Inhomogeneous percolation on ladder graphs

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joint work with Daniel Valesin

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Online probability seminar, Lyon

The inhomogeneous percolation framework

$\mathbb{G} = (\mathbb{V}, \mathbb{E})$ oriented/non-oriented graph, split the edge set into two disjoint sets: $\mathbb{E} = \mathbb{E}' \cup \mathbb{E}''$

$$\forall e \in \mathbb{E}' \text{ open with probability } p$$
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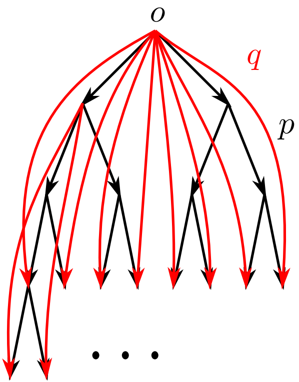
What can we say about $q \mapsto p_c(q)$?

Related work

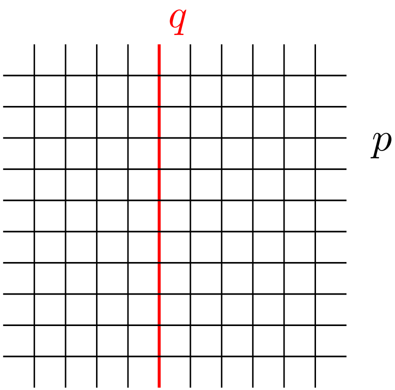
de Lima, Rolla, Valesin (17): oriented d-regular tree with additional edges of length k



$p_c(q)$ is continuous and strictly decreasing in the region where it is positive

Related work

Zhang (94): bond percolation on $\mathbb{Z}^2$



$p_c(q)$ is constant on $(0, 1)$

no infinite cluster at $p_c(q)$

Construction of ladder graphs

G is an arbitrary 'base graph' that is

- connected
- locally finite
- (infinite)

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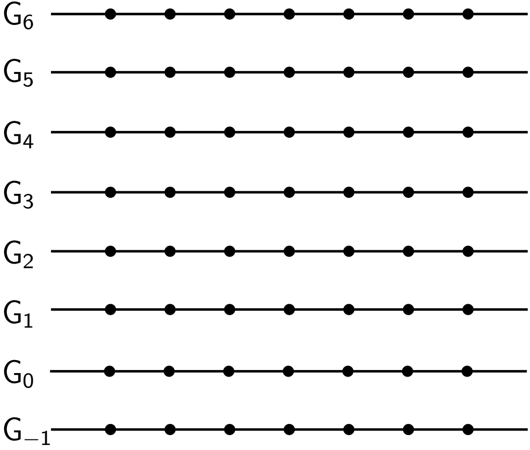

Construction

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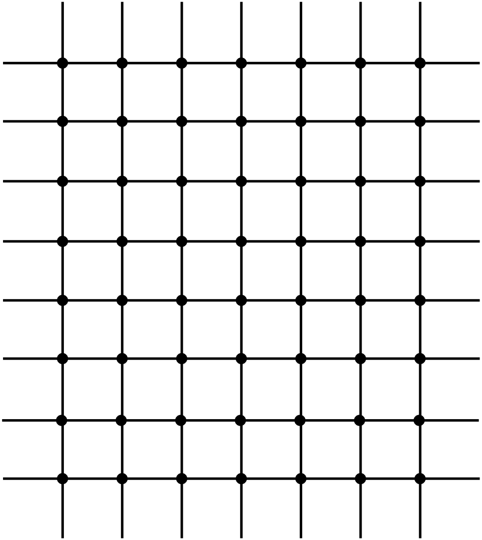

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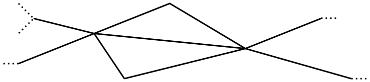

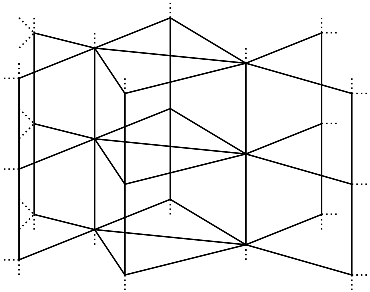

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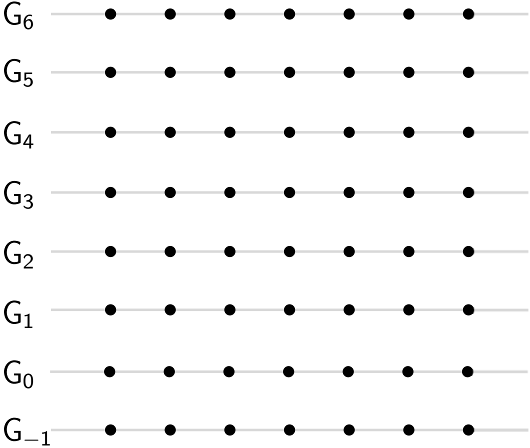

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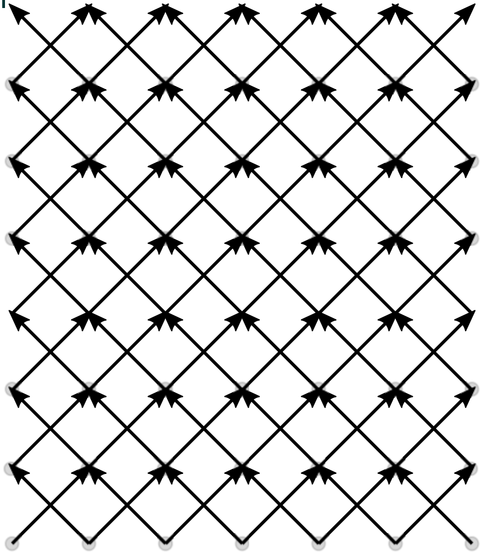

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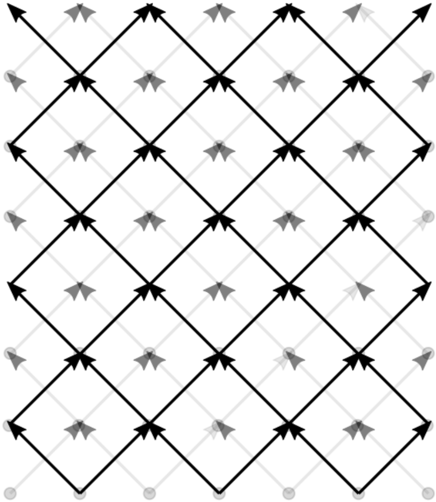

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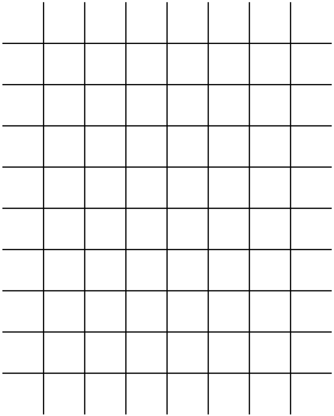

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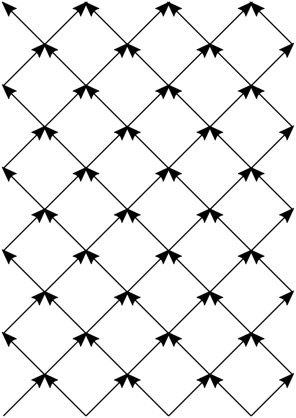
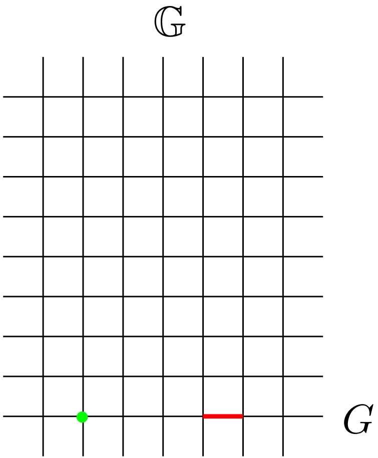


The inhomogeneous percolation model



Fix finitely many vertices and edges of the base graph G

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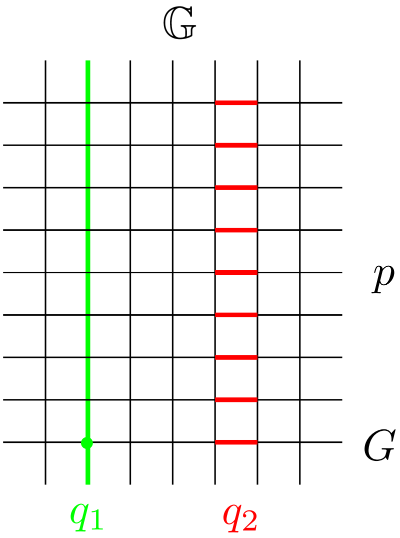
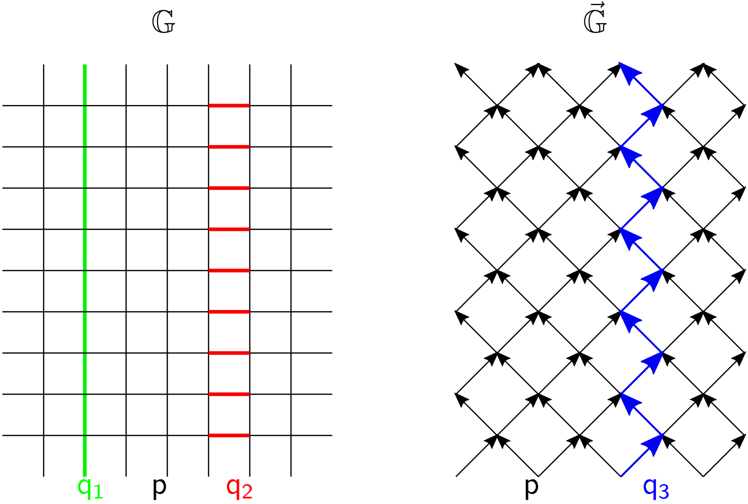


Fix finitely many vertices and edges of the base graph G

Let the edges of the coloumns corresponding to these edges and vertices be open with probability $q_1, q_2, \ldots$

The inhomogeneous percolation model



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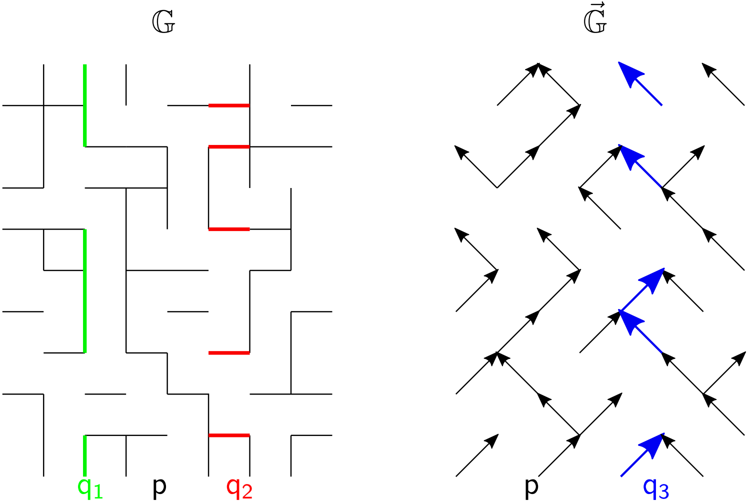
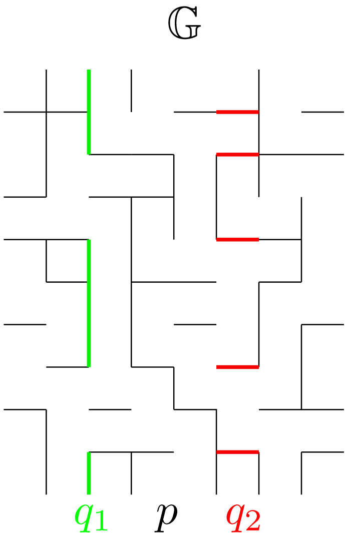


Result



$$\mathbf{q} := (q_1, q_2, \ldots, q_K)$$

Theorem

The function $\mathbf{q} \mapsto p_c(\mathbf{q})$ is continuous on $(0,1)^K$.

Motivation

Contact Process: model of epidemics on a graph with vertices in state 0 (healthy) or 1 (infected)

Transition rules:

- $1 \longrightarrow 0$ at rate 1
- $0 \longrightarrow 1$ at rate $\lambda \cdot \#\{\text{infected neighbors}\}$

$$\lambda_c = \sup\{\lambda : \text{ CP with parameter } \lambda \text{ a.s. dies out}\}$$

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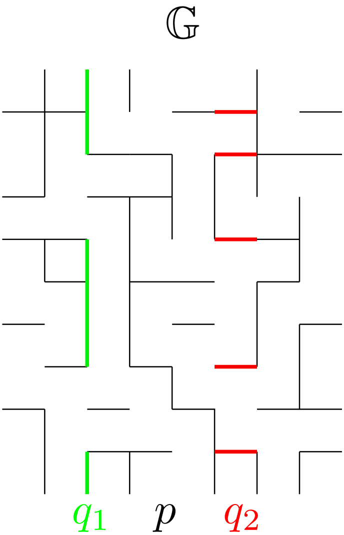
Conjecture (Pemantle and Stacey, 2001)

Changing the infection parameter on a finite set of edges does not affect λ_c.

Result



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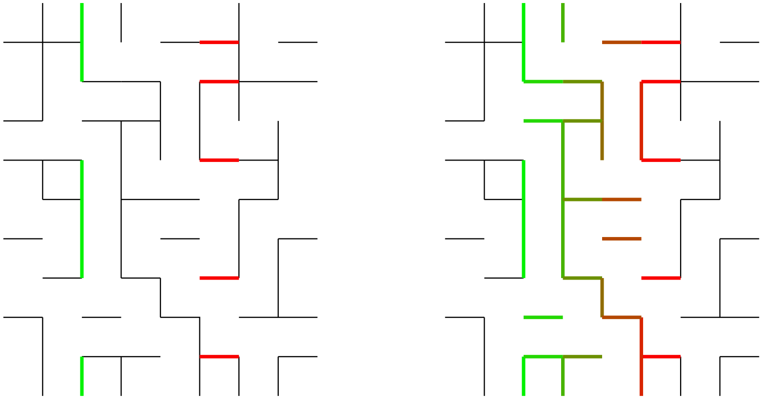
Result

If the theorem holds for a finite set of columns, than it holds for any subset of these columns.



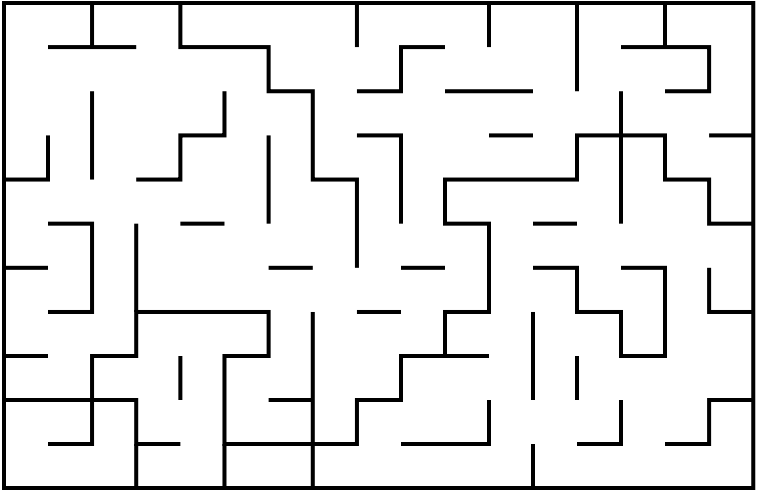
Coupling lemma

Lemma (de Lima, Rolla, Valesin '17)

Let $\{\mathbb{P}_\theta\}_{\theta \in \Theta}$ denote probability measures on a finite set S, parametrized by θ, such that $\theta \mapsto \mathbb{P}_\theta(x)$ is continuous for every $x \in S$. Assume that for some $\theta_1 \in \Theta$ and $\bar{x} \in S$ we have $\mathbb{P}_{\theta_1}(\bar{x}) > 0$. Then, for any θ_2 close enough to θ_1, there exists a coupling of two random elements X and Y of S such that $X \sim \mathbb{P}_{\theta_1}$, $Y \sim \mathbb{P}_{\theta_2}$ and

$$\mathbb{P}\left(\{X = Y\} \cup \{X = \bar{x}\} \cup \{Y = \bar{x}\}\right) = 1.$$

Toy example



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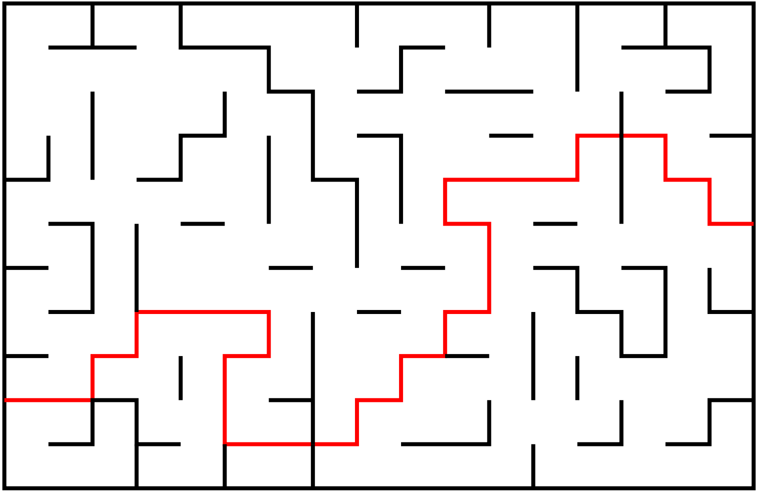

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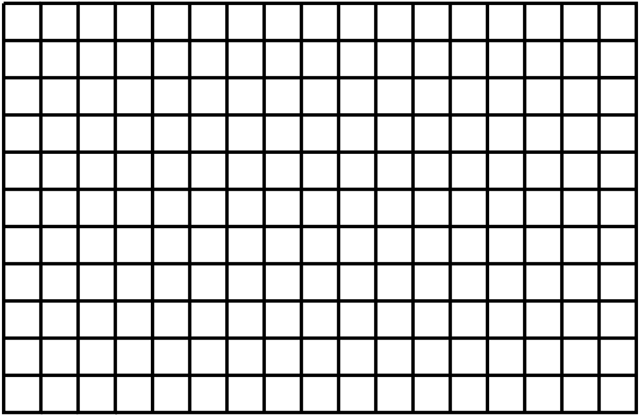

$\mathbb{P}_{p,p}(\text{crossing})$

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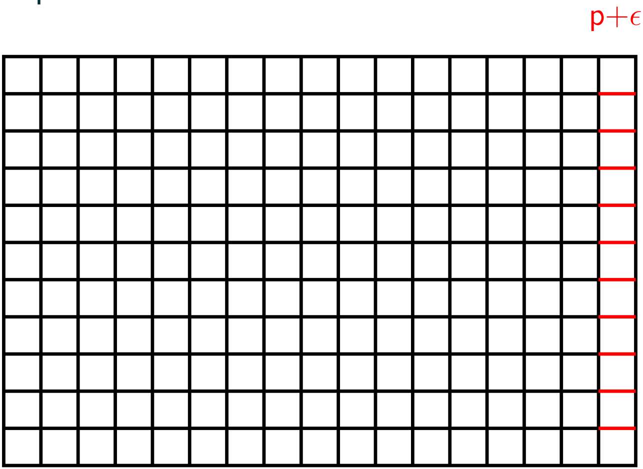

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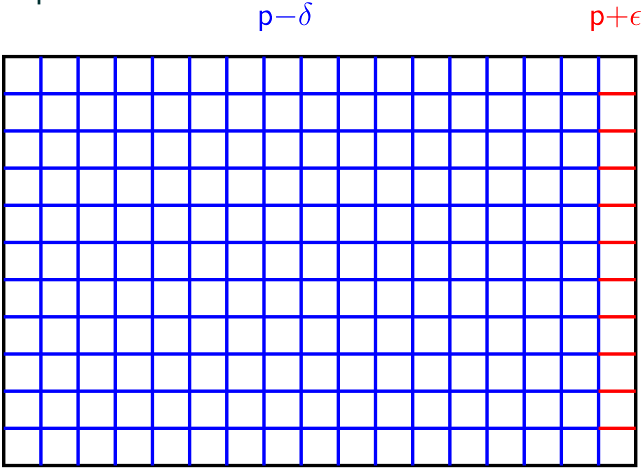

$$\forall p, \epsilon \ \exists \delta : \mathbb{P}_{p,p}(\text{crossing}) \leq \mathbb{P}_{p-\delta,p+\epsilon}(\text{crossing})$$

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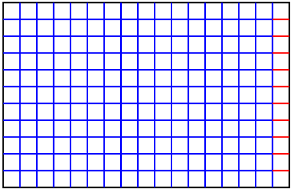

Sets of all possible configurations:
S_1, S_2

Goal: introduce a coupling of configurations (s, s') on $(S_1 \times S_2)^2$ such that $s \sim \mathbb{P}_{p,p}$, $s' \sim \mathbb{P}_{p-\delta, p+\epsilon}$ and

$$\text{crossing on } s \Rightarrow \text{crossing on } s'$$

Toy example

$$S := S_1 \times S_2 \times S_2$$
$$\theta_1 := (p, p, \tfrac{\epsilon}{1-p})$$
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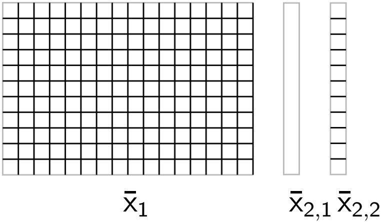

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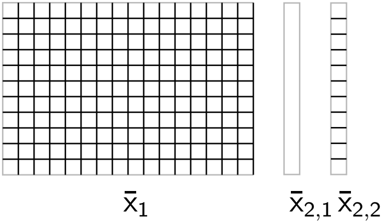

Then $X = (X_1, X_{2,1}, X_{2,2})$, $Y = (Y_1, Y_{2,1}, Y_{2,2}) \in S$,
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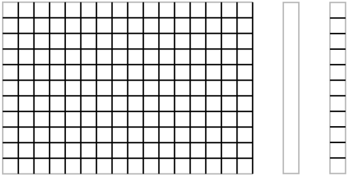

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$s = (s_1, s_2) := (X_1, X_{2,1}) \sim \mathbb{P}_{p,p}$

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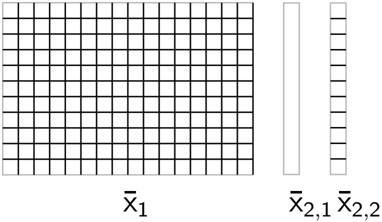

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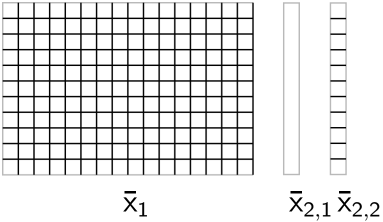

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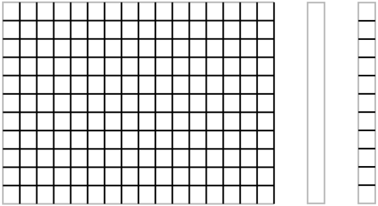
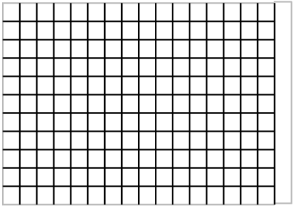
Need: crossing on $s \Rightarrow$ crossing on s'

Toy example



$\bar{x}_1$ $\bar{x}_{2,1}$ $\bar{x}_{2,2}$



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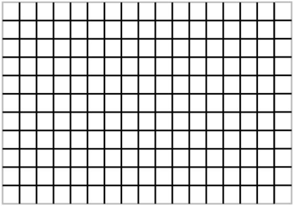

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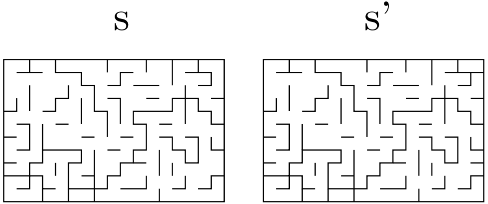
Toy example

$X = Y$



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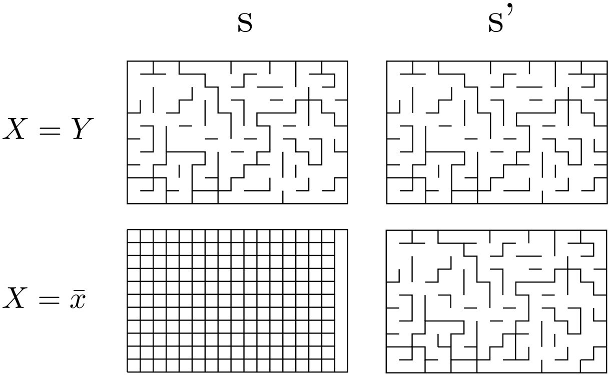

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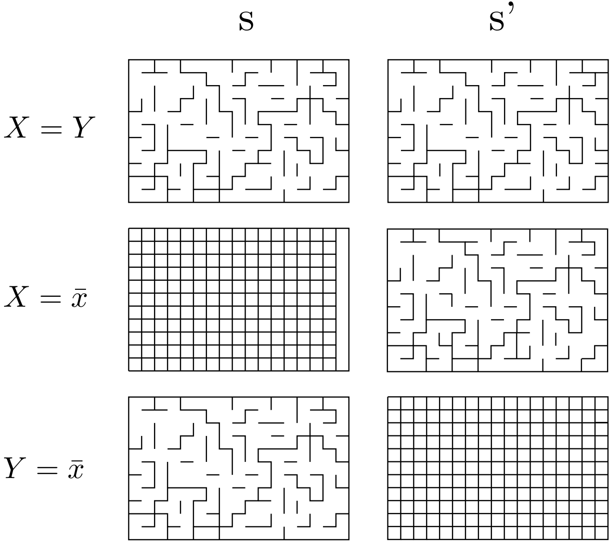
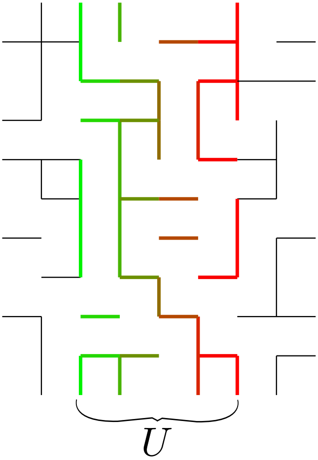


Proof of Theorem



$$\mathbf{q} := (q_1, q_2, \ldots, q_K)$$

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The function $\mathbf{q} \mapsto p_c(\mathbf{q})$ *is continuous on* $(0,1)^K$.

Proof of Theorem

Claim

For all $p \in (0,1)$, $\mathbf{q} \in \mathbb{R}^K$ *and* $\epsilon \in (0, 1-p)$ *if* $\boldsymbol{\delta} \in \mathbb{R}^K$ *is small enough, then*

$$\mathbb{P}_{p,\mathbf{q}}(C_\infty) \leq \mathbb{P}_{p+\epsilon,\mathbf{q}-\boldsymbol{\delta}}(C_\infty).$$

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Proof:

Goal: Introduce a coupling of configurations (ω, ω') such that $\omega \sim \mathbb{P}_{p,\mathbf{q}}$, $\omega' \sim \mathbb{P}_{p+\epsilon,\mathbf{q}-\boldsymbol{\delta}}$ and

$$C_\infty \text{ in } \omega \Rightarrow C_\infty \text{ in } \omega'.$$

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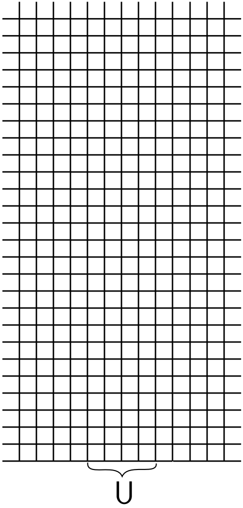

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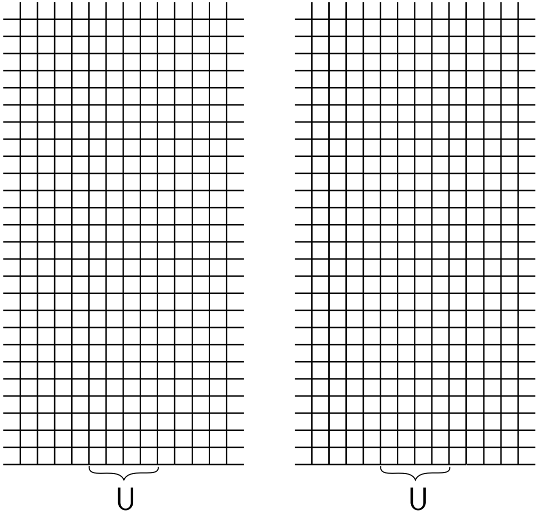

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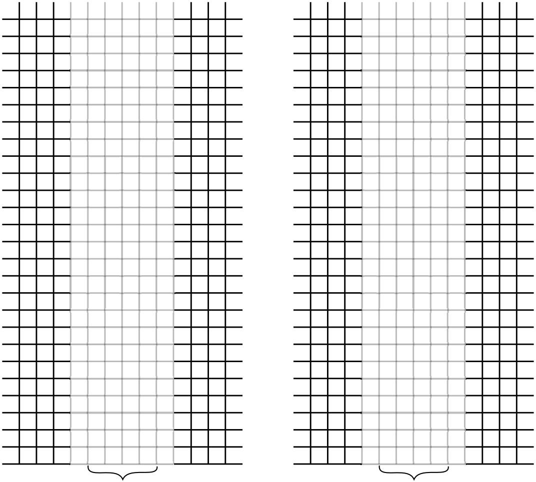

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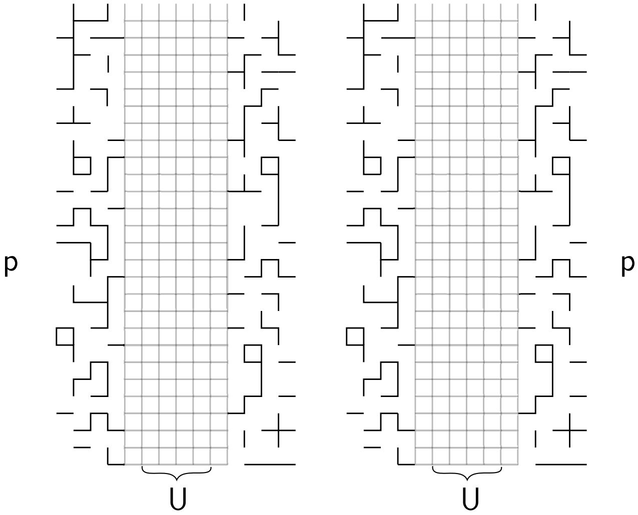

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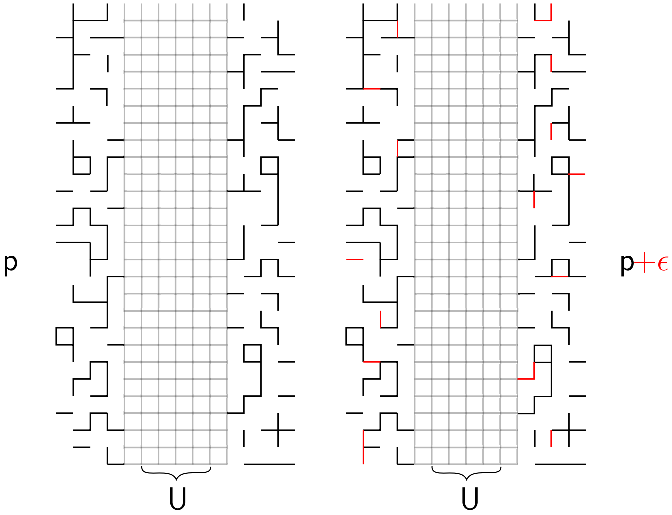

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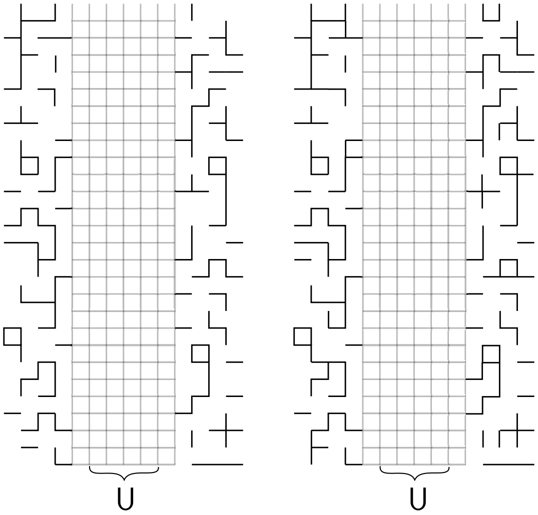
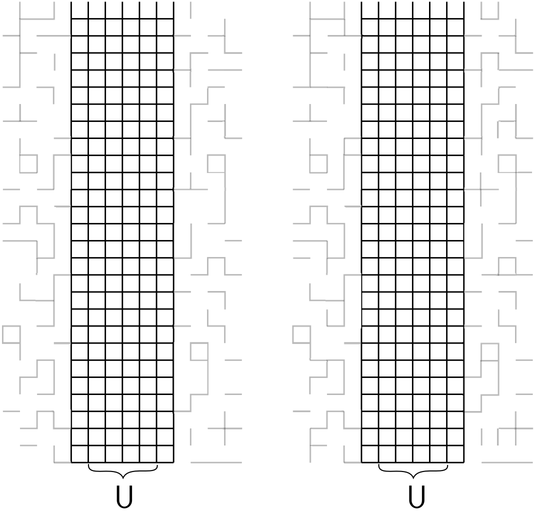


Proof of Claim



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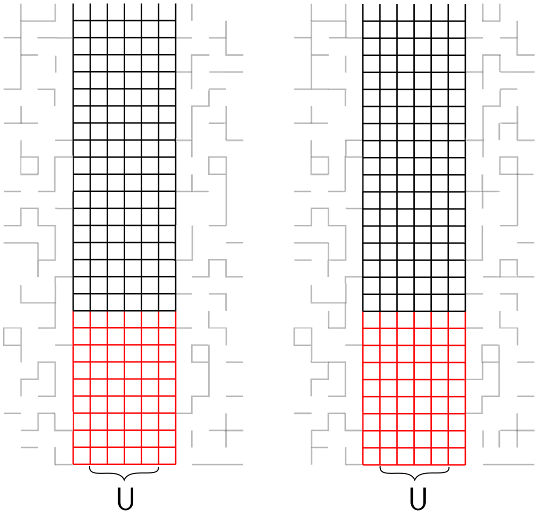

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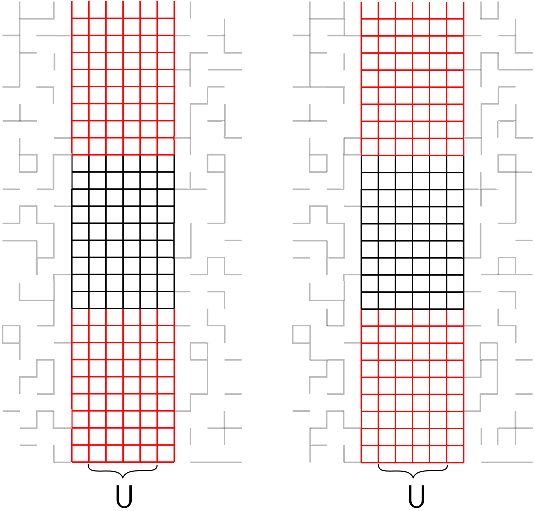

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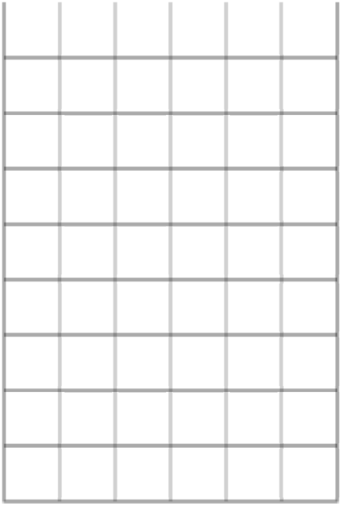
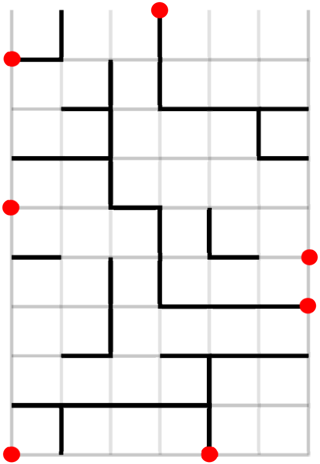


Proof of Claim

ω_0

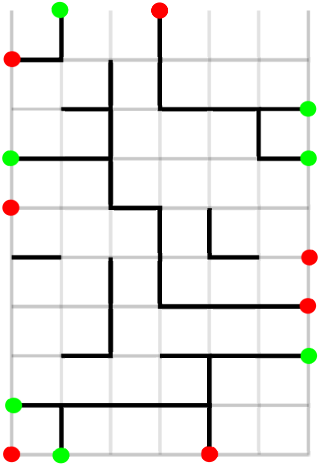
A



Proof of Claim

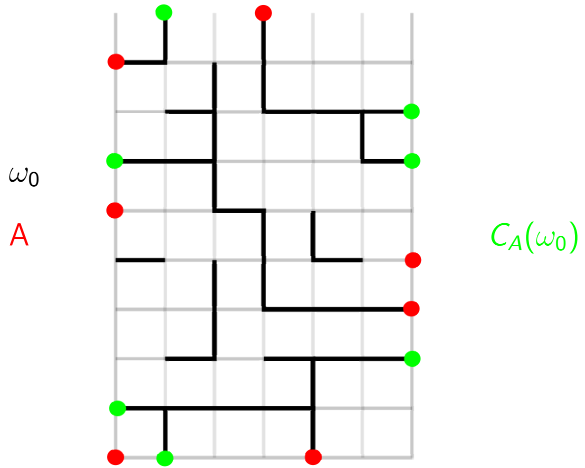
ω_0

A



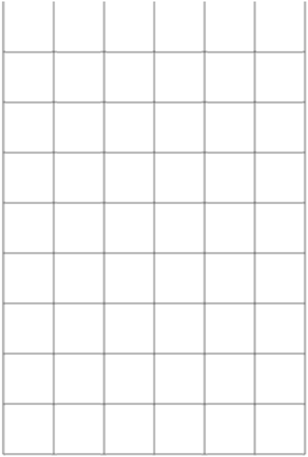
$C_A(\omega_0)$

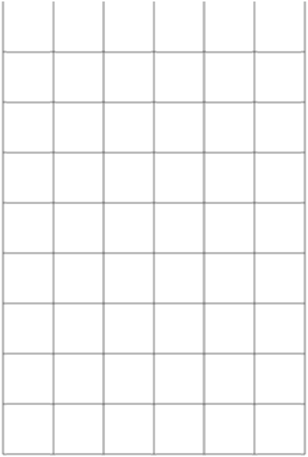
Proof of Claim



Goal: coupling (ω_0, ω_0') satisfying $C_A(\omega_0) \subseteq C_A(\omega_0')$ $\forall A$

Proof of Claim





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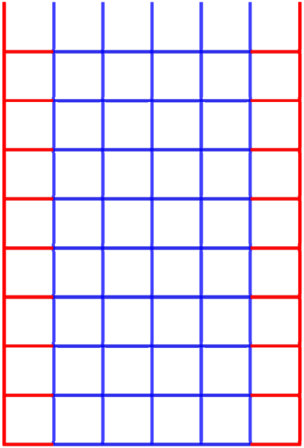
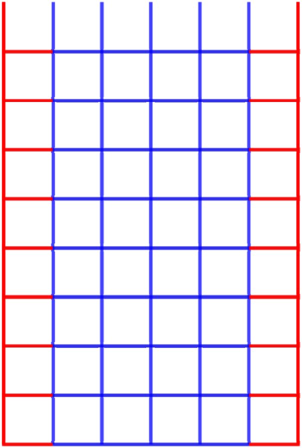
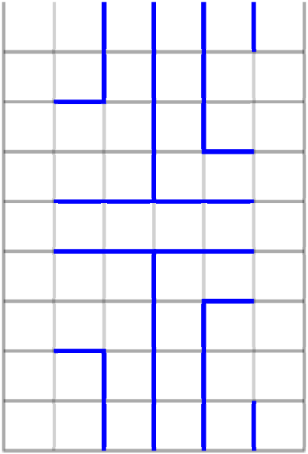


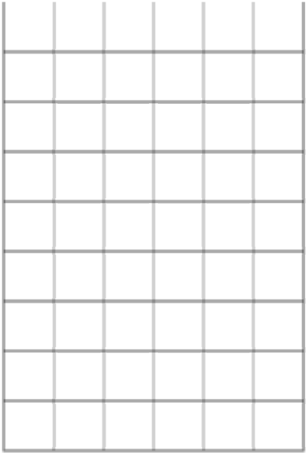
p $\underline{q}$



$\underline{q} - \underline{\delta}$ $p + \epsilon$

Proof of Claim





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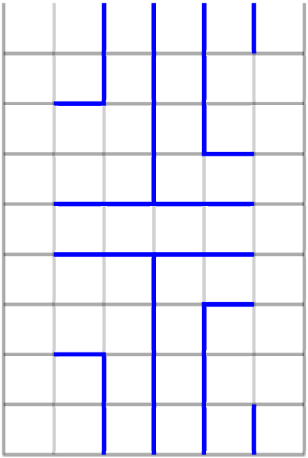

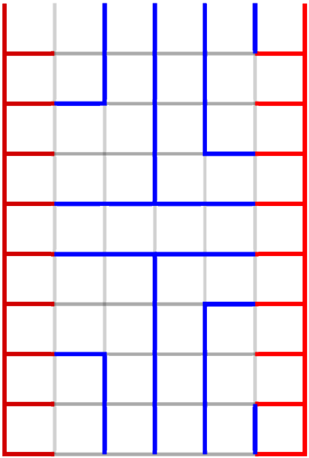

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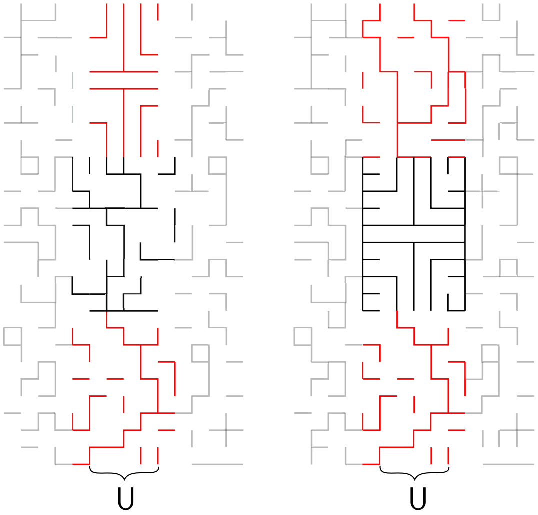

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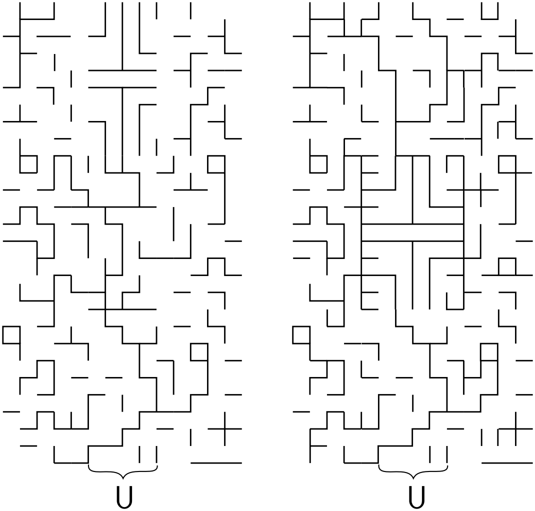

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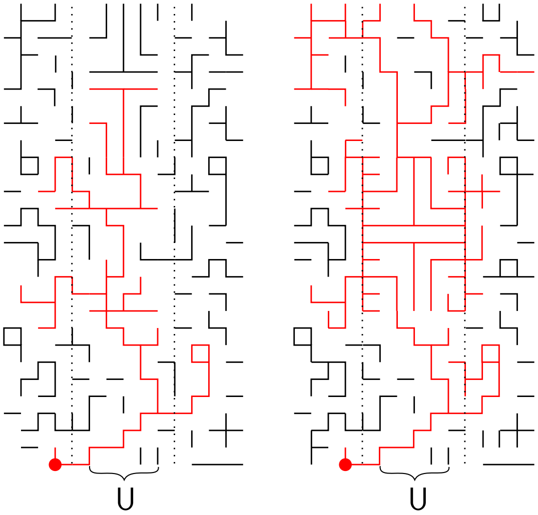
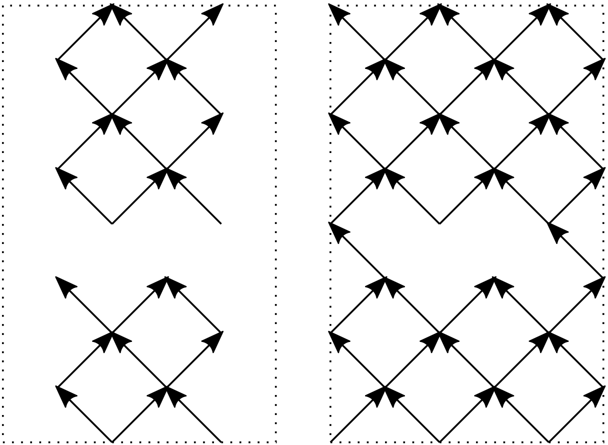
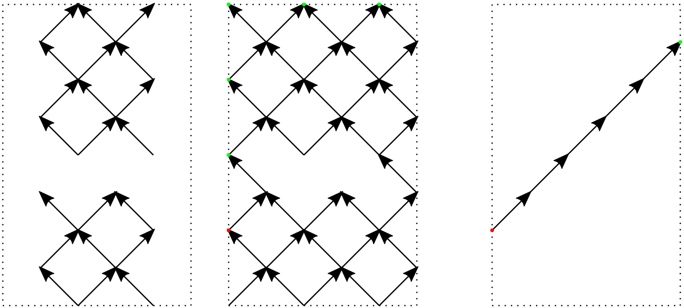


Thank you!

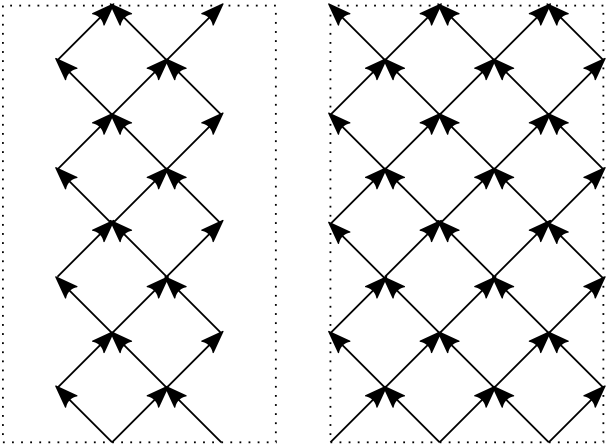
Oriented case



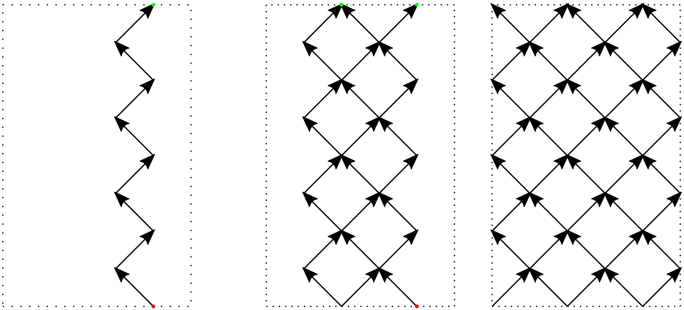
Oriented case



Oriented case



Oriented case



Oriented case