Subgroups of the additive group of a separably closed field

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Abstract

We study the infinitely definable subgroups of the additive group in a separably closed field of finite positive imperfection degree. We give some constructions of families of such subgroups which confirm the diversity and the richness of this class of groups. We show in particular that there exists a locally modular minimal subgroup such that the division ring of its quasi-endomorphisms is not a fraction field of the ring of its definable endomorphisms, and that in contrast there exist $2^{\aleph_0}$ pairwise orthogonal locally modular minimal subgroups whose induced structure is exactly that of an $\mathbb{F}_p$-vector space. We also show that there exist infinitely many pairwise orthogonal subgroups of infinite U-rank. Furthermore, these constructions are carried out in the additive group considered as a module over its ring of definable endomorphisms.

Introduction

We consider the class of infinitely definable subgroups (of a finite cartesian product) of the additive group in a separably closed field L of characteristic p and of finite positive imperfection degree ν (i.e. $[L : L^p] = p^\nu$). Note that every infinitely definable subgroup of a cartesian product of $(L, +)$ is definably isomorphic to a subgroup of $(L, +)$ because $(L^{\times p^{\nu k}}, +)$ is definably isomorphic to $(L, +)$ for every $k > 0$. Moreover all commutative infinitely definable subgroups of exponent p arise in this manner.

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Our basic motivation for studying this class is that it is quite rich, hence a good place to look for examples and counter-examples for some still mysterious phenomena in the context of stability. Recall that, in contrast with the cases of algebraically closed fields (of any characteristic) and of differentially closed fields of characteristic 0, this class includes locally modular minimal additive subgroups. Moreover, in [B-D2], E. Bouscaren and F. Delon answer positively the question of the existence of a non-thin minimal subgroup of $(L,+)$ (i.e. a minimal subgroup with infinite transcendence degree). Some of the main questions we had in mind when we began the study of this class were the following:

- What are the induced structures on these groups when they are one-based?
- What are the quasi-endomorphism division rings and the endomorphism subrings of these groups when there are minimal?
- Can we find $\aleph_0$-categorical groups?
- Can we find ranked groups of infinite U-rank?

In section 2, we show that, as in algebraically closed fields, every infinitely definable subgroup (of a cartesian product) of the additive group is defined by additive equations.

In section 3, we study the module structure of $(L,+)$ over the ring of all endomorphisms definable over an elementary substructure K. In particular, we observe that if G is a one-based infinitely definable subgroup (of a cartesian product) of $(L,+)$ over K, then the structure induced on G by the field L is exactly the structure induced by the module structure of $(L,+)$.

It is well known that the geometry associated to a locally modular minimal group G is the geometry of a vector space over the quasi-endomorphism ring of G. However the geometry does not give the full induced structure. In fact, the relation between the geometry and the induced structure turns out to be complicated in the case of a bounded exponent group (see [H-L]). In section 4, we show that there exists a locally modular minimal subgroup of $(L,+)$ whose quasi-endomorphism division ring is not a field of fractions of its endomorphism subring. In the class of minimal subgroups of $(L,+)$, we can see the local modularity on the quasi-endomorphism division ring but not necessarily on the endomorphism subring. Finally, we construct a family of $\aleph_0$-categorical minimal subgroups of $(L,+)$ whose induced structure is exactly that of an $\mathbb{F}_p$-vector space.

In section 5, we answer positively the question of the existence of ranked subgroups of infinite U-rank.

1 Preliminaries

We assume that the reader is familiar with the basic notions and facts of stability theory (see [Bu] or [Pi]). We will recall most of the definitions and results on stable groups which we use. Others, including basic facts about the generics in a stable group, can be found in [Po], [Pi] or [Wa].

First we make precise some notation and definitions. We denote definable closure and algebraic closure, in the usual sense of model theory, by dcl and acl, respectively. Fix a first order structure M. For $k < \omega$, $M^{\times k}$ denotes the cartesian product of M. An $\bigwedge\!\!\!\bigwedge$-definable subset D of $M^{\times k}$ is an **infinitely** definable subset, a possibly infinite intersection of fewer than $|M|$ definable subsets of $M^{\times k}$. For $l < \omega$, a **relatively** definable subset of $D^{\times l}$ is the intersection of $D^{\times l}$ with a definable subset of $M^{\times kl}$. If there is no risk of ambiguity, we will say a definable subset of $D^{\times l}$ instead of a relatively definable subset. The structure **induced** by M on D over a set A of parameters, is the set D together with all the relatively A-definable subsets of the cartesian powers of D. Let D be an $\bigwedge\!\!\!\bigwedge$-definable set in a stable structure, defined over a set A. We say D is **one-based** if for every $B \supset A$ and every finite tuple $\bar{a} \in D$, the canonical basis of the strong type of $\bar{a}$ over B is included in $\mathrm{acl}^{eq}(B)$.

Recall that if G is an $\bigwedge\!\!\!\bigwedge$-definable group in a stable structure, then G is an intersection of definable groups, and G^0 denotes the **connected component** of G, i.e. the intersection of all definable subgroups of finite index in G. We say that G is **connected** if G has no proper definable subgroup of finite index.

A field L is said to be **separably closed** if it has no proper separable algebraic extension. If K is a field, we denote its separable closure by K^{sep}. From now on we consider fields of fixed characteristic $p > 0$. Note that a separably closed field L is algebraically closed if and only if $L^p = L$. In the case where $[L : L^p]$ is finite and so equal to p^ν for some integer $\nu \geq 0$, ν is called **the degree of imperfection** of L. It is well known that the theory of separably closed fields of finite degree of imperfection $\nu \geq 1$ is complete [Er], stable non-superstable [Wo], but does not admit quantifier elimination, nor elimination of imaginaries in the pure language of rings $\{0, 1, +, -, .\}$. This contrasts with algebraically closed fields, where in particular the definable sets are the constructible sets of the Zariski topology. But separably closed fields do admit quantifier elimination and elimination of imaginaries in a natural language, and the definable sets are also the constructible sets in a related topology, the λ-topology. This topology is no longer Noetherian, but is a "limit" of Noetherian topologies. We will see this more precisely in the next two subsections. These results can also be found in [De 88], [De 98] or [Me 96].

1.1 p-basis and iterated components

From now on we fix a separably closed field L of finite degree of imperfection ν. A subset $B = \{b_0, \ldots, b_{\nu-1}\}$ of L is called a p-**basis** of L if the set of p-monomials of B,

$$\{m_i := \prod_{j=1}^{\nu} b_j^{i(j)}, i \in p^\nu\}$$

forms a basis of the L^p-vector space L.

We also fix a p-basis B of L.

Notation 1.1. For every $x \in L$, we denote by $(x_i)_{i\in p^\nu}$ the p-components of x of level one, i.e., the unique x_i's in L such that $x = \sum_{i\in p^\nu} x_i^p m_i$. For each $i \in p^\nu$, we denote by f_i the p-component map which sends x to x_i.

More generally, one can associate to $x \in L$ a tree of countable height indexed by $(p^\nu)^{<\omega} = \cup_{n\in\omega}(p^\nu)^n$, which we call the **tree of p-components** of x. (In particular $x_\emptyset = x$). For each $i \in (p^\nu)^n$, we define

$$f_i = f_{i(n-1)} \circ .. \circ f_{i(0)} \text{ and } m_i = m_{i(n-1)}^{p^{n-1}} \dots m_{i(0)}.$$

Then for each $n < \omega$ and each $x \in L$,

$$x = \sum_{i\in(p^\nu)^n} f_i(x)^{p^n} m_i = \sum_{i\in(p^\nu)^n} x_i^{p^n} m_i$$

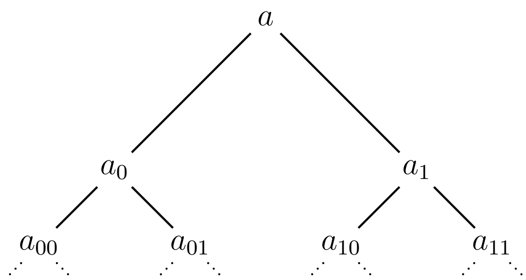
We define also, for each $n > 0$, the bijection λ_n from L onto $L^{\times p^{\nu n}}$ which associates to each $x \in L$ the tuple of its p-components of level n, $(x_i)_{i\in p^{\nu n}}$.

We let $p^\infty := (p^\nu)^{<\omega}$ and for each $x \in L$, $x_\infty := (x_i : i \in p^\infty)$ the sequence of p-components of x, and for each $n < \omega$, $x_{=n} := \lambda_n(x)$ the p-components of x of level n and $x_{\leq n} := (x_{=0}, \dots, x_{=n})$. In the natural way, one can extend all these notations to tuples of elements.

For a sequence $(a_j)_{j\in J}$ of elements (or tuples) of L, we denote by $a_{j,i}$ the element (or tuple) $f_i(a_j)$ for each $j \in J$ and each $i \in p^\infty$. For $i, j \in p^\infty$ we denote by x_{ij} the component $f_j \circ f_i(x)$.

In order to simplify notation in some technical proofs, it is useful to identify the set of maps from ν to p to the set of the integers strictly smaller than the integer p^ν. For this purpose, we choose the reverse lexicographic order on p^ν, that is $i < j$ in p^ν if there is $l < \nu$ such that $i(l) < j(l)$ and for every $k > l$, $i(k) = j(k)$. Then for $i < p^\nu$ we identify i to the $(i + 1)$ st elements in p^ν. With this convention, it is is easy to check that for every integer $i < p$, we have $m_i = b_0^i$.

Moreover we use the notation x_{01} for $f_1(f_0(x))$ and the notation x_{011} for $f_1(f_1(f_0(x)))$, etc. Again to simplify notation, we define $b := b_0$ and for $n < \omega$, we denote by $x_{\bar{0}(n)}$ the component $f_0^n(x)$ and by $x_{\bar{1}(n)}$ the component $f_1^n(x)$.



The tree of p-components of a for $p = 2$ and $\nu = 1$

The following results follow from the description of types in L by separable ideals, which we describe in the next subsection:

Fact 1.2. *The theory of L admits quantifier elimination and elimination of imaginaries in the language* $\mathcal{L}_{p,\nu} = \{0, 1, +, -, .\} \cup \{b_0, \ldots, b_{\nu-1}\} \cup \{f_i : i \in p^\nu\}$.

From now on we consider the structure L in the language $\mathcal{L}_{p,\nu}$ and we suppose that L is "sufficiently" saturated, that is, if we consider any subfield K of L, that L is $|K|^+$-saturated. (In fact, in our constructions of families of subgroups, we only use that L is $\aleph_1$-saturated.)

Fact 1.3. *Let K be an elementary substructure of L and A a subset of L.*
(i) $K\langle A\rangle := K(f_i(A) : i \in p^\infty)^{sep}$ is the prime model over $K \cup A$.
(ii) The algebraic closure $\mathrm{acl}(A)$ *of A (in the model theory sense) is equal to* $\mathbb{F}_p(B, f_i(A) : i \in p^\infty)^{sep}$ *and is the prime model over A.*
(iii) The definable closure $\mathrm{dcl}(A)$ *of A is equal to* $\mathbb{F}_p(B, f_i(A) : i \in p^\infty)$.

Observe that a definably closed subfield of L is an elementary substructure of L if and only if it is separably closed.

1.2 Separable ideals and λ-topology

In this subsection, we fix a definably closed subfield K of L. Note that $K \supset \mathbb{F}_p(B)$, and $f_i(K) \subset K$ for every $i \in p^\infty$.

Definition. For X an indeterminate (or a k-tuple of indeterminates), we denote by $K[X_\infty]$ the ring $K[X_i : i \in p^\infty]$ of (λ-)polynomials on K with infinitely many indeterminates.

This ring is the countable union of the Noetherian rings $K[X_{\leq n}] := K[X_i : i \in (p^\nu)^{\leq n}]$. Hence, each ideal of $K[X_\infty]$ is countably generated.

For each $n < \omega$ and each ideal I of $K[X_\infty]$, we define

$$K[X_{=n}] := K[X_i : i \in (p^\nu)^n],\ I_{\leq n} := I \cap K[X_{\leq n}] \text{ and } I_{=n} := I \cap K[X_{=n}].$$

For an indeterminate X, we denote by I^0_K (or I^0, if there is no risk of ambiguity), the ideal of $K[X_\infty]$ generated by the polynomials $X_i - \sum_{j\in p^\nu} X^p_{ij} m_j$, for all $i \in p^\infty$. For a k-tuple of indeterminates $X = (X_1, .., X_k)$, we denote also by I^0_K (or I^0) the ideal of $K[X_\infty]$ generated by the I^0's corresponding to each X_l.

For $x \in L$,

$$I(x, K) := \{P \in K[X_\infty] : P(x_\infty) = 0\}.$$

Definition. An ideal I of a K-algebra C is **separable** if given a sequence $(P_i)_{i\in p^\nu}$ of C such that $\sum_{i\in p^\nu} P^p_i m_i \in I$, $P_i \in I$ for each $i \in p^\nu$.

The quantifier elimination result follows from:

Fact 1.4. *[De 88, Proposition 33] Suppose that L is $|K|^+$-saturated.*

For each $k > 0$, the map which sends $x \in L^{\times k}$ to $I(x, K)$ induces a bijection between complete k-types over K and prime separable ideals of $K[X_\infty]$ containing I^0.

In order to describe the definable subsets of L, one introduces the λ-topology:

Definition. For every subset S of polynomials in $K[X_\infty]$, where X is a k-tuple of indeterminates, we define a **λ-closed set** of $L^{\times k}$ (over K),

$$V(S) := \{x \in L^{\times k} : S(x_\infty) = 0\}.$$

These sets form the closed sets of the **λ-topology** on $L^{\times k}$ (over K).
If $S \subset K[X_{\leq n}]$, we say that $V(S)$ is a **λ_n-closed set**. The λ-topology is the "limit" of the Noetherian λ_n-topologies.

Observe that, by quantifier elimination, every definable subset of $L^{\times k}$ is a λ_n-constructible set of $L^{\times k}$ for some $n < \omega$, i.e., a finite boolean combination of λ_n-closed sets.

Definition. To each subset A of $L^{\times k}$, we associate the ideal $I(A, K)$ (or $I(A)$, if there is no ambiguity), defined as

$$I(A, K) := \{P \in K[X_\infty] : P(a_\infty) = 0 \text{ for each } a \in A\}.$$

As in the case of algebraically closed fields, there is a "Nullstellensatz" theorem:

Fact 1.5. *[De 98, Proposition 3.3] The map which sends $A \subset L^{\times k}$ to $I(A, K)$ induces a bijection between λ-closed sets of $L^{\times k}$ over K and separable ideals of $K[X_\infty]$ containing I^0.*

Remark. *In [De 98], it is assumed that K is an elementary substructure of L. One can deduce the result for K definably closed from the result for K^{sep} and from the fact that if I is a separable ideal of $K[X_\infty]$ containing I^0 then $I \otimes_K K^{sep}$ is a separable ideal of $K^{sep}[X_\infty]$ containing $I^0_{K^{sep}}$ (see [Bl, Remarques 3.1.11, 3.1.13]).*

1.3 Additive subgroups

We use results on definable groups in separably closed fields from [B-D1]. For the sake of simplification, we concentrate on the definable subgroups of a finite cartesian product of the additive group, which are in particular subgroups of an algebraic group. The following facts follow from [B-D1].

Let $k > 0$, let G be an $\bigwedge$-definable subgroup of $(L^{\times k}, +)$ and let K be a definably closed subfield of L. We denote by $\bar{L}$ the algebraic closure of L, and by $\bar{G}$ the Zariski closure of G.

Fact 1.6.
(i) If G is definable over K then there is $n \in \omega$ such that G is λ_n-closed over K.
(ii) If G is $\bigwedge$-definable over K then G is λ-closed over K.

Fact 1.7. *By using the bijections λ_n's, we have that*
(i) G is definably isomorphic to a subgroup of $(L, +)$,
(ii) if G is λ_n-closed then G is definably isomorphic to a λ_0-closed subgroup of $(L^{\times kp^{\nu n}}, +)$.

Fact 1.8. *Suppose G is a λ-closed group. Then:*
(i) The **topological** *generics of the irreducible components of the λ-closed set G are the generics of the group G in the sense of the model theory of stable groups.*
(ii) G is connected if and only if G is irreducible (i.e. if and only if $I(G)$ is prime).
(iii) If G is λ_0-closed, then:
(a) $G = \bar{G}(L)$.
(b) If g is generic in G, then g is generic in $\bar{G}$.
(c) G is connected if and only if $I_{=0}(G)$ is prime, if and only if $\bar{G}$ is connected.
(d) $G^0 = (\bar{G})^0(L)$, and so G^0 is λ_0-closed.
(e) $[G : G^0]$ is finite.

Fact 1.9. *Suppose G is λ-closed. Then:*
(i) $V(I(G)_{=0}) = \bar{G}(L)$.
(ii) G is the intersection of the λ_n-closed groups $G_n := V(I(G)_{=n})$.
(iii) G is connected if and only if all G_n's are connected.

To finish the preliminaries, we recall that the λ-closed subgroups of the additive group are exactly the abelian $\bigwedge$-definable groups of exponent p:

Fact 1.10. *[B-D2, Proposition 2.1] An abelian $\bigwedge$-definable group of exponent p in L is definably isomorphic to a λ-closed subgroup of $(L, +)$.*

From now on we say an **additive subgroup** (over K) to mean a λ-closed subgroup (over K) of the additive group.

2 Additive subgroups and p-polynomials

In this section we prove that, as in the case of algebraically closed fields, every λ-closed subgroup (of a finite cartesian product) of the additive group is defined by additive equations (p-polynomial equations from the ring of λ-polynomials).

Definition. Let F be a field of characteristic p. A polynomial P of $F[X_i : i \in I]$ is a p**-polynomial** if P is a sum of monomials of the form $a_{i,j}X_i^{p^j}$.

Remark 2.1. *Let F be a field of characteristic p and let $k > 0$.*

(i) A p-polynomial P is **additive** *(i.e. for every $(a_i)_{i\in I} \in F$ and $(b_i)_{i\in I} \in F$, $P((a_i + b_i)_{i\in I}) = P((a_i)_{i\in I}) + P((b_i)_{i\in I})$).*

(ii) Conversely, if F is infinite, an additive polynomial is also additive on the algebraic closure of F, hence is a p-polynomial (see [Hum, Chapter VII 20]).

(iii) Furthermore, one can easily check that if F is infinite, C a cofinite subset of F, and R a rational function in $F(X)$ such that for every $x, y \in C$ with $x + y \in C$, $R(x + y) = R(x) + R(y)$, then R is equal to an additive polynomial in $F[X]$. In particular, it follows that if F is an algebraically closed field and if ϕ is a rational homomorphism from $(F^{\times k}, +)$ to $(F, +)$ defined over F, then ϕ is equal to a p-polynomial in $F[X_1, ..., X_k]$.

We recall first the classical case of algebraically closed fields:

Fact 2.2. *[Hum, Chapter VII 20] Let F be an algebraically closed field of characteristic p and let $k > 0$.*

(i) A connected closed subgroup G of $(F^{\times k}, +)$ is a **vector group***: there is an algebraic automorphism ϕ of $(F^{\times k}, +)$ defined by p-polynomials in $F[X_1, ..., X_k]$ such that ϕ^{-1} is also defined by p-polynomials in $F[X_1, ..., X_k]$, and such that $\phi(G) = F^{\times l} \times \{0\}^{\times k-l}$.*

(ii) Every closed subgroup of $(F^{\times k}, +)$ is a finite intersection of kernels of p-polynomials.

In the case of separably closed fields, every connected λ_0-closed subgroup of $(F^{\times k}, +)$ is **definably isogenous** (but not always definably isomorphic) to a vector group [B-D2, Lemma 2.4] and we are going to show that:

Proposition 2.3. *Let K be a definably closed subfield of L such that L is $|K|^+$-saturated.*

(i) For each $k > 0$, every λ_0-closed subgroup of $(L^{\times k}, +)$ over K is the kernel of a p-polynomial in $K[X_1, \ldots, X_k]$.

(ii) If G is an additive subgroup over K then there is a set S of p-polynomials of $K[X_\infty]$ such that $G = V(S)$, and such that $I(G, K)$ is generated by S.

Remark 2.4. *The inverses of the bijections λ_n are defined by p-polynomials. So by using these bijections, we can apply (i) to every definable subgroup of a power of $(L, +)$, and (ii) to every λ-closed subgroup of a power of $(L, +)$. Moreover, it* *is easy to check that in L, a finite intersection of kernels of p-polynomials is the kernel of one p-polynomial. (Note that this is not true in the case of algebraically closed fields).*

The proof will be broken into a series of lemmas. If K is separably closed, then (i) is a direct corollary of Fact 2.2 (ii) and Remark 2.4:

Lemma 2.5. *Let G be a λ_0-closed subgroup of $(L^{\times k}, +)$ over K. Then there exist p-polynomials P_1,..., P_n in $K^{sep}[X_1, \ldots, X_k]$ such that $G = V(\{P_1, \ldots, P_n\})$.*

Proof. By Fact 2.2 (ii), the Zariski closure $\bar{G}$ of G is the intersection of the kernels of some p-polynomials $Q_1, \ldots, Q_n$ of $\bar{K}[X_1, \ldots, X_k]$. Thus there is $m < \omega$ such that $Q_1^{p^m}$,..., $Q_n^{p^m}$ are in $K^{sep}[X_1, \ldots, X_k]$. So $G = \bar{G}(L) = V(\{Q_1^{p^m}, \ldots, Q_n^{p^m}\})$. □

Secondly, it is easy to check the case of finite groups:

Lemma 2.6. *Let G be a finite subgroup of $(L^{\times k}, +)$ definable over K. Then there is a p-polynomial Q in $K[X_0, \ldots, X_{k-1}]$ such that $G = V(\{Q\})$.*

Proof. Let n be such that $p^{\nu n} \geq k$ and let $H = \lambda_n^{-1}(G \times \{0\}^{p^{\nu n}-k})$. Then H is a finite subgroup of $(L, +)$, definable over K. Let $P(X) = \prod_{h \in H}(X - h)$. The set of the roots of P is fixed by automorphisms of L fixing K; since K is definably closed, P is in $K[X]$. The polynomial P is square-free and the set of zeros of P is a subgroup of $(L, +)$. Thus P is a p-polynomial (see [Hum, Chapter VII 20]). Let $i_0, \ldots, i_{k-1}$ be the first k elements of $p^{\nu n}$, and let $Q(X_0, \ldots, X_{k-1}) = P(X_0^{p^n} m_{i_0} + \ldots + X_{k-1}^{p^n} m_{i_{k-1}})$. Then Q is a p-polynomial and $G = V(\{Q\})$. □

Thirdly, we treat the case of connected groups. The second part of the following lemma will only be used in later sections.

Lemma 2.7. *Let G be a connected λ_0-closed subgroup of $(L^{\times k}, +)$ over K. Then:*

(i) There are p-polynomials $P_1, \ldots, P_n$ in $K[Z_1, \ldots, Z_k]$ such that

$$G = V(\{P_1, \ldots, P_n\}).$$

(ii) If g is a generic of G over K such that $g := (\bar{x}, y_1, \ldots, y_n)$ where $\bar{x}$ is a separating transcendence basis of $K(g)$ over K, then we can choose the polynomials $P_1, \ldots, P_n$ such that for all $i \in \{1, \ldots, n\}$,

(a) $P_i \in K[\bar{X}, Y_1, \ldots, Y_i]$ and P_i is unitary in Y_i,

(b) P_i is of minimal degree in Y_i among the p-polynomials of $K^{sep}[\bar{X}, Y_1, \ldots, Y_i]$ of degree > 0 in Y_i which vanish on g,

(c) for all $j < i$, $\deg_{Y_j}(P_i) < \deg_{Y_j}(P_j)$.

The polynomials P_i are the only polynomials which satisfy the above properties.

Proof. Let g be a realization of the generic of G over K. We can suppose that $g := (\bar{x}, \bar{y})$ where $\bar{x}$ is a separating transcendence basis of $K(g)$ over K. Let $m := l(\bar{x})$ and $n := l(\bar{y})$ $(k = m + n)$.

Note that if $n = 0$, then $G = (L^{\times k}, +)$.

Suppose that $n > 0$. Let $\bar{G}$ be the Zariski closure of G in $\bar{L}^{\times k}$. By Fact 1.8, g realizes the generic of $\bar{G}$ over $\bar{K}$. For any $i \in \{1, \ldots, n\}$, let $g_i := (\bar{x}, y_1, \ldots, y_i)$ and let $\bar{G}_i$ be the projection of $\bar{G}$ on $\bar{L}^{m+i}$. So every g_i realizes the generic of $\bar{G}_i$.

We construct, by induction on i, p-polynomials $P_i \in K^{sep}[\bar{X}, Y_1, \ldots, Y_i]$ such that P_i is unitary and of minimal degree in Y_i among the p-polynomials of $K^{sep}[\bar{X}, Y_1, \ldots, Y_i]$ of degree > 0 in Y_i which vanish on g_i, and such that for every $j < i$, $\deg_{Y_j}(P_i) < \deg_{Y_j}(P_j)$.

Suppose that either $n = 1$ or for $i < n$, $P_1, \ldots, P_i$ are defined. Since $\bar{G}_{i+1} \neq \bar{L}^{m+i+1}$, by (2.2), there is a p-polynomial of $\bar{K}[\bar{X}, Y_1, \ldots, Y_{i+1}]$ vanishing on g_{i+1} and of degree > 0 in Y_{i+1}. For some l, a p^l-power of this polynomial has its coefficients in K^{sep}. So there is a p-polynomial $P'_{i+1} \in K^{sep}[\bar{X}, Y_1, \ldots, Y_{i+1}]$ vanishing on g_{i+1} and which is unitary and of minimal degree in Y_{i+1}. Now if $n = 1$, take $P_1 := P'_1$. Otherwise we reduce successively the degree in Y_i, in Y_{i-1}, ..., to get a p-polynomial P_{i+1} such that for every $j \leq i$, $\deg_{Y_j}(P_{i+1}) < \deg_{Y_j}(P_j)$: indeed, if P is a unitary p-polynomial in $K[X]$ of degree p^m and Q is a p-polynomial in $K[X]$ of degree $p^{m'}$ such that $m' > m$ then there is a p-polynomial Q' such that $\deg(Q') < \deg(Q)$ and such that $Q' - Q \in \langle P \rangle$ (it suffices to replace $X^{p^{m'}}$ by $(X^{p^m} - P(X))^{p^{m'-m}}$ in Q).

In fact, this construction gives p-polynomials with coefficients in K. Every automorphism σ of L which fixes K point-wise also fixes the P_i's: since P_i is unitary in Y_i, $\deg_{Y_i}(\sigma(P_i) - P_i) < \deg_{Y_i}(P_i)$. But $\sigma(P_i)(g) = 0$ because G is connected, hence has a unique generic type over K. So $\sigma(P_i) - P_i$ has degree zero in Y_i, hence also in Y_{i-1}, ..., Y_1 since for every $j < i$, $\deg_{Y_j}(P_i) < \deg_{Y_j}(P_j)$. Finally, $\sigma(P_i) - P_i = 0$ because $\bar{x}$ is transcendental. In the same way one may check that the P_i's are the only polynomials which satisfy the above properties.

We prove now that $G = V(\{P_1, \ldots, P_n\})$. By Lemma 2.5, it suffices to check that every p-polynomial Q of $K^{sep}[\bar{X}, \bar{Y}]$ which vanishes on g must lie in $\langle P_1, \ldots, P_n \rangle$.

As above, it is easy to prove that

$$Q(\bar{X}, \bar{Y}) \equiv Q_0(\bar{X}) + Q_1(Y_1) + \ldots + Q_n(Y_n) (\text{mod } P_1, \ldots, P_n),$$

with $\deg(Q_i) < \deg_{Y_i}(P_i)$.

Since $Q(g) = 0$ it follows that $Q_n = \ldots = Q_0 = 0$. Therefore $Q \in \langle P_1, \ldots, P_n \rangle$. $\square$

Proof of proposition 2.3.

(i) Let G be a λ_0-closed subgroup of $(L^{\times k}, +)$ over K. Let G^0 be the connected component of G. Recall that by Fact 1.8, G^0 is also λ_0-closed over K.

By Lemma 2.7, there are p-polynomials $P_1, \ldots, P_n$ in $K[Z_1, \ldots, Z_k]$ such that $G^0 = V(\{P_1, \ldots, P_n\})$.

Let ϕ be the homomorphism from $(L^{\times k}, +)$ to $(L^{\times n}, +)$ which sends $\bar{z}$ to $(P_1(\bar{z}), \ldots, P_n(\bar{z}))$. Then $\phi(G)$ is finite, since $\ker(\phi) = G^0$ and G^0 is of finite index in G. So by Lemma 2.6, there is a p-polynomial $Q \in K[Y_1, \ldots, Y_n]$ such that $\phi(G) = V(\{Q\})$. It follows that $G = V(\{R\})$, where $R(\bar{Z}) := Q(P_1(\bar{Z}), \ldots, P_n(\bar{Z}))$.

(ii) Let G be a λ-closed subgroup of $(L, +)$ over K. By Fact 1.9, G is an intersection of λ_n-closed subgroups G_n of $(L, +)$ over K. By (i) and Remark 2.4, there is a set S of p-polynomials of $K[X_\infty]$ such that $G = V(S)$. By the following lemma, we can choose S such that S generates $I(G, K)$. $\square$

Lemma 2.8. *Let S be a set of p-polynomials of $K[X_\infty]$. Then $I(V(S), K)$ is generated by a set of p-polynomials.*

Proof. Let P be a polynomial in $K[X_{=n}]$. Then there are polynomials P_i in $K[X_{=n+1}]$ such that
$$P \equiv \sum_{i \in p^\nu} P_i^p m_i \pmod{I^0}.$$
It is easy to check that if P is a p-polynomial then so are the P_i's.

We can assume S is closed under this process (i.e., for $P \in S$ there are $P_i \in S$ such that $P \equiv \sum_{i \in p^\nu} P_i^p m_i \pmod{I^0}$).

Then $I = \langle S \cup I^0 \rangle$ is a separable ideal: indeed if $\sum_{i \in p^\nu} Q_i^p m_i \in I$, then modulo I^0 this sum equals a sum of terms of the form $R \cdot P$ where $P \in S$ and $R \in K[X_\infty]$. Then
$$R \cdot P \equiv (\sum_{i \in p^\nu} R_i^p m_i) \cdot (\sum_{i \in p^\nu} P_i^p m_i)$$
with $P_i \in S$.

It follows that there are $R'_{i,j} \in K[X_\infty]$ such that
$$R \cdot P \equiv \sum_{i \in p^\nu} \left(\sum_{j \in p^\nu} R'_{i,j} \cdot P_j \right)^p m_i.$$
Thus every Q_i is in $\langle S \cup I^0 \rangle$. By Fact 1.5, we conclude that $I(V(S)) = I$. $\square$

3 The additive group as a module

For this section we fix a definably closed subfield K of L. Recall that L is assumed to be $|K|^+$-saturated.

P. Dellunde, F. Delon and F. Point [D-D-P1, D-D-P2] have considered separably closed fields as modules over $\mathbb{F}_p(B)\{\alpha\}$, the ring of endomorphisms generated by $\mathbb{F}_p(B)$ and the Frobenius map α. They axiomatize theory and study its class of models. They expand the usual language for modules with the analogues of the p-component functions and show that the theory eliminates quantifiers in this language.

Here we adopt a different point of view: we study only the module structure of $(L,+)$ itself over the ring of all definable endomorphisms over K. One can remark easily that if $K = \mathbb{F}_p(B)$, the two different languages are $\emptyset$-interdefinable. Our point of view yields another proof of the elimination of quantifiers. Moreover we show that if K is separably closed, and G is a one-based additive subgroup over K, then the structure induced on G by the field L over K is equal to the structure induced by the module structure of $(L,+)$ over K. Finally, we show that any definable endomorphism of an additive subgroup is the restriction of a definable endomorphism of $(L,+)$.

3.1 The ring of definable endomorphisms

Recall that every definable endomorphism of the additive group of an algebraically closed field of characteristic p is the composition of a p-polynomial with a negative power of the Frobenius. In our case, we will see that every definable endomorphism of the additive group is defined by a p-polynomial of $K[X_\infty]$.

Proposition 3.1. *Let $\mathrm{E}_K(L,+)$ be the ring of definable endomorphisms of $(L,+)$ over K.*

(i) The Frobenius α and the component maps $\{f_i : i \in p^\nu\}$ generate $\mathrm{E}_K(L,+)$ over K.

(ii) Let R be the skew polynomial ring over K with indeterminates t and $(s_i)_{i\in p^\nu}$, and let I be the two-sided ideal of R generated by the following polynomials (as usual, the m_i's denote p-monomials of the fixed p-basis B). The ring $\mathrm{E}_K(L,+)$ has the following presentation.

1. $ta - a^p t$ *for every* $a \in K$.
2. $s_i m_i t - 1$ *and* $s_i m_j t$, *for each distinct pair* $i, j \in p^\nu$.
3. $s_i \cdot a - \displaystyle\sum_{j,l\in p^\nu : l+j\equiv i\ (p)} a_l (m_l m_j m_i^{-1})^{1/p} s_j$ *for each* $i \in p^\nu$ *and each* $a \in K$.
4. $\displaystyle\sum_{i\in p^\nu} m_i t s_i - 1.$

Then R/I is isomorphic to $\mathrm{E}_K(L,+)$ via a map sending the indeterminate t onto the Frobenius α and the s_i's onto the f_i's.

(iii) For every $r \in \mathrm{E}_K(L,+)$, there is an integer N such that for every $n \geq N$, r is equal to a unique p-polynomial of $K[X_{=n}]$. In fact, for every $n \geq N$ and $i \in p^{\nu n}$, $r = \sum_{i\in p^{\nu n}} r_i \cdot f_i$, where $r_i := r \cdot m_i \cdot \alpha^n \in K\{\alpha\}$.

(Here $K\{\alpha\}$ is the subring of $\mathrm{E}_K(L,+)$ generated by α over K, that is the skew polynomial ring $K[t;\alpha]$ where $ta = a^p t$.)

Proof.

(i) Let $g \in \mathrm{E}_K(L,+)$. Since g is a definable map, by compactness there exist an integer n and definable subsets $D_1, \dots, D_n$ over K, which cover L, such that $g_{|D_i} = P_i/Q_i$ with $P_i, Q_i \in K[X_\infty]$ and $Q_i \neq 0$ on D_i. (By Fact 1.3, $a \in L$, $\mathrm{dcl}(K \cup \{a\}) = K(a_\infty)$ for $a \in L$). So we can assume that $P_i, Q_i \in K[X_{=m}]$, and D_i is defined by $(R_i \neq 0) \wedge (S_i = 0)$ with R_i and S_i in $K[X_{=m}]$ for some choice of m.

For every $j \in (p^\nu)^m$ we define $g_j := g \circ m_j \circ \alpha^m$ (i.e. $g_j(x) = g(m_j.x^{p^m})$).

Since g is an endomorphism,

$$g = \sum_{j \in (p^\nu)^m} g_j \circ f_j.$$

Note that the g_j's are also in $\mathrm{E}_K(L,+)$.

We show that each g_j is equal to an additive polynomial in $K[X]$. Let $j \in (p^\nu)^m$. For every $T \in K[X_{=m}]$ we define

$$T^j(X_j) := T(0, ..., 0, X_j, 0, ..., 0).$$

Let D_i^j be the subset of L defined by $(R_i^j(X) \neq 0) \wedge (S_i^j(X) = 0)$, i.e., D_i^j is the projection on L of the set $\lambda_m(D_i) \cap (0 \times \dots \times 0 \times L \times 0 \times \dots \times 0)$.

Then $L = \cup_{i \in \{1,\dots,n\}} D_i^j$, and so there is $i \in \{1, ..., n\}$ such that D_i^j is cofinite in L. Since $g_{j|D_i^j} = P_i^j/Q_i^j$ for each $i \in \{1, ..., n\}$, g_j is equal to a rational function R on a cofinite set C of L. Then for every $x, y \in C$ with $x + y \in C$, $R(x+y) = R(x)+R(y)$. By Remark 2.1 (iii), R is equal to an additive polynomial P. It follows that g_j is equal to P.

Thus each g_j equals a p-polynomial in $K[X]$, and it follows that g is equal to a p-polynomial of $K[X_{=m}]$.

(ii) First note that the relations 1, 2, 3 and 4 are satisfied by α and the f_i's. So by (i) we have a surjective map ϕ from R/I onto $\mathrm{E}_K(L,+)$.

Now let $r \in \ker \phi$. By an easy induction, one deduces from relations 1, 2 and 3 that r can be written in the following form :

$$r = \sum_{i \in (p^\nu)^{<\omega},\, j} a_{i,j} t^j s_i,$$

where $a_{i,j} \in K$ and $s_i := s_{i(n-1)} \dots s_{i(0)}$ for $i \in p^{\nu n}$.

From relation 4, one finds an integer n such that

$$r = \sum_{i \in p^{\nu n}} \left(\sum_j a'_{i,j} t^j \right) s_i.$$

For each $i \in p^{\nu n}$, we see by 2 that

$$r m_i t^n = \sum_j a'_{i,j} t^j.$$

Thus $rm_it^n = 0$, because the p-polynomial $\sum_j a'_{i,j}X^{p^j}$ is equal to zero on L. Therefore $r = \sum_{i\in p^{\nu n}} rm_it^ns_i = 0$.

(iii) follows from (i) and the proof of (ii). □

Corollary 3.2. *The ring* $\mathrm{E}_K(L,+)$ *contains no non-trivial two-sided ideal.*

Proof. Let $r \in \mathrm{E}_K(L,+) \setminus \{0\}$. We show that the two-sided ideal $\langle r\rangle$ generated by r is all $\mathrm{E}_K(L,+)$.

By Proposition 3.1 (iii), there is an integer n such that for every $i \in p^{\nu n}$, $r \cdot m_i \cdot \alpha^n \in K\{\alpha\}$. So there is a non-zero polynomial P in $K[X]$ such that $P(\alpha) = \sum_j a_j\alpha^j \in \langle r\rangle$.

We prove by induction on the degree of P that $1 \in \langle r\rangle$. If $\deg(P) = 0$ then it is obvious that $1 \in \langle r\rangle$. If $\deg(P) = m+1$, there are two possibilities:

(i) If $a_0 = 0$, then by the relations 2 and 3 we see for every $l \in p^\nu$ that

$$f_l \cdot P(\alpha) = \sum_j \left(\sum_{i\in p^\nu} a'_{j,i}f_i\alpha^j\right) = \sum_j a'_{j,0}\alpha^{j-1}.$$

By the relation 4, there is $l \in p^\nu$ such that $f_l \cdot P(\alpha) \neq 0$, so there is a non-zero polynomial R in $K[X]$ such that $\deg(R) < \deg(P)$ and $R(\alpha) \in \langle r\rangle$.

(ii) Otherwise, by multiplication by a scalar we can suppose that $a_0 = 1$. Let $Q(\alpha) = P(\alpha) - 1$. As in (i), there is $l \in p^\nu$ such that $f_lQ(\alpha) \neq 0$. So there is R in $K[X]$ such that $\deg(R) < \deg(P)$ and $f_l + R(\alpha) \in \langle r\rangle$. Now take $i \neq l$. Then

$$(f_l + R(\alpha)) \cdot m_i \cdot \alpha = R(\alpha) \cdot m_i \cdot \alpha.$$

Thus there is S in $K[X]$ such that $\deg(S) \leq \deg(P)$ and $S(\alpha) = \sum_j a'_j\alpha^j \in \langle r\rangle$, with $a'_0 = 0$. To finish the proof we apply (i) to $S(\alpha)$. □

3.2 The $\mathrm{E}_K(L,+)$-module structure

Now we study the connections between the field structure L over K and the $\mathrm{E}_K(L,+)$-module structure. We denote by L_{K-mod} the module structure on $(L,+)$ over $\mathrm{E}_K(L,+)$. We work with the usual language for modules, that is, the language $\{0,+,r;\ r \in \mathrm{E}_K(L,+)\}$, where each r is a unary function interpreted as scalar multiplication by the endomorphism element r on $(L,+)$.

The following proposition is a direct consequence of Proposition 2.3:

Proposition 3.3.

(i) Let $k > 0$. Every subgroup of $(L^{\times k},+)$ definable over K in the field structure L is already quantifier-free definable over $\emptyset$ in the module structure L_{K-mod}. Thus every λ-closed subgroup of $(L^{\times k},+)$ over K is $\bigwedge\!\!\!\bigwedge$-definable over $\emptyset$ in L_{K-mod}.

(ii) L_{K-mod} has no non-trivial $\bigwedge\!\!\!\bigwedge$-definable submodule over $\emptyset$.

Remark. *This does not mean that every definable subgroup of $(L^{\times k}, +)$ in the field structure L is one-based. Indeed it is known that no definable set in the field structure is one-based, as follows from the fact that any non-algebraic formula contains a point having some generic p-component, i.e. a p-component which realizes the generic of $(L, +)$. Some λ-closed subgroups of $(L^{\times k}, +)$ over K are one-based; this is the case exactly when the structure induced by the field L coincides with the structure induced by the module L_{K-mod} (see Proposition 3.6 below). Neither does it mean that every λ-closed subgroup of $(L^{\times k}, +)$ is non-multidimensional. (In fact there exist multidimensional minimal additive subgroups, as can be seen by modifying slightly the construction of multidimensional minimal types of F. Delon [De 88, Lemme 71].)*

Proof.

(i) Let G be a subgroup of $(L^{\times k}, +)$ definable over K in the field structure L. By Proposition 2.3 and Remark 2.4, there is a p-polynomial P in $K[X_{1,\infty}, \ldots, X_{k,\infty}]$ such that $G = V(\{P\})$. Then $P(X_{1,\infty}, \ldots, X_{k,\infty}) = \sum_{j=1}^{k} P_j(X_{j,\infty})$ where the P_j's are p-polynomials of $K[X]$. For every j, let $r_j \in \mathrm{E}_K(L, +)$ be defined by P_j. Then the group G is defined in the module structure L_{K-mod} by

$$\sum_{j=1}^{k} r_j(x_j) = 0.$$

(ii) Let M be an $\bigwedge$-submodule of L_{K-mod} over $\emptyset$. Let $\mathrm{Ann}(M) := \{r \in \mathrm{E}_K(L, +) : r(m) = 0$ for every $m \in M\}$. Since M is a submodule, $\mathrm{Ann}(M)$ is a two-sided ideal of $\mathrm{E}_K(L, +)$. So by Corollary 3.2, $\mathrm{Ann}(M)$ is trivial; M is also trivial because M is an additive subgroup over K and by Proposition 2.3, $M := \{x \in L : r(x) = 0$ for every $r \in \mathrm{Ann}(M)\}$. □

We can deduce from (i),

Corollary 3.4. *The theory of L_{K-mod} admits quantifier elimination.*

Proof. In a module, each formula is equivalent to a finite boolean combination of *pp*-formulas [Zie]. Moreover a *pp*-formula without parameters defines a subgroup, so in L_{K-mod} a *pp*-formula defines a subgroup of $(L^{\times k}, +)$, definable over K in the field structure L. We conclude with Proposition 3.3 (i). □

Now we prove that the induced structure on a one-based additive subgroup in the field L reduces to that induced by L_{K-mod}.

We recall some general classical facts:

Fact 3.5. *[H-P, Pi] Let G be an $\bigwedge$-definable group over $\emptyset$ in a stable saturated structure.*

(i) G is **one-based** *if and only if for every $n < \omega$, every* $\mathrm{acl}^{eq}(\emptyset)$*-definable subset of G^n is a finite boolean combination of cosets of* $\mathrm{acl}^{eq}(\emptyset)$*-definable subgroups of G^n.*

(ii) If G is one-based, B is a set of parameters and $n < \omega$, then every B-definable subset of G^n is a finite boolean combination of $\mathrm{acl}^{eq}(B)$-definable cosets of $\mathrm{acl}^{eq}(\emptyset)$-definable subgroups of G^n.

(iii) If G is one-based and $n < \omega$, then every connected definable subgroup of G^n is definable over $\mathrm{acl}^{eq}(\emptyset)$.

Proposition 3.6. *Suppose here that K is separably closed (i.e., $K \prec L$).*

Let $k > 0$ and let G be a λ-closed subgroup of $(L^{\times k}, +)$ over K.

(i) If G is one-based and $n < \omega$, then every subset of G^n definable (with parameters) in L is already definable in L_{K-mod}.

(ii) The group G is one-based if and only if the structure induced on G by the field L over K is equal to the structure induced on G by the module L_{K-mod} over $\mathrm{acl}(\emptyset)$.

Proof. Note first that since $K \prec L$ and so $\mathrm{acl}^{eq}(K) = \mathrm{dcl}^{eq}(K)$, every definable subset of $L^{\times k}$ over $\mathrm{acl}^{eq}(K)$ in L is definable over K.

Suppose that G is one-based.

(i) By Fact 3.5, every definable subset D of G^n in L is a finite boolean combination of cosets of K-definable subgroups of G^n for each $n < \omega$. Thus, by Proposition 3.3, D is definable in L_{K-mod}.

(ii) Moreover, if D is K-definable in L, then by Fact 3.5, D is a finite boolean combination of K-definable cosets. A K-definable coset T in L of a K-definable group H is equal to $\bar{a}+H$ where $\bar{a} \in K$, because K is an elementary substructure of L. So by Proposition 3.3, H is $\emptyset$-definable in L_{K-mod} and T is definable in L_{K-mod} over K. But in L_{K-mod}, $K = \mathrm{acl}(\emptyset) = \mathrm{dcl}(\{1\})$ because $\mathbb{F}_p \subset \mathrm{acl}(\emptyset)$, $K = K \cdot 1$ and K is an elementary substructure of the reduced structure L_{K-mod}.

Conversely, suppose that the structure induced on G by the field L over K is equal to the structure induced on G by the module L_{K-mod} over $\mathrm{acl}(\emptyset)$. In a module M, every A-definable set of $M^{\times n}$ is a finite boolean combination of A-definable cosets of $\emptyset$-definable subgroups of $(M^{\times n}, +)$ by *pp*-elimination of quantifiers. Thus every K-definable subset of G^n in L is a finite boolean combination of K-definable cosets of K-definable subgroups of G^n in L. So by Fact 3.5, G is one-based. □

3.3 Definable endomorphisms of an additive subgroup

We have seen that every definable endomorphism of the additive group is defined by a p-polynomial. This is also true for the additive subgroups:

Proposition 3.7. *Let G be an additive subgroup defined over K. If ϕ is a K-definable homomorphism from G to $(L, +)$, then ϕ is defined by a p-polynomial P of $K^{sep}[X_\infty]$; i.e., ϕ is a restriction of an element of $E_{K^{sep}}(L, +)$. Moreover, if G is connected, we can choose $P \in K[X_\infty]$.*

The proof use the corresponding results for the irreducible closed subgroups of a power of the additive group of an algebraically closed field:

Lemma 3.8. *Let F be an algebraically closed field of characteristic p, and let F_0 be an algebraically closed subfield. Let G be an irreducible closed subgroup of $(F^{\times k}, +)$ over F_0, and let ϕ be an F_0-definable homomorphism (respectively, a rational homomorphism defined over F_0) from G to $(F, +)$. Then ϕ can be extended to an F_0-definable homomorphism (respectively, a rational homomorphism defined over F_0) from $(F^{\times k}, +)$ to $(F, +)$.*

Proof. By Fact 2.2(i), there are $l \leq k$ and an algebraic automorphism β of $(F^{\times k}, +)$, defined by p-polynomials in $F_0[X_1, ..., X_k]$, which sends G onto $F^{\times l} \times \{0\}^{k-l}$. Then $\beta = (\beta_1, \ldots, \beta_k)$ where the β_i's are p-polynomials in $F_0[X_1, ..., X_k]$.

Let $\delta := (\beta_1, \ldots, \beta_l, 0, \ldots, 0)$. Then $\beta^{-1} \circ \delta$ is an algebraic homomorphism from $F^{\times k}$ to G, defined over F_0, whose restriction to G is the identity. Let $\psi = \phi \circ \beta^{-1} \circ \delta$. Then ψ is an F_0-definable homomorphism (respectively a rational homomorphism defined over F_0) from $F^{\times k}$ to $\phi(G)$ which extends ϕ and which is F_0-definable. □

Proof of Proposition 3.7. Let G be an additive subgroup over K and let ϕ be a K-definable homomorphism from G to $(L, +)$.

First of all, we can assume that G is definable: indeed, by Fact 1.9, G is the intersection of λ_n-closed subgroups G_n of $(L, +)$. By compactness, there is $n < \omega$ such that ϕ is the restriction of a K-definable homomorphism from G_n to $(L, +)$. Moreover, if G is connected, G_n can be taken to be connected as well.

Suppose now that G is definable. Recall that the connected component G^0 of G is also K-definable. First, we prove that ϕ is defined on G^0 by a p-polynomial of $K[X_\infty]$.

Since ϕ is a K-definable map, by Fact 1.3(iii), there are $n < \omega$ and K-definable subsets $D_1, \ldots, D_n$ covering G, such that $\phi_{|D_i} = P_i/Q_i$ where $P_i, Q_i \in K[X_\infty]$ and $Q_i \neq 0$ on D_i.

Then there is $m < \omega$ such that G^0 is λ_m-closed, the subsets D_i are λ_m-constructible, and the P_i's and the Q_i's are in $K[X_{=m}]$.

Let $H := \lambda_m(G^0)$ and let ϕ' be the homomorphism from H to $(L, +)$ defined by $\phi'(\bar{x}) := \phi(\lambda_m^{-1}(\bar{x}))$, for every $\bar{x} \in H$. Then H is a λ_0-closed subgroup of $L^{\times p^{\nu m}}$. Let $D'_i := \lambda_m(D_i)$. Then for every i, $\phi'_{|D'_i}(\bar{x}) = P_i(\bar{x})/Q_i(\bar{x})$ where the P_i's and the Q_i's are seen as polynomials of $K[\bar{X}]$ with $\bar{X}$ a $p^{\nu m}$-tuple of variables.

Since H is connected, we can assume that the generic elements of H are all in D'_1.

Let $\bar{H}$ be the Zariski closure of H in $\bar{L}^{\times p^{\nu m}}$. Let Γ be the K-definable subset of $(\bar{H} \times \bar{L})$ defined by

$$\Gamma := \{(\bar{x}, y) : \bar{x} \in \bar{H},\ Q_1(\bar{x}) \neq 0 \text{ and } P_1(\bar{x}) = Q_1(\bar{x})y\}.$$

Since H is λ_0-closed, by Fact 1.8(iii), $H = \bar{H}(L)$ and the generic elements of H in L are also generic in $\bar{H}$.

It follows that Γ defines generically a rational homomorphism from $\bar{H}$ to $(\bar{L}, +)$. By a classical result, [Ro, Theorem 3], there is a rational homomorphism ψ from $\bar{H}$ to $(\bar{L}, +)$ defined over K and extending the homomorphism generically defined by Γ. On the generic elements of H, ψ agrees with ϕ'; since every element of H is a sum of two generics, ψ extends ϕ'.

Now by Lemma 3.8 and Remark 2.1(iii), there are $(a_{i,j})_{i\in\{1,\dots,p^{\nu m}\},j\in\omega}$ in $\bar{K}$, only finitely many of which are non-zero, such that for every $x \in H$,

$$\psi(\bar{x}) = \sum_{i=1}^{p^{\nu m}} \sum_{j\geq 0} a_{i,j}\alpha^j(x_i).$$

Thus there exist $r < \omega$ and a p-polynomial S of $K^{sep}[\bar{X}]$ such that $\alpha^r \circ \phi'(\bar{x}) = S(\bar{x})$ for every $\bar{x} \in H$. Since $\alpha^r \circ \phi'$ is K-definable, we can choose $S \in K[\bar{X}]$. Indeed, it suffices to consider the connected λ_0-closed subgroup of $(L^{\times p^{\nu m}} \times L, +)$ defined by the graph of $\alpha^r \circ \phi'$. By Lemma 2.7, there is a p-polynomial P of $K[\bar{X}, Y]$, unitary and of degree 1 in Y, such that $P(\bar{x}, S(\bar{x})) = 0$ for every $\bar{x} \in H$.

Now, $\phi' = f_0^r \circ \alpha^r \circ \phi'$, so $\phi'(\bar{x}) = S'(\bar{x}_{=r})$ for every $\bar{x} \in H$, where S' is a p-polynomial of $K[\bar{X}_{=r}]$. Thus $\phi(x) = T(x_{=m+r})$ for every $x \in G^0$, where T is a p-polynomial of $K[X_{=m+r}]$.

The subgroup G^0 is of finite index in G. So $G = \cup_{j=1}^{n}(a_j + G^0)$. It follows that for every j, $a_j + G^0$ is K^{sep}-definable because $K^{sep} \preccurlyeq L$. Thus, there are $k < n$ and $c_1, \dots, c_k \in K^{sep}$ such that

$$G^0 \subsetneq G^0 + \mathbb{F}_p c_1 \subsetneq \dots \subsetneq G^0 + \mathbb{F}_p c_1 + \dots + \mathbb{F}_p c_k = G.$$

For $i \in \{0, ..., k\}$, let $H_i := G^0 + \mathbb{F}_p c_1 + \dots + \mathbb{F}_p c_i$. Note that the H_i's are K^{sep}-definable. We prove by induction that for every i, there is a p-polynomial T_i of $K^{sep}[X_\infty]$ such that $\phi(x) = T_i(x_\infty)$ for all $x \in H_i$.

We have just proved the case $i = 0$ above.

Suppose that for $i < k$, there is a p-polynomial T_i of $K^{sep}[X_\infty]$ such that $\phi(x) = T_i(x_\infty)$ for every $x \in H_i$. By Proposition 2.3, there is a p-polynomial P of $K^{sep}[X_\infty]$ such that $H_i = V(\{P\})$. Since $c_{i+1} \notin H_i$, $P(c_{i+1,\infty}) \neq 0$. We define a p-polynomial T_{i+1} of $K^{sep}[X_\infty]$ by

$$T_{i+1} := T_i + \frac{\phi(c_{i+1}) - T_i(c_{i+1,\infty})}{P(c_{i+1,\infty})} P.$$

Because $T_{i+1} = T_i$ on H_i and $\phi(c_{i+1}) = T_{i+1}(c_{i+1,\infty})$, we conclude $\phi(x) = T_{i+1}(x_\infty)$ for every $x \in H_{i+1}$. $\square$

Remark. *Either by using the bijections λ_n or via the proof above - but with much more cumbersome notation - we obtain the analogous result for every λ-closed subgroup of a power of $(L, +)$: if G is a λ-closed subgroup of $(L^{\times k}, +)$*

and ϕ is a K-definable homomorphism from G to $(L,+)$, then there is a p-polynomial P of $K^{sep}[X_{1,\infty},\dots,X_{k,\infty}]$ such that $\phi(x) = P(x_{1,\infty},\dots,x_{k,\infty})$ for all $x=(x_1,\dots,x_k)\in G$,.

4 Minimal additive subgroups

In this section we construct families of minimal additive subgroups (over $\emptyset$) with specific properties. These constructions enable us to give partial answers to certain questions mentioned in the Introduction. We show (i) that there exists a minimal additive subgroup which is not locally modular and whose ring of definable endomorphisms is equal to $\mathbb{F}_p$, (ii) that there exist $2^{\aleph_0}$ pairwise orthogonal $\aleph_0$-categorical minimal additive subgroups over $\emptyset$ whose induced structures are exactly those of $\mathbb{F}_p$-vector spaces, and (iii) that there exists a locally modular minimal additive subgroup whose quasi-endomorphism division ring is infinite but whose endomorphism subring is equal to $\mathbb{F}_p$.

4.1 Preliminaries

First we recall some general definitions and facts on minimal groups in a stable structure (see [Pi] or [Wa]). We fix a stable sufficiently saturated structure M.

Recall that an $\bigwedge\!\!\bigwedge$-definable subset D of $M^{\times k}$ is **minimal**, if for every definable subset C of $M^{\times k}$, $C\cap D$ is finite or cofinite. For an $\bigwedge\!\!\bigwedge$-definable group G, G is minimal (in the sense above) if and only if G is connected and of U-rank 1.

We fix now an $\bigwedge\!\!\bigwedge$-definable minimal group G in M. Without loss of generality, we assume that G is $\bigwedge\!\!\bigwedge$-definable over $\emptyset$. Recall that G is **locally modular** (in the sense of the associated pregeometry) if and only if G is one-based. (See Fact 3.5 for the characterization of one-based groups.)

A minimal group is necessarily abelian. If it is locally modular as well, its induced structure is, "modulo $\mathrm{acl}(\emptyset)$", a vector space:

Definition-Fact 4.1. Let B be a set of parameters. Let $G_\emptyset := G\cap \mathrm{acl}(\emptyset)$ and $G_B := G\cap\mathrm{acl}(B)$.

i) A **quasi-endomorphism** of G over B is a minimal subgroup S of $G\times G$ which is definable over $\mathrm{acl}^{eq}(B)$ and whose projection on the first factor is all of G. We denote by $\mathrm{QsE}_B(G)$ the set of quasi-endomorphisms of G over B. For each quasi-endomorphism S we denote by $\mathrm{coker}(S)$ the finite subset $\{y\in G : (0,y)\in S\}$.

ii) A **quasi-definable endomorphism of** G/G_B is an endomorphism f of G/G_B for which there exists a formula $\psi(x,y)$ with parameters in $\mathrm{acl}^{eq}(B)$ such that for every a, $b\in G$: $f(a+G_B)=b+G_B$ if and only if there are $c\in a+G_B$ and $d\in b+G_B$ such that $\psi(c,d)$ holds.

iii) Let $S\in \mathrm{QsE}_B(G)$. Then $\{(a+G_B, b+G_B) : (a,b)\in S\}$ is the graph of a quasi-definable endomorphism f_S of G/G_B. Moreover, the map which sends

$S \in \mathrm{QsE}_B(G)$ to f_S is a bijection from $\mathrm{QsE}_B(G)$ onto the set of quasi-definable endomorphisms of G/G_B.

iv) The endomorphism ring of G/G_B induces the structure of a division ring on $\mathrm{QsE}_B(G)$.

v) Note that the ring of endomorphisms of G which are definable over $\mathrm{acl}^{eq}(B)$ is a subring of $\mathrm{QsE}_B(G)$.

Remark. *One can define, in several equivalent ways, the set of quasi-endomorphisms in contexts more general than that of minimal groups. This set is always a ring, but not necessarily a division ring. (See [Bu], [Wa], or [Bl].)*

Remark 4.2. *If G is locally modular then $\mathrm{QsE}_B(G) = \mathrm{QsE}_\emptyset(G)$ because, by Fact 3.5(iii), every definable minimal subgroup of $G \times G$ is definable over $\mathrm{acl}^{eq}(\emptyset)$.*

Fact 4.3. *[Hr 85] If G is locally modular then the pregeometry on G is given by the structure of $\mathrm{QsE}_\emptyset(G)$-vector space on $G/G_\emptyset$: for every $g, g_1, \ldots, g_n \in G$, $g \in \mathrm{acl}(g_1, \ldots, g_n)$ if and only if there are $S_1, \ldots, S_n \in \mathrm{QsE}_\emptyset(G)$ such that $g + G_\emptyset = f_{S_1}(g_1 + G_\emptyset) + \ldots + f_{S_n}(g_n + G_\emptyset)$.*

Now we consider the relation between non-orthogonality and isogenies for minimal groups. We fix a second $\mathbb{M}$-definable minimal group H in M (over $\emptyset$).

Recall that if G and H are locally modular and **non-orthogonal**, then their generics over $\emptyset$ are equialgebraic ([Pi, Corollary 2.5.5]).

Definition. An **isogeny** is a surjective homomorphism from one group onto another with finite kernel. We say that the first group is isogenous to the second.

We have given a proof of the following fact for stable abelian connected groups in [Bl]:

Fact 4.4. *If G is definably isogenous to H, then $\mathrm{QsE}_\emptyset(G)$ is isomorphic to $\mathrm{QsE}_\emptyset(H)$.*

Obviously if G is definably isogenous to H then G and H are non-orthogonal. Conversely, one can check the following in the locally modular case:

Fact 4.5. *Suppose that G and H are locally modular and non-orthogonal.*

There are $\mathrm{acl}^{eq}(\emptyset)$-definable finite subgroups G', H' of G, H, resp., such that G/G' and H/H' are $\mathrm{acl}^{eq}(\emptyset)$-definably isomorphic. In particular $\mathrm{QsE}_\emptyset(G)$ and $\mathrm{QsE}_\emptyset(H)$ are isomorphic.

Thus G is definably isogenous to H/H'. If H is divisible, then G is definably isogenous to H.

By Fact 4.3, one knows the pregeometry of a locally modular minimal group, but it is still quite difficult to describe the actual structure induced on the group. This complication comes from $\mathrm{acl}(\emptyset)$. In particular:

Remark 4.6. *If G is locally modular and $G_\emptyset = \{0\}$, then the induced structure on G is exactly the* $\mathrm{QsE}_\emptyset(G)$*-vector space.*

In the general case, a reasonable description can be obtained in the case of locally modular minimal groups of unbounded exponent, up to almost interdefinability:

Fact 4.7. *[Lo] Suppose that G is of unbounded exponent.*

(i) The division ring $\mathrm{QsE}_\emptyset(G)$ *is a field of fractions of the subring of endomorphisms of G in* $\mathrm{QsE}_\emptyset(G)$*.*

(ii) Let R be the subring of endomorphisms of G in $\mathrm{QsE}_\emptyset(G)$*. If G is locally modular, then the structure induced on G by M over* $\mathrm{acl}^{eq}(\emptyset)$ *is almost interdefinable with the structure* $(G,\ +,\ r,\ c:\ r \in R, c \in G_\emptyset)$*.*

The situation is much more complicated for locally modular minimal groups of bounded exponent (see [H-L]). In particular, E. Hrushovski and J. Loveys have given an example of such a group but which satisfies neither conclusion in Fact 4.7.

Although the quasi-endomorphism division rings are of interest primarily in the locally modular case, we examine first the non-locally modular case for minimal additive subgroups, to show that one cannot see the locally modularity from the subring of endomorphisms.

In the following subsections, we fix an elementary substructure K of L. If G is a minimal additive subgroup over K, we denote by $\mathrm{E}_K(G)$ the subring of endomorphisms of G in $\mathrm{QsE}_K(G)$. Observe that every quasi-endomorphism in $\mathrm{QsE}_K(G)$ is definable over K, because K is an elementary substructure; thus $\mathrm{E}_K(G)$ is the ring of endomorphisms of G definable over K.

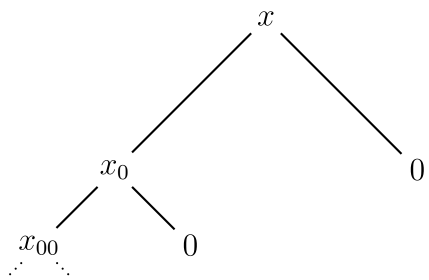
4.2 $(L^{p^\infty}, +)$ and the non-locally modular case

In this subsection, we observe in particular that we can recognize the non-local modularity from the quasi-endomorphism division ring. In fact, all non-locally modular minimal additive subgroups have the same quasi-endomorphism division ring. We display here a non-locally modular minimal additive subgroup whose endomorphism subring is $\mathbb{F}_p$.

Recall that the subfield $L^{p^\infty} := \bigcap_{n\in\omega} L^{p^n}$ is a minimal $\bigwedge\!\!\bigwedge$-definable subset of L; it is the maximal algebraically closed subfield of L, and the unique minimal algebraically closed subfield of L, up to definable endomorphism ([Hr 96], using the results about fields definable in separably closed fields [Me 94]). The field L^{p^∞} is a pure field in the following sense:

Fact 4.8. *(see [De 98, Proposition 2.14]) Let K be an elementary substructure of L.*

Let k be an integer and D be a K-definable subset of $L^{\times k}$. Then $D \cap (L^{p^\infty})^{\times k}$ is a constructible subset of $(L^{p^\infty})^{\times k}$, definable with parameters from K^{p^∞}.



The field L^{p^∞}

Clearly the additive group $(L^{p^\infty}, +)$ is a non-locally modular minimal additive subgroup. It is unique up to definable isogeny:

Fact 4.9. *[Hr 96] or [B-D2, Proposition 4.7].*
Let G be a minimal additive subgroup. The following properties are equivalent:
- G is non-locally modular,
- G is non-orthogonal to L^{p^∞},
- G is definably isogenous to $(L^{p^\infty}, +)$.

From now on K is an elementary substructure of L, where L is $|K|^+$-saturated. Recall that α denotes the Frobenius map.

Proposition 4.10.
(i) $\mathrm{E}_K(L^{p^\infty}) = K^{p^\infty}\{\alpha, \alpha^{-1}\}$.
(ii) $\mathrm{QsE}_K(L^{p^\infty})$ *is the (unique) field of fractions of* $\mathrm{E}_K(L^{p^\infty})$.
(iii) Let G be a minimal additive subgroup over K. If $|K| > \aleph_0$, then $\mathrm{QsE}_K(G)$ *is isomorphic to* $\mathrm{QsE}_K(L^{p^\infty})$ *if and only if G is non-locally modular.*

Proof.
(i) From Fact 4.8, we deduce that $\mathrm{E}_K(L^{p^\infty})$ is the ring of K^{p^∞}-definable endomorphisms of the additive group of the algebraically closed field L^{p^∞}.

(ii) Let $S \in \mathrm{QsE}_K(L^{p^\infty})$. Then $\mathrm{coker}(S)$ is a K-definable finite subgroup of L^{p^∞}. So $\mathrm{coker}(S) \subseteq K^{p^\infty}$ and $P(X) := \prod_{r \in \mathrm{coker}(S)}(X - r)$ is a p-polynomial of $K^{p^\infty}[X]$. Thus $x \mapsto P(x)$ defines an endomorphism $f \in \mathrm{E}_K(L^{p^\infty})$ such that $f \circ S \in \mathrm{E}_K(L^{p^\infty})$.

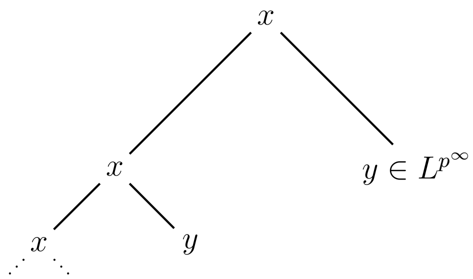
Moreover it is easy to see that $K^{p^\infty}\{\alpha, \alpha^{-1}\}$ is an Ore domain, and hence $\mathrm{E}_K(L^{p^\infty})$ has a unique field of fractions [Co].

(iii) If G is non-locally modular then by Fact 4.9, G is definably isogenous to L^{p^∞}, so by Fact 4.4, $\mathrm{QsE}_K(G)$ is isomorphic to $\mathrm{QsE}_K(L^{p^\infty})$.

If G is locally modular, then $\mathrm{QsE}_K(G)$ is countable. Indeed, G is λ-closed, so there exists a countable elementary submodel $K_0 \preccurlyeq K$ such that G is λ-closed over K_0. By Remark 4.2, $\mathrm{QsE}_K(G) = \mathrm{QsE}_{K_0}(G)$, which is countable. $\square$

All non-locally modular minimal additive subgroups have the same quasi-endomorphism division ring, but we will see that they can have very different endomorphism subrings. In [B-D2] the authors give an example of a minimal additive subgroup which is definably isogenous but not definably isomorphic to $(L^{p^\infty},+)$. We show that this example has a finite endomorphism ring.

Proposition 4.11. *Let* $G := \{x : \exists y \in L^{p^\infty},\ x = x^p + by^p\}$.



The group G

The endomorphism ring $\mathrm{E}_K(G)$ *is equal to* $\mathbb{F}_p$.

Proof. We use the notation described in section 1.1: in particular for every $x \in L$,

$$x = x_0^p + x_1^p b + \ldots + x_{p-1}^p b^{p-1} + \sum_{p \le i < p^\nu} x_i^p m_i$$

and so

$$G = \{x : x_0 = x,\ x_1 \in L^{p^\infty},\ x_i = 0 \text{ for every } i \in p^\nu \setminus \{0,1\}\}.$$

Let $y := x_1$ for $x \in G$.

Let ϕ be a K-definable endomorphism of G.

Claim 1. *There are* a_j*'s and* c_j*'s in* K *such that for every* $x \in G$, $\phi(x) = \sum_{j=0}^n a_j x^{p^j} + \sum_{j=0}^m c_j y^{p^{-j}}$.

$\triangleright$ It follows from Proposition 3.7, that there exist an integer $m \ge 0$ and a p-polynomial P of $K[X,Y]$ such that $\phi(x) = P(x, y^{p^{-m}})$ for every $x \in G$. By using the relation $y^p = (x - x^p)/b$ we verify the claim. $\triangleleft$

Claim 2. *For* $0 < j \le n$, $a_j = 0$, $a_0 = a_{0,0}$, *and* $c_0 = a_{0,p-1}b$; *for* $0 < j \le m$, $c_j = c_{j-1,0}$, *and* $c_{m,0} = 0$.

$\triangleright$ We use the equation $f_0 \circ \phi(x) = \phi(x)$ for every $x \in G$ to get

$$f_0 \circ \phi(x) = a_{0,0}x + a_{0,p-1}by + \sum_{j=1}^n a_{j,0} x^{p^{j-1}} + \sum_{j=0}^m c_{j,0} y^{p^{-j-1}}.$$

So we have the following equality on G:

$$(1)\quad \sum_{j=0}^{n} a_j x^{p^j} + \sum_{j=0}^{m} c_j y^{p^{-j}} = a_{0,0}x + \sum_{j=1}^{n} a_{j,0} x^{p^{j-1}} + a_{0,p-1} by + \sum_{j=0}^{m} c_{j,0} y^{p^{-j-1}}.$$

Now it is easy to check that if we have an equation over G of the following type

$$\sum_{j=0}^{l} d_j x^{p^j} + \sum_{j=0}^{r} e_j y^{p^{-j}} = 0,$$

then the coefficients d_j and e_j are all zero. (It suffices to take the equation to the power p^{r+1} and to use the relation $y^p = (x - x^p)/b$.) The claim now follows trivially. $\triangleleft$

Claim 3. *For every $i \neq 0$, $a_{0,i} = 0$.*

$\triangleright$ On the one hand, for every $i \neq 0,\ 1$ and for every $x \in G$,

$$f_i \circ \phi(x) = 0 = a_{0,i}x + \text{terms in } y,\ y^{p^{-1}},\ y^{p^{-m}}.$$

So $a_{0,i} = 0$.

On the other hand, $f_1(\phi(x)) \in L^{p^\infty}$ for all $x \in G$. Thus

$$a_{0,1}x + a_{0,0}y + \sum_{j=0}^{m} c_{j,1} y^{p^{-j-1}} \in L^{p^\infty}.$$

Setting $x = 1$ we get $a_{0,1} \in L^{p^\infty}$. Now for every $k > 0$ and $i \in p^{\nu k} \setminus \{0\}$,

$$f_i(a_{0,1}x + a_{0,0}y + \sum_{j=0}^{m} c_{j,1} y^{p^{-j-1}}) = a_{0,1}^{p^{-k}} f_i(x) + a_{0,0i} y^{p^{-k}} + \sum_{j=0}^{m} c_{j,1i} y^{p^{-j-1-k}} = 0.$$

But $f_i(x)$ equals 0 or x or y,..., or $y^{p^{-k+1}}$. Therefore $a_{0,0i} = 0$ and $c_{j,1i} = 0$ for all j.

Thus $a_{0,1}$, $a_{0,0}$ and all the $c_{j,1}$'s lie in L^{p^∞}. So $a_{0,1} = 0$, otherwise $G \subseteq L^{p^\infty}$.$\triangleleft$

It follows from claims 2 and 3 that the $c_j's$ are equal to zero and $a_0 = a_0^p$. Therefore for every $x \in G$, $\phi(x) = a_0 x$ with $a_0 \in \mathbb{F}_p$. $\square$

We finish this section with a property of those minimal additive subgroups which are not definably isomorphic to $(L^{p^\infty}, +)$:

Proposition 4.12. *Let G be a minimal additive subgroup over K. Either $K \cap \mathrm{E}_K(G)$ is a finite field, or $K \cap \mathrm{E}_K(G) = K^{p^\infty}$ and $G = gL^{p^\infty}$ for each $g \in G \setminus \{0\}$. If G is locally modular, then $K \cap \mathrm{E}_K(G)$ is finite.*

Proof. Let G be a minimal additive subgroup over K.

For every $l < \omega$, let $G_l := V(I(G)_{=l})$. Each G_l is connected, and by Fact 1.9, $G = \cap G_l$.

First we check that

$$K \cap \mathrm{E}_K(G) = K \cap \bigcap_{l \in \omega} \mathrm{E}_K(G_l).$$

It suffices to show that if $k \in K \cap \mathrm{E}_K(G) \setminus \{0\}$, then $kG_l = G_l$ for any $l < \omega$. We have

$$I(kG)_{=l} = \{Q(X_{=l}) : Q(X_{=l}) := P(k^{-1}X_{\leq l}) \text{ and } P(X_{=l}) \in I(G)_{=l}\}.$$

So $V(I(kG)_{=l}) = kV(I(G)_{=l}) = kG_l$. Moreover, by minimality of G, $kG = G$. Thus $G_l = V(I(kG)_{=l}) = kG_l$.

We now examine $K \cap \mathrm{E}_K(G_l)$ for a fixed integer l. Let H be the connected λ_0-closed subgroup $\lambda_l(G_l)$ of $(L^{\times p^{\nu l}}, +)$ over K. Let h be a generic of H over K. We can assume that $h := (\bar{x}, y_1, \ldots, y_n)$ where $\bar{x}$ is a separating transcendence basis of $K(h)$ over K. By Lemma 2.7, there are p-polynomials $P_1, \ldots, P_n$ such that $H = V(\{P_1, \ldots, P_n\})$, and with $P_i \in K[\bar{X}, Y_1, \ldots, Y_i]$, P_i unitary and of minimal degree p^{n_i} in Y_i among the p-polynomials of $K[\bar{X}, Y_1, \ldots, Y_i]$ vanishing on h and of degree > 0 in Y_i, for $i \in \{1, ..., n\}$, and such that $\deg_{Y_j}(P_i) < \deg_{Y_j}(P_j)$ for all $j < i$.

Using the bijection λ_l, we can consider these polynomials as p-polynomials in $K[Z_{=l}]$, such that

$$G_l = \{z \in L : P_1(z_{=l}) = \ldots = P_n(z_{=l}) = 0\}.$$

Let $k \in K \cap \mathrm{E}_K(G_l)$. Then $k^{p^l} \in \mathrm{E}_K(G_l)$, so $k \in \mathrm{E}_K(H)$. Thus for each $i \in \{1, \ldots, n\}$ we have:

$$P_i(k\bar{x}, ky_1, \ldots, ky_i) - k^{p^{n_i}} P_i(\bar{x}, y_1, \ldots, y_i) = 0,$$

and for each $j \in \{1, ..., i\}$, $\deg_{Y_j}(P_i(k\bar{X}, kY_1, \ldots, kY_i) - k^{p^{n_i}} P_i(\bar{X}, Y_1, \ldots, Y_i)) < \deg_{Y_j}(P_j)$. By the property of minimality of the polynomials $P_1, \ldots, P_n$, it follows as in the proof of Lemma 2.7 for each $i \in \{1, \ldots, n\}$, that

$$P_i(k\bar{X}, kY_1, \ldots, kY_i) = k^{p^{n_i}} P_i(\bar{X}, Y_1, \ldots, Y_i).$$

If for some i, there is an integer $m \neq n_i$ with a variable occurring in P_i to the power p^m, then $k^{p^m} = k^{p^{n_i}}$. In this case $k \in \mathbb{F}_{p^{|n_i - m|}}$, hence $K \cap \mathrm{E}_K(G_l)$ is finite. Otherwise we have for all i that

$$P_i(Z_{=l}) = \sum_{j \in p^{\nu l}} a_j Z_j^{p^{n_i}}.$$

Let $g \in G$. Let us check that $gL^{p^\infty} \subseteq G_l$. Let $c \in L^{p^\infty}$. Then

$$P_i((cg)_{=l}) = \sum_{j \in p^{\nu l}} a_j((cg)_j)^{p^{n_i}} = \sum_{j \in p^{\nu l}} a_j(c^{p^{-l}} g_j)^{p^{n_i}} = c^{p^{n_i - l}} P_i(g) = 0$$

for all $i \in \{1, \dots, n\}$. So $cg \in G_l$.

Consequently, either there is an integer l such that $K \cap \mathrm{E}_K(G_l)$ is finite, in which case $K \cap \mathrm{E}_K(G)$ is a finite field, or $gL^{p^\infty} \subseteq G_l$ for every $l < \omega$ and every $g \in G$, and so $gL^{p^\infty} \subseteq G$. In the second case $G = gL^{p^\infty}$ for every $g \in \setminus\{0\}$ by minimality of G. It follows that $K \cap \mathrm{E}_K(G) = K \cap \mathrm{E}_K((L^{p^\infty}, +)) = K^{p^\infty}$.

If G is locally modular, $G \neq gL^{p^\infty}$ for every g, and we are necessarily in the first case.

□

Remark. *There exist additive groups definably isomorphic to $(L^{p^\infty}, +)$ and satisfying the first conclusion of Proposition 4.12. For example, take $G := \{y + by^p : y \in L^{p^\infty}\}$. Then G is definably isomorphic to $(L^{p^\infty}, +)$, but $K \cap \mathrm{E}_K(G) = \mathbb{F}_p$.*

4.3 A family of locally modular minimal additive subgroups of RT-rank one

We consider a very simple family of additive subgroups of RT-rank one which lies at the basis of our constructions. We show that there are $2^{\aleph_0}$ pairwise orthogonal $\aleph_0$-categorical groups inside this family, and that there is an additive subgroup isogenous to a group of this family, and whose quasi-endomorphism division ring is not a field of fractions of its endomorphism ring.

In this section, all the groups which we consider are $\bigwedge$-definable over $\emptyset$. Recall that K is an elementary substructure of L such that L is $|K|^+$-saturated. Since we are interested in locally modular minimal groups, by Fact 3.5 and Remark 4.2 we can assume that $K = \mathbb{F}_p(B)^{sep}$.

First recall the definition of the transcendence rank RT:

Definition.
(i) For $x \in L$, $\mathrm{RT}(x; K) := \text{tr.degree}(K(X_\infty), K)$.
(ii) For an $\bigwedge$-definable set D over K, $\mathrm{RT}(D; K) := \max\{\mathrm{RT}(x; K) : x \in D\}$.

We have the following fact about orthogonality of types of RT-rank one:

Fact 4.13. *[B-D2, Proof of lemma 2.7] Let p and q be two non-orthogonal types over K of RT-rank one. In this case, p and q are interalgebraic. Then there are integers n and k such that if x realizes p and y realizes q, then*

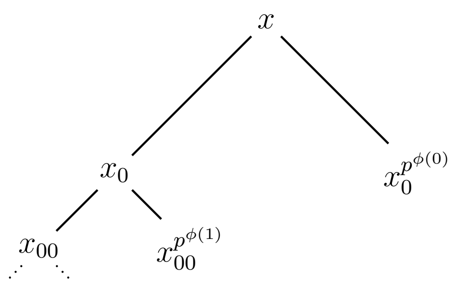
$$[K(y_{\leq i}) : K(y)] \leq k[K(x_{\leq i+n}) : K(x)] \text{ for all } i.$$

Using the above fact, the authors construct infinitely many pairwise orthogonal minimal additive subgroups of RT-rank one in [B-D2]. These, together with the constructions in [C-W], yield the existence of a minimal additive subgroup of infinite RT-rank.

Here we introduce a very simple family of minimal additive subgroups of RT-rank one, one which contains $2^{\aleph_0}$ pairwise orthogonal groups. (For the notation, see Notation 1.1.):

Definition. For ϕ a sequence of integers, we define an additive subgroup G_ϕ over $\emptyset$ by the following equations:

$$G_\phi := \{x \in L : x_{\bar{0}(n)} = x_{\bar{0}(n+1)}^{p} + x_{\bar{0}(n+1)}^{p^{\phi(n)+1}} b,\ n < \omega\}.$$



The groups G_ϕ

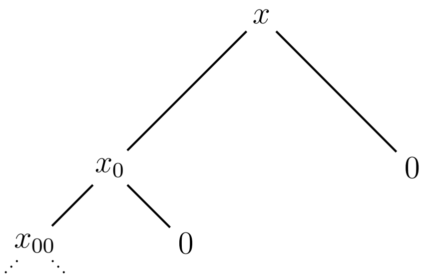
Fact 4.14. *[B-D2, Proof of lemma 2.7] For every ϕ, G_ϕ is connected and of* RT*-rank one, hence minimal. If x is generic in G_ϕ over K, then for all n,*

$$[K(x_{\leq n}) : K(x)] = \prod_{0 \leq i < n} p^{\phi(i)+1}.$$

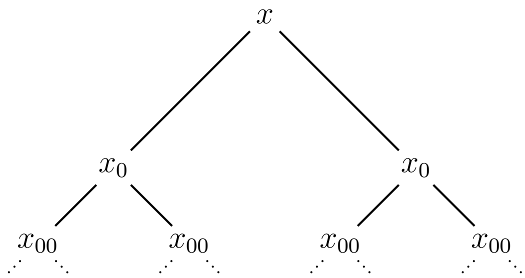
From the two facts above, the following remarks follow easily:

Remark 4.15.

(i) If ϕ is the zero sequence, then G_ϕ is isomorphic to $(L^{p^\infty}, +)$, and for any $g \in G_\phi \setminus \{0\}$, $gL^{p^\infty} \subseteq G_\phi$.



The group $(L^{p^\infty}, +)$



The group $G_{(0)}$

(ii) If $\phi(n) \neq 0$ *for infinitely many* n*, then* G_ϕ *is orthogonal to* $(L^{p^\infty}, +)$*, and thus locally modular by Fact 4.9.*

(iii) There are $2^{\aleph_0}$ *pairwise orthogonal groups* G_ϕ*'s : we take a family* $(O_\beta)_{\beta<2^\omega}$ *of subsets of* ω*, such that if* $\beta < \gamma < 2^\omega$ *then* $O_\beta \subset O_\gamma$ *and* $O_\gamma \setminus O_\beta$ *is infinite, and define* ϕ_β *by* $\phi_\beta(n) = 1$ *if* $n \in O_\beta$ *and* $\phi_\beta(n) = 0$ *otherwise. If* $\beta < \gamma < 2^\omega$*, the groups* G_{ϕ_β} *and* G_{ϕ_γ} *are orthogonal.*

We fix some notation for the two following subsections:

Notation. Let k be an integer and ϕ a sequence of integers.

If ϕ is the sequence with constant value k, we denote G_ϕ by $G_{(k)}$.

We denote by $G_{k\phi}$ the group G_ψ with $\psi(n) = k\phi(n)$ for all n.

We denote by $G_{\phi_{+k}}$ the group G_ψ with $\psi(n) = \phi(n+k)$ for all n. ($G_{\phi_{+k}} = f_{\bar{0}(k)}(G_\phi)$.)

4.3.1 $2^{\aleph_0}$ pairwise orthogonal $\aleph_0$-categorical minimal additive subgroups

To prove that there exists ϕ such that the induced structure on G_ϕ is $\aleph_0$-categorical, we use two propositions. In the first, we show that one can easily choose ϕ so that $G_\phi \cap \mathrm{acl}(\emptyset) = \{0\}$. In the second, we give a simple condition on ϕ so that $\mathrm{E}_K(G_\phi) = \mathbb{F}_p$. Thus, by Remark 4.6, we obtain an integer sequence ϕ such that the induced structure on G_ϕ is an $\mathbb{F}_p$-vector space.

Proposition 4.16. *There exists an integer sequence* ψ *such that* $G_\phi \cap \mathrm{acl}(\emptyset) = \{0\}$ *for every integer sequence* ϕ *satisfying* $\phi(n) \neq \psi(n)$ *for all* n*.*

Proof. Let (a_n) be an enumeration of $\mathrm{acl}(\emptyset) = \mathbb{F}_p(B)^{sep}$. For every n, we define $\psi(n)$ to be the least integer l such that $a_{n,\bar{0}(n)1} = a_{n,\bar{0}(n)0}^{p^l}$ if such exists, and 0 otherwise.

Let ϕ be a sequence of integers such that $\phi(n) \neq \psi(n)$ for all n. Suppose that $a_n \in G_\phi$ for some $n < \omega$. Then $a_{n,\bar{0}(n)1} = a_{n,\bar{0}(n)0}^{p^{\phi(n)}} = a_{n,\bar{0}(n)0}^{p^{\psi(n)}}$. It follows that $a_{n,\bar{0}(n)0} = a_{n,\bar{0}(n)0}^{p^{\phi(n)-\psi(n)}}$ and thus $a_{n,\bar{0}(n)01} = 0$. It is easy to check that for every $x \in G_\phi$, if $x_{\bar{0}(k)1} = 0$ for some k,then $x = 0$. Thus $a_n = 0$. $\square$

Proposition 4.17. *Let ϕ be a sequence of integers such that $\phi(n) \neq 0$ for all n, with the $\phi(n)$'s pairwise distinct and such that for every n, there is $m > n$ with $\phi(n)$ and $\phi(m)$ relatively prime. Then $\mathrm{E}_K(G_\phi) = \mathbb{F}_p$. Moreover, $\mathrm{E}_K(G_{k\phi}) = \mathbb{F}_{p^k}$ for every $k > 0$.*

Proof.
Claim 1. *If g is an endomorphism of G_ϕ given by $g(x) = \sum_{j=0}^m a_j x^{p^j}$, then $a_0 \in \mathbb{F}_p$.*

▷ For every $x \in G_\phi$, we have:

$$\begin{aligned}
(g(x))_0 &= a_{0,0}x_0 + a_{0,p-1}bx_0^{p^{\phi(0)}} + \sum_{j=1}^m a_{j,0}x^{p^{j-1}}, \\
(g(x))_1 &= a_{0,1}x_0 + a_{0,0}x_0^{p^{\phi(0)}} + \sum_{j=1}^m a_{j,1}x^{p^{j-1}}, \\
(g(x))_i &= 0 = a_{0,i}x_0 + \text{ terms in } x_0^{p^{\phi(0)}} \text{ and in } x^{p^j}\text{, for } 1 < i < p^\nu\text{, and} \\
(g(x))_1 &= (g(x))_0^{p^{\phi(0)}}.
\end{aligned}$$

We replace x by $x_0^p + bx_0^{p^{\phi(0)+1}}$, and obtain polynomial equations in the powers of x_0. From the equations $((g(x))_i = 0$, we conclude that $a_{0,i} = 0$ for all $i > 1$. In the equation $(g(x))_1 = (g(x))_0^{p^{\phi(0)}}$, $a_{0,1}$ is the only term in x_0, so $a_{0,1} = 0$. Now we have the following equation :

$$0 = (a_{0,0} - a_{0,0}^{p^{\phi(0)}})x_0^{p^{\phi(0)}} + \text{ terms in } x^{p^j}.$$

Since $x = x_0^p + bx_0^{p^{\phi(0)+1}}$, we can deduce that the terms in x^{p^j} are zero and so $a_{0,0} = a_{0,0}^{p^{\phi(0)}}$. Thus $a_0 \in \mathbb{F}_{p^{\phi(0)}}$.

We can now consider the endomorphism of $G_{\phi_{+1}}$ defined by $x_0 \mapsto f_0(g(x_0^p + bx_0^{p^{\phi(0)+1}}))$. This endomorphism is defined by a p-polynomial of $K[x_0]$ such that the coefficient of x_0 is $a_{0,0}$. Thus $a_{0,0} \in \mathbb{F}_{p^{\phi(1)}}$ and so $a_0 \in \mathbb{F}_{p^{\phi(1)}}$. In the same way, by an easy induction, we can check that $a_0 \in \mathbb{F}_{p^{\phi(n)}}$ for all n. Since $\phi(n)$ is prime to $\phi(0)$ for some n, we conclude that $a_0 \in \mathbb{F}_p$. ◁

Claim 2. *For $l > 0$, let g be a homomorphism from $G_{\phi_{+l}}$ to G_ϕ given by $g(x) = \sum_{j=0}^m a_j x^{p^j}$. Then $a_0 = 0$.*

▷ For every $x \in G_{\phi_{+l}}$ we have:

$$\begin{aligned}
(g(x))_0 &= a_{0,0}x_0 + a_{0,p-1}bx_0^{p^{\phi(l)}} + \sum_{j=1}^m a_{j,0}x^{p^{j-1}}, \\
(g(x))_1 &= a_{0,1}x_0 + a_{0,0}x_0^{p^{\phi(l)}} + \sum_{j=1}^m a_{j,1}x^{p^{j-1}},
\end{aligned}$$

$$\begin{aligned}(g(x))_i &= 0 = a_{0,i}x_0 + \text{ terms in } x_0^{p^{\phi(l)}} \text{ and in } x^{p^j}, \text{ for } 1 < i < p^\nu, \text{ and}\\ (g(x))_1 &= (g(x))_0^{p^{\phi(0)}}.\end{aligned}$$

We replace x by $x_0^p + bx_0^{p^{\phi(l)+1}}$, and as before conclude that $a_{0,i} = 0$ for $i > 1$ and $a_{0,1} = 0$.

Now, considering the equation $(g(x))_1 = (g(x))_0^{p^{\phi(0)}}$, we obtain:

$$a_{0,0}x_0^{p^{\phi(l)}} + \sum_{j=1}^{m} a_{j,1}x^{p^{j-1}} = a_{0,0}^{p^{\phi(0)}} x_0^{p^{\phi(0)}} + \sum_{j=1}^{m} a_{j,0}^{p^{\phi(0)}} x^{p^{j-1+\phi(0)}}.$$

We again replace x by $x_0^p + bx_0^{p^{\phi(l)+1}}$ but keep only the terms in $x_0^{p^{\phi(0)+t\phi(l)}}$:

$$\begin{aligned}&\sum_{t\geq 0} a_{\phi(0)+t\phi(l),1}(x_0^{p^{\phi(0)+t\phi(l)}} + b^{p^{\phi(0)+t\phi(l)-1}} x_0^{p^{\phi(0)+(t+1)\phi(l)}})\\ &= a_{0,0}^{p^{\phi(0)}} x_0^{p^{\phi(0)}} + \sum_{t\geq 1} a_{t\phi(l),0}^{p^{\phi(0)}}(x_0^{p^{\phi(0)+t\phi(l)}} + b^{p^{\phi(0)+t\phi(l)-1}} x_0^{p^{\phi(0)+(t+1)\phi(l)}}).\end{aligned}$$

(Since $\phi(l) \neq \phi(0) \neq 0$, the term $a_{0,0}x_0^{p^{\phi(l)}}$ does not appear.) Thus, we obtain a polynomial equation of the following type:

$$a_{0,0}^{p^{\phi(0)}} x_0^{p^{\phi(0)}} + \sum_{t\geq 0} c_t(x_0^{p^{\phi(0)+t\phi(l)}} + b^{p^{\phi(0)+t\phi(l)-1}} x_0^{p^{\phi(0)+(t+1)\phi(l)}}) = 0.$$

It follows that all the c_t's are zero and $a_{0,0} = 0$.

Thus $a_0 = 0$. $\triangleleft$

Claim 3. *For $l > 0$, let g be a homomorphism from G_ϕ to $G_{\phi_{+l}}$ given by $g(x) = \sum_{j=0}^m a_j x^{p^j}$. Then $a_0 = 0$.*

$\triangleright$ The proof is analogous to that of claim 2. (In this case, it suffices to consider only the terms in $x_0^{p^{\phi(l)+t\phi(0)}}$.) $\triangleleft$

We deduce the following claim:

Claim 4.

For $l \geq 0$, let g be a homomorphism from G_ϕ to $G_{\phi_{+l}}$ given by $g(x) = \sum_{j=1}^m a_j x^{p^j}$. Then $g = 0$.

$\triangleright$ We prove, by induction on j, for every $l \geq 0$ and every such homomorphism g from G_ϕ to $G_{\phi_{+l}}$ that $a_j = 0$.

Let $k > 0$ and suppose that the result is true for $j < k$. Let $g(x) = \sum_{j=k}^m a_j x^{p^j}$ from G_ϕ to $G_{\phi+l}$, As in the previous proofs, it is easy to check that $a_{k,i} = 0$ for $i > 0$. Then we can consider the homomorphism $f_0 \circ g$ from G_ϕ to $G_{\phi_{+(l+1)}}$ and we have $f_0(g(x)) = \sum_{j=k}^m a_{j,0} x^{p^{j-1}}$. If $k > 1$, $a_{k,0} = 0$ by induction. If $k = 1$, $a_{k,0} = 0$ by claim 3. Thus $a_k = 0$. $\triangleleft$

Now let g be a definable endomorphism of G_ϕ over K. By Proposition 3.7, g is defined by a p-polynomial in $K[X_\infty]$. Thus, $g(x) = P(x_{\bar{0}(n)})$ for some integer n and p-polynomial P of $K[X_{\bar{0}(n)}]$.

First, we prove that the endomorphism of $G_{\phi_{+n}}$ sending $x_{\bar{0}(n)}$ to $f_0^n \circ P(x_{\bar{0}(n)})$ is defined by a p-polynomial of $K[X_{\bar{0}(n)}]$.

Indeed if $n > 0$, we can consider the homomorphism from $G_{\phi_{+n}}$ to G_ϕ defined by P. By claim 2, $P \in K[X^p_{\bar{0}(n)}]$. Thus, the homomorphism which sends $x_{\bar{0}(n)}$ to $f_0 \circ P(x_{\bar{0}(n)})$ is defined by a p-polynomial of $K[X_{\bar{0}(n)}]$. By iterating n times, we obtain an endomorphism of $G_{\phi_{+n}}$ defined by a p-polynomial Q of $K[X_{\bar{0}(n)}]$ which sends $x_{\bar{0}(n)} \in G_{\phi_{+n}}$ to $Q(x_{\bar{0}(n)})$. By claim 1, $Q(X_{\bar{0}(n)}) = aX_{\bar{0}(n)} + R(X_{\bar{0}(n)})$ with $a \in \mathbb{F}_p$. Thus, the map which sends $x_{\bar{0}(n)} \in G_{\phi_{+n}}$ to $R(x_{\bar{0}(n)})$ is an endomorphism of $G_{\phi_{+n}}$, and by claim 4, $R = 0$.

We have shown that for every $x \in G_\phi$, $(g(x))_{\bar{0}(n)} = ax_{\bar{0}(n)}$ for some $a \in \mathbb{F}_p$. Now there is a p-polynomial S in $K[X_{\bar{0}(n)}]$ such that $x = S(x_{\bar{0}(n)})$ for all $x \in G_\phi$. Thus,

$$g(x) = S((g(x))_{\bar{0}(n)}) = S(ax_{\bar{0}(n)}) = aS(x_{\bar{0}(n)}) = ax.$$

The proof is analogous for the groups $G_{k\phi}$. □

From the previous proposition it follows that we can not necessarily see the local modularity from the ring of definable endomorphisms:

Corollary 4.18. *There are minimal additive subgroups G_1 and G_2 over $\emptyset$, locally modular and non-locally modular, respectively, such that $\mathrm{E}_{K'}(G_1) = \mathrm{E}_{K'}(G_2) = \mathbb{F}_p$ for any elementary substructure K' of L .*

Proof. This follows from Propositions 4.11 and 4.17. □

We use the following lemma to obtain $2^{\aleph_0}$ pairwise orthogonal $\aleph_0$-categorical minimal additive subgroups:

Lemma 4.19. *There is a family of subsets of ω, $(O_\alpha)_{\alpha \in 2^\omega}$ such that if $\alpha < \beta < 2^\omega$, then $O_\alpha \subset O_\beta$, and $O_\beta \setminus O_\alpha$ contains arbitrary long segments.*

Proof. We construct, by induction on the length of $\varepsilon \in 2^{<\omega}$, a family

$$(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_\varepsilon, E_\varepsilon)_{\varepsilon \in 2^{<\omega}}$$

of partitions of ω, such that A_ε, B_ε, C_ε, D_ε and E_ε contains arbitrary long segments for all $\varepsilon \in 2^{<\omega}$, and

$$\begin{aligned} A_{\varepsilon\frown 0} &= A_\varepsilon \cup C_\varepsilon \cup E_\varepsilon, \\ B_{\varepsilon\frown 0} &\supseteq B_\varepsilon, \\ A_{\varepsilon\frown 1} &\supseteq A_\varepsilon, \\ B_{\varepsilon\frown 1} &= B_\varepsilon \cup D_\varepsilon \cup E_\varepsilon \text{ and} \\ \{0, \ldots, l(\varepsilon) - 1\} &\subset A_\varepsilon \cup B_\varepsilon. \end{aligned}$$

Start with a partition $(A_\emptyset, B_\emptyset, C_\emptyset, D_\emptyset, E_\emptyset)$ of ω such that each of $A_\emptyset$, $B_\emptyset$, $C_\emptyset$, $D_\emptyset$ and $E_\emptyset$ contains segments of arbitrary length.

Let $\varepsilon \in 2^{<\omega}$ and suppose that $(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_\varepsilon, E_\varepsilon)$ is constructed.
We construct $(A_{\varepsilon\frown 0}, B_{\varepsilon\frown 0}, C_{\varepsilon\frown 0}, D_{\varepsilon\frown 0}, E_{\varepsilon\frown 0})$ as follows: let $A_{\varepsilon\frown 0} := A_\varepsilon \cup C_\varepsilon \cup E_\varepsilon$. We take a partition of D_ε into four parts $D_\varepsilon \cap \{0, \ldots, l(\varepsilon)\}$, C, D, E such that C, D, E contain segments of arbitrary length.

Let $B_{\varepsilon\frown 0} := B_\varepsilon \cup (D_\varepsilon \cap \{0, \ldots, l(\varepsilon)\})$, $C_{\varepsilon\frown 0} := C$, $D_{\varepsilon\frown 0} := D$ and $E_{\varepsilon\frown 0} := E$. We construct in an analogous way $(A_{\varepsilon\frown 1}, B_{\varepsilon\frown 1}, C_{\varepsilon\frown 1}, D_{\varepsilon\frown 1}, E_{\varepsilon\frown 1})$.

Now for $\alpha \in 2^\omega$ we define

$$O_\alpha := \bigcup_{n\in\omega} B_{\alpha_{|n}} = \omega \setminus (\bigcup_{n\in\omega} A_{\alpha_{|n}}).$$

If $\alpha < \beta$ then there are $\varepsilon \in 2^{<\omega}$, α' and β' such that $\alpha = \varepsilon \frown 0 \frown \alpha'$ and $\beta = \varepsilon \frown 1 \frown \beta'$. Thus

$$O_\alpha \subseteq \omega \setminus A_{\varepsilon\frown 0} = B_\varepsilon \cup D_\varepsilon \subset B_\varepsilon \cup D_\varepsilon \cup E_\varepsilon = B_{\varepsilon\frown 1} \subseteq O_\beta.$$

So $O_\beta \setminus O_\alpha$ contains E_ε and thus segments of arbitrary length. $\square$

Corollary 4.20. *There are $2^{\aleph_0}$ pairwise orthogonal $\aleph_0$-categorical minimal additive subgroups over $\emptyset$ each of whose induced structure is exactly that of an $\mathbb{F}_p$-vector space.*

Proof. Let ψ be an integer sequence as in Proposition 4.16, such that for every integer sequence ϕ with $\phi(n) \neq \psi(n)$ for all n, then $G_\phi \cap \operatorname{acl}(\emptyset) = \{0\}$. Take two sequences of prime integers, $(p_n)_{n\in\omega}$ and $(q_n)_{n\in\omega}$ such that $q_n \geq 2n + \sum_{i\leq 2n} p_i$, $p_n \neq \psi(n)$ and $q_n \neq \psi(n)$ for all n.

Let $(O_\alpha)_{\alpha\in 2^\omega}$ be a family of subsets of ω with the properties of Lemma 4.19.

For every $\alpha \in 2^\omega$, let ϕ_α be the integer sequence defined by $\phi_\alpha(n) = q_n$ if $n \in O_\alpha$, and $\phi_\alpha(n) = p_n$ otherwise.

By Propositions 4.16, 4.17 and Remark 4.6, the induced structure on G_{ϕ_α} is an $\mathbb{F}_p$-vector space, so G_{ϕ_α} is $\aleph_0$-categorical for all $\alpha \in 2^\omega$.

Now suppose that there are $\alpha < \beta$ such that G_{ϕ_α} and G_{ϕ_β} are non-orthogonal. Let x be generic in G_{ϕ_α} and y be generic in G_{ϕ_β}. Then by Fact 4.13, there are integers n and k such that for every $i < \omega$,

$$[K(y_{\leq i}) : K(y)] \leq k[K(x_{\leq i+n}) : K(x)].$$

Let $m > n$ such that $p^m > k$. By Lemma 4.19 there is $j \geq m$ such that $\{j, \ldots, j+n\} \subset O_\beta \setminus O_\alpha$. Then, by Fact 4.14

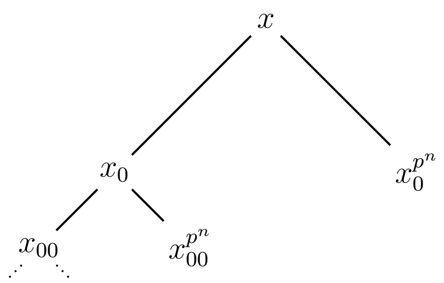
$$\begin{aligned}
[K(y_{\leq j+1}) : K(y)] &= [K(y_{\leq j}) : K(y)]p^{1+q_j} \\
&\geq [K(x_{\leq j}) : K(x)]p^m p^{n+1} \prod_{j\leq l\leq j+n} p^{p_l} \\
&\geq p^m [K(x_{\leq j}) : K(x)] \prod_{j\leq l\leq j+n} p^{\psi_\alpha(l)+1} \\
&\geq p^m [K(x_{\leq j+n+1}) : K(x)] > k[K(x_{\leq j+n+1}) : K(x)].
\end{aligned}$$

This yields a contradiction. $\square$

Remark. *All the groups we have constructed have RT-rank one. We do not know if it is possible to construct $\aleph_0$-categorical groups of arbitrary RT-rank.*

4.3.2 A minimal additive subgroup whose quasi-endomorphism division ring is not a field of fractions of its endomorphism subring

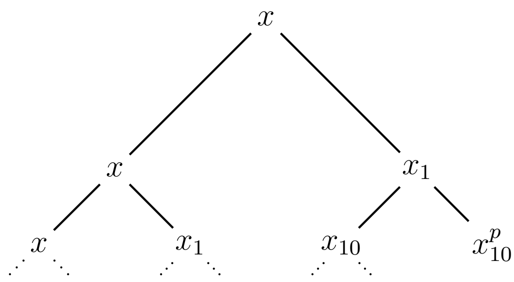
There are minimal locally modular additive subgroups with infinite definable endomorphism rings: let $n \geq 1$, then the group $G_{(n)}$ is minimal locally modular by Remark 4.15, and with $f_0, f_0^2, \dots, f_0^l, \dots$ an infinite sequence of automorphisms of $G_{(n)}$. In fact we can prove that $\mathrm{E}_K(G_{(n)}) = \mathbb{F}_{p^n}\{f_{0|G_{(n)}}, f_{0|G_{(n)}}^{-1}\}$ (see [Bl, Proposition 3.4.17]). (Note that the group defined by $\{x : x_{\bar{1}(l)} = x_{\bar{1}(l+1)}^{p^{n+1}} + b x_{\bar{1}(l+1)}^{p},\ l < \omega\}$ has as definable endomorphism ring $\mathbb{F}_{p^n}\{f_{1|H_{(n)}}, f_{1|H_{(n)}}^{-1}\}$.) But we do not know if the quasi-endomorphism ring of $G_{(n)}$ is a field of fractions of $\mathrm{E}_K(G_{(n)})$.



The group $G_{(n)}$

We see now that there is a group isogenous to $G_{(1)}$, whose endomorphism ring is finite. This example is quite similar to the non-locally modular example of Proposition 4.11.

Proposition 4.21. *Let $H := \{x : \exists y \in G_{(1)},\ x = x^p + by^p\}$. The group H is a minimal group definably isogenous to $G_{(1)}$, with $\mathrm{E}_K(H) = \mathbb{F}_p$.*



The group H

Proof. Let (Q_n) be the sequence of p-polynomials in $K[Y]$ defined by the following induction:

$$Q_0(Y) = Y^p \text{ and } Q_{n+1}(Y) = Q_n(Y^p + bY^{p^2}).$$

Let P_n be the sequence of p-polynomials in $K[X, Y]$ defined by

$$P_n(X, Y) = X^p - X + bQ_n(Y).$$

It is clear that $x \in H$ if and only if $P_n(x, x_{1\bar{0}(n)}) = 0$ for all $n \in \omega$.

First we prove in three steps that the polynomials P_n are irreducible.

Claim 1. *For every n, there are $(s_{i,n}) \in \mathbb{F}_p(B)^{\times(n+1)}$ and $t_n \in \omega$ such that $Q_n(Y) = \sum_{0 \leq i \leq n} s_{i,n} Y^{p^{n+1+i}}$ and $s_{n,n} = b^{pt_n}$.*

$\triangleright$ We prove this claim by induction on n:

For $n = 0$, $t_0 = 0$.

Suppose the result holds for n.

$$\begin{aligned} Q_{n+1}(Y) &= Q_n(Y^p + bY^{p^2}) \\ &= \sum_{i=0}^{n} s_{i,n}(Y^{p^{(n+2)+i}} + b^{p^{n+1+i}} Y^{p^{(n+2)+1+i}}). \end{aligned}$$

In particular $s_{n+1,n+1} = s_{n,n} b^{p^{2n+1}}$. Thus $s_{n+1,n+1} = b^{p(t_n + p^{2n})}$ and $t_{n+1} = t_n + p^{2n}$. $\triangleleft$

Claim 2. *For every n, there is no polynomial A in $\bar{K}[Y]$ such that $A - A^p = bQ_n(Y)$.*

$\triangleright$ Suppose that for $n < \omega$ there is a polynomial A such that $A - A^p = bQ_n(Y)$. It is easy to check that A is a p-polynomial of the following form: $A(Y) = \sum_{i=0}^{n-1} c_i Y^{p^{n+1+i}}$. Then $bQ_n(Y) = \sum_{i=0}^{n-1}(c_i Y^{p^{n+1+i}} - c_i^p Y^{p^{n+2+i}})$. So $bs_{0,n} = c_0$, $bs_{i,n} = c_i - c_{i-1}^p$ for $0 < i < n$, and $bs_{n,n} = -c_{n-1}^p$.

It follows that $\sum_{i=0}^{n}(bs_{i,n})^{p^{n-i}} = 0$. Take $a = \sum_{i=0}^{n-1}(bs_{i,n})^{p^{n-1-i}}$. We obtain the equation $a^p + bb^{pt_n} = 0$, but b is not in L^p. $\lhd$

Claim 3. *The polynomials P_n are irreducible in $\bar{K}[X,Y]$.*

$\rhd$ Suppose that P_n is reducible. Consider the polynomial $R(X) = X^p - X + bQ_n(Y)$ as a polynomial in $\bar{K}(Y)[X]$. Then by the Artin-Schreier theorem, R has a root in $\bar{K}(Y)$. Since the ring $\bar{K}[Y]$ is factorial, this root lies in $\bar{K}[Y]$. We obtain a contradiction to claim 2. $\lhd$

Now we deduce that H is minimal. First of all, $\mathrm{UR}(H) = 1$ since H is definably isogenous to a minimal group. So it suffices to show that H is connected, i.e., that $I(H)$ is prime by Fact 1.8. Let J be the ideal of $K[X_\infty]$ generated by the polynomials P_n and I^0. Since $H = V(J)$, by Fact 1.5 it is sufficient to prove that J is separable and prime.

For every $n \in \omega$, consider the ideal J_{n+1} of $K[X_{\leq n+1}]$ generated by $I^0_{\leq n+1}$ and P_n.

$$K[X_{\leq n+1}]/J_{n+1} \simeq K[X/J_{n+1}, X_{1\bar{0}(n)}/J_{n+1}].$$

Since $Z := X_{1\bar{0}(n)}/J_{n+1}$ is transcendental over K, and $T := X/J_{n+1}$ is separably algebraic over $K[Z]$ via the irreducible polynomial $P_n(T,Z) = T - T^p + bQ_n(Z)$, J_{n+1} is prime separable.

We now prove that $\mathrm{E}_K(H) = \mathbb{F}_p$. Recall that $H := \cap H_n$ where $H_n := V(I(H)_{\leq n}) = V(J_n)$. We use the following claim:

Claim 4. *Let $n \in \omega$, $Y := X_{1\bar{0}(n)}$ and let R be a p-polynomial in $K[X,Y] \cap I(H)_{\leq n+1}$. Then $R \in \langle P_n \rangle$, and in particular either $\deg_Y R \geq \deg Q_n$ or $R = 0$.*

$\rhd$ Let x realize the generic of H_{n+1} over K, $y := x_{1\bar{0}(n)}$ and let R be a p-polynomial in $K[X,Y] \cap I(H)_{\leq n+1}$.

First we show by induction on $\deg_X R$ that there is $U \in K[X,Y]$ such that $\deg_X(R - UP_n) \leq 1$. Suppose that $\deg_X R = p^{m+1}$. Then there is $R' \in K[X,Y]$ such that $\deg_X R' \leq p^m$ and $R(X,Y) = R'(X,Y) + aX^{p^{m+1}}$. But $(X^p)^{p^m} = (X + P_n(X,Y) - bQ_n(Y))^{p^m}$. This yields

$$R(X,Y) = R'(X,Y) + a(X + P_n(X,Y) - bQ_n(Y))^{p^m},$$

with $\deg_X(R - aP_n^{p^m}) \leq p^m$ and the induction hypothesis applies.

Let $U \in K[X,Y]$ such that $\deg_X(R - UP_n) \leq 1$. Then $R - UP_n = 0$ because $(R - UP_n)(x,y) = 0$, $x \notin K[y]$ and $y \notin \bar{K}$. $\lhd$

Let τ be a non-zero endomorphism of H. By Proposition 3.7 we can consider τ to be an endomorphism of $(L,+)$ such that $\tau(H) \subseteq H$.
By compactness, there is $m > 0$ such that $\tau(H_m) \subseteq H_2$.

Moreover there are $n \geq m$ and a p-polynomial $R \in K[X_{=n+1}]$, such that $\tau(x) = R(x_{=n+1})$ for each $x \in L$.

Thus there is a p-polynomial S in $K[X,Y]$ such that for all $x \in H_{n+1}$, $\tau(x) = S(x,y)$, where $y := x_{1\bar{0}(n)}$. By induction on $\deg_Y S$, one can easily check that there is a p-polynomial $U \in K[Z]$ such that $\deg_Y(S - U(P_n)) < \deg_Y(P_n)$. So we can assume $\deg_Y(S) < \deg_Y(P_n) = \deg(Q_n)$.

Then for every $x \in H_{n+1}$,

$$\tau(x) = \sum_{i=0}^{l} a_i x^{p^i} + \sum_{j=0}^{k} c_j y^{p^j}, \text{ where } p^k < \deg Q_n (= p^{2n+1}).$$

Moreover, since $\tau(H_{n+1}) \subseteq H_1$, we have the following equation on H_{n+1}:

$$\tau = \tau^p + b(f_1 \circ \tau)^p.$$

Thus on H_{n+1},

$$f_0 \circ \tau = \tau \text{ and}$$

$$f_i \circ \tau = 0 \text{ for each } i \in p^\nu \setminus \{0,1\}.$$

Furthermore (as is needed for the case $p = 2$), since $\tau(H_{n+1}) \subseteq H_2$ and using the definition of $G_{(1)}$, we have on H_{n+1} that

$$f_1 \circ f_1 \circ \tau = (f_0 \circ f_1 \circ \tau)^p.$$

Now we consider τ on H_{n+2} and obtain $y\ (= x_{1\bar{0}(n)}) = y_0^p + by_0^{p^2}$ and $x = x^p + bx_1^p$ for all $x \in H_{n+2}$.

Therefore, for every $x \in H_{n+2}$,

$$f_0 \circ \tau(x) = a_{0,0}x + a_{0,p-1}bx_1 + \sum_{i=1}^{l} a_{i,0}x^{p^{i-1}} + c_{0,0}y_0 + c_{0,p-1}by_0^p + \sum_{j=1}^{k} c_{j,0}y^{p^{j-1}}.$$

(Once more we use the notation: $a_{0,0} = f_0(a_0)$, $a_{0,p-1} = f_{p-1}(a_0)$,.... (see Notation 1.1)).

Take $Q'_{n+1} \in K[Y]$ such that $Q_{n+1}(Y) = Q_0(Q'_{n+1}(Y)) = (Q'_{n+1}(Y))^p$. Thus $x_1\ (= f_1(x)) = Q'_{n+1}(y_0)$ as $x = x^p + bx_1^p = x^p + b(Q'_{n+1}(y_0))^p$.

Replace y by $y_0^p + by_0^{p^2}$ and x_1 by $Q'_{n+1}(y_0)$ in the equations:

$$\begin{aligned} f_0 \circ \tau(x) &= a_{0,0}x + a_{0,p-1}bQ'_{n+1}(y_0) + \sum_{i=1}^{l} a_{i,0}x^{p^{i-1}} \\ &\quad + c_{0,0}y_0 + c_{0,p-1}by_0^p + \sum_{j=1}^{k} c_{j,0}(y_0^p + by_0^{p^2})^{p^{j-1}} \text{ and} \\ \tau(x) &= \sum_{i=0}^{l} a_i x^{p^i} + \sum_{j=0}^{k} c_j (y_0^p + by_0^{p^2})^{p^j}. \end{aligned}$$

With the equation $f_0 \circ \tau = \tau$, we have (recall that $x_{1\bar{0}(n+1)} = y_0$),

$$T(x, x_{1\bar{0}(n+1)}) = 0 \text{ for all } x \in H_{n+2},$$

where

$$\begin{aligned} T(X,Y) &= \sum_{i=0}^{l} a_i X^{p^i} - (a_{0,0}X + \sum_{i=1}^{l} a_{i,0} X^{p^{i-1}}) \\ &\quad + \sum_{j=0}^{k} c_j (Y^p + bY^{p^2})^{p^j} \\ &\quad - (a_{0,p-1} b Q'_{n+1}(Y) + c_{0,0}Y + c_{0,p-1} bY^p + \sum_{j=1}^{k} c_{j,0}(Y^p + bY^{p^2})^{p^{j-1}}). \end{aligned}$$

Since $p^k < \deg Q_n$, $\deg_Y(T) \le \max\{p^{k+2}, \deg Q'_{n+1}\} < \deg(Q_{n+1}) = p^2 \deg(Q_n)$. So by claim 4, $T = 0$.

Since $T = 0$, it is easy to check that for every $i > 0$, $a_i = 0$ and $a_0 = a_{0,0}$. Thus $\tau(x) = a_0 x + \sum_{j=0}^{k} c_j y^{p^j}$ for all $x \in H_{n+1}$.

We will deduce from the following claim that $\tau(x) = a_0 x$ for all $x \in H_{n+1}$.

Claim 5. $a_{0,p-1} = 0$.

$\triangleright$ For $p > 2$, this is easy: it suffices to consider $f_{p-1} \circ \tau$ which is equal to zero. For $x \in H_{n+2}$,

$$f_{p-1} \circ \tau(x) = a_{0,p-1}x + a_{0,p-2}x_1 + \text{“terms in powers of } y_0 \text{ of degree } < \deg(Q_{n+1})\text{”}.$$

Again by claim 4, $a_{0,p-1} = 0$.

For $p = 2$, we work on H_{n+3}. For every $x \in H_{n+3}$, $y_0 = y_{00}^p + by_{00}^{p^2}$. We use the following relation:

$$f_1 \circ f_1 \circ \tau = (f_0 \circ f_1 \circ \tau)^p.$$

Then

$$f_1 \circ \tau(x) = a_{0,1}x + a_{0,0}x_1 + c_{0,1}y_0 + c_{0,0}y_0^p + \sum_{j=1}^{k} c_{j,1} y^{p^{j-1}}.$$

Replace y by $y_0^p + by_0^{p^2}$ and x_1 by $Q'_{n+1}(y_0)$:

$$f_1 \circ \tau(x) = a_{0,1}x + S(y_0),$$

where S is a p-polynomial in $K[Y]$ such that $\deg S < \deg Q_{n+1}$. Moreover, using the equation $y_0 = y_{00}^p + by_{00}^{p^2}$, we obtain:

$$f_1 \circ f_1 \circ \tau(x) = a_{0,11}x + a_{0,10}x_1 + S_1(y_{00}).$$

with $p\deg(S_1) < \deg(Q_{n+2})$. In the same way:

$$f_0 \circ f_1 \circ \tau(x) = a_{0,10}x + a_{0,11}bx_1 + S_0(y_{00}).$$

with $p\deg(S_0) < \deg(Q_{n+2})$. So

$$(f_0 \circ f_1 \circ \tau(x))^p = a_{0,10}^p x^p + a_{0,11}^p b^p x_1^p + S_0^p(y_{00}).$$

On the one hand $x_1 = Q'_{n+2}(y_{00})$ with $\deg(Q_{n+2}) = p\deg(Q'_{n+2})$, and on the other hand $bx_1^p = x - x^p$. With these relations and the equalities $f_1 \circ f_1 \circ \tau = (f_0 \circ f_1 \circ \tau)^p$ we obtain the following equation on H_{n+3}:

$$a_{0,11}x + a_{0,10}Q'_{n+2}(y_{00}) + S_1(y_{00}) = a_{0,10}^p x^p + ba_{0,11}^p(x - x^p) + S_0^p(y_{00}).$$

The degrees of Q'_{n+2}, S_1 and S_0^p are strictly less than the degree of Q_{n+2}, so by claim 4, $a_{0,11}X - a_{0,10}^p X^p - ba_{0,11}^p(X - X^p) = 0$. Thus $a_{0,10}^p = ba_{0,11}^p$.

In the same way, the relation $f_i \circ f_1 \circ \tau = 0$ implies that $a_{0,1i} = 0$ for each $i \in 2^\nu \setminus \{0, 1\}$.

Thus $a_{0,1} = a_{0,10}^p + ba_{0,11}^p = 2a_{0,10}^p = 0$. $\triangleleft$

Go back to the equation $T(X,Y) = 0$. Since $a_{0,p-1} = 0$, we obtain:

$$\sum_{j=0}^{k} c_j(Y^p + bY^{p^2})^{p^j} - (c_{0,0}Y + c_{0,p-1}bY^p + \sum_{j=1}^{k} c_{j,0}(Y^p + bY^{p^2})^{p^{j-1}}) = 0.$$

It follows that $c_j = 0$ for all j, and thus $\tau(x) = a_0 x$ for all $x \in H_{n+1}$.

Now we use the relation $\tau = \tau^p + b(f_1 \circ \tau)^p$ on H_{n+1} :

$$\begin{aligned}
a_0 x &= (a_0 x)^p + b(a_{0,1}x + a_{0,0}x_1)^p \\
&= a_0^p x^p + ba_{0,1}^p x^p + a_0^p(bx_1^p) \\
&= a_0^p x^p + ba_{0,1}^p x^p + a_0^p(x - x^p).
\end{aligned}$$

Then $a_0 = a_0^p$, and so $a_0 \in \mathbb{F}_p$. $\square$

Corollary 4.22. *The group H is an example of a locally modular minimal additive subgroup whose quasi-endomorphism division ring is not a field of fractions of its endomorphism ring.*

Proof. H is definably isogenous to $G_{(1)}$, so H is locally modular and has the same quasi-endomorphism division ring as $G_{(1)}$. Hence we know that $\mathrm{QsE}_K(H) = \mathrm{QsE}_K(G_1)$ is infinite, because it contains $\mathrm{E}_K(G_1) = \mathbb{F}_p\{f_{0|G_{(1)}}, f_{0|G_{(1)}}^{-1}\}$. But $\mathrm{E}_K(H)$ is finite. $\square$

Remark. *Every non-locally modular minimal additive subgroup is definably isogenous to a minimal additive subgroup (the group $(L^{p^\infty}, +)$) whose quasi-endomorphism division ring is a field of fractions of its endomorphism ring. We do not know whether this holds also in the locally modular case. It is true for every minimal additive subgroup G over K such that $G \cap K$ is finite ([Bl, Proposition 3.5.17]).*

4.3.3 Further remarks and questions

Scanlon has suggested studying the following class of additive subgroups, in connection with some questions about Drinfeld modules.

Let θ be an endomorphism of $(L,+)$ in $K\{\alpha\} \setminus \{0\}$. (Recall that α denotes the Frobenius.) We define

$$\theta^\sharp := \bigcap_{n\in\omega} \theta^n(L).$$

Lemma 4.23. *If $\theta = \sum_{i=1}^m a_i\alpha^i$, then $\theta^\sharp$ is a minimal additive subgroup of RT-rank one. Otherwise $\theta^\sharp = L$.*

Proof. For every $x \in \theta^\sharp$ and $n \in \omega$, there is $y \in \theta^\sharp$ such that $x = \theta^n(y)$. First, suppose that $\theta = \sum_{i=1}^m a_i\alpha^i$. So $K(x_{\leq n}) \subseteq K(y)$ and tr.degree$(K(x_{\leq n}), K) \leq 1$. It follows that $\mathrm{RT}(\theta^\sharp; K) = 1$ since $\theta^\sharp$ is infinite as a decreasing intersection of infinite definable groups. For every $n \in \omega$, $\theta^n(L)$ is connected because $(L,+)$ is connected and $\theta^n(L)$ is a definable homomorphism. Thus $\theta^\sharp$ is connected, as a decreasing intersection of connected groups, and thus minimal.

Now suppose that $\theta = \sum_{i=0}^m a_i\alpha^i$ with $a_0 \neq 0$. Let P be the p-polynomial $\sum_{i\geq 0} a_i X^{p^i}$. Then for every $l \in L$, $P - l$ is separable. Thus $\theta(L) = L$ and so $\theta^\sharp = L$. □

From now on, we assume that $\theta^\sharp$ is minimal, that is $\theta = \sum_{i=1}^m a_i\alpha^i$. We characterize such groups which are also locally modular.

Proposition 4.24. *The additive subgroup $\theta^\sharp$ is locally modular if and only if there is no $a \in K^*$ such that $a^{-1}\theta a \in K^p\{\alpha\}$.*

Proof. Suppose that there is $a \in K^*$ such that $a^{-1}\theta a \in K^p\{\alpha\}$. Then $(a^{-1}\theta a)(L) \subseteq L^p$. Thus $(a^{-1}\theta a)^\sharp \subseteq L^{p^\infty}$. Since L^{p^∞} is minimal, $(a^{-1}\theta a)^\sharp = L^{p^\infty}$. But $\theta^\sharp = a(a^{-1}\theta a)^\sharp = aL^{p^\infty}$, so $\theta^\sharp$ is not locally modular.

Conversely, let P be the p-polynomial in $K[Y]$ which defines θ, and let N be such that $\deg P = p^N$. Let $x \in \theta^\sharp$ and $y \in \theta^\sharp$ such that $x = \theta(y)$. Note that if x is transcendental on K, then y is also transcendental on K, with $[K(y) : K(x)] = \deg P = p^N$ since $X - P(Y)$ is irreducible in $K[X,Y]$.

Claim 1. *There exists a p-polynomial R in $K[Y]$ such that $1 \leq \deg R \leq p^{N-2}$ and $R(y) \in K[x_{\leq 1}]$.*

$\triangleright$ There are p-polynomials P_i in $K[Y]$ such that $P = \sum_{i\in p^\nu} P_i^p m_i$, $x_i = P_i(y)$ for every $i \in p^\nu$, and $\deg P_i \leq p^{N-1}$.
Suppose there is i such that $P_i \neq 0$ and $\deg P_i \leq p^{N-2}$. In this case we can take such a P_i.
If not, then for every $i \in p^\nu$, $\deg P_i = p^{N-1}$ or $P_i = 0$. In this case there are two P_i's which are not collinear and a linear combination of these polynomials gives the p-polynomial R. Otherwise there would exist $i_0 \in p^\nu$ and c_i's in K such that $P_i = c_i P_{i_0}$ for each $i \in p^\nu$. Thus $P = \sum_i m_i c_i^p P_{i_0}^p$, hence there would be $c \in K$

and $\theta' \in K\{\alpha\}$ such that $\theta = c\alpha\theta'$. Then if we take a such that $a^{p-1} = c^{-1}$ we get $a^{-1}\theta a \in K^p\{\alpha\}$. $\triangleleft$

So there is

$$R(Y) = \sum_{i=n_0}^{N-2} d_i Y^{p^i}$$

with $d_{n_0} = 1$ and $R(y)$ in $K[x_{\leq 1}]$. Let

$$S(Y) := \sum_{i=n_0}^{N-2} d_{i,\bar{0}(n_0)} Y^{p^{i-n_0}}.$$

Then $S(y) \in K[x_{\leq n_0+1}]$, so $[K(x_{\leq n_0+1}, y) : K(x_{\leq n_0+1})] \leq p^{N-n_0-2}$ and $y_{\leq m} \in K[x_{\leq n_0+1+m}, y]$ for all $m \in \omega$.

Observe that the above properties do not depend on the choice of x.

Claim 2. *For every $m \in \omega$ and every $x \in \theta^\sharp$, if x is transcendental on K then $[K(x_{\leq m(n_0+1)}) : K(x)] \geq p^{m(n_0+2)} = p^m p^{m(n_0+1)}$.*

$\triangleright$ We prove this claim by induction on m. It is trivially true for $m = 0$.

Suppose the result true for m. Let $x \in \theta^\sharp$ such that x is transcendental on K. Then there is $y \in \theta^\sharp$ such that $\theta(y) = x$ and y is also transcendental on K. So

$$[K(y_{\leq m(n_0+1)}) : K(y)] \geq p^{m(n_0+2)}.$$

By the above properties, $y_{\leq m(n_0+1)} \in K[x_{\leq (m+1)(n_0+1)}, y]$. Thus

$$[K(x_{\leq (m+1)(n_0+1)}, y) : K(y)] \geq p^{m(n_0+2)}.$$

Since $[K(y) : K(x)] = p^N$, we get

$$[K(x_{\leq (m+1)(n_0+1)}, y) : K(x)] \geq p^{N+m(n_0+2)}.$$

Since $[K(x_{\leq n_0+1}, y) : K(x_{\leq n_0+1})] \leq p^{N-n_0-2}$ we have also

$$[K(x_{\leq (m+1)(n_0+1)}, y) : K(x_{\leq (m+1)(n_0+1)})] \leq p^{N-n_0-2}.$$

It follows that

$$[K(x_{\leq (m+1)(n_0+1)}) : K(x)] \geq p^{(m+1)(n_0+2)}.$$

$\triangleleft$

Now, if $\theta^\sharp$ were not locally modular then, by Fact 4.9, $\theta^\sharp$ could not be orthogonal to L^{p^∞}. Thus by Fact 4.13, there would exist $k \in \omega$ such that for every x generic in $\theta^\sharp$, $[K(x_{\leq n}) : K(x)] \leq kp^n$. But this contradicts claim 2. $\square$

Question 4.25 (T. Scanlon). *Suppose that $\theta^\sharp$ is locally modular and S is a quasi-endomorphism of $\theta^\sharp$. Must there exists a positive integer m such that S commutes with θ^m?*

Remark. *Let $n \in \omega$. The group $G_{(n)}$ is equal to $\theta_n^\sharp$ where $\theta_n = \alpha + b\alpha^{n+1}$. We noted in the previous subsection that $\mathrm{E}_K(G_{(n)}) = \mathbb{F}_{p^n}\{f_{0|G_{(n)}}, f^{-1}_{0|G_{(n)}}\}$. Observe that if $n > 0$ and $a \in \mathbb{F}_{p^n} \setminus \mathbb{F}_p$ then $\theta_n \circ a \neq a \circ \theta_n$, but on the other hand $\theta_n^n \circ a = a \circ \theta_n^n$.*

5 Additive subgroups of U-rank ω

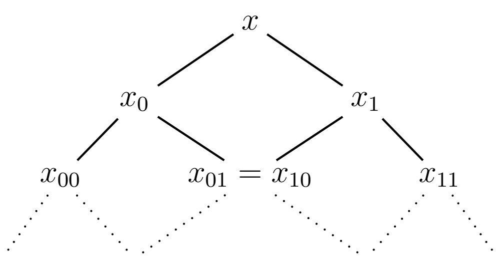
In [CCSSW] the authors construct trivial types of U-rank ω. Here we give a construction of additive subgroups of U-rank ω. In particular this construction gives infinitely many one-based pairwise orthogonal additive subgroups (over $\emptyset$) of U-rank ω.

Recall that K is an elementary substructure of L. For this section we suppose that K is countable. All groups and types are defined over K. In fact we can take as K the prime model $\mathbb{F}_p(B)^{sep}$.

First, we consider a particular additive subgroup, the "**Grill**". It will be easy to show that it cannot have finite U-rank. However we do not know if it is ranked by U-rank, and even if it is, whether it has rank ω:

Definition. The grill is defined by the following equations: $x \in$ Grill if for every $i \in (p^\nu)^{<\omega}$, $x_{i01} = x_{i10}$, and for $i \notin \{0,1\}^{<\omega}$, $x_i = 0$. (Recall that $x_i = f_i(x)$, $x_{i01} = f_1 \circ f_0(x_i)$ and $x_{i10} = f_0 \circ f_1(x_i)$ (see Notation 1.1).)

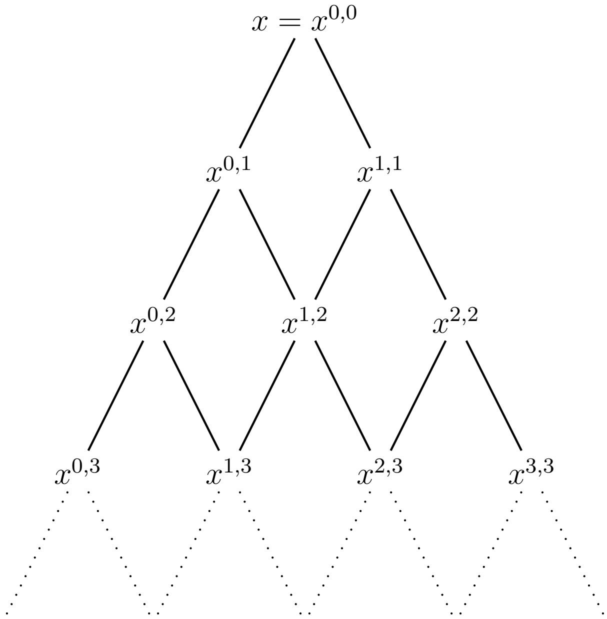
The grill :



Observe that $(L^{p^\infty}, +)$ is a subgroup of the grill, hence the grill is not one-based.

By the definition of the grill, we can ignore the components x_i for the i's which are not in $\{0,1\}^{<\omega}$. We denote the components x_i for the i's which are in $\{0,1\}^{<\omega}$ in the following way: for every level $n < \omega$ and every $j \le n$, $x^{j,n} := f_1^j \circ f_0^{n-j}(x)$.

The p-component tree of the grill with this new notation:



In the following, we say that elements are **independent**, or **free**, if they are independent in the model theoretic sense and that they are **algebraically independent** if they are independent in the field theoretic sense, which means that there is no algebraic relation among them.

It is easy to check that for every level n, the grill does not impose algebraic relations on the $n+1$ components $x^{0,n}, \ldots, x^{n,n}$. So the equations "$x \in \text{Grill}$" determine a complete type over K, which we denote p_{Grill}, and which is the unique generic type of the group. In particular the grill is a connected group.

It is easy to check the following:

Remark 5.1. *Let $a \models p_{\text{Grill}}$. For every $n < \omega$, the components $a^{0,n}, \ldots, a^{n,n}$ are not independent in the model theoretic sense. In fact, for every $0 \leq j \leq j' \leq n$, every $m \geq n$ and every $j \leq k \leq j' + m - n$:*

(i) $a^{k,m} \in K\langle a^{j,n}_{\infty}, a^{j',n}_{\infty}\rangle (= \text{dcl}(K, a^{j,n}, a^{j',n})$ by Fact 1.3),

(ii) $a^{j,n} \notin \text{acl}(K, a^{j+1,n}, \ldots, a^{n,n})$ and $a^{j,n} \notin \text{acl}(K, a^{0,n}, \ldots, a^{j-1,n})$.

Now we check that the grill does not have finite U-rank:

Lemma 5.2. *Let $a \models p_{\text{Grill}}$. For every $0 \leq j \leq j' \leq n$, $\text{UR}(a/K \cup \{a^{j,n}, a^{j',n}\}) = n - (j' - j) = \text{RT}(a/K\langle a^{j,n}, a^{j',n}\rangle)$.*

Proof. By Remark 5.1(i), for every $m \geq n$,

$$\text{tr.degree}(K(a_{=m})/K\langle a^{j,n}, a^{j',n}\rangle) \leq n - (j' - j).$$

So

$$\text{UR}(a/K \cup \{a^{j,n}, a^{j',n}\}) \leq \text{RT}(a/K\langle a^{j,n}, a^{j',n}\rangle) \leq n - (j' - j).$$

On the other hand, we prove by induction on $n - j'$ that

$$\mathrm{UR}(a/K, a^{0,n}, \ldots, a^{j',n}) \geq n - j'.$$

It is trivially true for $n-j' = 0$. Suppose the result true for $n-j' < n$. Since $a^{j',n}$ is in $\mathrm{dcl}(a)$ but not in $\mathrm{acl}(K \cup \{a^{0,n}, \ldots, a^{j'-1,n}\})$ (see Remark 5.1 ii), we conclude by forking symmetry that $\mathrm{tp}(a/K, a^{0,n}, \ldots, a^{j',n})$ forks over $K, a^{0,n}, \ldots, a^{j'-1,n}$. So

$$\mathrm{UR}(a/K, a^{0,n}, \ldots, a^{j'-1,n}) > \mathrm{UR}(a/K, a^{0,n}, \ldots, a^{j',n}) \geq n - j'.$$

By the same kind of induction on j, we check that for every $0 \leq j \leq j' \leq n$, $\mathrm{UR}(a/K \cup \{a^{j,n}, a^{j',n}\}) \geq n - (j' - j)$. $\square$

Remark. *Let a be a realization of p_{Grill} in an elementary extension of L. Then $\mathrm{tp}(a/L)$ forks over K if and only if there is $n < \omega$ such that there is an algebraic relation over L between the components $a^{0,n}, \ldots, a^{n,n}$. Note that if the forking of $\mathrm{tp}(a/L)$ over K were to imply that a component $a^{j,m}$ is in L, then the U-rank of a over L would be finite, by the above lemma.*

To obtain an additive subgroup G of U-rank ω, we use the grill and infinitely many pairwise orthogonal minimal types of RT-rank one, such that a realizes a forking extension over L of the generic type of G if and only if a (non-trivial) component of a is in L. This is the same idea as the construction of a trivial type of U-rank ω in [CCSSW], but in the trivial case the authors must work with only one minimal trivial type of RT-rank one.

Notation 5.3. Let q be a complete type over K. For every $n \in \omega$ and every realization a of q, we denote by

$$q_{\leq n} := \{P(x_{\leq n}) = 0, Q(x_{\leq n}) \neq 0 : P, Q \in K[X_{\leq n}],\ P(a_{\leq n}) = 0,\ Q(a_{\leq n}) \neq 0\}.$$

By quantifier elimination (see Fact 1.4), the type q is determined by the conjunction of the partial types $q_{\leq n}$'s.

We use the following easy lemma:

Lemma 5.4. *Let $p_0, \ldots, p_n$ be pairwise orthogonal minimal types over K.*

Then for every $Q \in K[X_0, \ldots, X_n, \bar{Y}]$, there is $m(Q) \leq \omega$ such that for every $(c_j)_{j \in \{1,\ldots,n\}} \in L$, every $\bar{d} \in L$ and every $I \subseteq \{1, \ldots, n\}$:

$$\bigwedge_{i \in I} p_{i, \leq m(Q)}(x_i) \ \vdash \ \Big(Q((X_i)_{i\in I}, (c_j)_{j \notin I}, \bar{d}) \neq 0 \bigwedge Q((x_i)_{i \in I}, (c_j)_{j\notin I}, \bar{d}) = 0 \Big)$$
$$\Rightarrow \bigvee_{i \in I} \text{"} x_i \in \mathrm{acl}(K \cup \bar{d} \cup \{c_j : j \notin I\}) \text{"}.$$

(The notation $Q((X_i)_{i\in I}, (c_j)_{j\notin I}, \bar{d})$ means that we have replaced the indeterminate X_j by the constant c_j for all $j \notin I$, and the indeterminates $\bar{Y}$ by the constants $\bar{d}$, and likewise for $Q((x_i)_{i\in I}, (c_j)_{j\notin I}, \bar{d})$.)

Proof. Note that it suffices to prove the result for $I = \{1, \dots, n\}$. Now let $x_1, \dots, x_n \in L$ be realizations of $p_1, \dots, p_n$ and $\bar{d} \in L$. By induction on $j \leq n$, we check that if every $x_i \notin \mathrm{acl}(K\bar{d})$ then $x_0, \dots, x_j$ are independent over $K\bar{d}$: indeed by orthogonality, if $x_0, \dots, x_j$ are free over $K\bar{d}$ and x_{j+1} is not in $\mathrm{acl}(K\bar{d})$ then $\mathrm{tp}(x_0, \dots, x_j/K\bar{d})$ is orthogonal to $\mathrm{tp}(x_{j+1}/K\bar{d})$, hence $x_0, \dots, x_{j+1}$ are free over $K\bar{d}$. We deduce that for every $Q \in K[X_0, \dots, X_n, \bar{Y}]$,

$$p_0(x_0) \bigwedge \dots \bigwedge p_n(x_n) \ \vdash \ \Big(Q(X_0, \dots, X_n, \bar{d}) \neq 0 \bigwedge Q(x_0, \dots, x_n, \bar{d}) = 0\Big)$$
$$\Rightarrow \bigvee_{i=0}^{n} \text{"} x_i \in \mathrm{acl}(K\bar{d}) \text{"}.$$

But the disjunction of formulas "$x_i \in \mathrm{acl}(K\bar{d})$" is equivalent to one formula in a closed set, so by compactness we find $m(Q)$. □

Proposition 5.5. *Given infinitely many pairwise orthogonal minimal types of RT-rank one over K, infinitely many pairwise orthogonal types of U-rank ω over K can be constructed.*

Proof. First, we give a construction of types of U-rank ω over K. Let $(p_j)_{j\in\omega}$ be a family of pairwise orthogonal minimal types of RT-rank one over K. To simplify the construction we suppose in addition that for every $j, n \in \omega$, if $x \models p_j$ then for every $i \in (p^\nu)^{\leq n}$, $x_i \in K[x_{\bar{0}(n)}]$. This last assumption is not necessary; we could make a construction by induction in a way similar to the construction of minimal types of infinite RT-rank in [C-W]. But with this assumption, we can start with a family $(p_\beta)_{\beta\in\omega^\omega}$ such that for every $\beta \in \omega^\omega$, p_β has properties similar to the properties of p_{Grill}. Then we define by induction $\beta \in \omega^\omega$, so that U-rank of p_β is equal to ω.

For every $j, n \in \omega$, recall that $p_{j,\leq n}$ is the partial type p_j restricted to the level n (see Notation 5.3).

Let β be a map from ω to ω. We define p_β as follows: we begin with some $x \models p_{0,\leq\beta(0)}$, next we use a first step of the grill , then we require of the two new algebraically independent components y, z that $y \models p_{0,\leq\beta(1)}$ and $z \models p_{1,\leq\beta(1)}$ and so on. More precisely, we define $\gamma(l, n) \in \{0,1\}^{<\omega}$ for every $l \leq n$. Let $\gamma(0, 0) = \emptyset$ and for $l \leq n$, $\gamma(l, n+1) = \gamma(l, n)\ \bar{0}(\beta(n) + 1)$ and $\gamma(n+1, n+1) = \gamma(n, n)\ \bar{0}(\beta(n))\ 1$.

We define p_β by the following relations:
for every $l \leq n$,

$$x_{\gamma(l,n)} \models p_{l,\leq\beta(n)} \text{ and } x_{\gamma(l,n)\bar{0}(\beta(n))} = x^p_{\gamma(l,n+1)} + b x^p_{\gamma(l+1,n+1)},$$

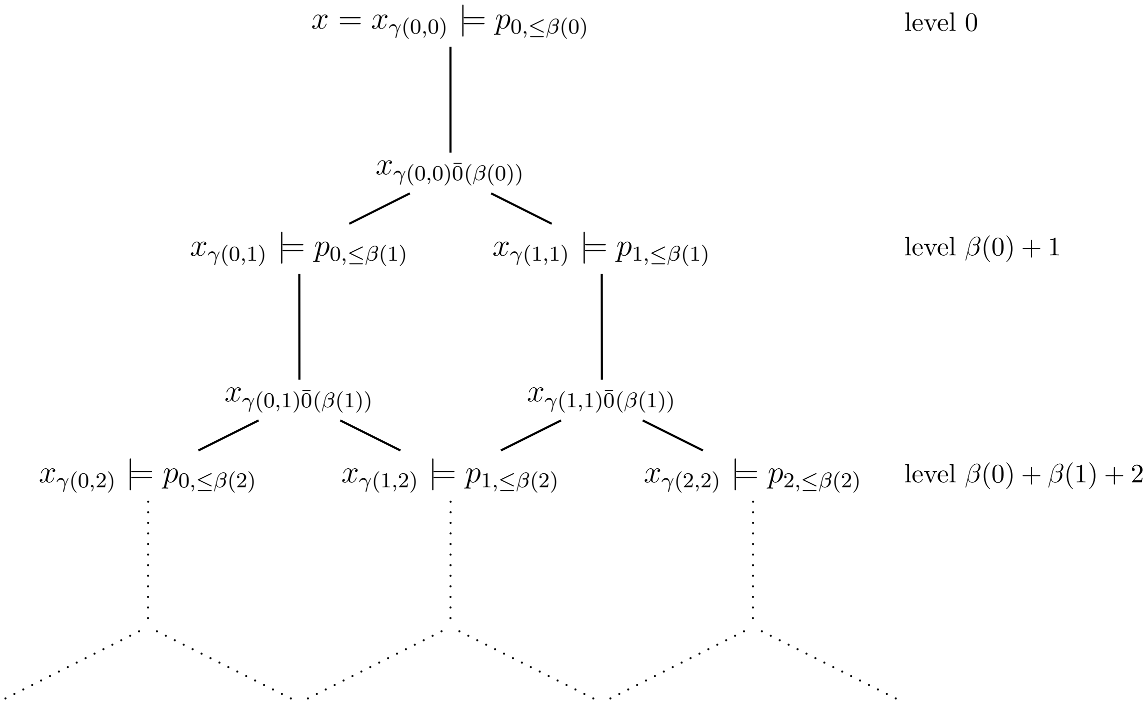
and for every n,

$$x_{\gamma(0,n)}, \dots, x_{\gamma(n,n)} \text{ are algebraically independent over K.}$$

Using the extra assumption on the family $(p_j)_{j\in\omega}$, we observe that for every $l \leq n$ and $i \in (p^\nu)^{\leq \beta(n)}$,

$$x_{\gamma(l,n)i} = f_i(x_{\gamma(l,n)}) \in K[x_{\gamma(l,n)\bar{0}(n)}].$$

Hence we represent only the algebraically independent components in the p-component tree of p_β:



It is easy to check that for every $x \models p_\beta$, the components $x_{\gamma(0,n)}, \ldots, x_{\gamma(n,n)}$ have the same properties as the components $x^{0,n}, \ldots, x^{n,n}$ for $x \models p_{\mathrm{Grill}}$: for every $a \models p_\beta$ and for every $j \leq j' \leq n$, $\mathrm{UR}(a/K \cup \{a_{\gamma(j,n)}, a_{\gamma(j',n)}\}) = n - (j'-j) = \mathrm{RT}(a/K\langle a_{\gamma(l,n)}, a_{\gamma(j',n)}\rangle)$ and so $\mathrm{UR}(p_\beta) \geq \omega$. (The proof is the same as the proof of Lemma 5.2.)

Now we choose β such that $\mathrm{UR}(p_\beta) \leq \omega$. For every $n < \omega$, we denote by p_β^n the partial type $p_{\beta,\leq n+\sum_{i<n}\beta(i)}$ (i.e the restriction of p_β to the level of the length of $\gamma(0,n)$).

First, note that for every $n < \omega$, p_β^n depends only on the restriction of β to $n = \{0, \ldots, n-1\}$, and that for every $l \leq m \leq n$, there is $Q_{\beta_{|n},l,m} \in K[X_0, \ldots, X_n]$ such that for every $x \models p_\beta^n$,

$$x_{\gamma(l,m)} = Q_{\beta_{|n},l,m}(x_{\gamma(0,n)}, \ldots, x_{\gamma(n,n)}).$$

We want to construct β such that for every $\bar{c} \in L$, if there is an algebraic relation over $K\bar{c}$ between the components at any level of the tree of p_β, then there are $l \leq n < \omega$, such that $a_{\gamma(l,n)}$ is in $\mathrm{acl}(K\bar{c})$. For this let $\bar{Y}$ be a tuple of infinitely many countable indeterminates and $(P_n)_{n\in\omega}$ an enumeration of the polynomials in

$$\bigcup_{m\in\omega} K[X_{0,m},\dots,X_{m,m},\bar{Y}].$$

We can choose this enumeration such that for every $n<\omega$,

$$P_n \in K[X_{0,m},\dots,X_{m,m},\bar{Y}] \text{ with } m\le n.$$

Then we define β by induction on $n<\omega$, so that for every $n\in\omega$, every $\bar{c}\in L$, and every $a\models p_\beta^{n+1}$, if $P_n(a_{\gamma(0,m)},\dots,a_{\gamma(m,m)},\bar{c})=0$ and $P_n(X_{0,m},\dots,X_{m,m},\bar{c})\neq 0$ then there is $l\le n$ such that $a_{\gamma(l,n)}$ is in $\mathrm{acl}(K\bar{c})$.

For $n=0$ we can choose $\beta(0)=0$.
Suppose that β is defined on $n=\{0,\dots,n-1\}$: let

$$R_n(X_0,\dots,X_n,\bar{Y}) := P_n(Q_{\beta_{|n},0,m}(X_0,\dots,X_n),\dots,Q_{\beta_{|n},m,m}(X_0,\dots,X_n),\bar{Y}).$$

By the orthogonality of the types $p_0,\dots,p_n$ we define $\beta(n) := m(R_n)$ as in Lemma 5.4.

Then if $a\models p_\beta^{n+1}$ such that

$$P_n(a_{\gamma(0,m)},\dots,a_{\gamma(m,m)},\bar{c})=0 \text{ and } P_n(X_{0,m},\dots,X_{m,m},\bar{c})\neq 0,$$

there is $l\le n$ such that $a_{\gamma(l,n)}\in\mathrm{acl}(K\bar{c})$: indeed, we have

$$R_n(a_{\gamma(0,n)},\dots,a_{\gamma(n,n)},\bar{c})=0 \text{ and } R_n(X_0,\dots,X_n,\bar{c})\neq 0.$$

This holds because if a' realizing p_β^{n+1} is such that $a'_{\gamma(0,m)},\dots,a'_{\gamma(m,m)}$ are algebraically free over $K\bar{c}$, then

$$R_n(a'_{\gamma(0,n)},\dots,a'_{\gamma(n,n)},\bar{c}) = P_n(a'_{\gamma(0,m)},\dots,a'_{\gamma(m,m)},\bar{c})\neq 0.$$

Thus we can apply Lemma 5.4.

Let us check that $\mathrm{UR}(p_\beta)\le\omega$. Let a be a realization of a forking extension of p_β over L. Then there is $m<\omega$ such that there is an algebraic relation between $a_{\gamma(0,m)},\dots,a_{\gamma(m,m)}$ over L.

So there is $n\ge m$ and $\bar{c}\in L$ such that $P_n(a_{\gamma(0,m)},\dots,a_{\gamma(m,m)},\bar{c})=0$.

By the choice of β, there is $l\le n$ such that $a_{\gamma(l,n)}$ is in $\mathrm{acl}(K,\bar{c})$. Thus $a_{\gamma(l,n)}\in L$ and $\mathrm{UR}(a/L)\le\mathrm{UR}(a/K,a_{\gamma(l,n)})\le n$.

We can now give a description of the extensions of p_β: let $K_1\supset K$ be an algebraically closed subset of L (in the model theoretic sense). Note that by Fact 1.3, K_1 is an elementary substructure of L. Let a be a realization of p_β. Then the type of a over K_1 is determined by the $a_{\gamma(l,n)}$'s which lie in K_1. Indeed, let b be another realization of p_β such that for every $l\le n$, $b_{\gamma(l,n)}\in K_1$ if and only if $a_{\gamma(l,n)}\in K_1$, and in this case $b_{\gamma(l,n)}=a_{\gamma(l,n)}$. Suppose that a and b do not have the same type over K_1. Then one can suppose that there is $\bar{c} \in K_1$ and P_n such that $P_n(a_{\gamma(0,m)}, \ldots, a_{\gamma(m,m)}, \bar{c}) = 0$ and $P_n(b_{\gamma(0,m)}, \ldots, b_{\gamma(m,m)}, \bar{c}) \neq 0$.

Let R_n be the polynomial associated with P_n and $I := \{i : a_{\gamma(i,n)} \notin K_1\}$. Then

$$R_n(a_{\gamma(0,n)}, \ldots, a_{\gamma(n,n)}, \bar{c}) = P_n(a_{\gamma(0,m)}, \ldots, a_{\gamma(m,m)}, \bar{c}) = 0$$

and

$$R_n((X_{i,n})_{i\in I}, (a_{\gamma(i,n)})_{i\notin I}, \bar{c}) \neq 0$$

because

$$R_n(b_{\gamma(0,n)}, \ldots, b_{\gamma(n,n)}, \bar{c}) = P_n(b_{\gamma(0,m)}, \ldots, b_{\gamma(m,m)}, \bar{c}) \neq 0.$$

Therefore by Lemma 5.4, there is $i \in I$ such that $a_{\gamma(i,n)} \in \mathrm{acl}(K_1 \cup \{a_{\gamma(j,n)} : j \notin I\} = K_1$. This is a contradiction.

To construct infinitely many pairwise orthogonal types of U-rank ω we use a family $(p_{s,n})_{s\in\omega,n\in\omega}$ of pairwise orthogonal minimal types of RT-rank one. As above, for every $s \in \omega$ and $\beta \in \omega^\omega$, we can construct $p_{s,\beta}$ with the family $(p_{s,n})_{n\in\omega}$. By Lemma 5.4 we can choose β such that for every $s \neq s'$ and every set of parameters C, if a_s realize $p_{s,\beta}$, and $a_{s'}$ realize $p_{s',\beta}$, and if there is $m < \omega$ and an algebraic relation over C between the components $a_{s,\gamma(0,m)}, \ldots, a_{s,\gamma(m,m)}$ and the components $a_{s',\gamma(0,m)}, \ldots, a_{s',\gamma(m,m)}$, then there are $l \leq n$ such that $a_{s,\gamma(l,n)}$ or $a_{s',\gamma(l,n)}$ is in $\mathrm{acl}(C)$.

Now, for $s \neq s'$, let a_s be a realization of the non-forking extension of $p_{s,\beta}$ over C and $a_{s'}$ a realization of the non-forking extension of $p_{s',\beta}$ over C. Then for every, $m < \omega$, $a_{s,\gamma(0,m)}, \ldots, a_{s,\gamma(m,m)}, a_{s',\gamma(0,m)}, \ldots, a_{s',\gamma(m,m)}$ are algebraically independent over $\mathrm{acl}(C)$; thus a_s and $a_{s'}$ are independent over C. $\square$

Corollary 5.6.

(i) There are infinitely many pairwise orthogonal additive subgroups of U-rank ω over $\emptyset$.

(ii) There are additive subgroups of U-rank ω over $\emptyset$ which are not one based.

(iii) There are infinitely many pairwise orthogonal one-based additive subgroups of U-rank ω over $\emptyset$.

Proof. In this proof, we assume that $K = \mathbb{F}_p(B)^{sep} = \mathrm{acl}(\emptyset)$.

First, observe that in the construction of the previous proposition, if the minimal types are generic types of additive subgroups, then the types p_β are also generic types of additive subgroups.

So for (i), it suffices to take a family of generic types of pairwise orthogonal minimal additive subgroups of RT-rank one over $\emptyset$. By Corollary 4.20, such a family exists.

For (ii), it suffices to take a family of types as above, such that the first type is the generic type of $(L^{p^\infty}, +)$.

For (iii), we will use a family $(p_n)_{n\in\omega}$ of generic types of pairwise orthogonal minimal additive subgroups of RT-rank one over $\emptyset$ whose induced structures are $\mathbb{F}_p$-vector spaces. We know that such families exists by Corollary 4.20.

Note that for all n, all l and all realizations $a_0, ..., a_l$ of p_n, the set $\{a_0, ..., a_l\}$ is free over K if and only if no non-trivial linear combination of $a_0, ..., a_l$ over $\mathbb{F}_p$ is in K. So with this family $(p_n)_{n\in\omega}$, we deduce the following:

Claim 1. *For all l, $n < \omega$ and $Q \in K[X_0^0, \ldots, X_0^l, \ldots, X_n^0, \ldots, X_n^l, \bar{Y}]$ there is $k(Q) < \omega$ such that for all $\bar{d} \in L$:*

$$\bigwedge_{i=0}^{n} \bigwedge_{j=0}^{l} p_{i,\leq k(Q)}(x_i^j) \vdash \big(Q((X_i^j)_{i\in\{0,\ldots,n\},j\in\{0,\ldots,l\}}, \bar{d}) \neq 0 \bigwedge Q((x_i^j), \bar{d}) = 0\big)$$

$$\Rightarrow \bigvee_{i=0}^{n} \text{“}CL((x_i^j)_{j\in\{0,\ldots,l\}}) \in \operatorname{acl}(\bar{d})''.$$

(“$CL(x_1, \ldots, x_k) \in A$” means that there is a non trivial linear combination of $x_1, \ldots, x_k$ in A; i.e. there is $(\lambda_1, \ldots, \lambda_k) \in \mathbb{F}_p^k \setminus \{(0, \ldots, 0)\}$) such that $\lambda_1 x_1 + \ldots + \lambda_k x_k \in A$.)

$\triangleright$ The claim follows from the fact that, if for all $i < \omega$ and for all realizations $x_i^0, \ldots, x_i^l$ of p_i, there is no non-trivial linear combination of $x_i^0, \ldots, x_i^l$ in $\operatorname{acl}(\bar{d})$, then for all i, $x_i^0, \ldots, x_i^l$ are independent over $\operatorname{acl}(\bar{d})$ and thus the types $\operatorname{tp}(x_0^0, \ldots, x_0^l/\operatorname{acl}(\bar{d}))$, ..., $\operatorname{tp}(x_n^0, \ldots, x_n^l/\operatorname{acl}(\bar{d}))$ are pairwise orthogonal. $\triangleleft$

Furthermore, observe that for all n, all l, all elementary substructure K_1 of L and all realizations $a_0, ..., a_l$ of p_n, the type of $(a_0, ..., a_l)$ over K_1 is given by knowing which linear combinations of $a_0, ..., a_l$ over $\mathbb{F}_p$ lie in K_1. Thus, in order to have an easy description of the types realized in the cartesian products of the group G_β which we plan to construct, we use an improvement of claim 1:

Claim 2. *For all l, $n < \omega$ and $Q \in K[X_0^0, \ldots, X_0^l, \ldots, X_n^0, \ldots, X_n^l, \bar{Y}]$ there is $k(Q) < \omega$ such that for all $(c_i^j)_{i\in\{0,\ldots,n\},j\in\{0,\ldots,l\}}$ in L, all $\bar{d}$ in L, all $J_0 \subseteq \{0, \ldots, l\}, \ldots, J_n \subseteq \{0, \ldots, l\}$ and all $\lambda_{i,j,j'} \in \mathbb{F}_p$ for $i \in \{0, \ldots, n\}$, $j' \in J_i$ and $j \in \{0, \ldots, l\} \setminus J_i$, if we denote by $Q'((Z_i^j), \bar{Y})$ the polynomial Q where we replace X_i^j by Z_i^j if $j \in J_i$, and by $Z_i^j + \sum_{j'\in J_i} \lambda_{i,j,j'} Z_i^{j'}$ otherwise, then:*

$$\bigwedge_{i=0}^{n} \bigwedge_{j\in J_i} p_{i,\leq k(Q)}(z_i^j) \vdash \left(\begin{array}{c} Q'((Z_i^j)_{i\in\{0,\ldots,n\},j\in J_i}, (c_i^j)_{i\in\{0,\ldots,n\},j\notin J_i}, \bar{d}) \neq 0 \\ \bigwedge \\ Q'((z_i^j)_{i\in\{0,\ldots,n\},j\in J_i}, (c_i^j)_{i\in\{0,\ldots,n\},j\notin J_i}, \bar{d}) = 0 \end{array}\right)$$

$$\Rightarrow \bigvee_{i=0}^{n} \text{”}CL((z_i^j)_{j\in J_i}) \in \operatorname{acl}(\bar{d} \cup \{c_k^j : k \in \{0, \ldots, n\} \text{ and } j \notin J_k\})\text{”}.$$

(The notation $Q'((Z_i^j)_{i\in\{0,\ldots,n\},j\in J_i}, (c_i^j)_{i\in\{0,\ldots,n\},j\notin J_i}, \bar{d})$ means that we have replaced the indeterminate Z_i^j by the constant c_i^j for each $i \in \{0, ..., n\}$ and $j \notin J_i$, and the indeterminates $\bar{Y}$ by the constants $\bar{d}$.)

Note that if $l = 0$, this is Lemma 5.4.

$\triangleright$ Since there are only finitely many choices of J_i's and $\lambda_{i,j,j'}$'s, we can apply claim 1. $\triangleleft$

Again we consider the types p_β. We define β by induction such that p_β is the generic type of a one-based additive subgroup. For this let $\bar{Y}$ be a tuple of infinitely many countable indeterminates and take an enumeration $(P_n)_{n\in\omega}$ of the polynomials in

$$\bigcup_{l\in\omega}\bigcup_{m\in\omega} K[X^0_{0,m},\ldots,X^l_{0,m},\ldots,X^0_{m,m},\ldots,X^l_{m,m},\bar{Y}]$$

We choose this enumeration such that for every $n<\omega$,

$$P_n \in K[X^0_{0,m},\ldots,X^l_{0,m},\ldots,X^0_{m,m},\ldots,X^l_{m,m},\bar{Y}] \text{ with } m\le n \text{ and } l\in\omega.$$

We define β in the following way:

Let $\beta(0) := k(P_0)$ as defined in claim 2.

If β is defined on n, let $\beta(n) := k(R_n)$ as defined in claim 2, where R_n is the polynomial P_n with the indeterminates $X^j_{i,m}$ replaced by $Q_{\beta_{|n},i,m}(X^j_0,\ldots,X^j_n)$.

Since claim 2 implies Lemma 5.4, it follows that $\mathrm{UR}(p_\beta)=\omega$.

Let G_β be the additive group whose generic type is p_β (i.e. $G_\beta = V(I(p_\beta))$). We see that claim 2 gives an easy description of the types realized in $G_\beta^{\times(l+1)}$ for $l<\omega$:

Claim 3. *Let K_1 be an algebraically closed (in the model theoretic sense) subset of L and let $(a^0,\ldots,a^l)\in G_\beta^{\times(l+1)}$. Then the type of $(a^0,\ldots,a^l)$ over K_1 is given by knowing which linear combinations of $a^0_{\gamma(i,n)},\ldots,a^l_{\gamma(i,n)}$ over $\mathbb{F}_p$ lie in K_1, for every $n\in\omega$ and $i\in\{0,\ldots,n\}$.*

$\triangleright$ Let $a=(a^0,\ldots,a^l)\in(G_\beta)^{\times(l+1)}$ and $b=(b^0,\ldots,b^l)\in(G_\beta)^{\times(l+1)}$ be such that, for all $n\in\omega$, all $i\in\{0,\ldots,n\}$ and all $\lambda_0,\ldots,\lambda_l\in\mathbb{F}_p$,

$$\sum_{j=0}^{l}\lambda_j a^j_{\gamma(i,n)}\in K_1 \text{ if and only if } \sum_{j=0}^{l}\lambda_j b^j_{\gamma(i,n)}\in K_1,$$

and such that if $\sum_j \lambda_j a^j_{\gamma(i,n)}\in K_1$ then

$$\sum_{j=0}^{l}\lambda_j a^j_{\gamma(i,n)} = \sum_{j=0}^{l}\lambda_j b^j_{\gamma(i,n)}.$$

Suppose that a and b do not have the same type over K_1. Then we can assume that there are $n\in\omega$ and $\bar{d}\in K_1$ such that $P_n((a^j_{\gamma(i,m)}),\bar{d})=0$ and $P_n((b^j_{\gamma(i,m)}),\bar{d})\neq 0$. Consider the polynomial R_n associated to P_n. Then

$$R_n((a^j_{\gamma(i,n)}),\bar{d})=0 \text{ and } R_n((b^j_{\gamma(i,n)}),\bar{d})\neq 0.$$

For each $i \in \{0, \ldots, n\}$, let $J_i \subseteq \{0, \ldots, l\}$ be such that $(a^j_{\gamma(i,n)})_{j \in J_i}$ is a basis of the $\mathbb{F}_p$-vector space generated by $a^0_{\gamma(i,n)}, \ldots, a^l_{\gamma(i,n)}$ over K_1. Observe that for all $i \in \{0, \ldots, n\}$ and $j \in J_i$, $a^j_{\gamma(i,n)}$ satisfies $p_{i, \leq \beta(n)}$, otherwise $a^j_{\gamma(i,n)} \in K$.

By the above assumptions, for all $i \in \{0, \ldots, n\}$, the set $(b^j_{\gamma(i,n)})_{j \in J_i}$ is also a basis of the $\mathbb{F}_p$-vector space generated by $b^0_{\gamma(i,n)}, \ldots, b^l_{\gamma(i,n)}$ over K_1, and there exist $\lambda_{i,j,j'} \in \mathbb{F}_p$ for every $i \in \{0, \ldots, n\}$, $j' \in J_i$ and $j \in \{0, \ldots, l\} \setminus J_i$, such that for all i and $j \notin J_i$,

$$a^j_{\gamma(i,n)} - \sum_{j' \in J_i} \lambda_{i,j,j'} a^{j'}_{\gamma(i,n)} = b^j_{\gamma(i,n)} - \sum_{j' \in J_i} \lambda_{i,j,j'} b^{j'}_{\gamma(i,n)} = c^j_i \in K_1.$$

We use now the polynomial R'_n associated to R_n in claim 2.

For every $i \in \{0, \ldots, n\}$ and $j \in \{0, \ldots, l\}$, define a'^j_i to be $a^j_{\gamma(i,n)}$ if $j \in J_i$, and otherwise to be c^j_i, and define (b'^j_i) in the same way. Then

$$R'_n((a'^j_i), \bar{d}) = P_n((a^j_{\gamma(i,m)}), \bar{d}) = 0$$

and

$$R'_n((b'^j_i), \bar{d}) = P_n((b^j_{\gamma(i,m)}), \bar{d}) \neq 0.$$

So by claim 2, we deduce that for some $i \in \{0, \ldots, n\}$, there is a non-trivial linear combination of $(a^j_{\gamma(i,n)})_{j \in J_i}$ in K_1. This contradicts the definition of J_i. $\triangleleft$

Claim 4. *Let K_1 be an elementary substructure of L and $a = (a^0, \ldots, a^l) \in G_\beta^{\times(l+1)}$.*

(i) The type of a over L does not fork on K_1 if and only if for all $n \in \omega$, all $i \in \{0, \ldots, n\}$ and all $\lambda_0, ..., \lambda_l \in \mathbb{F}_p$, if $\sum_j \lambda_j a^j_{\gamma(i,n)} \in L$ then $\sum_j \lambda_j a^j_{\gamma(i,n)} \in K_1$.

(ii) The canonical basis of a over K_1 is in

$$\operatorname{acl}\left(\left\{\sum_{j=0}^{l} \lambda_j a^j_{\gamma(i,n)} \in K_1 : n \in \omega,\ i \in \{0, \ldots, n\},\ \lambda_0, ..., \lambda_l \in \mathbb{F}_p\right\}\right),$$

a subset of $\operatorname{acl}(a)$.

$\triangleright$ (i) First suppose that there are $n \in \omega$, $i \in \{0, \ldots, n\}$ and $\lambda_0, ..., \lambda_l \in \mathbb{F}_p$, such that $c = \sum_j \lambda_j a^j_{\gamma(i,n)} \in L$ but $c \notin K_1$. If f and g are K_1-automorphisms of L such that $f(c) \neq g(c)$, then, by claim 3, the types of $f(a)$ and of $g(a)$ over L are distinct. So the type of a over L forks over K_1.

Thus if the type of a over L does not fork on K_1, then for all $n \in \omega$, all $i \in \{0, \ldots, n\}$ and all $\lambda_0, ..., \lambda_l \in \mathbb{F}_p$, if $\sum_j \lambda_j a^j_{\gamma(i,n)} \in L$ then $\sum_j \lambda_j a^j_{\gamma(i,n)} \in K_1$. But by claim 3, this determines the unique non-forking extension of the type of a over K_1.

Part (ii) follows from (i) and claim 3. $\triangleleft$

We conclude, by claim 4 (ii), that the group G_β is one-based.

In the same way as in the previous proposition, we see easily that there are infinitely many pairwise orthogonal such groups. $\square$

Remarks and questions.

(i) The one-based additive subgroups of U-rank ω which we have constructed above are such that the associated pregeometries are "limits" of $\mathbb{F}_p$-vector spaces: if G_β is such a group and $a^0, \ldots, a^l$ are realizations of the generic, then $a^0, \ldots, a^l$ are independent if and only if for every $i \leq n$, $a^0_{\gamma(i,n)}, \ldots, a^l_{\gamma(i,n)}$ are linearly independent over $\mathbb{F}_p$.

(ii) We have constructed additive subgroups of U-rank ω which are not one-based, but we do not know if there is an additive subgroup of U-rank ω whose generic is not locally modular.

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