Talk I: One dimensional Convex Integration

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Convex Integration Theory is a powerful tool for solving differential relations. It was introduced by M. Gromov in his thesis dissertation in 1969, then published in an article [2] in 1973 and eventually generalized in a book [3] in 1986. Nevertheless, reading Gromov is often a challenge since important details are not provided explicitely. Fortunately, there is a good reference that leaves no details in the shadow : the Spring's book [5]. My understanding of Convex Integration Theory primarily comes from this book. I owe it much in this presentation.

1 Two introductory examples

1.1 A first example

Let us consider the following elementary problem.

Problem 1.– Let

$$f_0: \begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{R}^3 \\ t & \longmapsto & (0,0,t) \end{array}$$

be the linear application mapping the segment $[0,1]$ vertically in $\mathbb{R}^3$. The problem is to find $f:[0,1]\stackrel{C^1}{\longrightarrow}\mathbb{R}^3$ such that:

i) $\forall\, t\in[0,1],\quad |\cos(f'(t),e_3)|<\epsilon$
ii) $\|f-f_0\|_{C^0}<\delta$
where $\epsilon>0$ and $\delta>0$ are given.

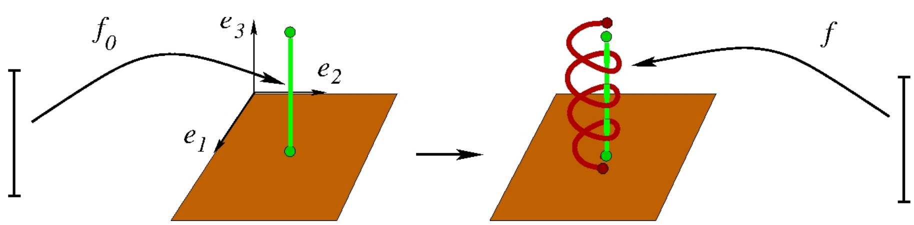
Solution.– At a first glance, the problem seems hopeless since condition i says that the slope is small and then the image has to move far away from the segment before reaching the desired height. After a few seconds of extra thinking, the solution occurs. It is good enough to move along an helix spiralling around the vertical axis:

$$\begin{array}{rccl} f: & [0,1] & \longrightarrow & \mathbb{R}^3 \\ & t & \longmapsto & \left\{ \begin{array}{l} \delta \cos 2\pi N t \\ \delta \sin 2\pi N t \\ t \end{array} \right. \end{array}$$

where $N \in \mathbb{N}^*$ is the number of spirals. We have

$$\langle \frac{f'}{\|f'\|}, e_3 \rangle = \frac{1}{\sqrt{1+4\pi^2 N^2 \delta^2}}.$$

Therefore, if N is large enough, f fulfills conditions i and ii.

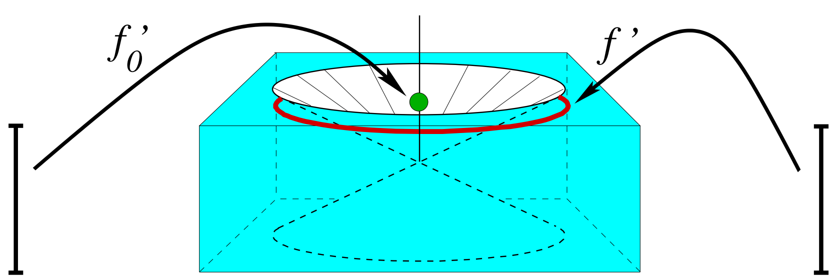


The image of f_0 is the green vertical segment, the solution f is the red helix.

Rephrasing.– The above problem was pretty easy, it will become very informative with a rephrasing of the two conditions. Condition (i) means that the image of f' lies inside the cone:

$$\mathcal{R} = \{v \in \mathbb{R}^3 \setminus \{O\} \mid \left|\langle \frac{v}{|v|}, e_3 \rangle\right| < \epsilon\} \cup \{O\}.$$

By extension, that cone $\mathcal{R}$ is called the *differential relation* of our problem.



The cone $\mathcal{R}$ is pictured in blue, the image of f' is the red circle and the constant image of f_0' the green point outside the cone.

The C^0-closeness required in the second condition, is a consequence of a geometric property of the derivative of f. Indeed, the image of f' in that cone is a circle whose center is the constant image of f_0'. Therefore, the average of f' for each spiral of f is $f_0'(t)$:

$$\frac{1}{Long(I_k)} \int_{I_k} f'(u)du = f_0'(t)$$

where $I_k = [\frac{k}{N}, \frac{k+1}{N}]$ the preimage of one spiral by f. Therefore, when integrating, the two resulting maps are closed together.

1.2 An more general example

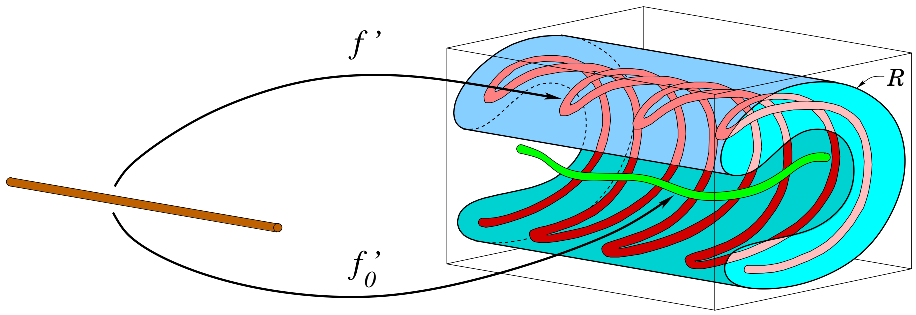
Problem.– Let $\mathcal{R} \subset \mathbb{R}^3$ be a path-connected subset (=our differential relation) and $f_0 : [0,1] \xrightarrow{C^1} \mathbb{R}^3$ be a map such

$$\forall t \in [0,1], \quad f_0'(t) \in \ IntConv(\mathcal{R})$$

where $IntConv(\mathcal{R})$ denotes the interior of the convex hull of $\mathcal{R}$. The problem is to find $f : [0,1] \xrightarrow{C^1} \mathbb{R}^3$ such that :

i) $\forall t \in [0,1], \quad f'(t) \in \mathcal{R}$
ii) $\|f - f_0\|_{C^0} < \delta$
with $\delta > 0$ given.

Solution.– From the hypothesis, the image of f_0' lies in the convex hull of $\mathcal{R}$. The idea is to build f' with an image lying inside $\mathcal{R}$ and such that, on average, it looks like the derivative of f_0. One way to do that is to choose a the f'-image to resemble to a kind of spring. In the spring, each arc as the same effect, on average, as a small piece of the image of the initial map f_0'. So, when integrating, the resulting map will be close to the initial map. As before, we will improve the closeness of f to f_0 by incresing the number of spirals.



The green bended spaghetti[1] pictures the image of f'_0, the half of a spring in rep/pink is the chosen image for f'.

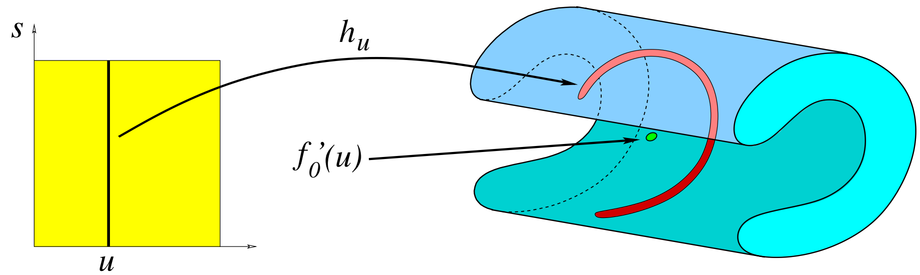
To formally construct a solution f of the problem, it is enough to choose a continuous family of loops of $\mathcal{R}$:

$$\begin{array}{rccc} h: & [0,1] & \longrightarrow & C^0(\mathbb{R}/\mathbb{Z}, \mathcal{R}) \\ & u & \longmapsto & h_u \end{array}$$

such that

$$\forall u \in\ [0,1], \quad \int_{[0,1]} h_u(s)ds = f'_0(u)$$

$i.e$ the average of the loop h_u is $f'_0(u)$.



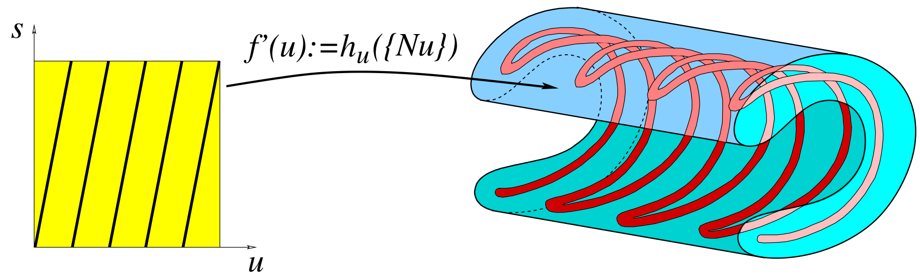
The image of the loop h_u. In that picture, this image is an arc. This loop is a round-trip starting at one of the endpoint of the arc and arriving at the same endpoint.

Then, the map f' is extracted from that familly of loops by a simple diagonal process

$$\forall t \in\ [0,1], \quad f'(t) := h_t(\{Nt\})$$

where $N \in \mathbb{N}^*$ and $\{Nu\}$ is the fractional part of Nt.

[1]Spaghetto ?



The image of f'.

Eventually, it remains to integrate to obtain a solution to our problem:

$$f(t) := f_0(0) + \int_0^t h_u(\{Nu\})du.$$

We say that f is obtained from f_0 by a **convex integration process**.

2 Finding the loops

In the above problem, we were wilfully blind to the question of the existence of the family of loops $(h_u)_{u\in[0,1]}$ needed to build the solution. We now deal with that issue.

Notation.– Let $A \subset \mathbb{R}^n$ and $a \in A$. We denote by $IntConv(A, a)$ the interior of the convex hull of the connected component of A to which a belongs.

Definition.– A (continuous) loop $g : [0,1] \to \mathbb{R}^n$, $g(0) = g(1)$, *strictly surrounds* $z \in \mathbb{R}^n$ if

$$IntConv(g([0,1])) \supset \{z\}.$$

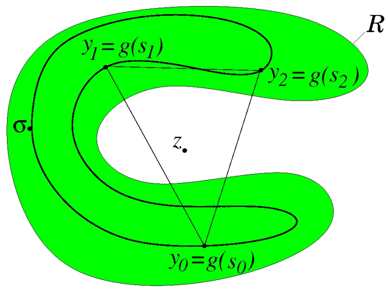
Fundamental Lemma.– *Let $\mathcal{R} \subset \mathbb{R}^n$ be an open set, $\sigma \in \mathcal{R}$ and $z \in IntConv(\mathcal{R}, \sigma)$ There exists a loop $h : [0,1] \xrightarrow{C^0} \mathcal{R}$ with base point σ that strictly surrounds z and such that:*

$$z = \int_0^1 h(s)ds.$$

Proof.– Since $z \in IntConv(\mathcal{R}, \sigma)$, there exists a n-simplex Δ whose vertices $y_0, ..., y_n$ belong to $\mathcal{R}$ and such that z lies in the interior of Δ. Therefore, there also exist

$$(\alpha_0, ..., \alpha_n) \in \,]0,1[^{\,n+1}$$

such that $\sum_{k=0}^{n} \alpha_k = 1$ and $z = \sum_{k=0}^{n} \alpha_k y_k$. Every loop $g : [0,1] \to \mathcal{R}$ with base point σ and passing through $y_0, ..., y_n$ satisfies $IntConv(g([0,1]) \supset \{z\}$ i. e. g surrounds z.



In general

$$z \neq \int_0^1 g(s)ds.$$

Let $s_0, ..., s_n$ be the times for which $g(s_k) = y_k$ and let $f_k : [0,1] \to \mathbb{R}_+^*$ be such that :

i) $f_k < \eta_1$ sur $[0,1] \setminus [s_k - \eta_2, s_k + \eta_2]$,

ii) $\int_0^1 f_k = 1$,

with η_1, η_2 two small positive numbers. We set:

$$z_k := \int_0^1 g(s) f_k(s) ds.$$

The number $\epsilon > 0$ being given, we can choose η_1, η_2 such that:

$$\forall k \in \{0, ..., n\}, \qquad \|z_k - g(s_k)\| \leq \epsilon.$$

Since $\mathcal{R}$ in open and $z \in Int\ \Delta$, for ϵ small enough we have

$$z \in IntConv(z_0, ..., z_n).$$

Therefore, there exist $(p_0, ..., p_n) \in]0,1[^{\,n+1}$ such that $\sum_{k=0}^{n} p_k = 1$ and:

$$\begin{aligned} z &= \sum_{k=0}^{n} p_k z_k &&= \sum_{k=0}^{n} p_k \int_0^1 g(s) f_k(s) ds \\ &= \int_0^1 g(s) \sum_{k=0}^{n} p_k f_k(s) ds &&= \int_0^1 g(s) \varphi'(s) ds \end{aligned}$$

where we have set

$$\varphi'(s) := \sum_{k=0}^{n} p_k f_k(s)$$

and

$$\begin{array}{rccc} \varphi : & [0,1] & \longrightarrow & [0,1] \\ & s & \longmapsto & \displaystyle\int_0^s \varphi(u)du. \end{array}$$

We have $\varphi'(s) > 0$, $\varphi(0) = 0$, $\varphi(1) = 1$. Thus φ is a strictly increasing diffeomorphism of $[0,1]$. Let us employ the change of coordinates $s = \varphi^{-1}(t)$, that is $t = \varphi(s)$, we have

$$dt = \varphi'(s)ds$$

therefore:

$$z = \int_0^1 g(s)\varphi'(s)ds = \int_0^1 g \circ \varphi^{-1}(t)dt.$$

Thus $h = g \circ \varphi^{-1}$ is our desired loop. $\square$

Remark.– *A priori* $h \in \Omega_\sigma(\mathcal{R})$, but it is obvious that we can choose h among "round-trips" *i.e* the space:

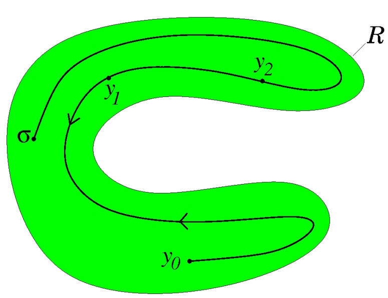
$$\Omega_\sigma^{AR}(\mathcal{R}) = \{h \in \Omega_\sigma(\mathcal{R}) \mid \forall s \in [0,1] \ \ h(s) = h(1-s)\}.$$

The point is that the above space is contractible. For every $u \in [0,1]$ we then denote by $h_u : [0,1] \longrightarrow \mathcal{R}$ the map defined by

$$h_u(s) = \left\{ \begin{array}{lll} h(s) & \text{if} & s \in [0, \frac{u}{2}] \cup [1 - \frac{u}{2}] \\ h(u) & \text{if} & s \in [\frac{u}{2}, 1 - \frac{u}{2}]. \end{array} \right.$$

This homotopy induces a deformation retract of $\Omega_\sigma^{AR}(\mathcal{R})$ to the constant map

$$\begin{array}{rccc} \widetilde{\sigma} : & [0,1] & \longrightarrow & \mathcal{R} \\ & s & \longmapsto & \sigma. \end{array}$$



Parametric version of the Fundamental Lemma. − *Let P be a compact manifold, $E = P \times \mathbb{R}^n \xrightarrow{\pi} P$ be a trivial bundle, and $\mathcal{R} \subset E$ be a set such that*

$$\forall p \in P, \quad \mathcal{R}_p := \pi^{-1}(p) \cap \mathcal{R} \ \textit{ is an open set of } \mathbb{R}^n$$

Let $\sigma \in \Gamma(\mathcal{R})$ and $z \in \Gamma(E)$ such that:

$$\forall p \in P, \ z(p) \in IntConv(\mathcal{R}_p, \sigma(p)).$$

Then, there exists $h : P \times [0,1] \xrightarrow{C^\infty} \mathcal{R}$ such that:

$$h(.,0) = h(.,1) = \sigma \in \Gamma^\infty(\mathcal{R}), \quad \forall p \in P, \ h(p,.) \in \Omega^{AR}_{\sigma(p)}(\mathcal{R}_p)$$

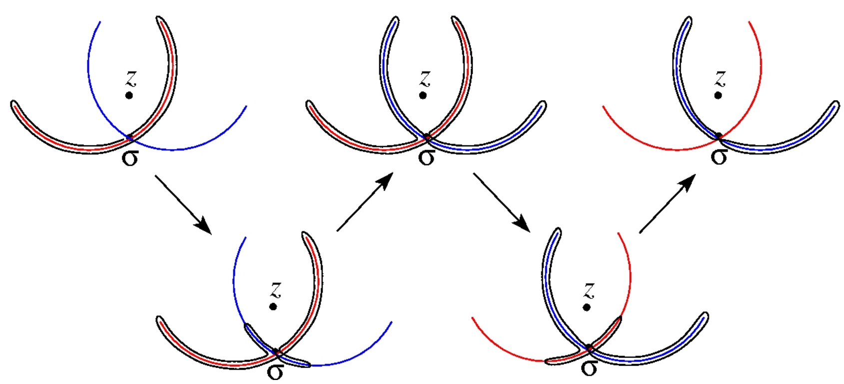
and

$$\forall p \in P, \ z(p) = \int_0^1 h(p,s)ds.$$

Proof.– The proof is rather long and technical. The main problem is the following: the result of the previous lemma rests on the existence of points $y_0, ..., y_n$ of $\mathcal{R}$ such that $z \in \ IntConv(\{y_0, ..., y_n\})$. If we want to mimic the previous proof while adding, we need to be able to follow continuously the points over P, that is, we need to show the existence of $(n+1)$ continuous maps $y_0, ..., y_n : P \longrightarrow \mathbb{R}^n$ such that

$$\forall p \in P, \quad z(p) \in \ IntConv(\{y_0(p), ..., y_n(p)\}).$$

Locally, it is easy to obtain maps $h_{\mathcal{U}} : \mathcal{U} \times [0,1] \xrightarrow{C^\infty} \mathcal{R}$ over open sets $\mathcal{U}$, the true problem is to glue them together. In order to do that, we take advantage of the contractibility of the round-trip loops. The following sequence of pictures should be enlightning.



A homotopy among loops surrounding z and joining $h_{\mathcal{U}}$ (red) to $h_{\mathcal{V}}$ (blue).

We then obtain a globally defined continuous map $h : P \times [0,1] \xrightarrow{C^\infty} \mathcal{R}$ such that

$$\forall p \in P, \quad z(p) \in \; IntConv(h(p,[0,1]))$$

and

$$h(.,0) = h(.,1) = \sigma \in \Gamma^\infty(\mathcal{R}), \quad \forall p \in P,\; h(p,.) \in \Omega^{AR}_{\sigma(p)}(\mathcal{R}_p).$$

It eventually remains to reparametrize the map h so that

$$\forall p \in P,\; z(p) = \int_0^1 h(p,s)ds.$$

For more details, see [5] p. 29-31. □

C^∞ parametric version of the Fundamental Lemma. – *Let P be a compact manifold, $E = P \times \mathbb{R}^n \xrightarrow{\pi} P$ a trivial bundle and $\mathcal{R} \subset E$ be a set such that*

$$\forall p \in P, \quad \mathcal{R}_p := \pi^{-1}(p) \cap \mathcal{R} \;\; \textit{is an open set of}\; \mathbb{R}^n$$

Let $\sigma \in \Gamma^\infty(\mathcal{R})$ and $z \in \Gamma^\infty(E)$ such that

$$\forall p \in P,\; z(p) \in IntConv(\mathcal{R}_p, \sigma(p)).$$

Then there exists $h : P \times [0,1] \xrightarrow{C^\infty} \mathcal{R}$ such that

$$h(.,0) = h(.,1) = \sigma \in \Gamma(\mathcal{R}), \quad \forall p \in P,\; h(p,.) \in \Omega^{AR}_{\sigma(p)}(\mathcal{R}_p)$$

and

$$\forall p \in P,\; z(p) = \int_0^1 h(p,s)ds.$$

Proof.– Let $(\rho_\epsilon : [0,1] \longrightarrow \mathbb{R})_{\epsilon>0}$ be a sequence of mollifiers. For every $p \in P$ we define a C^∞ map by the formula

$$\begin{array}{cccc} h_\epsilon(p,.) : & [0,1] & \longrightarrow & \mathbb{R}^n \\ & t & \longmapsto & (h(p,.) * \rho_\epsilon)(t). \end{array}$$

We set

$$z_\epsilon(p) := \int_0^1 h_\epsilon(p,t)dt$$

and we define $H_\epsilon : P \times \mathbb{R} \longrightarrow \mathbb{R}^n$ by

$$H_\epsilon(p,t) := h_\epsilon(p,t) + z(p) - z_\epsilon(p).$$

We have

$$\int_0^1 H_\epsilon(p,t)dt = z(p).$$

If ϵ is small enough, the image of the map $t \longmapsto H_\epsilon(p,t)$ lies inside $\mathcal{R}_p$. Thanks to the compactness of P the choice of the ϵ can be made independently of $p \in P$. □

3 C^0-density

Let $\mathcal{R} \subset \mathbb{R}^n$ be a arc-connected subset, $f_0 \in C^\infty(I, \mathbb{R}^n)$ be a map such that $f_0'(I) \subset \ IntConv(\mathcal{R})$. From the C^∞ parametric version of the Fundamental Lemma there exists a C^∞-map $h : I \times \mathbb{E}/\mathbb{Z} \longrightarrow \mathcal{R}$ such that

$$\forall t \in I, \ \ f_0'(t) = \int_0^1 h(t,u)du.$$

We set

$$\forall t \in I, \qquad F(t) := f_0(0) + \int_0^t h(s, Ns)ds$$

with $N \in \mathbb{N}^*$.

Definition.– We say that $F \in C^\infty(I, \mathbb{R}^n)$ is obtained from f_0 by an *convex integration process.*

Obviously $F'(t) = h(t, Nt) \in \mathcal{R}$ and thus F is a solution of the differential relation $\mathcal{R}$. One crucial property of the convex integration process is that the solution F can be made arbitrarily close to the initial map f_0..

Proposition (C^0-density).– *We have*

$$\|F - f_0\|_{C^0} \leq \frac{1}{N}\left(2\|h\|_{C^0} + \|\frac{\partial h}{\partial t}\|_{C^0}\right)$$

where $\|g\|_{C^0} = \sup_{p\in D} \|g(p)\|_{\mathbb{E}^3}$ *denotes the* C^0 *norm of a function* $g : D \rightarrow \mathbb{E}^3$.

Proof.– Let $t \in [0,1]$. We put $n := [Nt]$ (the integer part of Nt) and set $I_j = [\frac{j}{N}, \frac{j+1}{N}]$ for $0 \le j \le n-1$ and $I_n = [\frac{n}{N}, t]$. We write

$$F(t) - f(0) = \sum_{j=0}^{n} S_j \quad \text{and} \quad f_0(t) - f_0(0) = \sum_{j=0}^{n} s_j$$

with $S_j := \int_{I_j} h(v, Nv)\mathrm{d}v$ and $s_j := \int_{I_j}\int_0^1 h(x,u)\mathrm{d}u\mathrm{d}x$. By the change of variables $u = Nv - j$, we get for each $j \in [0, n-1]$

$$S_j = \frac{1}{N}\int_0^1 h(\frac{u+j}{N}, u)\mathrm{d}u = \int_{I_j}\int_0^1 h(\frac{u+j}{N}, u)\mathrm{d}u\mathrm{d}x.$$

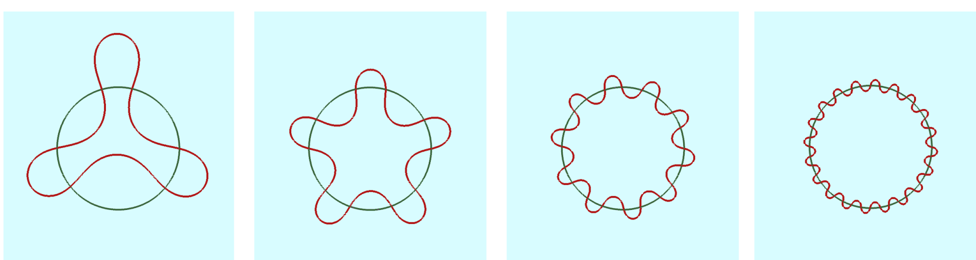
It ensues that

$$\|S_j - s_j\|_{\mathbb{E}^3} \le \frac{1}{N^2}\|\frac{\partial h}{\partial t}\|_{C^0}.$$

The proposition then follows from the obvious inequalities

$$\|S_n - s_n\|_{\mathbb{E}^3} \le \frac{2}{N}\|h\|_{C^0} \quad \text{and} \quad \|F(t) - f_0(t)\|_{\mathbb{E}^3} \le \sum_{j=0}^{n} \|S_j - s_j\|_{\mathbb{E}^3}.$$

□



The increase of the C^0 closeness with N.

In a multi-variables setting, the convex integration formula take the following natural form:

$$f(c_1, ..., c_m) := f_0(c_1, ..., c_{m-1}, 0) + \int_0^{c_m} h(c_1, ..., c_{m-1}, s, Ns)ds$$

where $(c_1, ..., c_m) \in [0,1]^m$. This expression is nothing else but the parametric formula of a convex integration process with parameter space $P =$

$[0,1]^{m-1}$. It turns out that the above C^0-density property can then be enhanced to a $C^{1,\widehat{m}}$-density property where the notation $C^{1,\widehat{m}}$ means that the closeness is measured with the following norm

$$\|f\|_{C^{1,\widehat{m}}} = \max(\|f\|_{C^0}, \|\frac{\partial f}{\partial c_1}\|_{C^0}, ..., \|\frac{\partial f}{\partial c_{m-1}}\|_{C^0}),$$

that is the C^1-norm without the $\|\frac{\partial f}{\partial c_m}\|_{C^0}$ term.

Proposition ($C^{1,\widehat{m}}$-density).– *Let $\mathcal{R} \subset \mathbb{R}^n$ be an open set, $E = C \times \mathbb{R}^n \xrightarrow{\pi} C$ be the trivial bundle over the cube $C = [0,1]^m$, $\sigma \in \Gamma(\mathcal{R})$ and let $f_0 : C \longrightarrow \mathbb{R}^n$ be a map such that:*

$$\forall c = (c_1, ..., c_m) \in [0,1]^m, \quad \frac{\partial f_0}{\partial c_m}(c) \in \ IntConv(\mathcal{R}_c, \sigma(c))$$

where $\mathcal{R}_c = \pi^{-1}(c) \cap \mathcal{R}$. Then, for every $\epsilon > 0$, there exists $f : C \longrightarrow \mathbb{R}^n$ such that:

i) $\frac{\partial f}{\partial c_m} \in \Gamma(\mathcal{R})$

ii) $\frac{\partial f}{\partial c_m}$ *is homotopic to σ in $\Gamma(\mathcal{R})$*

iii) $\|f - f_0\|_{C^{1,\widehat{m}}} = O\left(\frac{1}{N}\right)$.

Proof.– We have

$$\frac{\partial f}{\partial c_m}(c_1, ..., c_m) = h(c_1, ..., c_{m-1}, c_m, Nc_m) \in \mathcal{R}_c$$

and $\frac{\partial f}{\partial c_m}(c_1, ..., c_m)$ is homotopic to $\sigma(c)$ via

$$\sigma_u(c) := h_u(c_1, ..., c_{m-1}, c_m, Nc_m)$$

where h_u is the contracting map described just below the proof of the Fundamental Lemma. Mimicking the proof of the C^0-density property, it is easy to show that

$$\|\frac{\partial f}{\partial c_j} - \frac{\partial f_0}{\partial c_j}\|_{C^0} = O\left(\frac{1}{N}\right)$$

for every $j \in \{1, ..., m-1\}$. □

Remark.– Even if $f_0(0) = f_0(1)$, the map F obtained by a convex integration from f_0 does not satisfy $F(0) = F(1)$ in general. This can be easily corrected by defining a new map f with the formula

$$\forall t \in [0,1] \ , \ f(t) = F(t) - t\,(F(1) - F(0))\,.$$

The following proposition shows that the C^0-density property still holds for f and, provided N is large enough, that the map f is still a solution of $\mathcal{R}$.

Proposition.– *We have*

$$\|f - f_0\|_{C^0} \leq \frac{2}{N}\left(2\|h\|_{C^0} + \|\frac{\partial h}{\partial t}\|_{C^0}\right)$$

and $f'(\mathbb{R}/\mathbb{Z}) \subset \mathcal{R}$.

Proof.– The first inequality is obvious. Indeed, from

$$F(1) - F(0) = F(1) - f_0(0) = F(1) - f_0(1)$$

we deduce

$$\|f(t) - f_0(t)\| \leq \|F(t) - f_0(t)\| + \|F(1) - f_0(1)\| \leq 2\|F - f_0\|_{C^0}.$$

Derivating f we have $f'(t) = F'(t) - (F(1) - F(0))$ thus

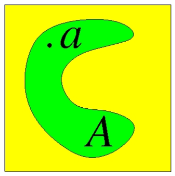
$$\|f' - F'\|_{C^0} \leq \|F - f_0\|_{C^0} = O\left(\frac{1}{N}\right).$$

Since $\mathcal{R}$ is open, if N is large enough $f'(\mathbb{R}/\mathbb{Z}) \subset \mathcal{R}$. □

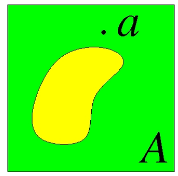
Remark.– It is of course easy to produce a parametric version of that proposition.

4 One dimensional h-principle

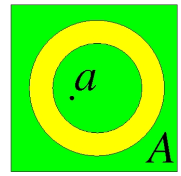
Definition.– A subset $A \subset \mathbb{R}^n$ is *ample* if for every $a \in A$ the interior of the convex hull of the connected component to which a belongs is $\mathbb{R}^n$ *i. e.* : $IntConv(A, a) = \mathbb{R}^n$ (in particular $A = \emptyset$ is ample).



A is not ample



A is ample



A is not ample.

Example.– The complement of a linear subspace $F \subset \mathbb{R}^n$ is ample if and only if Codim $F \geq 2$.

Definition.– Let $E = P \times \mathbb{R}^n \xrightarrow{\pi} P$ be a fiber bundle, a subset $\mathcal{R} \subset E$ is said to be *ample* if, for every $p \in P$, $\mathcal{R}_p := \pi^{-1}(p) \cap \mathcal{R}$ is ample in $\mathbb{R}^n$.

Remark.– If $\mathcal{R} \subset E$ is ample, then, for every $p \in P$, the condition $z(p) \in Conv(\mathcal{R}_p, \sigma(p))$ necessarily holds.

Proposition.– *Let $E = \mathbb{R}/\mathbb{Z} \times \mathbb{R}^n \xrightarrow{\pi} \mathbb{R}/\mathbb{Z}$ be a trivial bundle and let $\mathcal{R} \subset E$ be an open and ample differential relation. Then, for every $\sigma \in \Gamma(\mathcal{R})$, there exists $f : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}^n$ such that*

i) $f' \in \Gamma(\mathcal{R})$, i. e. $f \in \mathcal{S}ol(\mathcal{R})$,

ii) f' is homotopic to σ in $\Gamma(\mathcal{R})$.

Remark.– As a consequence, the natural

$$\pi_0(\mathcal{S}ol(\mathcal{R})) \longrightarrow \pi_0(\Gamma(\mathcal{R}))$$

is onto.

Proof.– Let $f_0 : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}^n$ be a C^1 map. Since $\mathcal{R}$ is ample, we have

$$\forall t \in \mathbb{R}/\mathbb{Z}, \quad f_0'(t) \in \mathbb{R}^n = IntConv(\mathcal{R}_t, \sigma(t)).$$

If N is large enough, the map $f : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}^n$ obtained from f_0 by a convex integration (with gluing)

$$\forall t \in [0,1], \quad f(t) := f_0(0) + \int_0^t h(s, Ns)ds - t\int_0^1 h(s, Ns)ds$$

is a solution of $\mathcal{R}$. Thus, the point i. For all $u \in [0,1]$, we define $f_u : [0,1] \longrightarrow \mathbb{R}^n$ by

$$\forall t \in [0,1], \quad f_u(t) := f_0(0) + \int_0^t h_u(s, Ns)ds - u.t\int_0^1 h(s, Ns)ds$$

where $h_u : \mathbb{R}/\mathbb{Z} \times [0,1] \longrightarrow \mathcal{R}$ is the natural deformation retract

$$h_u(t,s) = \begin{cases} h(t,s) & \text{if} \quad s \in [0, \frac{u}{2}] \cup [1 - \frac{u}{2}] \\ h(t,u) & \text{if} \quad s \in [\frac{u}{2}, 1 - \frac{u}{2}]. \end{cases}$$

Of course $h_1(t,s) = h(t,s)$ and $h_0(t,s) = \sigma(t)$. The map f_u does not descend to the quotient $\mathbb{R}/\mathbb{Z} = [0,1]/\partial[0,1]$. But its derivative

$$f'_u(t) = h_u(t, Nt) - u \int_0^1 h(s, Ns)ds$$

induces a map from $\mathbb{R}/\mathbb{Z}$ in $\mathbb{R}^n$ since

$$f'_u(0) = h_u(0,0) - \int_0^1 h_u(s, Ns)ds = \sigma(0) - u \int_0^1 h(s, Ns)ds$$

$$f'_u(1) = h_u(1,N) - \int_0^1 h_u(s, Ns)ds = \sigma(1) - u \int_0^1 h(s, Ns)ds$$

and thus $f'_u(0) = f'_u(1)$ because $\sigma(0) = \sigma(1)$. Hence, $\sigma_u := f'_u$ is a homotopy joining $f' = f'_1$ to σ. Since

$$\left\| \int_0^1 h(s, Ns)ds \right\| = \|F(1) - f_0(1)\| = O\left(\frac{1}{N}\right)$$

for every $u \in [0,1]$ and $t \in \mathbb{R}/\mathbb{Z}$, the point $\sigma_u(t)$ is as close as desired to $h_u(t, Nt) \in \mathcal{R}$. Since $\mathcal{R}$ is open, it exists N such that, for all $u \in [0,1]$, we have $\sigma_u \in \Gamma(\mathcal{R})$. This shows the point *ii*. □

A parametric version of that proof allows to obtain the following theorem:

Theorem (One-dimensional h-principle).– *Let $E = \mathbb{R}/\mathbb{Z} \times \mathbb{R}^n \xrightarrow{\pi} \mathbb{R}/\mathbb{Z}$ be a un trivial bundle and let $\mathcal{R} \subset E$ be a open and ample differential relation, then the map*

$$J : \mathcal{S}ol(\mathcal{R}) \longrightarrow \Gamma(\mathcal{R})$$

is a weak homotopy equivalence.

Observation.– Obviously, in the above theorem, $\mathbb{R}/\mathbb{Z}$ can be replaced by an interval.

5 Two applications of one-dimensional convex integration

5.1 Whitney-Graustein Theorem

Whitney-Graustein Theorem (1937). – *We have :* $\pi_0(I(\mathbb{S}^1, \mathbb{R}^2)) \simeq \mathbb{Z}$, *with an identification given by the tangential degree.*

Proof.– The theorem is a direct application of the 1-dimensional h-principle with $n = 2$ and $\mathcal{R} = \mathbb{R}/\mathbb{Z} \times \big(\mathbb{R}^2 \setminus \{(0,0)\}\big)$ which is open and ample. We then have

$$\mathcal{S}ol(\mathcal{R}) = I(\mathbb{S}^1, \mathbb{R}^2), \qquad \Gamma(\mathcal{R}) = C^0(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2 \setminus \{(0,0)\})$$

and

$$\begin{array}{rccc} J: & \mathcal{S}ol(\mathcal{R}) & \longrightarrow & \Gamma(\mathcal{R}) \\ & \gamma & \longmapsto & \gamma' \end{array}$$

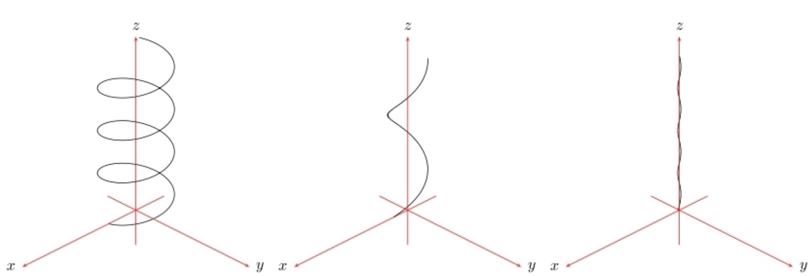
induces a bijection at the π_0-level. Note that the components of $C^0(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2 \setminus \{(0,0)\})$ are in one to one correspondance with $\mathbb{Z}$, the bijection being given by the turning number. It ensues that $\pi_0(J)$ is the tangential degree. □

5.2 A theorem of Ghomi

Theorem (Ghomi 2007).– *Let* $f_0 \in I(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ *be a curve with curvature function* k_0 *and let* c *be a real number such that* $c > max\ k_0$*. Then, for every* $\epsilon > 0$, *there exists* $f_1 \in I(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ *of constant curvature* c *and such that*

$$\|f_1 - f_0\|_{C^1} = \|f_1 - f_0\|_{C^0} + \|f_1' - f_0'\|_{C^0} \leq \epsilon.$$

An example.– How to C^1 approximate a line by curve with an arbitrarily large constant curvature ? The answer lies in an picture :



Just a little comment however (from [4]: let us parametrize the line as a vertical segment in the three dimensional Euclidean space

$$f_0(t) = \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix}$$

with $t \in\ [0,1]$. The theorem asserts that there exists a curve with constant curvature c which is C^1-close to f_0. A starting point is to begin by approximating the segment with an helix, for instance:

$$f_1(t) = \begin{pmatrix} \epsilon \cos \alpha t \\ \epsilon \sin \alpha t \\ t \end{pmatrix}$$

where $\alpha > 0$ and $\epsilon > 0$. The C^0 closeness of f_1 to f_0 is ruled by ϵ. Regarding the curvature, it is constant and can be made as large as we want by decreasing α. However, as the number α is becoming large, the derivative moves far away from the derivative of the initial function. It ensues that the helix is not C^1 close to f_0. To correct that point, we need to reduce the horizontal variations of the function. Let $k > 0$ and τ be two numbers, we set

$$f_{k,\tau}(t) = \begin{pmatrix} \dfrac{k}{k^2+\tau^2} \cos \sqrt{k^2+\tau^2}\ t \\ \dfrac{k}{k^2+\tau^2} \sin \sqrt{k^2+\tau^2}\ t \\ \dfrac{\tau}{\sqrt{k^2+\tau^2}}\ t \end{pmatrix}.$$

This is an helix with constant curvature k and constant torsion τ. It is then visible that we have to choose a torsion notably bigger to the curvature to ensure a quasi-vertical derivative.

Skecth of the proof.– This is a good example of use of the 1-dimensional convex integration even if it is not an direct application of the 1-dimensional h-principle theorem. Here are the main steps:

1) First, reduce the problem to the case where the parametrization of f_0 is given by the arc-length.Then, the curvature is the norm of the second derivative, that is the speed of $T_0 := f_0' : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{S}^2$.

2) Find $T_1 : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{S}^2$ with constant speed (in order to have a constant curvature) which is C^0-close to T_0 (to ensure that $\|f_1' - f_0'\|_{C^0}$ is small)

and close in average to T_0 (to get a small norm $\|f_1 - f_0\|_{C^0}$).

3) Technically, T_1 should complete small loops with constant speed in a neighborhood of $T_0(\mathbb{R}/\mathbb{Z})$ in $\mathbb{S}^2$ and such that the average on each loop is close to the one of T_0 in the corresponding interval.

For more details, see [1]. □

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