi + h$. $h|_U = a$. Cameron – Martin. $\square$

Rmk:

- true on $\tilde{\mathcal{G}}$ or $\mathcal{G}$
- kill immediately $\rightsquigarrow$ Margulis – Russo.

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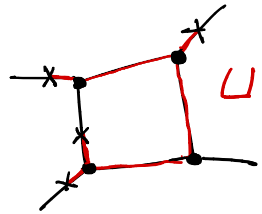

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$\widetilde{\text{Lemma}}$ (on $\tilde{\mathcal{G}}$). If —″—, then

$$\boxed{-\frac{d}{da}\, \mathbb{E}_U[F(K^a)] = a\, \mathbb{E}_U[\mathrm{cap}(K^a)\, F(K^a)]}\ .$$

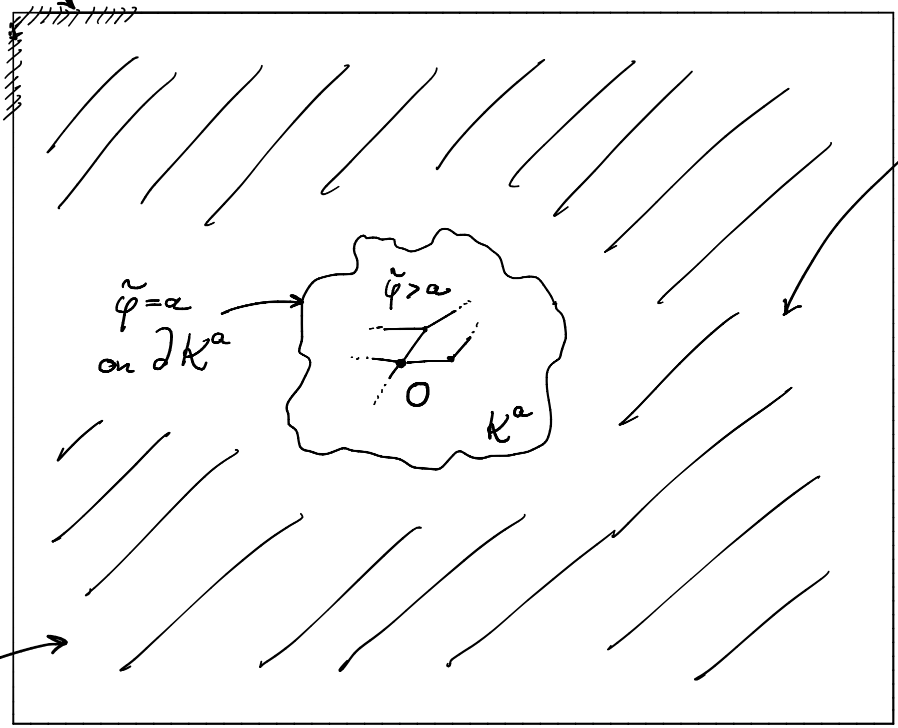
Pf of $\widetilde{\text{Lemma}}$:

$$\mathbb{E}_U\left[M_U \, F(K^a) \right] \overset{\text{Claim}}{=} a\, \mathbb{E}_U\left[\operatorname{cap}(K^a)\, F(K^a) \right]$$

'Explore' $K^a = K^a \cap U$ (on $\{\underbrace{\emptyset \neq}_{\text{add}} K^a \subset U\}$). under $\mathbb{P}_U$

add $(F(\emptyset) = 0)$

$$\sum_{x \in \partial U} e_U(x)\, \varphi_x = M_U$$



strong Markov:

$$\mathbb{E}\left[M_U \,\middle|\, \text{`}\tilde\varphi|_{K^a}\text{'} \right]$$

$$= \sum_{x \in \partial U} e_U(x)\, h_x^{K^a}$$

harm. extension of $\tilde\varphi|_{K^a}$

$$= a \sum_{x \in \partial U} e_U(x)\, P_x[\text{hit } K^a]$$

$$\underset{\text{balayage}}{=} a \sum_{x \in \partial K^a} e_{K^a}(x) = a \operatorname{cap}(K^a)$$

$\square$

Cluster capacity observable

Lemma (on $\tilde{\mathcal{G}}$). $-\frac{d}{da}\mathbb{E}_u[F(\mathcal{K}^a)] = a\,\mathbb{E}_u[\mathrm{cap}(\mathcal{K}^a)\,F(\mathcal{K}^a)]$.

$\boxed{\mathrm{cap}(\mathcal{K}^a) \text{ has appeared!}}$

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$$\boxed{\mathrm{cap}(\mathcal{K}^a) \text{ has appeared!}}$$

$\longrightarrow$ take $F(\mathcal{K}^a) = 1\{t-\varepsilon < \mathrm{cap}(\mathcal{K}^a) \le t\}$, $\varepsilon > 0$, $t > \frac{1}{g}$ $\quad g = g(0,0)$

$\overset{\text{Lemma}}{\underset{\mathcal{U}\nearrow\tilde{\mathcal{G}}}{\Longrightarrow}}$ $-\frac{d}{da}\log \mu_a((t-\varepsilon, t]) \in (a(t-\varepsilon), at]$ $\quad$ μ_a: law of $\mathrm{cap}(\mathcal{K}^a)1\{\emptyset \neq \mathcal{K}^a \text{ bdd}\}$

$\underset{\int \text{ and } \varepsilon\downarrow 0}{\Longrightarrow}$ $\frac{d\mu_a}{d\mu_b}(t) = e^{-\frac{1}{2}(a^2-b^2)t}$ $\quad (*)$

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Thm 1 $a \in \mathbb{R}$, $u \geq 0$. $\Phi(\cdot) = \mathbb{P}[\varphi_0 < \cdot]$

$$\mathbb{E}[e^{-u\operatorname{cap}(\mathcal{K}^a)}1\{\mathcal{K}^a \text{ bounded}\}] = \Phi(a) + \mathbb{P}[\emptyset \neq \mathcal{K}^{\sqrt{2u+a^2}} \text{ bdd}]$$

$$\overset{\text{if (SIGN)}}{=} \Phi(a) + 1 - \Phi(\sqrt{2u+a^2})$$

$$= \Phi(a) + \int e^{-ut}d\mu_a(t) \overset{(*)}{=} \Phi(a) + \int d\mu_{\sqrt{a^2+2u}}(t)$$

Consequences

Thm 1bis If (SIGN) then $\mathbb{E}\left[e^{-u\,\mathrm{cap}(K^a)} \right] = \Phi(a) + \Phi\left(\sqrt{a^2+2u}\right)$, $a, u \geq 0$

$$\left(\text{with density: } \frac{1}{2\pi} \frac{1}{t\sqrt{g(t-g^{-1})}} e^{-a^2 t} \, 1\{t > g^{-1}\} \right)$$

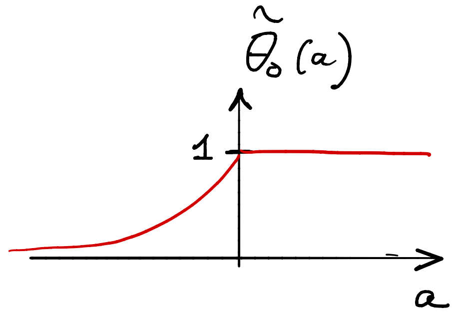
In particular $\mathbb{P}\left[\mathrm{cap}(K^0) \geq r \right] \sim r^{-\frac{1}{2}}$, $r \to \infty$

$$\mathbb{E}\left[\mathrm{cap}(K^a) \right] = \frac{\Phi'(a)}{|a|} \quad a \to 0^+$$

Cor 2 If (SIGN) then

$$\left(\mathbb{P}[K^a \text{ bdd}] = \right) \qquad \tilde{\theta}_0(a) = 2\,\Phi(a \wedge 0), \quad a \in \mathbb{R}$$

In particular $\displaystyle \lim_{a \to 0^-} \frac{1 - \tilde{\theta}_0(a)}{|a|} = \sqrt{\frac{2}{\pi g}}$



cf. Werner – Lupu

Pf: $u \downarrow 0$ in Thm 1 $\square$

Rmk: $1 - \tilde{\theta}_0(a) \sim |a|^{\beta}$ as $a \to 0^+$ with $\boxed{\beta = 1}$. Mean-field ???

More conditions on G

Parameters $\alpha > 2$, $0 < \nu \leq \alpha - 2$

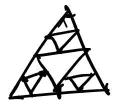
Assume: $\exists\, d$ = distance on G $\left(\overset{\text{eg.}}{=} d_{gr} \right)$

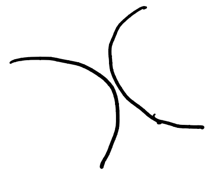
(V_α) : $\lambda\big(B(x,R)\big) \sim R^{\alpha}$, $x \in G$, $L \geq 1$

(G_ν) : $g(x,y) \sim d(x,y)^{-\nu}$

$\left(+ \text{ ellipticity } + \dots 1\{ d = d_{gr} \} \right)$

Ex: $\mathbb{Z}^d$: $\alpha = d$, $\nu = d-2$





bottlenecks at every scale.

à la Bass, Barlow, Kumagai, ...

Rmk: • read $\beta \overset{\text{def}}{=} \alpha - \nu$ as $E_x\left[T_{\text{exit } B(x,R)} \right] \sim R^\beta$

• $d \neq d_{gr}$ possible

• natural: (for $d = d_{gr}$)

$\iff$ $p_t(x,y) \sim t^{-\frac{\alpha}{\beta}} e^{-c\left(\frac{d(x,y)^\beta}{t} \right)^{\frac{1}{\beta - 1}}}$

$(\beta = \alpha - \nu)$

(Grigor'yan – Telcs

Cluster radius $\quad \psi(a,r) \overset{\text{def.}}{=} \mathbb{P}[\, r \leq \mathrm{rad}(K^a) < \infty \,] \quad a \in \mathbb{R},\ r \geq 1.$

Thm 3 $\quad (V_\alpha),\ (G_\nu),\ + \varepsilon$

Let $\quad \xi(a) \overset{\text{def.}}{=} |a|^{-\frac{2}{\nu}}. \qquad \exists\ c_i = c_i(\nu,\alpha) > 0,\ i=1,\dots,4$ s.t. $\forall a \in \mathbb{R},\ r > 1$

i) If $\nu < 1$: $\quad c_1 r^{-\frac{\nu}{2}} \exp\left\{ -c_2 \left(\frac{r}{\xi(a)}\right)^{\nu} \right\} \leq \psi(a,r) \leq c_3 r^{-\frac{\nu}{2}} \exp\left\{ -c_4 \left(\frac{r}{\xi(a)}\right)^{\nu} \right\}$

Pf: Use $\quad \{ c r^{\nu} \leq \mathrm{cap}(K^a) < \infty \} \subset \{ r \leq \mathrm{rad}(K^a) < \infty \} \subset \{ c' f_\nu(r) \leq \mathrm{cap}(K^a) < \infty \},\ \nu > 0$

$+$ Thm 1^{bis} (SIGN holds) $\square$

$$f_\nu(r) \overset{\text{def}}{=} \begin{cases} r^{\nu}, & \nu < 1 \\ r/\log r, & \nu = 1 \\ r, & \nu > 1 \end{cases}$$

Cluster radius $\quad \psi(a,r) \stackrel{\text{def.}}{=} \mathbb{P}[\, r \leq \mathrm{rad}(\mathcal{K}^a) < \infty \,] \qquad a \in \mathbb{R}, \; r \geq 1.$

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$+$ Thm 1^{bis} (SIGN holds) $\square$

$\boxed{\text{Critical exponents:}}$

Exponent	α_c	β	γ	δ	Δ	ς	ν_c	η
		Thm 2				Thm 3 ($\nu<1$)	Thm 3 ($\nu<1$)	Titus (loop soups)
Value	$2 - \frac{2\alpha}{\nu}$	1	$\frac{2\alpha}{\nu} - 2$	$\frac{2\alpha}{\nu} - 1$	$\frac{2\alpha}{\nu} - 1$	$\frac{2}{\nu}$	$\frac{2}{\nu}$	$\nu - \alpha + 2$

- All relations satisfied!
- All exponents rational fn's of α & ν! $\qquad \nu \to 4, \; \alpha \to 6$: mean-field.
- 'Constraints' (rigidity of ψ)

The case $\nu = 1$

Thm 3 d'd (V_α), (G_ν), $+\ \varepsilon$

Let $\xi(a) \overset{\text{def.}}{=} |a|^{-\frac{2}{\nu}}$.

$$\psi(a,r) = \mathbb{P}[0 \overset{\geq a}{\longleftrightarrow} \partial B_r,\ 0 \overset{\geq a}{\not\longleftrightarrow} \infty]$$

replace by

$$\tilde{\psi}(a,r) = \quad B_{\xi(a)} \qquad B_{\xi(a)}, \quad \frac{r}{\xi(a)} \geq 1$$

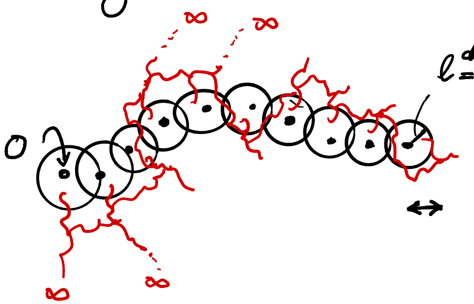
ii) If $\nu = 1$:

$$\psi(a,r) \leq \psi(0,r) \exp\left\{-c_4 \frac{(r/\xi(a))}{\log r}\right\}, \quad a \in \mathbb{R},\ r \geq 2$$

$$\tilde{\psi}(a,r) \geq \tilde{\psi}(0,r) \exp\left\{-c_2 \frac{(r/\xi(a))}{\log(r/\xi(a))}\right\}, \quad |a| \leq 1,\ \frac{r}{\xi(a)} \geq 2$$

(w. Goswami, Severo: on $\mathbb{Z}^3$: $a \neq a_*$ fix $\lim_{r\to\infty} \frac{\log r}{r} \log \psi(a,r) = -\frac{\pi^2}{6}(a-a_*)^2$)

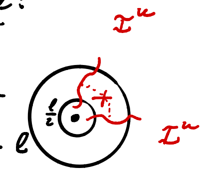
L.B.: Change measure $a>0 \to -a$



$\ell \overset{\text{def.}}{=} \frac{c}{u}$ ok because

- $\mathbb{P}[\ \text{"}\mathcal{I}^u \text{ hits } B_\ell\text{"}\ ] = 1 - e^{-u\,\mathrm{cap}(B_\ell)} \geq 1 - e^{-cu\ell^\nu}$

- Lemma:



$$\mathbb{P}[\ \text{"2 arms}_{\frac{\ell}{2},\ell}\text{"}\ ] \geq 1 - e^{-c(u\ell^\nu)^{c'}} \qquad \square$$

$$\mathcal{I}^u \big|_{u = \frac{a^2}{2}} \supset \{\tilde{\varphi} > -a\}$$

Thank you !