Automorphism groups of trivial strongly minimal structures

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Abstract

We study automorphism groups of trivial strongly minimal structures. First we give a characterization of structures of bounded valency through their groups of automorphisms.

Then we characterize the triplets of groups which can be realized as the automorphism group of a non algebraic component, the subgroup stabilizer of a point and the subgroup of strong automorphisms in a trivial strongly minimal structure, and also we give a reconstruction result.

Finally, using HNN extensions we show that any profinite group can be realized as the stabilizer of a point in a strongly minimal structure of bounded valency.

Introduction

In this paper we study automorphism groups of trivial strongly minimal structures. In particular, we give a complete characterization of the pair of topological groups which can be realized in such a structure as the automorphism group of a component and the stabilizer of a point in this component. Our original motivation for such a study came from a conjecture about finitely axiomatized strongly minimal structures and some results of A. Ivanov on the subject.

Let G be an infinite finitely presented group containing elements $g_1, \ldots, g_n$ such that for every $g \in G$ there is $m > 0$ such that g^m is conjugated with one of the g_i's. Then the Cayley graph of G is a finitely axiomatizable trivial strongly minimal structure. But the existence of a group as above is an open problem, as is the existence of a finitely axiomatizable strongly minimal theory. We know from results of Hrushovski [Hru 94] that a finitely axiomatizable $\aleph_1$-categorical theory must be locally modular. There are examples of finitely axiomatizable

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$\aleph_1$-categorical theories of Morley Rank 2, which all have trivial pregeometries. For the trivial strongly minimal case, Ivanov [Iva 93] has conjectured that the existence of a finitely axiomatizable trivial strongly minimal theory is equivalent to the existence of an infinite finitely presented group containing elements $g_1, \ldots, g_n$ such that for every $g \in G$ there is $m > 0$ such that g^m is conjugated with one of the g_i's.

Ivanov [Iva 93] has been working towards showing this conjecture. First he showed that the existence of a finitely axiomatizable trivial strongly minimal theory implies the existence of a finitely axiomatizable strongly minimal structure M of bounded valency . If C is the component of a non algebraic point $c \in M$ (i.e. $C := \mathrm{acl}(c) \setminus \mathrm{acl}(\emptyset)$) in such a structure, then C is also a finitely axiomatizable strongly minimal structure of bounded valency which is very similar to a connected graph of bounded valency. We can now summarize his main results:

Theorem. *[Iva 89, Iva 93] Let C be the component of a non algebraic point c in a trivial finitely axiomatizable strongly minimal structure of bounded valency, and let G be the automorphism group of C and H the stabilizer of c in G. Then*

(i) G is finitely presented if and only if H is finite,

(ii) there are $g_1, \ldots, g_n$ such that for every $g \in G$ there is $m > 0$ and $h \in H$ such that g^m is conjugated with some hg_i.

Thus if H is finite then G is a finitely presented group containing elements $g_1, \ldots, g_n \in G$ such that for every $g \in G$ there is $m > 0$ such that g^m is conjugated with one of the g_i's. Unfortunately H is not necessarily finite: the existence of a finitely axiomatizable trivial strongly minimal theory implies the existence of such a structure where the stabilizer of a point is infinite [Blo, Proposition 2.6.7.]. On the other hand, it can be easily deduced from Ivanov's results that a finitely axiomatizable trivial strongly minimal theory is unimodular [Blo, Proposition 2.6.5.].

Our original motivation for looking at automorphism groups of trivial structures was to understand whether these conditions, bounded valency and unimodularity, imposed any restrictions on the profinite groups corresponding to the stabilizer of a point. We will see that this is not the case.

First, in section 2 we show that a saturated trivial strongly minimal structure is of bounded valency if and only if the automorphism group of this structure is the product of the automorphism group of algebraic points with the wreath product of the automorphism group of a non algebraic component by the symmetric group on the set of non-algebraic components.

Then, in section 3 we characterize the triplets of groups which can be realized as the automorphism group of a non algebraic component, the subgroup stabilizer of a point and the subgroup of strong automorphisms in a trivial strongly minimal structure. Moreover we see that we can reconstruct a trivial strongly minimal structure without algebraic point from these groups.

In section 4 we present some examples of trivial strongly minimal structures which are not of bounded valency.

For any structure there is a natural topology on the automorphism group. With this topology the stabilizer of a point in the automorphism group of the component of this point is always a profinite group. In section 5 we show conversely that any profinite group can be realized as the stabilizer of a non algebraic point in a unimodular strongly minimal structure of bounded valency. On the other hand, we can assume that the above structure is non-unimodular if and only if the profinite group has two homeomorphic open subgroups of distinct indices.

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1 Notation and definitions

In this paper we consider structures in a first order language (not necessarily countable). We often identify the structure M with its universe. We denote definable closure and algebraic closure, in the usual sense of model theory, respectively by dcl and acl (i.e. $a \in \mathrm{dcl}(A)$ if there is a first order formula ϕ with parameters in A such that a is the unique element which satisfies ϕ, and $a \in \mathrm{acl}(A)$ if there is a first order formula ϕ with parameters in A such that a satisfies ϕ and only finitely many elements satisfy ϕ).

Let M_1 and M_2 be two structures with the same universe. We say that the structures M_1 and M_2 are **equidefinable** (or their languages are equidefinable) if they have same $\emptyset$-definable relations.

Components of a structure. Let M be a structure.

• For every $a \in M$ and $A \subset M$, we denote by $\mathrm{iacl}_M(a)$ (or $\mathrm{iacl}(a)$ if there is no risk of ambiguity) the set of elements interalgebraic with a in the structure M (i.e. the set of elements b such that $b \in \mathrm{acl}(a)$ and $a \in \mathrm{acl}(b)$) and $\mathrm{iacl}_M(A)$ (or $\mathrm{iacl}(A)$) the set of elements interalgebraic with some element of A.

• For every $a \in M$, we call $\mathrm{iacl}(a)$ a **component** of M. Thus M is the disjoint union of its components.

• We say that a component is non algebraic if it is different from $\mathrm{acl}(\emptyset)$.

Strong types. Let M be a stable structure.

If $\overline{a}$, $\overline{b} \in M^n$ and $A \subset M$, $\overline{a}$ and $\overline{b}$ realize the same **strong type** over A if whenever E is an A-definable equivalence relation over M^n which has only a finite number of classes, $M \models E(\overline{a}, \overline{b})$. (We call such an equivalence relation E a **finite equivalence relation**.) It follows that $\overline{a}$ and $\overline{b}$ realize the same type over

A. Moreover if A is an elementary substructure of M then $\overline{a}$ and $\overline{b}$ realize the same strong type over A if and only if they realize the same type over A.

A type is **stationary** if it realizes only one strong type.

A **strong automorphism** of M is an automorphism which fixes setwise each class of each $\emptyset$-definable finite equivalence relation. An automorphism of M which fixes pointwise an elementary substructure of M is strong.

Groups of automorphisms and topological groups. Let M be a structure and X a subset of M.

We denote by $\mathrm{Aut}(M)$ the group of automorphisms of M and by $\mathrm{FixC}(M)$ the normal subgroup of automorphisms which fixes setwise each component of M.

We denote by $\mathrm{Sym}(X)$ the symmetric group on X and by $\mathrm{Aut}(X)$ the subgroup of permutations of X which preserve each type of tuples in X (i.e. $\mathrm{Aut}(X)$ is the automorphism group of the structure induced on X by M).

The following are well known facts about groups of automorphisms. There is a natural topology on $\mathrm{Sym}(X)$: give X the discrete topology and let $\mathrm{Sym}(X)$ have the coarsest topology that makes the action $\mathrm{Sym}(X) \times X \to X$ continuous. Then, the stabilizers $\mathrm{Sym}_{\overline{a}}(X) = \{g \in \mathrm{Sym}(X);\ g(\overline{a}) = \overline{a}\}$ as $\overline{a}$ ranges over the finite tuples in X form a basis of open-closed neighbourhoods of the identity. This topology is clearly Hausdorff and totally disconnected.

We make any permutation group on X into a topological group by giving it the relative topology. Then $\mathrm{Aut}(X)$ is closed subgroup of $\mathrm{Sym}(X)$ and every closed subgroup G of Sym(X) is compact if and only if all its orbits on X are finite.

If M is stable, we denote by $\mathrm{Autf}(X)$ the group of permutations of X which preserve each strong type of tuples in X. Then $\mathrm{Autf}(X)$ is a closed normal subgroup of $\mathrm{Aut}(X)$.

Semidirect products and wreath products.

Let N and H be two groups and ϕ an homomorphism from H to the automorphism group of N. The **semidirect product** of N with H denoted by $N \rtimes_\phi H$ is the set $N \times H$ with the following group law:

$$(n, h) \cdot (n', h') = (n \cdot (\phi(h)(n')), h \cdot h').$$

Let G be a group, I a set and H a permutation group of I. The **wreath product** of G by H is the canonical semidirect product $G^I \rtimes_\phi H$ where ϕ is defined by $\phi(h)((g_i)_{i\in I}) = (g_{h(i)})_{i\in I}$ for every $h \in H$.

We will denote by S_I the group of all permutations of I and we denote $G^I \rtimes S_I$ the wreath product of G by S_I.

Strongly minimal structures. A structure M is **strongly minimal** if M is infinite and in every structure elementarily equivalent to M, every definable subset is finite or cofinite (i.e. over any set of parameters there is only one non algebraic type). So all the non algebraic components of a strongly minimal structure are isomorphic.

It is well known that algebraic closure defines a pregeometry on strongly minimal structures (see for example [Pil]).

A strongly minimal structure is **trivial** if algebraic closure is **disintegrated** (i.e. for every elementarily equivalent structure N and every $A \subset N$, $\mathrm{acl}(A) = \cup_{a\in A}\mathrm{acl}(\{a\})$).

The dimension, in the sense of the pregeometry, of a trivial strongly minimal structure M, $\dim M$, is equal to the cardinality of the set of the non algebraic components of the structure.

Remark. *Let M be a strongly minimal structure and C a component of M. It is easy to check that* $\mathrm{Aut}(C)$ *is equal to the group of permutations of C which extend to automorphisms of M and* $\mathrm{Autf}(C)$ *is equal to the group of permutations of C which extend to strong automorphisms of M.*

2 Structures of bounded valency

We use the notion of structures of bounded valency in the sense introduced by Ivanov in [Iva 93]. In a trivial strongly minimal structure, any tuple of independent non algebraic elements has stationary type. But this is not necessarily the case for a tuple of non independent elements. In fact we will see this condition exactly characterizes structures of bounded valency (lemma 2.2). In section 4 we will see some examples which illustrate this.

Definition. Let M be a structure in a language L.

• A relation $R \subset M^k$ has **bounded valency** in M if there is some finite m such that for every $a \in M$, the number of tuples in R containing a is bounded by m.

• We denote by $L_{bound}(M)$ (or L_{bound} if there is no risk of ambiguity) the set of all $\emptyset$-definable relations of bounded valency in M.

• The structure M is of **bounded valency** if any formula in the language L is equivalent in M to a formula in the language L_{bound} (i.e. M is equidefinable to its L_{bound}-structure). Note that this is a property of the elementary theory of the structure M.

Examples.

• Finite structures are of bounded valency.

• A structure in a language with only unary relations is of bounded valency.

• The Cayley graph of a group G (i.e. the structure $(G, R_g; g \in G)$ such that $R_g = \{(h, hg);\ h \in G\}$) is of bounded valency. Note that, for every generating set X of G, the structures $(G, R_g; g \in G)$ and $(G, R_g; g \in X)$ are equidefinable.

• For examples of trivial strongly minimal structures which are not of bounded valency, see section 4.

Our main result in this section will be

Theorem 1. *Let M be a saturated trivial strongly minimal structure and C one non algebraic component of M. The following are equivalent.*

(i) M is of bounded valency

(ii) $\mathrm{Aut}(C) = \mathrm{Autf}(C)$

(iii) $\mathrm{FixC}(M) = \mathrm{Aut}(\mathrm{acl}(\emptyset)) \times \mathrm{Aut}(C)^{\dim(M)}$.

The proof will be decomposed in a series of three lemmas. The following lemma shows in particular that (i) implies (iii) in Theorem 1.

Lemma 2.1.

(i) Let $\overline{a}$ be a finite tuple in a structure M in a language L. If $\overline{a}$ satisfies a relation $R \in L_{bound}$ then $\overline{a}$ is a tuple of interalgebraic elements. Conversely if $\overline{a}$ is a tuple of interalgebraic elements then the type of $\overline{a}$ in the L-structure M is determined by its atomic type in the L_{bound}-structure M.

(ii) An infinite structure N of bounded valency is ω-strongly homogeneous (i.e. if $\overline{a}$ and $\overline{b}$ are finite tuples in N which realize the same type, there is an automorphism f of N such that $f(\overline{a}) = \overline{b}$), admits elimination of quantifiers in the language L_{bound} and has U-rank equal to one.

(iii) Furthermore an infinite structure N of bounded valency such that $\mathrm{Aut}(N)$ *acts transitively on N is strongly minimal.*

(iv) An infinite structure is of bounded valency if and only if in any elementary extension of this structure, for every finite tuple $\overline{a} = \overline{a}_1 \ldots \overline{a}_n$ such that the $\overline{a}_i$'s are in different components the type of $\overline{a}$ is determined by the types of the $\overline{a}_i$'s.

(v) Let $(C_i)_{i \in I}$ be the components of a structure M of bounded valency then $\mathrm{FixC}(M) = \prod_{i \in I} \mathrm{Aut}(C_i)$.

(vi) A strongly minimal structure M of bounded valency is trivial and if C is one non algebraic component of M, $\mathrm{Aut}(M) = \mathrm{Aut}(\mathrm{acl}(\emptyset)) \times (\mathrm{Aut}(C)^{\dim(M)} \rtimes S_{\dim(M)})$.

Proof. (i) It follows from definitions that $\overline{a}$ is a tuple of interalgebraic elements if and only if $\overline{a}$ satisfies some bounded valency relation. So if $\overline{a}$ is a tuple of interalgebraic elements, there is a bounded valency relation R such that $M \models R(\overline{a})$, and for each $\emptyset$-definable relation S in L, $R \wedge S$ has bounded valency and $M \models S(\overline{a})$ if and only if $M \models R(\overline{a}) \wedge S(\overline{a})$.

(ii) Let N be an infinite structure of bounded valency. We will consider N as an L_{bound}-structure. Let M be any structure elementarily equivalent to N.

Let $\overline{a}$ and $\overline{b}$ be finite tuples of M with the same atomic type. We are going to show that there is an automorphism f of M such that $f(\overline{a}) = f(\overline{b})$. This means both that M is ω-strongly homogeneous and that the theory of N admits elimination of quantifiers.

We can assume that $\overline{a} = \overline{a}_1 \dots \overline{a}_n$ such that for every i, $\overline{a}_i$ is a tuple of interalgebraic elements with $\overline{a}_i$ and $\overline{a}_j$ in different components of M for $i \neq j$. Then $\overline{b} = \overline{b}_1 \dots \overline{b}_n$ with the same properties and by (i) for every i, $\mathrm{tp}(\overline{a}_i) = \mathrm{tp}(\overline{b}_i)$.

It follows immediately that there exist partial (elementary) isomorphisms f_i of $\mathrm{iacl}(\overline{a}_i)$ onto $\mathrm{iacl}(\overline{b}_i)$. Let f be the union of the f_i's. Since M is an L_{bound}-structure, by (i) f is a partial isomorphism of $\mathrm{iacl}(\overline{a})$ onto $\mathrm{iacl}(\overline{b})$. We extend f to a partial bijective isomorphism of $\mathrm{iacl}(\overline{a}) \cup \mathrm{iacl}(\overline{b})$ onto itself by finite induction : for any component C of M, there exists the same number of components isomorphic to C in $\mathrm{iacl}(\overline{a})$ as in $\mathrm{iacl}(\overline{b})$. Thus if C is a component in $\mathrm{iacl}(\overline{b}) \setminus \mathrm{iacl}(\overline{a})$ then there exist an isomorphic component C' in $\mathrm{iacl}(\overline{a}) \setminus \mathrm{iacl}(\overline{b})$ and we can extend f to a partial isomorphism of $\mathrm{iacl}(\overline{a}) \cup C$ onto $\mathrm{iacl}(\overline{b}) \cup C'$.

We then obtain an automorphism of M by extending the partial isomorphism of $\mathrm{iacl}(\overline{a}) \cup \mathrm{iacl}(\overline{b})$ onto itself by the identity on $M \setminus (\mathrm{iacl}(\overline{a}) \cup \mathrm{iacl}(\overline{b}))$, again because M is an L_{bound}-structure.

In the same way, we can show that for every a, $b \in M \setminus \mathrm{acl}(\emptyset)$ which realize the same type over the empty set and for every $A \subset M$ such that $a \notin \mathrm{acl}(A)$ and $b \notin \mathrm{acl}(A)$, there is an automorphism of M which fixes pointwise $\mathrm{acl}(A)$ and sends a to b, and so a and b realize the same type over A. It follows that the U-rank of M is equal to one.

(iii) Since $\mathrm{Aut}(N)$ acts transitively on N, $\mathrm{acl}(\emptyset) = \emptyset$ and in any structure elementarily equivalent to N there is only one type over the empty set. So (iii) follows from (ii).

(iv) Follows immediately from (i) and the elimination of quantifiers in (ii).

(v) and (vi) follow from (iv). $\square$

Remark. *By Lemma 2.1 (i), if C is a component of an L-structure M, the automorphism group of C in the sense of the L-structure M is equal to the automorphism group of C in the sense of the L_{bound}-structure M which is of bounded valency.*

We now prove the equivalence of (i) and (ii) in Theorem 1.

Lemma 2.2. *Let M be a trivial strongly minimal structure with a non algebraic component. The following are equivalent.*

(i) M is of bounded valency.

(ii) For every tuple $\overline{a}$ in an elementary extension of M without element in $\mathrm{acl}(\emptyset)$*, the type of $\overline{a}$ is stationary.*

(iii) For every tuple $\overline{a}$ in a non algebraic component of M, the type of $\overline{a}$ is stationary. (Note that this is a property of the elementary theory of the structure M since all the non algebraic components of a strongly minimal structure are isomorphic.)

Proof. (ii) $\Rightarrow$ (iii) is clear.

Let N be a saturated elementary extension of M.

(i) $\Rightarrow$ (ii) : let $\overline{a}$, $\overline{b}$ be two finite tuples in N without algebraic element which realize the same type. By Lemma 2.1 (iv), $\overline{a}$ and $\overline{b}$ realize the same type over $N \setminus (\mathrm{iacl}(\overline{a}) \cup \mathrm{iacl}(\overline{b}))$ which is an elementary submodel of N since N is saturated.

(iii) $\Rightarrow$ (i) : by Lemma 2.1 (iv), it suffices to prove that for any two tuples $\overline{a} = \overline{a}_0\overline{a}_1 \ldots \overline{a}_n$ and $\overline{b} = \overline{b}_0\overline{b}_1 \ldots \overline{b}_n$ in N such that $\overline{a}_0 \in \mathrm{acl}(\emptyset)$, the $\overline{a}_i$'s are in distinct components, the $\overline{b}_i$'s are in distinct components, and for every i, $\overline{a}_i$ and $\overline{b}_i$ realize the same type, then $\overline{a}$ and $\overline{b}$ have the same type.

By hypothesis for every $i > 0$, the type of $\overline{a}_i$ is stationary. Since N is a trivial strongly minimal structure, the tuples $\overline{a}_1, \ldots, \overline{a}_n$ are independent and the tuples $\overline{b}_1, \ldots, \overline{b}_n$ are independent, so $\overline{a}_1 \ldots \overline{a}_n$ and $\overline{b}_1 \ldots \overline{b}_n$ realize the same type which is stationary. This type does not fork over $\mathrm{acl}(\emptyset)$ and $\overline{a}_0$ realizes the same type as $\overline{b}_0$ hence $\overline{a}$ realizes the same type as $\overline{b}$. $\square$

It remains to show that (iii) implies (i) in Theorem 1.

Lemma 2.3. *Let M be a trivial strongly minimal structure of dimension κ and C a non algebraic component of M. Let S_κ be the symmetric group on κ.*

(i) $\mathrm{Aut}(M)$ *is a semidirect product of* $\mathrm{FixC}(M)$ *with* S_κ

(ii) If M is saturated and $\mathrm{FixC}(M) = \mathrm{Aut}(\mathrm{acl}(\emptyset)) \times \mathrm{Aut}(C)^\kappa$ *then M is of bounded valency.*

Proof. (i) Let $(C_i)_{i\in\kappa}$ be the non-algebraic components of M and p be the projection from $\mathrm{Aut}(M)$ to S_κ.

For every i, take $c_i \in C_i$.

Since M is a trivial strongly minimal structure, for every $\sigma \in S_\kappa$, the tuples $(c_i)_{i\in\kappa}$ and $(c_{\sigma(i)})_{i\in\kappa}$ realize the same strong type. So $p(\mathrm{Aut}(M)) = S_\kappa$.

For every i, let ϕ_i be a strong automorphism such that $\phi_i(c_0c_i) = c_ic_0$. We define an homomorphism s from S_κ to $\mathrm{Aut}(M)$ such that $p \circ s$ is the identity on S_κ : for every $\sigma \in S_\kappa$, let $s(\sigma)(x) = \phi_{\sigma(i)} \circ \phi_i^{-1}(x)$ if $x \in C_i$ and $s(\sigma(x)) = x$ if $x \in \mathrm{acl}(\emptyset)$.

(ii) By Lemma 2.2 we only have to prove that for $\overline{a}$ in a non algebraic component of M, the type of $\overline{a}$ is stationary. Let $\overline{b} \in M$ realize the same type as $\overline{a}$. Moving $\overline{b}$ by a strong automorphism, we can suppose that $\overline{b}$ is in the same component as $\overline{a}$. Let M' be the union of the components not containing $\overline{a}$. Then M' is infinite and algebraically closed, so M' is an elementary substructure of M. We will show that $\overline{a}$ and $\overline{b}$ have the same type over M'.

Let ϕ be an automorphism of M such that $\phi(\overline{a}) = \overline{b}$. By (i), there exists an automorphism ψ which sends $\phi(C_i)$ on C_i for every C_i and which fixes pointwise the component of $\overline{a}$. Let $\bar{m}$ be an enumeration of M'. Since $\mathrm{FixC}(M) = \mathrm{Aut}(\mathrm{acl}(\emptyset)) \times \mathrm{Aut}(C)^\kappa$, there exists an automorphism θ which again fixes pointwise the component of $\overline{a}$ and such that $\theta \circ \psi \circ \phi(\bar{m}) = \bar{m}$. Thus $\overline{a}$ and $\overline{b}$ have the same type over M'. $\square$

Remark.

(i) If M is not $\aleph_0$-categorical we can replace, in Lemma 2.3 (ii), the assumption that M is saturated by $\dim(M) \geq 2$.

(ii) A saturated strongly minimal structure M where $\mathrm{FixC}(M) = \mathrm{Aut}(\mathrm{acl}(\emptyset)) \times \mathrm{Aut}(C)^{\dim(M)}$ *is necessarily trivial.*

Remark. *Let M be a trivial strongly minimal structure. If* $\mathrm{acl}(\emptyset)$ *is infinite then* $\mathrm{acl}(\emptyset)$ *is an elementary substructure of M. So if* $\mathrm{acl}(\emptyset) = \mathrm{dcl}(\emptyset)$ *is infinite then, by Lemma 2.2, M is of bounded valency. In particular, if one expands a trivial strongly minimal structure by naming the elements of some elementary substructure, then the resulting structure is of bounded valency.*

In [Iva 93] Ivanov proved a finer result, which does not follow directly from the above remark:

Proposition. *[Iva 93, Theorem1.4]*

Assume M is a trivial strongly minimal structure with a finite language and such that $\mathrm{acl}(\emptyset)$ *is infinite. Then there is a finite set* $C \subset \mathrm{acl}(\emptyset)$ *such that the expansion $(M, c)_{c \in C}$ is of bounded valency.*

Remark. *In fact we have shown in [Blo, Corollaire 2.2.15], that one does not need the assumption that the algebraic closure of the empty set is infinite and that any trivial strongly minimal structure with a finite language has an expansion by a finite number of constants which is of bounded valency. This follows from an analysis of the same type as Ivanov's. We show [Blo, Proposition 2.2.14] that the strong types are determined by the finite equivalence relations on tuples satisfying some bounded valency relation. And then that a trivial strongly minimal structure such that* $\mathrm{acl}(\emptyset) = \mathrm{dcl}(\emptyset)$ *or such that* $\mathrm{acl}(\emptyset)$ *is infinite is equidefinable with its structure in a "canonical" language consisting of bounded valency relations and of finite equivalence relations on tuples satisfying some bounded valency relation. If the language is finite, then this canonical language can be taken to be also finite, and it is sufficient to fix a finite set of equivalence classes in order that all types of tuples of non algebraic elements become stationary.*

3 Automorphisms of a component

In this section we characterize the groups which can be realized as groups of automorphisms of a component in a trivial strongly minimal set and give some reconstruction results.

We recall some equivalent definitions of a profinite group which will be useful to us (see for example [Hig]):

Definition. A topological group G is profinite if it satisfies the following equivalent properties:

(i) G is compact, Hausdorff and totally disconnected.

(ii) G is compact, Hausdorff and the set of open normal subgroups of G is a fundamental system of open neighbourhoods of 1.

(iii) If $(N_i)_{i\in I}$ is the family of open normal subgroups of G of finite index then $G = \varprojlim G/N_i$.

A profinite group G is separable if it satisfies the following equivalent properties:

(i) G has a countable dense subset.

(ii) G has a countable basis of open sets.

(iii) The set of open normal subgroups of G is countable.

(iv) The set of open subgroups of G is countable.

We recall also that a topological group is locally compact if it has a system of open compact neighbourhoods of the identity.

Let M be a structure, C a component of M and c in M such that $C = \mathrm{iacl}(c)$.

Recall that the stabilizers $\mathrm{Stab}(\overline{a}) = \{g \in \mathrm{Aut}(C);\ g(\overline{a}) = \overline{a}\}$ as $\overline{a}$ ranges over the finite tuples in C form a basis of open-closed neighbourhoods of the identity for the topological group $\mathrm{Aut}(C)$. For every non-empty finite tuple $\overline{a}$ in C, all orbits of $\mathrm{Stab}(\overline{a})$ on C are finite, so $\mathrm{Stab}(\overline{a})$ is compact. Thus, $\mathrm{Aut}(C)$ is locally compact and $\mathrm{Stab}(c)$ is an open profinite subgroup of $\mathrm{Aut}(C)$.

If C is the algebraic component (i.e. $C = \mathrm{acl}(\emptyset)$) then $\mathrm{Aut}(C)$ is profinite. We have shown conversely that any profinite group can be realized as $\mathrm{Aut}(\mathrm{acl}(\emptyset))$ in a (trivial) strongly minimal structure of bounded valency (we do not give the proof here).

Proposition. *[Blo, Proposition 2.4.2] Let G be a profinite group. There is a strongly minimal structure M of bounded valency such that for every subset A of M, $\mathrm{acl}(A) = A \cup \mathrm{acl}(\emptyset)$ and such that $\mathrm{Aut}(\mathrm{acl}(\emptyset))$ is homeomorphic to G.*

Assume now that M is a strongly minimal structure and C is a non algebraic component in M. It is easy to check that the set of finite intersections of the conjugates of $\mathrm{Stab}(c)$ by $\mathrm{Aut}(C)$ is a system of open neighbourhoods of the identity since M is strongly minimal structure and $\mathrm{Aut}(C)$ is transitive on C. Moreover, $\mathrm{Autf}(C)$ is a closed normal subgroup of $\mathrm{Aut}(C)$ and $\mathrm{Aut}(C) = \mathrm{Autf}(C)\cdot\mathrm{Stab}(c)$.

We will prove that these necessary conditions on the groups are in fact sufficient.

Theorem 2. *Let G be a topological group, H an open profinite subgroup of G and K a closed normal subgroup of G such that the set of finite intersections of the conjugates of H by G is a system of open neighbourhoods of 1 and such that $G = HK$.*

(i) There exists a trivial strongly minimal structure M with a non algebraic component $C = \mathrm{iacl}(c)$ such that $\mathrm{acl}(\emptyset) = \emptyset$ and such that $\mathrm{Aut}(C)$, $\mathrm{Stab}(c)$ and $\mathrm{Autf}(C)$ are homeomorphic respectively to G, H and K.

(ii) If N is a trivial strongly minimal structure such that $\mathrm{acl}(\emptyset) = \emptyset$, we can reconstruct N, in the following sense, from its dimension and the groups $\mathrm{Aut}(D)$, $\mathrm{Autf}(D)$, $\mathrm{Stab}(d)$ where $D = \mathrm{acl}(d)$ is a non algebraic component of N. Let C' be the structure $(\mathrm{Aut}(D)/\mathrm{Stab}(d), R_\Theta, E_\Theta)$ where for each orbit Θ of n-tuples ($n \geq 2$) of $\mathrm{Aut}(D)/\mathrm{Stab}(d)$ under the action of $\mathrm{Aut}(D)$ by left multiplication, R_Θ is the n-ary relation satisfied by the tuples in Θ and E_Θ is an equivalence relation on Θ whose classes are the $\mathrm{Autf}(D)$-orbits on R_Θ. Then N is equidefinable up to isomorphism to $\widehat{N} = \cup_{i \in \dim(N)} C' \times \{i\}$ with the following structure: for every $\overline{a}$, $\overline{b} \in C'$, every orbit Θ, and every $i, i' \in \dim(N)$, we have $\widehat{N} \models R_\Theta(\overline{a}, i)$ if and only if $\overline{a} \in \Theta$ and $\widehat{N} \models E_\Theta((\overline{a}, i), (\overline{b}, i'))$ if and only if $C' \models E_\Theta(\overline{a}, \overline{b})$.

Remark. *By Theorem 1, M is of bounded valency if and only if $G = K$.*

We begin by a construction which involves no topological considerations.

For G a group, H a subgroup of G and $g \in G$, we denote by H^g the conjugate $g^{-1}Hg$ of H by g.

Lemma 3.1. *Let G be a group and H, K subgroups of G such that for every $g \in G$, $[H : H \cap H^g]$ is finite, K is normal in G and $G = HK$.*

Let C be the structure $G/H = \{gH : g \in G\}$ with, for each $n \geq 2$ and for each orbit Θ of n-tuples of G/H under the action of G by left multiplication, an n-ary relation R_Θ satisfied by the tuples in Θ. Then the relations R_Θ have bounded valency in C, G acts as an automorphism group on C and $\mathrm{Aut}(C)$ acts transitively on C.

Let C' be the expansion of C where, for each orbit Θ, there is an equivalence relation $E_\Theta \subseteq R_\Theta \times R_\Theta$ on Θ whose classes are the K-orbits on R_Θ. Then G acts also as an automorphism group on C' and K is transitive on C'.

Proof. Let $g \in G$, then g acts as an automorphism on the structure C. Hence $\mathrm{Aut}(C)$ acts transitively on C.

Let Θ be the orbit of a tuple $H, g_2H, \ldots, g_nH$. The number of tuples $\overline{a} = (a_1, \ldots, a_n) \in \Theta$ such that $a_1 = H$ is finite because for every $i \geq 2$, the number of conjugates of g_iH by the action of H is equal to $[H : H \cap H^{g_i^{-1}}]$. It follows that R_Θ has bounded valency.

For every $k \in K$, k acts as an automorphism on the structure C' and fixes setwise the equivalence classes defined by the relations E_Θ. Since K is a normal subgroup of G, G acts on C' : let $\overline{a}$, $\overline{b}$ in an orbit Θ be such that $E_\Theta(\overline{a}, \overline{b})$. Then there is $k \in K$ such that $\overline{b} = k\overline{a}$. Let $g \in G$. There is $k' \in K$ such that $gk = k'g$. Thus $g\overline{b} = k'(g\overline{a})$ and $E_\Theta(g\overline{a}, g\overline{b})$. Since $G = HK$ and G is transitive on C', the subgroup K is transitive on C'. $\square$

The following lemma will be used for the proof of the first part of Theorem 2 and for the construction of examples of trivial strongly minimal structures of non bounded valency in section 4.

Lemma 3.2. *Let M be an infinite structure of bounded valency. Let M' be the expansion of M by some relations $(E_j)_{j\in J}$, such that, for every $j \in J$, there is a relation R_j in $L_{bound}(M)$ such that $E_j \subseteq R_j \times R_j$ and E_j defines an equivalence relation on the tuples satisfying R_j. Let K be the subgroup of $\mathrm{Aut}(M')$ fixing setwise the equivalence classes defined by the relations E_j in $R_j \times R_j$. Suppose that K acts transitively on M'. Then*

(i) the structure M is trivial strongly minimal and if C is a component of M, the structure M' is equidefinable up to isomorphism to $\widetilde{M} = \cup_{i\in\dim(M)} C \times \{i\}$, with the following structure: for every $\overline{a}$, $\overline{b} \in C$, every $\emptyset$-definable relation R of bounded valency in M, and every i, $i' \in \dim(M)$, we have that $\widetilde{M} \models R((\overline{a}, i), (\overline{b}, i))$ if and only if $M \models R(\overline{a}, \overline{b})$, and, for every $j \in J$, $\widetilde{M} \models E_j((\overline{a}, i), (\overline{b}, i'))$ if and only if $M' \models E_j(\overline{a}, \overline{b})$,

(ii) the structure M' is trivial strongly minimal and algebraic closure in M' corresponds to algebraic closure in M,

(iii) $K = \mathrm{Autf}(M')$, and

(iv) the structure M' is of bounded valency if and only if $K = \mathrm{Aut}(M')$.

Proof. For the proof, we consider M as an $L_{bound}(M)$-structure and so M' as an $L_{bound}(M) \cup \{E_j : j \in J\}$-structure. Note first that, as K acts transitively on M', $\mathrm{acl}_M(\emptyset) = \mathrm{acl}_{M'}(\emptyset) = \emptyset$, by Lemma 2.1, M is trivial strongly minimal, and for every $j \in J$, since R_j is an n_j-ary bounded valency relation, E_j can be seen as a finite equivalent relation on M^{n_j} (we can assume that the set $M^{n_j} \setminus R_j$ corresponds to one equivalence class).

Let $c \in M'$ and $C = \mathrm{iacl}_M(c)$. Let $\bar{c} = (c_k)_{k\in\lambda}$ be an enumeration of C. Let p be the type of $\overline{c}$ in M and let C' be the expansion of C by the relations E_j.

Since K acts transitively on M', for every $a \in M'$ there exists $\overline{a} = (a_k)_{k\in\lambda} \in M'$ such that $\overline{a}$ realizes p, $a_0 = a$ and for every $j \in J$ and $k_1, \ldots, k_l \in \lambda$, if $(c_{k_1} \ldots c_{k_l})$ satisfies R_j, then $(c_{k_1} \ldots c_{k_l})$ and $(a_{k_1} \ldots a_{k_l})$ are E_j-equivalent. Note that $\overline{a}$ is an enumeration of $\mathrm{iacl}_M(a)$ which we will call a $\overline{c}$-enumeration.

Let N' be a structure elementarily equivalent to M' and N the reduct of N' to the language of M. We will show that N' is isomorphic to $\widetilde{N}$ the disjoint union of $\dim(N)$ copies of C', $\cup_{i\in\dim(N)} C' \times \{i\}$, with the following structure : for every $\overline{a}$, $\overline{b} \in C'$, every $\emptyset$-definable relation R of bounded valency in M, and every i, $i' \in \dim(N)$, we have that $\widetilde{N} \models R((\overline{a}, i), (\overline{b}, i))$ if and only if $C' \models R(\overline{a}, \overline{b})$, and, for every $j \in J$, $\widetilde{N} \models E_j((\overline{a}, i), (\overline{b}, i'))$ if and only if $C' \models E_j(\overline{a}, \overline{b})$.

We can suppose that M' and N' have a common sufficiently saturated elementary extension W'. For every $a \in W'$, there exists a $\overline{c}$-enumeration of $\mathrm{iacl}_W(a)$ where W is the reduct of W' to the language of M.

Let $(a^i)_{i\in\dim(N)}$ be a family of representatives of components of N. Let $(\overline{a}^i)_{i\in\dim(N)}$ be such that for every $i \in \dim(N)$, $\overline{a}^i$ is a $\overline{c}$-enumeration of $\mathrm{iacl}_W(a^i)$

which is equal to $\mathrm{iacl}_N(a^i)$. Then N' is equal to the disjoint union of the $\overline{a}^i$'s so we have a bijection between N' and $\cup_{i\in\dim(N)} C' \times \{i\}$. Moreover for every $j \in J$ and $k_1, \dots, k_l \in \lambda$, if $W \models R_j(c_{k_1}, \dots, c_{k_l})$ then for every $i, i' \in \dim(N)$, $W' \models E_j((a^i_{k_1}, \dots, a^i_{k_l}), (a^{i'}_{k_1}, \dots, a^{i'}_{k_l}))$ since E_j is an equivalence relation. Thus N' is isomorphic to the structure $\widetilde{N} = \cup_{i\in\dim(N)} C' \times \{i\}$. In particular, this shows (i).

Now we will prove (ii). For this, we show that for every subset A of N', $\mathrm{iacl}_N(A)$ is equal to $\mathrm{acl}_{N'}(A)$ and that there is only one non algebraic type over A. Let $a,\ b \in N' \backslash \mathrm{iacl}_N(A)$. Let $\overline{a}$ and $\overline{b}$ be $\overline{c}$-enumerations of respectively $\mathrm{iacl}_N(a)$ and $\mathrm{iacl}_N(b)$. The map which sends $\overline{a}$ onto $\overline{b}$ defines a partial isomorphism which fixes the equivalence classes defined by the relations E_j. If a and b are in distinct components of N, the map which sends $\overline{a}\overline{b}$ onto $\overline{b}\overline{a}$ is a partial isomorphism of $\mathrm{iacl}_N(a) \cup \mathrm{iacl}_N(b)$ onto itself. We then obtain an automorphism of N' by extending the partial isomorphism of $\mathrm{iacl}_N(a) \cup \mathrm{iacl}_N(b)$ onto itself by the identity on $N' \setminus \mathrm{iacl}_N(\overline{a}\overline{b})$, since this map fixes the E_j-classes for every $j \in J$. Thus a and b have the same type over A in N'.

Now we will prove (iii). Since for every j, E_j is a finite equivalence relation, $\mathrm{Autf}(M') \subseteq K$. Conversely if $f \in K$ and if V' is any elementary extension of M' such that $V' \setminus M'$ is infinite, then we obtain an automorphism of V' by extending f by the identity on $V' \setminus M'$ which is an elementary substructure of V' because V' is strongly minimal and $\mathrm{acl}(\emptyset) = \emptyset$. Thus f is a strong automorphism of M'.

(iv) follows from Lemma 2.2: since $\mathrm{acl}(\emptyset) = \emptyset$, M' is of bounded valency if and only if all types over the empty set are stationary if and only if $\mathrm{Autf}(M') = \mathrm{Aut}(M')$. $\square$

Proof of Theorem 2 (i).

Let G be a topological group, H an open profinite subgroup of G and K a closed normal subgroup of G such that the set of finite intersections of the conjugates of H by G is a system of open neighbourhoods of 1 and such that $G = HK$.

Since H is a profinite subgroup of G and the set of finite intersections of the conjugates H^g is a system of open neighbourhoods of 1, for every $g \in G$, the open subgroup $H \cap H^g$ has finite index in H. So the triplet (G, H, K) satisfies the hypothesis of Lemma 3.1. Let C and C' be the associated structures: recall that C is the structure $G/H = \{gH;\ g \in G\}$ with, for each $n \geq 2$ and for each orbit Θ of n-tuples of G/H under the action of G, an n-ary relation R_Θ satisfied by the tuples in Θ, and that C' is the expansion of C where, for each orbit Θ, there is an equivalence relation $E_\Theta \subseteq R_\Theta \times R_\Theta$ on Θ, given by the action of K on Θ.

We are going to show that G and H are homeomorphic respectively to $\mathrm{Aut}(C)$ and $\mathrm{Stab}(H)$ in the structure C.

By Lemma 3.1, G acts as an automorphism group on C. Since H is profinite, $\cap_{g\in G} H^g = \{1\}$. Thus if $g \in G$ acts as the identity on C then $g = 1$. Moreover for every $g \in G$, $\mathrm{Stab}(gH) \cap G = H^{g^{-1}}$. So we can see G as a topological subgroup of $\mathrm{Aut}(C)$.

Let $f \in \mathrm{Stab}(H)$, $\overline{a} \in C$ and $\overline{b} = f(\overline{a})$ such that $a_0 = H = b_0$. Let Θ be the orbit of $\overline{a}$ by the action of G. Then both $\overline{a}$ and $\overline{b}$ satisfy R_Θ. Thus there is $g \in G$ such that $g\overline{a} = \overline{b}$ and since $a_0 = H = b_0$, $g \in H$. It follows that the stabilizer of H in C is a subset of $\overline{H}$.

Since H is compact, so closed in $\mathrm{Aut}(C)$, $\mathrm{Stab}(H) = H$. It follows that $G = \mathrm{Aut}(C)$.

We have seen in Lemma 3.1 that G acts as an automorphism group on the structure C', so $\mathrm{Aut}(C) = G \subseteq \mathrm{Aut}(C') \subseteq \mathrm{Aut}(C) = G$, and H is also the stabilizer of H in C'.

We now use that K is closed. Let $g \in G$ fix setwise the equivalence classes defined by the relations E_Θ. Let $\overline{a}$ be in an orbit Θ. Then $C' \models E_\Theta(\overline{a}, g\overline{a})$ so there is $k \in K$ such that $k\overline{a} = g\overline{a}$. Thus $g \in \overline{K} = K$. So K is equal to the subgroup of $G = \mathrm{Aut}(C')$ fixing setwise the equivalence classes defined by the relations E_Θ.

Note that G is finite if and only if $[G : H]$ is finite: indeed if $[G : H]$ is finite then G contains elements $g_1, \ldots, g_k$ such that $\cap_{1\leq i\leq k} H^{g_i} = \{1\}$. Then H is finite because H is profinite. So, if G is infinite, then C' is infinite and by Lemmas 3.1 and 3.2, C' is a trivial strongly minimal structure such that G, H and K are respectively homeomorphic to $\mathrm{Aut}(C')$, $\mathrm{Stab}(H)$ and $\mathrm{Autf}(C')$. If G is finite, we consider the disjoint union of ω copies of C', $M = \cup_{i\in\omega} C' \times \{i\}$ with the following structure : for every $\overline{a}$, $\overline{b} \in C'$, every orbit Θ, and every $i, i' \in \omega$, we have $M \models R_\Theta(\overline{a}, i)$ if and only if $\overline{a} \in \Theta$ and $M \models E_\Theta((\overline{a}, i), (\overline{b}, i'))$ if and only if $C' \models E_\Theta(\overline{a}, \overline{b})$. In this case we obtain an $\aleph_0$-categorical structure since C' is finite. $\square$

For the second part of Theorem 2, the reconstruction problem, we show:

Lemma 3.3. *Let M be a trivial strongly minimal structure such that* $\mathrm{acl}(\emptyset) = \emptyset$.

(i) If p is the type of a finite tuple of interalgebraic elements, there is a $\emptyset$-definable bounded valency relation R_p which isolates p and a $\emptyset$-definable equivalence relation E_p on the tuples realizing R_p which partitions p into the different strong types which extend it.

(ii) Then M has the same $\emptyset$-definable relations as the structure M_0, the structure M reduced to the relations R_p and E_p, for every type p of interalgebraic tuples.

Proof. (i) As $\mathrm{acl}(\emptyset) = \emptyset$, every type p of a finite tuple of interalgebraic elements has Morley rank equal to one and so this type p is isolated by an $\emptyset$-definable relation R_p which is of bounded valency. Moreover in a strongly minimal structure there are only finitely many strong types which extend a fixed type, so for each relation R_p, there is an $\emptyset$-definable equivalence relation E_p on the tuples realizing R_p which partitions p into the different strong types which extend it. (Observe that the number of strong types which extend p is bounded by the valency of R_p, where the valency of a bounded valency relation R is the smallest m such that for every $a \in M$, the number of tuples in R containing a is bounded by m.)

(ii) Let N be a saturated elementary extension of M and N_0 its reduced structure. Let $\overline{a}$ and $\overline{b}$ be two finite tuples in N which realize the same type in N_0. We will show that $\overline{a}$ and $\overline{b}$ realize the same type in N.

It is clear that $\mathrm{iacl}_{N_0} = \mathrm{iacl}_N$. We can assume that $\overline{a} = \overline{a}_1 \ldots \overline{a}_n$ such that for every i, $\overline{a}_i$ is a tuple of interalgebraic elements and for distinct i, j, $\overline{a}_i$ and $\overline{a}_j$ are in distinct components of N. Then $\overline{b} = \overline{b}_1 \ldots \overline{b}_n$ with the same properties. An automorphism of N_0 which sends $\overline{a}$ to $\overline{b}$ sends $\mathrm{iacl}(\overline{a}_1)$ to $\mathrm{iacl}(\overline{b}_1)$. Thus we can assume that $\overline{a}_1$ is an enumeration of $\mathrm{iacl}(\overline{a}_1)$ and that $\overline{b}_1$ is an enumeration of $\mathrm{iacl}(\overline{b}_1)$.

Since the elements of $\overline{a}_1$ are interalgebraic, $\overline{a}_1$ and $\overline{b}_1$ realize the same type in N. By an automorphism of N, we can replace $\overline{b}$ with $\overline{b}'$ such that $\overline{b}$ and $\overline{b}'$ have the same type in N and $\overline{b}'_1 = \overline{a}_1$. Then for every $i > 1$, $\overline{a}_i$ and $\overline{b}'_i$ realize the same strong type in N: indeed for every $i > 1$, there is $\overline{d}_i \subseteq \overline{a}_0$ such that $\overline{d}_i$ and $\overline{a}_i$ have the same strong type in N, so if p_i is the type of $\overline{a}_i$ and $\overline{b}'_i$, we have $E_{p_i}(\overline{a}_i, \overline{b}'_i)$. Moreover $\overline{a}_1, \ldots, \overline{a}_n$ are independent, $\overline{b}'_1, \overline{b}'_2, \ldots, \overline{b}'_n$ are independent, and thus $\overline{a}$ and $\overline{b}'$ realize the same strong type in N.

So $\overline{a}$ and $\overline{b}$ have the same type in N. $\square$

Proof of Theorem 2 (ii).

Let N be a trivial strongly minimal structure such that $\mathrm{acl}(\emptyset) = \emptyset$. Let $d \in N$ and $D = \mathrm{acl}(d)$. By lemma 3.3, we can assume that N is the structure with the relations R_p and E_p for the types p realized in D. Now, by Lemma 3.2 (i), N is isomorphic to $\widetilde{N} = \cup_{i \in \dim(N)} D \times \{i\}$, with the following structure: for every $\overline{a}$, $\overline{b} \in D$, every type p realized in D, and every i, $i' \in \dim(N)$, we have that $\widetilde{N} \models R_p((\overline{a}, i), (\overline{b}, i))$ if and only if $N \models R_p(\overline{a}, \overline{b})$, and $\widetilde{N} \models E_p((\overline{a}, i), (\overline{b}, i'))$ if and only if $N \models E_p(\overline{a}, \overline{b})$.

Let C' be the structure $(\mathrm{Aut}(D)/\mathrm{Stab}(d), R_\Theta, E_\Theta)$ where for each orbit Θ of n-tuples ($n \geq 2$) of $\mathrm{Aut}(D)/\mathrm{Stab}(d)$ under the action of $\mathrm{Aut}(D)$ by left multiplication, R_Θ is the n-ary relation satisfied by the tuples in Θ and E_Θ is an equivalence relation on Θ whose classes are the $\mathrm{Autf}(D)$-orbits on R_Θ.

Let ϕ be the bijection between D and C' such that $\phi(g(d)) = g\mathrm{Stab}(d)$ for all $g \in \mathrm{Aut}(D)$. Thus for each type p of a finite tuple $\overline{a}$ in D, $\phi(R_p) = R_\Theta$ and $\phi(E_p) = E_\Theta$ where Θ is the orbit of $\phi(\overline{a})$. It follows that $\widetilde{N}$ is equidefinable up to isomorphism to $\widehat{N} = \cup_{i \in \dim(N)} C' \times \{i\}$ with the following structure: for every $\overline{a}$, $\overline{b} \in C'$, every orbit Θ, and every $i, i' \in \dim(N)$, we have $\widehat{N} \models R_\Theta(\overline{a}, i)$ if and only if $\overline{a} \in \Theta$ and $\widehat{N} \models E_\Theta((\overline{a}, i), (\overline{b}, i'))$ if and only if $C' \models E_\Theta(\overline{a}, \overline{b})$. $\square$

Corollary 3.4. *A trivial strongly minimal structure N such that algebraic closure* *is equal to definable closure and* $\mathrm{dcl}(\emptyset) = \emptyset$*, is equidefinable up to isomorphism to a disjoint union of copies of the Cayley graph of the automorphism group* G *of one component (i.e. to the structure* $(\cup_{i \in \dim(N)} G \times \{i\},\ R_g;\ g \in G)$ *where we have* $R_g((g_1, i_1), (g_2, i_2))$ *if and only if* $g_1 = g_2 g$ *and* $i_1 = i_2$*).*

Proof. Let $d \in N$ and $D = \mathrm{iacl}(d)$. Note first that N is of bounded valency and $\mathrm{Stab}(d) = \{1\}$. Let $\overline{a} = a_0 \ldots a_n$ be a finite tuple in D. Let p be the type of $\overline{a}$ and for every $1 \leq i \leq n$, p_i the type of $a_0 a_i$. Then $R_p(x_0, \ldots, x_n)$ is equivalent to the conjunction of the relations $R_{p_i}(x_0, x_i)$. The corollary follows from lemma 3.3 and Theorem 2 (ii). $\square$

We now observe that the unimodularity of a trivial strongly minimal structure can be seen on the group of the automorphisms of a non algebraic component.

Several equivalent definitions of unimodularity for a strongly minimal structure have been given by Hrushovski. Recall that a locally compact topological group is called unimodular if its right-invariant Haar integrals are left-invariant. (See for example [Hig].)

Definition-Proposition 3.5. [Hru 92]
Let M be a strongly minimal saturated structure.

(i) For $\overline{a}$ and $\overline{b}$ in M, $\mathrm{mult}(\overline{a}/\overline{b})$ is the number of strong types which extend the type of $\overline{a}$ over $\overline{b}$.

(ii) M is unimodular if whenever $\overline{a},\ \overline{b} \in M^n$, $\mathrm{acl}(\overline{a}) = \mathrm{acl}(\overline{b})$ and $\dim(\mathrm{acl}(\overline{a})) = n$, then $\mathrm{mult}(\overline{a}/\overline{b}) = \mathrm{mult}(\overline{b}/\overline{a})$.

(iii) If $\overline{a}$ is a finite tuples of independent elements in M (i.e. $\overline{a} \in M^n$ and $\dim(\mathrm{acl}(\overline{a})) = n$), the locally compact $\mathrm{Aut}(\mathrm{acl}(\overline{a}))$ is unimodular if and only if the locally compact $\mathrm{Aut}(\mathrm{iacl}(\overline{a}))$ is unimodular if and only if for every $f \in \mathrm{Aut}(\mathrm{acl}(\overline{a}))$, $\mathrm{mult}(f(\overline{a})/\overline{a}) = \mathrm{mult}(\overline{a}/f(\overline{a}))$.

(iv) M is unimodular if and only if for each $X \subset M$ such that $X = \mathrm{acl}(X)$ and $\dim(X)$ is finite, the locally compact group $\mathrm{Aut}(X)$ is unimodular.

We will use the following lemma in section 5.

Lemma 3.6. *Let* M *be a saturated trivial strongly minimal structure and* $C = \mathrm{iacl}(c)$ *a non algebraic component of* M*. The following are equivalent.*

(i) M *is unimodular.*

(ii) For every $g \in \mathrm{Aut}(C)$,

$$[\mathrm{Stab}(c) : \mathrm{Stab}(c) \cap \mathrm{Stab}(c)^g] = [\mathrm{Stab}(c)^g : \mathrm{Stab}(c) \cap \mathrm{Stab}(c)^g].$$

(iii) $\mathrm{Aut}(C)$ *is unimodular.*

Proof. First, note that for every a be in a non algebraic component C and $g \in \mathrm{Aut}(C)$,

$$\mathrm{mult}(a/g(a)) = [\mathrm{Stab}(a) : \mathrm{Stab}(a\ g(a))] = [\mathrm{Stab}(a) : \mathrm{Stab}(a) \cap \mathrm{Stab}(a)^g].$$

So ((i) $\Rightarrow$ (ii)) follows from (3.5 (ii)) and ((ii) $\Leftrightarrow$ (iii)) follows from (3.5 (iii)).

We will now prove ((ii) $\Rightarrow$ (i)). Let $\overline{a}$, $\overline{b} \in M^n$ such that $\mathrm{acl}(\overline{a}) = \mathrm{acl}(\overline{b})$ and $\dim(\mathrm{acl}(\overline{a})) = n$. We have to show that $\mathrm{mult}(\overline{a}/\overline{b}) = \mathrm{mult}(\overline{b}/\overline{a})$. We can assume that $\overline{a} = a_1 \dots a_n$ and $\overline{b} = b_1 \dots b_n$ such that for every i, a_i and b_i are interalgebraic. So by (ii), for every i, $\mathrm{mult}(a_i/b_i) = \mathrm{mult}(b_i/a_i)$.

We will show that $\mathrm{mult}(\overline{a}/\overline{b}) = \mathrm{mult}(a_1 \dots a_{n-1} b_n / b_1 \dots b_{n-1} a_n)$. It follows by induction that $\mathrm{mult}(\overline{a}/\overline{b}) = \mathrm{mult}(\overline{b}/\overline{a})$.

First, observe that the type of $\overline{b}$ is stationary. It follows that

$$\mathrm{mult}(b_1 \dots b_{n-1}/a_n b_n) = \mathrm{mult}(b_1 \dots b_{n-1}/b_n) = 1.$$

In the following equations we use that if $\overline{x}$ and $\overline{y}$ are independent over A then $\mathrm{mult}(\overline{x}\overline{y}/A) = \mathrm{mult}(\overline{x}/A\overline{y}) \cdot \mathrm{mult}(\overline{y}/A)$.
$\mathrm{mult}(a_n b_1 \dots b_{n-1}/b_n) = \mathrm{mult}(a_n/\overline{b}) \cdot \mathrm{mult}(b_1 \dots b_{n-1}/b_n) = \mathrm{mult}(a_n/\overline{b})$ and
$\mathrm{mult}(a_n b_1 \dots b_{n-1}/b_n) = \mathrm{mult}(b_1 \dots b_{n-1}/a_n b_n) \cdot \mathrm{mult}(a_n/b_n) = \mathrm{mult}(a_n/b_n)$.

Thus $\mathrm{mult}(a_n/\overline{b}) = \mathrm{mult}(a_n/b_n) = \mathrm{mult}(b_n/a_n) = \mathrm{mult}(b_n/a_n b_1 \dots b_{n-1})$.

It follows

$$\begin{aligned}
\mathrm{mult}(\overline{a}/\overline{b}) &= \mathrm{mult}(a_n/\overline{b}) \cdot \mathrm{mult}(a_1 \dots a_{n-1}/a_n \overline{b}) \\
&= \mathrm{mult}(b_n/a_n b_1 \dots b_{n-1}) \cdot \mathrm{mult}(a_1 \dots a_{n-1}/a_n b_1 \dots b_{n-1} b_n) \\
&= \mathrm{mult}(a_1 \dots a_{n-1} b_n / b_1 \dots b_{n-1} a_n).
\end{aligned}$$

□

In the same spirit, we have remarked in certain cases that the small index property can be seen on the automorphism group of a component:

Proposition. *[Blo, Corollaire 2.4.13 and Corollaire 2.4.14] Let M be a countable trivial strongly minimal structure which contains a non algebraic point c. If M is of bounded valency or has a finite language then M has the small index property if and only if the topological group* $\mathrm{Aut}(\mathrm{acl}(c))$ *has the small index property.*

4 Examples

In this section we will present some examples of trivial strongly minimal structures in the three following cases : the $\aleph_0$-categorical case, the non-$\aleph_0$-categorical unimodular case and the non-unimodular case. In each case we construct in several steps an example which is not of bounded valency. (These examples are all constructed as double covers of very simple structures. We refer the reader for finite or double covers to [E-I-M].)

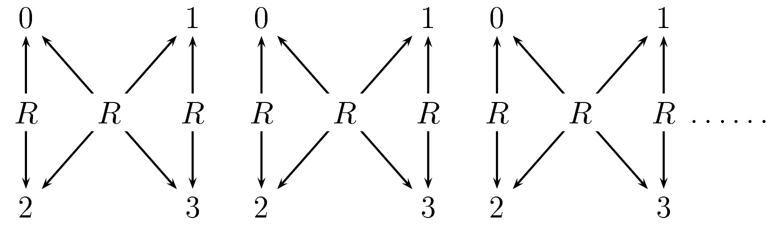
We begin by $\aleph_0$-categorical examples which are double covers of the infinite set with only one equivalence relation such that all classes have cardinal two.

Examples 4.1 ($\aleph_0$-categorical). Let I be an infinite set.

(i) Let $M_0 = (\mathbb{Z}/4\mathbb{Z} \times I, R)$ where R is the binary relation defined by

$$R = \{((2m+\varepsilon, i), (2m+2+\varepsilon', i));\ i \in I,\ m, \varepsilon, \varepsilon' \in \{0,1\}\}.$$

Structure M_0

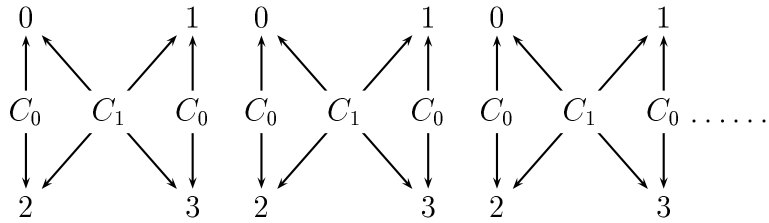


The relation R is of bounded valency and it is easy to check that the group $\mathrm{Aut}(M_0)$ acts transitively on M_0. So by Lemma 2.1, M_0 is trivial strongly minimal. Let C be a component of M_0. Then $C = \mathbb{Z}/4\mathbb{Z} \times \{i\}$ for $i \in I$, $\mathrm{Aut}(C) = (\mathbb{Z}/2\mathbb{Z})^2 \rtimes S_2$ the wreath product of $\mathbb{Z}/2\mathbb{Z}$ by S_2, $\mathrm{Stab}((0,i)) = \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Aut}(M_0) = \mathrm{Aut}(C) \rtimes S_I$ the wreath product of $\mathrm{Aut}(C)$ by S_I.

(ii) Let M_1 be the expansion of M_0 by an equivalence relation E on R with two classes C_0 and C_1 defined by

$$C_0 := \{((2m+\varepsilon, i), (2m+2+\varepsilon, i))\} \text{ and } C_1 := \{((2m+\varepsilon, i), (2m+2+(1-\varepsilon), i))\}.$$

Structure M_1



It is clear that K the subgroup of automorphisms which fixes setwise the classes C_0 and C_1, also acts transitively on M_1. So by 3.2, M_1 is trivial strongly minimal.

The component C in M_0 is still a component in M_1 and $\mathrm{Aut}(C)$ and $\mathrm{Stab}((0,i))$ are unchanged inside M_1. But in this case, $\mathrm{Autf}(C) = \mathbb{Z}/2\mathbb{Z} \times S_2$ so M_1 is not of bounded valency.

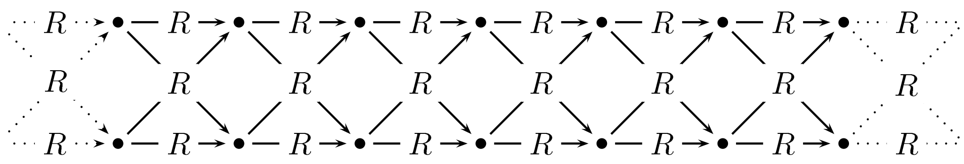
The following examples are all reduced to one non algebraic component.

We are now presenting unimodular non-$\aleph_0$-categorical examples.

Examples 4.2 (Unimodular non-$\aleph_0$-categorical).
(i) Let M_2 be the structure $\mathbb{Z}$ with the successor function. This structure is trivial strongly minimal of bounded valency, $\mathrm{Aut}(M_2) = \mathbb{Z}$ and $\mathrm{Stab}(0) = \{0\}$.

The following examples M_3, M_4 and M_5 are double covers of M_2.
(ii) Let $M_3 = (\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z}), R)$ with R the binary relation defined by

$$R := \{((m, \varepsilon), (m+1, \varepsilon'));\ m \in \mathbb{Z},\ \varepsilon, \varepsilon' \in \mathbb{Z}/2\mathbb{Z}\}.$$



Structure M_3

The group $\mathrm{Aut}(M_3)$ acts transitively on M_3 and R has bounded valency, so by Lemma 2.1 M_3 is trivial strongly minimal of bounded valency. We can see that $\mathrm{Aut}(M_3)$ is the wreath product $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}} \rtimes \mathbb{Z}$ and $\mathrm{Stab}((0,0)) = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^*}$.

(iii) Let M_4 be the expansion of M_3 by a binary relation S defined by

$$S := \{((m, \varepsilon), (m+2, \varepsilon));\ m \in \mathbb{Z},\ \varepsilon \in \mathbb{Z}/2\mathbb{Z}\}.$$

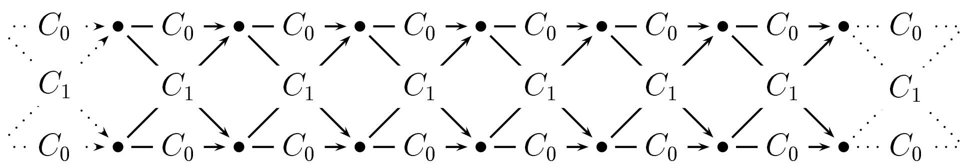
Again $\mathrm{Aut}(M_4)$ acts transitively on M_4 and S has bounded valency, so by Lemma 2.1 M_4 is trivial strongly minimal of bounded valency. Here $\mathrm{Aut}(M_4)$ is the semidirect product

$$(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \rtimes_{\psi} \mathbb{Z}$$

such that $\psi(2m) = id$ and $\psi(2m+1)(\theta_0, \theta_1) = (\theta_1, \theta_0)$, and $\mathrm{Stab}((0,0))$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})$.

(iv) Let M_5 be the expansion of M_3 by an equivalence relation E on R with two classes C_0 and C_1 defined by

$$C_0 := \{((m, \varepsilon), (m+1, \varepsilon))\} \text{ and } C_1 := \{((m, \varepsilon), (m+1, \varepsilon+1))\}.$$



Structure M_5

Note first that the binary relation S is definable in M_5. Hence M_5 is an expansion of M_4.

It is easy to check that K the subgroup of automorphisms which fixes setwise the classes C_0 and C_1 acts transitively on M_5. So by 3.2, M_5 is trivial strongly minimal.

We can see easily that $\mathrm{Aut}(M_5) = \mathrm{Aut}(M_4)$ and $\mathrm{Stab}((0,0)) = (\mathbb{Z}/2\mathbb{Z})$. So, by the reconstruction theorem (Theorem 2), the reduct of M_5 to the bounded valency language is equidefinable with M_4.

Moreover $\mathrm{Autf}(M_5) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$ so M_5 is not of bounded valency.

We finish this section by some non unimodular examples.

Examples 4.3 (Non unimodular).

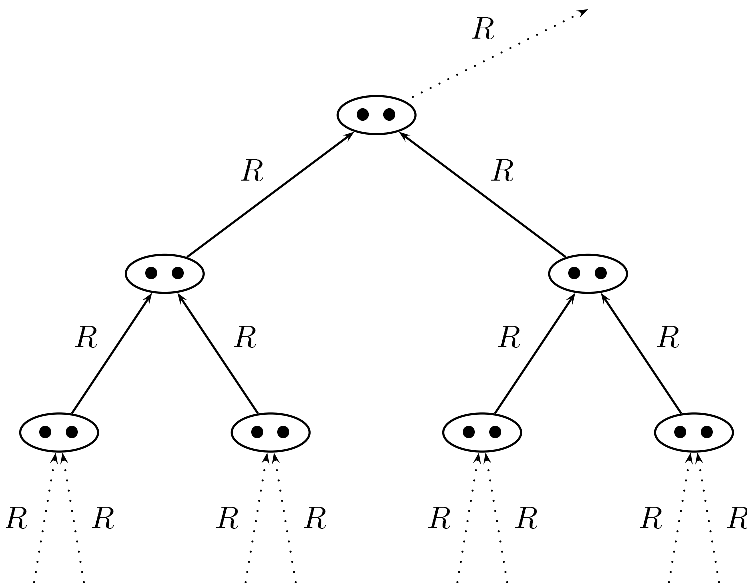
(i) Let Γ be the binary tree without a root (i.e. a connected structure with a function f such that any element has exactly two antecedents by f and f has no cycle).

This structure Γ is a very simple example of non-unimodular trivial strongly minimal structures.

We will now consider some double covers of Γ.

(ii) Let $M_6 = (\Gamma \times \mathbb{Z}/2\mathbb{Z}, R)$ where R is the binary relation defined by

$$\{((x,\varepsilon),(f(x),\varepsilon'));\ x \in \Gamma,\ \varepsilon,\varepsilon' \in \mathbb{Z}/2\mathbb{Z}\}.$$



Structure M_6

The relation R has bounded valency and since $\mathrm{Aut}(\Gamma)$ acts transitively on Γ, $\mathrm{Aut}(M_6)$ also acts transitively on M_6. So by Lemma 2.1, M_6 is trivial strongly minimal of bounded valency.

Let x be a point of Γ. It is easy to check that $\mathrm{Aut}(M_6)$ is equal to $(\mathbb{Z}/2\mathbb{Z})^\Gamma \rtimes \mathrm{Aut}(\Gamma)$, the wreath product of $\mathbb{Z}/2\mathbb{Z}$ by $\mathrm{Aut}(\Gamma)$ and $\mathrm{Stab}((x,0))$ is equal to $(\mathbb{Z}/2\mathbb{Z})^{\Gamma\setminus\{x\}} \rtimes \mathrm{Stab}(x)$ (where $\mathrm{Stab}(x)$ is the stabilizer of x in Γ).

(iii) Let M_7 be the expansion of M_6 by a binary relation S defined by

$$\{((x,\varepsilon),(f^2(x),\varepsilon));\ x \in \Gamma,\ \varepsilon \in \mathbb{Z}/2\mathbb{Z}\}.$$

The relation S has bounded valency (it defines a bijection) and $\mathrm{Aut}(M_7)$ acts again transitively since for every $\gamma \in \mathrm{Aut}(\Gamma)$ and all $\mu \in \mathbb{Z}/2\mathbb{Z}$, the function $f : (x, \varepsilon) \mapsto (\gamma(x), \varepsilon + \mu)$ is an automorphism. So M_7 is strongly minimal of bounded valency.

Let $\gamma \in \mathrm{Aut}(\Gamma)$. We say that γ is even if for all (one) $x \in \gamma$ the distance between x and $\gamma(x)$ is even. It is easy to check that:

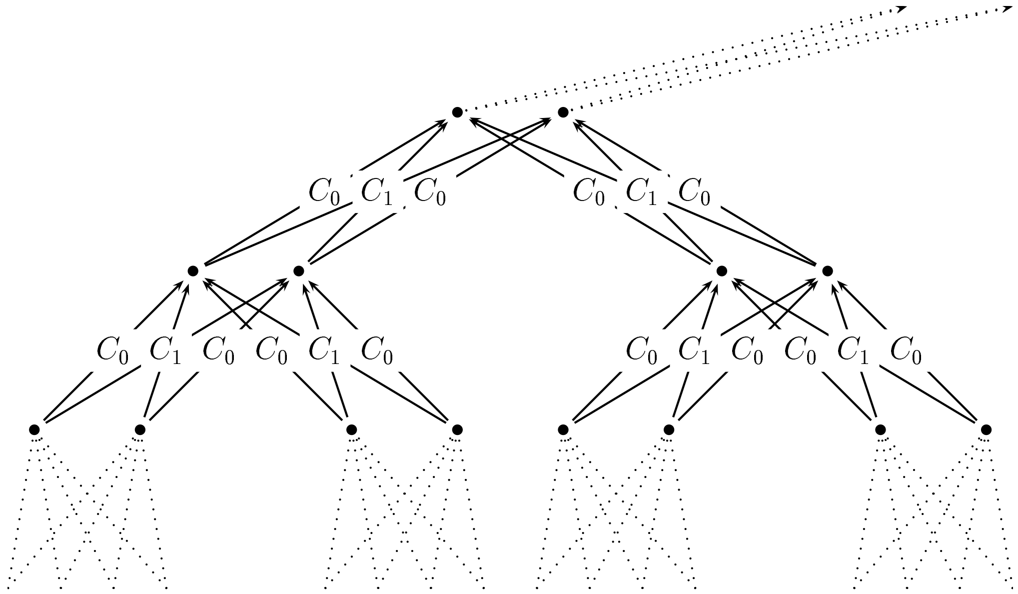
$$\mathrm{Aut}(M_7) = (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \rtimes_\psi \mathrm{Aut}(\Gamma)$$

where $\psi(\gamma)$ is identity if γ is even and $\psi(\gamma)(\varepsilon, \varepsilon') = (\varepsilon', \varepsilon)$ otherwise. And for $x \in \Gamma$,

$$\mathrm{Stab}((x, 0)) = \mathbb{Z}/2\mathbb{Z} \times \mathrm{Stab}(x).$$

(iv) Let M_8 be the expansion of M_6 by an equivalence relation E on R which two classes C_0 and C_1 defined by

$$C_0 := \{((x, \varepsilon), (f(x), \varepsilon))\} \text{ and } C_1 := \{((x, \varepsilon), (f(x), \varepsilon + 1))\}.$$



Structure M_8

Note that the binary relation S is definable in M_8. Hence M_8 is an expansion of M_7.

It is easy to check that the subgroup of automorphisms which fixes setwise the classes C_0 and C_1 is equal to $\mathbb{Z}/2\mathbb{Z} \times \mathrm{Aut}(\Gamma)$ (i.e. $\{(x, \varepsilon) \mapsto (\gamma(x), \varepsilon + \mu);\ \gamma \in \Gamma,\ \mu \in \mathbb{Z}/2\mathbb{Z}\}$). So it acts transitively on M_8 and by 3.2, M_8 is trivial strongly minimal.

We can see easily that $\mathrm{Aut}(M_8) = \mathrm{Aut}(M_7)$ and for $x \in \Gamma$, $\mathrm{Stab}((x, 0))$ is unchanged. So, by the reconstruction theorem, the reduct of M_8 to the bounded valency language is equidefinable with M_7.

Moreover $\mathrm{Autf}(M_8) = \mathbb{Z}/2\mathbb{Z} \times \mathrm{Aut}(\Gamma)$ is not equal to $\mathrm{Aut}(M_7)$, so M_8 is not of bounded valency.

5 Stabilizer of a point

In all this section we consider the stabilizer of a point inside a component, that is, the subgroup which fixes this point inside the automorphism group of its component.

As we said in the introduction we will show that any profinite group can be realized as the stabilizer of a non algebraic point in a unimodular strongly minimal structure of bounded valency.

Theorem 3. *Let H be a profinite group.*

(i) There exists a unimodular strongly minimal structure of bounded valency such that H is homeomorphic to the stabilizer of a non algebraic point.

(ii) There exists a non-unimodular strongly minimal structure of bounded valency such that H is homeomorphic to the stabilizer of a non algebraic point if and only if H has two homeomorphic open subgroups A_1 and A_2 of H such that $[H : A_1] \neq [H : A_2]$.

(iii) The above structures can be chosen to have exactly one non-algebraic component.

(iv) If H is separable, the above structures can be chosen to be countable.

For the proof of this theorem, we will use HNN extensions. We are going to recall briefly the mains facts about HNN extensions which we will be using. For more details see for example [Coh].

Let H be a group and let $(A_i, A_{-i})_{i\in I}$ be a family of pairs of isomorphic subgroups of H. For every $i \in I$, let α_i be an isomorphism between A_i and A_{-i}. Let P be the free group on $(p_i)_{i\in I}$ and let N be the normal subgroup of the free product $H * P$ generated by $\{p_i^{-1}a_ip_i(\alpha_i(a_i))^{-1};\ i \in I,\ a_i \in A_i\}$.

Then $G = (H * P)/N$ is called the HNN **extension** of the base group H with stable letters (p_i) and associated pairs of subgroups (A_i, A_{-i}). We denote this HNN extension by $\langle H, p_i;\ p_i^{-1}A_ip_i = A_{-i},\ i \in I\rangle$.

We say that an element $h_0p_{i_0}^{\varepsilon_0}h_1 \dots h_{n-1}p_{i_{n-1}}^{\varepsilon_{n-1}}h_n$ of $H*P$, where $n \geq 0$, $\varepsilon_j = \pm 1$, $h_j \in H$ has a **pinch** if there is some r such that $i_r = i_{r-1}$, $\varepsilon_r = -\varepsilon_{r-1}$ and $h_r \in A_{\varepsilon_r i_r}$.

We recall the Reduced and Normal Form Theorem [Coh, Theorems 31 and 32]:

Proposition (Britton's Lemma and Normal Form Theorem).

Let $G = \langle H, p_i;\ p_i^{-1}A_ip_i = A_{-i},\ i \in I\rangle$ be an HNN *extension of H. For every $i \in I$, let S_i and S_{-i} be left transversals of A_i and A_{-i} in H (i.e. sets of left coset representatives), all containing* 1.

(i) The canonical homomorphism from H to G is injective, so we can see H as a subgroup of G.

(ii) Any $g \in G$ can be written as an element of $H * P$ without pinch.

(iii) If an element of G can be written as two elements of $H*P$ without pinch

$$h_0 p_{i_0}^{\varepsilon_0} h_1 \dots h_{n-1} p_{i_{n-1}}^{\varepsilon_{n-1}} h_n \text{ and } h'_0 p_{i'_0}^{\varepsilon'_0} h'_1 \dots h'_{n'-1} p_{i'_{n'-1}}^{\varepsilon'_{n'-1}} h'_{n'}$$

then $n = n'$ and for every $j \in \{0, \dots, n-1\}$, $i_j = i'_j$ and $\varepsilon_j = \varepsilon'_j$.

(iv) If $g \in G$ has an expression as above with no pinch and $n > 0$ then $g \notin H$.

(v) **Normal Form.** *Any $g \in G$ can be uniquely written as*

$$g = h_0 p_{i_0}^{\varepsilon_0} h_1 \dots h_{n-1} p_{i_{n-1}}^{\varepsilon_{n-1}} h_n$$

where $n \geq 0$, $\varepsilon_j = \pm 1$, $h_n \in H$, $h_j \in S_{\varepsilon_j i_j}$ for $j < n$ and if $i_{j-1} = i_j$ and $\varepsilon_{j-1} = -\varepsilon_j$ then $h_j \neq 1$.
(for $n = 0$ the expression is just $g = h_0$).

We denote by $l(g)$ the integer n in the above expression.

It follows immediately that

Lemma 5.1. *Let $g \in G$ be written under the normal form*

$$g = h_0 p_{i_0}^{\varepsilon_0} h_1 \dots h_{n-1} p_{i_{n-1}}^{\varepsilon_{n-1}} h_n.$$

(i) Let $g' \in G$. If $l(g') < n$ then $gg' \notin H$.

(ii) g^{-1} is written under the normal form

$$g^{-1} = h'_n p_{i_{n-1}}^{-\varepsilon_{n-1}} \dots h'_1 p_{i_0}^{-\varepsilon_0} h'_0.$$

(iii) If $g' \in G$ can be written as $g' = h'_0 p_{i'_0}^{\varepsilon'_0} h'_1 \dots h'_{n-1} p_{i'_{n-1}}^{\varepsilon'_{n-1}} h'_n$ and $gg' \in H$ then $h_n h'_0 \in A_{\varepsilon'_0 i'_0}$. (We can remark that $h'_0 p_{i'_0}^{\varepsilon'_0} h'_1 \dots h'_{n-1} p_{i'_{n-1}}^{\varepsilon'_{n-1}} h'_n$ has no pinch since $gg' \in H$ and so $l(g') = l(g)$.)

The following technical lemma will be used for the proof of Theorem 3:

Lemma 5.2. *Let $g \in G$ be written under the normal form*

$$h_0 p_{i_0}^{\varepsilon_0} h_1 \dots h_{n-1} p_{i_{n-1}}^{\varepsilon_{n-1}} h_n.$$

with $n > 0$.

(i) Let $K_n = A_{-\varepsilon_{n-1} i_{n-1}}$. For $1 \leq j < n$, let

$$K_j = h_j (\alpha_{i_j}^{-\varepsilon_j} (K_{j+1})) h_j^{-1} \cap A_{-\varepsilon_{j-1} i_{j-1}}$$

and let $K_0 = h_0 (\alpha_{i_0}^{-\varepsilon_0} (K_1)) h_0^{-1}$.

Then $H \cap H^{g^{-1}} = K_0$.

(ii) If for every $i \in I$, $A_{-i} = A_i$, A_i is a normal subgroup of H and α_i is the identity then

$$H \cap H^{g^{-1}} = A_{i_0} \cap A_{i_1} \dots \cap A_{i_{n-1}} \text{ and } H \cap H^g = H \cap H^{g^{-1}}.$$

(iii) If for every $i \in I$, $A_{-i} = A_i$ and α_i is the identity then

$$[H : H \cap H^g] = [H : H \cap H^{g^{-1}}].$$

(iv) If H is a topological group such that for every i, the subgroups A_i and A_{-i} are open and α_i is an homeomorphism, then for every $g \in G$, $H \cap H^g$ is an open subgroup of H.

Proof. (i) For $0 < j \leq n$, let $g_j = h_0 p_{i_0}^{\varepsilon_0} h_1 \dots h_{j-1} p_{i_{j-1}}^{\varepsilon_{j-1}}$.

Let $h \in H$ such that $ghg^{-1} \in H$.

Let $k_n = h_n h h_n^{-1}$ in H. Then $g_n k_n g_n^{-1} = ghg^{-1}$. By Lemma 5.1 (iii) $k_n \in A_{-\varepsilon_{n-1} i_{n-1}} = K_n$ because $g_n = h_0 p_{i_0}^{\varepsilon_0} h_1 \dots h_{n-1} p_{i_{n-1}}^{\varepsilon_{n-1}}$ is written under the normal form, $k_n g_n^{-1} = k_n p_{i_{n-1}}^{-\varepsilon_{n-1}} h_{n-1}^{-1} \dots h_1^{-1} p_{i_0}^{-\varepsilon_0} h_0^{-1}$ and $g_n k_n g_n^{-1} \in H$.

It follows by induction that for $1 \leq j < n$, $k_j := h_j(\alpha_{i_j}^{-\varepsilon_j}(k_{j+1}))h_j^{-1}$ is in $h_j(\alpha_{i_j}^{-\varepsilon_j}(K_{j+1}))h_j^{-1}$, satisfies $g_j k_j g_j^{-1} = ghg^{-1}$ and hence, again by Lemma 5.1 (iii), that $k_j \in A_{-\varepsilon_{j-1} i_{j-1}}$ because $g_j = h_0 p_{i_0}^{\varepsilon_0} h_1 \dots h_{j-1} p_{i_{j-1}}^{\varepsilon_{j-1}}$ is written under the normal form, $k_j g_j^{-1} = k_j p_{i_{j-1}}^{-\varepsilon_{j-1}} h_{j-1}^{-1} \dots h_1^{-1} p_{i_0}^{-\varepsilon_0} h_0^{-1}$ and $g_j k_j g_j \in H$. Then for every $j > 0$, $k_j \in K_j$.

Thus $ghg^{-1} = h_0 p_{i_0}^{\varepsilon_0} k_1 p_{i_0}^{-\varepsilon_0} h_0^{-1}$ with $k_1 \in K_1$ and then we have $ghg^{-1} \in K_0$.

So we have shown that $H^{g^{-1}} \cap H = gHg^{-1} \cap H \subseteq K_0$.

Conversely let $k_0 \in K_0$. We define by induction for $0 < j \leq n$, $k_j := \alpha_{i_{j-1}}^{\varepsilon_{j-1}}((h_{j-1} k_{j-1} h_{j-1}^{-1}))$. Then $g_j k_j g_j^{-1} = k_0$. Thus $g_n k_n g_n^{-1} = k_0$ and so $k_0 = g k_n^{h_n} g^{-1}$.

(ii) With the hypothesis that for every $i \in I$, $A_{-i} = A_i$, A_i is a normal subgroup of H and α_i is the identity, we can see that $K_n = A_{i_{n-1}}$ and that for every $j > 1$, $K_{j-1} = K_j \cap A_{i_{j-2}}$. By (i), $H \cap H^{g^{-1}} = K_0 = \cap_{0<j<n} A_{i_j}$.

By Lemma 5.1 (ii), g^{-1} is written under the normal form $h'_n p_{i_{n-1}}^{-\varepsilon_{n-1}} \dots h'_1 p_{i_0}^{-\varepsilon_0} h'_0$. So the sets K'_j associated to g^{-1} by (i) satisfy $K'_n = A_{i_0}$ and for every $j > 1$, $K'_{j-1} = K'_j \cap A_{i_{n-j+1}}$. So $H \cap H^g = K'_0 = \cap_{0<j<n} A_{i_j} = H \cap H^{g^{-1}}$.

(iii) Observe that $[H : H \cap H^g] = [H : H \cap H^{h_0^{-1} g h_n^{-1}}]$ and that $[H : H \cap H^{g^{-1}}] = [H : H \cap H^{(h_0^{-1} g h_n^{-1})^{-1}}]$. So we can assume that $h_0 = h_n = 1$.

Then $g^{-1} = p_{i_{n-1}}^{-\varepsilon_{n-1}} h_{n-1}^{-1} \dots h_1^{-1} p_{i_0}^{-\varepsilon_0}$ and by Lemma 5.1 (ii) g^{-1} is written under the normal form

$$g^{-1} = p_{i_{n-1}}^{-\varepsilon_{n-1}} h'_{n-1} \dots h'_1 p_{i_0}^{-\varepsilon_0} h'_0.$$

We will check by induction on j that for $0 < j < n$,

$$p_{i_0}^{\varepsilon_0} h_1 \dots h_{n-j-1} p_{i_{n-j-1}}^{\varepsilon_{n-j-1}} h_{n-j} \dots h_{n-1} h'_{n-1} \dots h'_{n-j} p_{i_{n-j-1}}^{-\varepsilon_{n-j-1}} h'_{n-j-1} \dots h'_1 p_{i_0}^{-\varepsilon_0} h'_0 = 1$$

and

$$h_{n-j} \dots h_{n-1} h'_{n-1} \dots h'_{n-j} \in A_{i_{n-j-1}}.$$

For $j = 1$, we have,

$$1 = gg^{-1} = p_{i_0}^{\varepsilon_0} h_1 \ldots h_{n-2} p_{i_{n-2}}^{\varepsilon_{n-1}} h_{n-1} h'_{n-1} p_{i_{n-2}}^{-\varepsilon_{n-2}} h'_{n-2} \ldots h'_1 p_{i_0}^{-\varepsilon_0} h'_0.$$

Then $p_{i_0}^{\varepsilon_0} h_1 \ldots h_{n-2} p_{i_{n-2}}^{\varepsilon_{n-1}} h_{n-1} h'_{n-1} p_{i_{n-2}}^{-\varepsilon_{n-2}} h'_{n-2} \ldots h'_1 p_{i_0}^{-\varepsilon_0} h'_0$ has a pinch and so

$$h_{n-1} h'_{n-1} \in A_{-\varepsilon_{n-2} i_{n-2}} = A_{i_{n-2}}.$$

Let $1 < j < n$ and suppose the result for $j - 1$. Then

$$\begin{aligned} 1 &= p_{i_0}^{\varepsilon_0} h_1 \ldots h_{n-j} p_{i_{n-j}}^{\varepsilon_{n-j}} h_{n-j+1} \ldots h_{n-1} h'_{n-1} \ldots h'_{n-j+1} p_{i_{n-j}}^{-\varepsilon_{n-j}} h'_{n-j} \ldots h'_1 p_{i_0}^{-\varepsilon_0} h'_0 \\ &= p_{i_0}^{\varepsilon_0} h_1 \ldots h_{n-j-1} p_{i_{n-j-1}}^{\varepsilon_{n-j-1}} h_{n-j} \ldots h_{n-1} h'_{n-1} \ldots h'_{n-j} p_{i_{n-j-1}}^{-\varepsilon_{n-j-1}} h'_{n-j-1} \ldots h'_1 p_{i_0}^{-\varepsilon_0} h'_0 \end{aligned}$$

because $h_{n-j+1} \ldots h_{n-1} h'_{n-1} \ldots h'_{n-j+1} \in A_{-\varepsilon_{n-j} i_{n-j}}$.
Then $p_{i_0}^{\varepsilon_0} h_1 \ldots h_{n-j-1} p_{i_{n-j-1}}^{\varepsilon_{n-j-1}} h_{n-j} \ldots h_{n-1} h'_{n-1} \ldots h'_{n-j} p_{i_{n-j-1}}^{-\varepsilon_{n-j-1}} h'_{n-j-1} \ldots h'_1 p_{i_0}^{-\varepsilon_0} h'_0$ has a pinch and so $h_{n-j} \ldots h_{n-1} h'_{n-1} \ldots h'_{n-j} \in A_{i_{n-j-1}}$.

Thus for every $1 \leq j < n$:

$$A_{i_{j-1}}^{{h'}_j^{-1} \ldots {h'}_{n-1}^{-1}} = A_{i_{j-1}}^{h_j \ldots h_{n-1}}.$$

With the assumptions, the subsets K_j, associated to g by (i), satisfy $K_n = A_{i_{n-1}}$, for $1 \leq j < n$, $K_j = K_{j+1}^{h_j^{-1}} \cap A_{i_{j-1}}$ and $K_0 = K_1$. So

$$H \cap H^{g^{-1}} = A_{i_{n-1}}^{h_{n-1}^{-1} \ldots h_1^{-1}} \cap \ldots \cap A_{i_1}^{h_1^{-1}} \cap A_{i_0}.$$

In the same way,

$$H \cap H^g = A_{i_0}^{{h'}_1^{-1} \ldots {h'}_{n-1}^{-1}} \cap \ldots \cap A_{i_{n-1}}.$$

So

$$(H \cap H^{g^{-1}})^{h_1 \ldots h_{n-1}} = A_{i_{n-1}} \cap A_{i_{n-2}}^{h_{n-1}} \cap \ldots \cap A_{i_0}^{h_1 \ldots h_{n-1}} = H \cap H^g.$$

Thus $[H : H \cap H^{g^{-1}}] = [H : H \cap H^g]$.

(iv) Follows immediately from an induction on the subgroups $K_n, \ldots, K_0$ defined in (i).

□

We can now prove Theorem 3.

Proof of Theorem 3.

Let H be a profinite group and let $(N_i)_{i \in I}$ be the family of the normal open subgroups of H.

(i) Let G be the HNN extension $\langle H, p_i;\ p_i^{-1} N_i p_i = N_i,\ i \in I \rangle$.

Let $g \in G \setminus H$ have normal form

$$g = h_0 p_{i_0}^{\varepsilon_0} h_1 \ldots h_{n-1} p_{i_{n-1}}^{\varepsilon_{n-1}} h_n.$$

Then by Lemma 5.2 (ii)

$$H \cap H^{g^{-1}} = H \cap H^g = N_{i_0} \cap \ldots \cap N_{i_{n-1}}.$$

Give G the topology such that the set of finite intersections of H^g is a system of open neighbourhoods of 1. Then, by Theorem 2 with $K = G$, there is a trivial strongly minimal structure M with a non algebraic component $C = \mathrm{iacl}(c)$ such that $\mathrm{Stab}(c)$ is homeomorphic to H and $\mathrm{Aut}(C) = \mathrm{Autf}(C)$ is homeomorphic to G. Since $\mathrm{Aut}(C) = \mathrm{Autf}(C)$, by Theorem 1, M is of bounded valency. Moreover, since for every $g \in G$, $H \cap H^{g^{-1}} = H \cap H^g$, by Lemma 3.6 M is unimodular.

(ii) The implication follows from Lemma 3.6.

Conversely assume there are two homeomorphic open subgroups A_1 and A_2 of H such that $[H : A_1] \neq [H : A_2]$. Let α be an homeomorphism between A_1 and A_2 and let G be the HNN extension $\langle H, p_i;\ q^{-1}A_1 q = \alpha(A_1),\ p_i^{-1} N_i p_i = N_i,\ i \in I\rangle$.

By Lemma 5.2 (iv), for every $g \in G$, $H \cap H^g$ is an open subgroup of H. As in the proof of (i), there is a strongly minimal structure M of bounded valency with a non algebraic component $C = \mathrm{iacl}(c)$ such that $\mathrm{Stab}(c)$ is homeomorphic to H and $\mathrm{Aut}(C)$ is homeomorphic to G. Since $H \cap H^q = A_2$, $H \cap H^{q^{-1}} = A_1$ and $[H : A_1] \neq [H : A_2]$, by Lemma 3.6, M is non-unimodular.

(iii) In (i) and (ii) G is infinite, so the structure given by Theorem 1 has infinite non algebraic components and any of these components is an elementary substructure (since $\mathrm{acl}(\emptyset) = \emptyset$).

(iii) If H is separable, I is countable and hence G/H is also countable (in (i) and (ii)). So the components of the associated structures are countable.

□

Remark.

(i) If H is finite, we can choose directly for G the free product $H * \mathbb{Z}$.

(ii) Not every automorphism group $\mathrm{Aut}(C)$ *is an* HNN *extension of* $\mathrm{Stab}(c)$ *for $C = \mathrm{iacl}(c)$ a non algebraic component of a strongly minimal structure of bounded valency. In particular, we can easily check that it is not the case for the structure M_4 in Examples 4.2.*

(iii) For every finite group K, the group K^ω satisfies Theorem 3 (ii) and can hence be realized as the stabilizer of a point in a non-unimodular strongly minimal structure of bounded valency.

We now remark that there is a strongly minimal structure reduced to one non algebraic component such that any separable profinite group can be realized as the stabilizer of some subset in the automorphism group of this structure.

Proposition 5.3. *There are two trivial strongly minimal structures C_1 and C_2 of bounded valency such that C_1 is unimodular, C_2 is non-unimodular, and such that C_1 and C_2 have dimension 1, are countable and for every separable profinite group T there are $X_1 \subseteq C_1$ and $X_2 \subseteq C_2$ such that T,* $\mathrm{Stab}(X_1)$ *and* $\mathrm{Stab}(X_2)$ *are homeomorphic.*

Proof. Let $(H_i)_{i\in\omega}$ be an enumeration up to isomorphism of finite groups and let H be the separable profinite group $\prod_{i\in\omega} H_i$. Let S be an infinite separable profinite group. Then it is easy to see that there are a strictly increasing sequence $(S_i)_{i\in\omega}$ of finite groups such that S is homeomorphic to a closed subgroup of $\prod_{i\in\omega} S_i$. It follows that any separable profinite group is homeomorphic to a closed subgroup of H which is an intersection of open subgroups of H.

Let C_1 be a countable unimodular trivial strongly minimal structure of bounded valency given by Theorem 3 (i), such that H is homeomorphic to the stabilizer of a non algebraic point. By Theorem 3 (iii), we can assume that C_1 is reduced to one non algebraic component. Let T be a separable profinite group. Then T is homeomorphic to a closed subgroup of the stabilizer of a point in C_1, which is the stabilizer of some subset of C_1.

For C_2, take $H' = (\mathbb{Z}/2\mathbb{Z})^\omega \times H$. Then H' has two homeomorphic open subgroups of distinct indices. So we can apply Theorem 3 (ii), and we can conclude in the same way because any separable profinite group is also homeomorphic to a closed subgroup of H'. □

The above constructions in Theorem 3 are all of bounded valency. We do not know how to construct a structure, which is not of bounded valency, corresponding to any non trivial profinite group but only to any non trivial abelian and separable profinite group. These constructions are generalizations of the structure M_5 in Examples 4.2.

Proposition 5.4. *Let H be a non trivial abelian separable profinite group. There is a trivial unimodular strongly minimal structure which is not of bounded valency and such that the stabilizer of a point is homeomorphic to H.*

Proof. Let H be a non trivial abelian separable profinite group. Since H is separable and abelian, we can see H as the projective limit of a family of abelian finite groups $(H_n)_{n\in\omega}$ with, for every $n \in \omega$, a projection p_n from H_{n+1} onto H_n.

Let C be the free product $*_{n\in\omega}(H_n \times \mathbb{Z})$ with the following language: for every $n \in \omega$ and every $h \in H_n$,

- the binary relation $R_n := \{(x, (k, 1) \cdot x);\ x \in C,\ k \in H_n\}$,
- the function g_h defined by $g_h(x) := (h, 0) \cdot x$ for any $x \in C$, and
- the binary function $P_n(x, (k, 1) \cdot x) = (p_n(k), 1) \cdot x$, for any $x \in C$ and any $k \in H_{n+1}$.

It is easy to see that C is of bounded valency: for every $n < \omega$ and $h \in H_n$, R_n is of bounded valency because H_n is finite, the graph of g_h defined a bounded valency relation because g_h is bijective, and the graph of P_n defined also a bounded valency relation because H_{n+1} is finite and for every $y \in C$, $P_n^{-1}(\{y\}) = \{((p_n(k), -1)y, (k, 1)(p_n(k), -1)y);\ k \in H_{n+1}\}$.

For every x, y in C, the bijection $f_{x,y}$ of C which sends t onto $t \cdot x^{-1} \cdot y$ is an automorphism, hence $\mathrm{Aut}(C)$ acts transitively on C. By Lemma 2.1, C is trivial strongly minimal.

Let C' be the expansion of C by for every $n \in \omega$, an equivalence relation E_n on R_n with $|H_n|$ classes $(C_h)_{h\in H_n}$ defined by

$$C_h := \{(x, (h,1)\cdot x);\ x \in C\}.$$

For every x, y in C, the bijection $f_{x,y}$ is also an automorphism of C' and moreover $f_{x,y}$ fixes setwise the classes C_h's. By Lemma 3.2, C' is trivial strongly minimal.

We will now prove that H is homeomorphic to Stab(1) in C': let ϕ the map defined by

$$\phi : \begin{array}{rcl} \mathrm{Stab}(1) & \to & \prod_{n\in\omega} H_n \\ f & \mapsto & (h_n)_{n\in\omega};\ f((1_n,1)) = (h_n,1). \end{array}$$

• ϕ is an homomorphism: let f_1 and f_2 in Stab(1) such that $\phi(f_1) = (h_{1,n})$ and $\phi(f_2) = (h_{2,n})$. Then

$$\begin{array}{rcl} f_1 \circ f_2((1_n,1)) & = & f_1((h_{2,n},1)) \\ & = & f_1(g_{h_{2,n}}((1_n,1))) \\ & = & g_{h_{2,n}}(f_1((1_n,1))) \\ & = & (h_{2,n},0)\cdot(h_{1,n},1) \\ & = & (h_{2,n}h_{1,n},1). \end{array}$$

• $\phi(\mathrm{Stab}(1)) \subseteq H$: let $f \in \mathrm{Stab}(1)$. For every $n \in \omega$, $P_n(1,(1_{n+1},1)) = (1_n,1)$, so $P_n(1, f(1_{n+1},1)) = f(1_n,1)$, and hence $p_n(\phi(f)_{n+1}) = \phi(f)_n$.

• $\phi(\mathrm{Stab}(1)) \supseteq H$: let $h \in H$. We are going to define $f \in \mathrm{Stab}(1)$ such that $\phi(f) = h$.

Any $x \in C$ can be written under the normal form: $x = (k_1,n_1)\ldots(k_m,n_m)$ such that for every $i \in \{1,\ldots,m\}$, $(k_i,n_i) \in H_{j_i} \times \mathbb{Z} \setminus \{(1_{j_i},0)\}$ and $j_i \neq j_{i+1}$.

Then for such a element x, we define $f(x) := (k_1 h_{j_1}^{n_1}, n_1)\ldots(k_m h_{j_m}^{n_m}, n_m)$.

Since H is abelian, it is easy to check that for every $x \in C$ and $k \in H_n$, $f((k,0)\cdot x) = (k,0)\cdot f(x)$ and $f((k,1)\cdot x) = (kh_n,1)\cdot f(x)$. It follows that f is an automorphism of C' such that $\phi(f) = h$ and such that for every $k \in H_n$, f sends the class C_k onto the class C_{kh_n}.

• By definition of ϕ, $\ker\phi = \mathrm{Stab}(\{1\} \cup \{(h,1);\ n \in \omega,\ h \in H_n\})$. Hence, if $f \in \ker\phi$, f fixes 1 and the classes C_h's and so f is the identity.

Thus ϕ defines an isomorphism from Stab(1) onto H. Furthermore, ϕ is an homeomorphism : let $x \in C$ be written under the normal form $x = (k_1,n_1)\ldots(k_m,n_m)$ such that for every $i \in \{1,\ldots,m\}$, $k_i \in H_{j_i}$ then $\phi(\mathrm{Stab}(x)) = \{h \in H;\ h_{j_i}^{n_i} = 1\}$, and, conversely, for every $n \in \omega$, $\phi^{-1}(\{h \in H;\ h_n = 1\}) = \mathrm{Stab}(\{1,\ (1_n,1)\})$.

We can remark that $\mathrm{Stab}(1) \cap \mathrm{Autf}(C')$ is trivial, so by Theorem 1, C' is not of bounded valency.

The fact that C and C' are unimodular, can be easily deduced from the fact that for every $x \in C$, for every $n \in \omega$ and for every $k \in H_n$, $\mathrm{mult}((k,1)\cdot x/x) = |H_n| = \mathrm{mult}(x/(k,1)\cdot x)$ in C as in C'. $\square$

Finite languages: we finish with a remark on finite languages. This remark is motivated by a a result of V. Nekrashevych on locally finite graphs, which was recently brought to our attention:

Nekrashevych ([Nek, Theorem 4.1]) has proved that a group can be realized as the stabilizer of a point of **a** transitive action on a locally finite connected graph (or tree) if and only if it is commensuratively sieveable. (In a topological sense, a group H is commensuratively sieveable if and only if it can be seen as a topological group with a finite set Φ of homeomorphisms between open subgroups of finite index of H such that $\cap_{n\in\omega} \cap_{p_1,\ldots,p_n\in\Phi} \operatorname{dom} p_1 \circ \ldots \circ p_n = \{1\}$.)

A transitive locally finite connected graph can be seen as a component of a strongly minimal structure with a language reduced to one binary bounded valency relation. More generally, it is easy to check that the stabilizer of a point of a transitive action on one component of a strongly minimal structure of bounded valency in a finite language is commensuratively sieveable. On the other hand, we do not know if every commensuratively sieveable profinite group can be realized as the stabilizer of **the full** automorphism group of a strongly minimal structure of bounded valency in a finite language.

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