L6 - Theillière’s Formula

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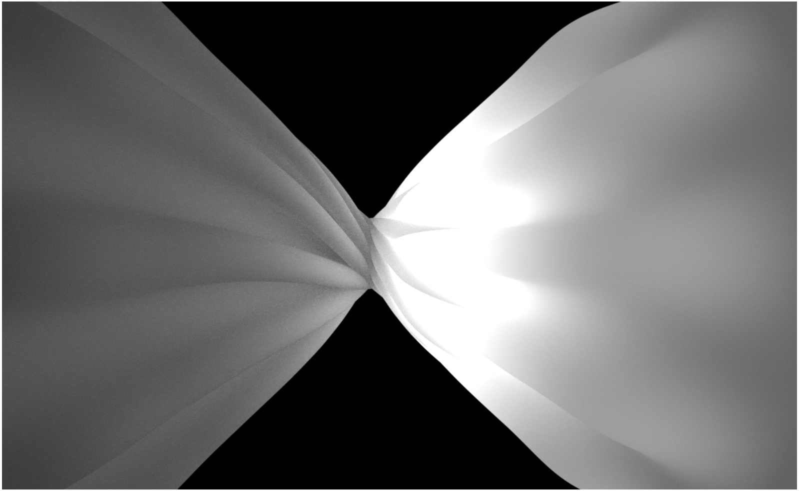
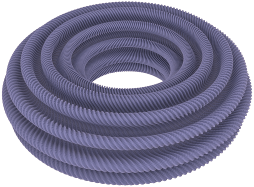


Theillière's Corrugation Formula

- It took 5 years to construct a C^1-isometric embedding of a flat torus



Hevea team: $\sim$, F. Lazarus, S. Jabrane, B. Thibert

Theillière's Corrugation Formula

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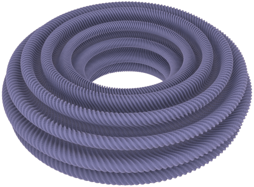

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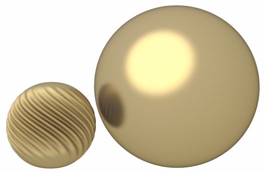
- It took 5 other years to construct a reduced sphere



Hevea team: E. Bartzos, $\sim$, R. Denis, F. Lazarus, D. Rohmer, B. Thibert

Theillière's Corrugation Formula

Mélanie Theillière

- Regarding construction of other explicit C^1-isometric maps, a more effective version of the Convex Integration would be valuable.

Theillière's Corrugation Formula

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Theillière's Corrugation Formula

Mélanie Theillière

- Regarding construction of other explicit C^1-isometric maps, a more effective version of the Convex Integration would be valuable.
- The search for a more effective version led Mélanie Theillière to the discovery of a new formula.
- In some specific but significant cases, this new formula produces solutions having very simple analytical expressions: roughly speaking, the integrals disappear!

Theillière's Corrugation Formula

Definition (Theillière).– Let

- $\gamma : [0,1]^m \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$ be a family of loops
- $f_0 : [0,1]^m \to \mathbb{R}^n$ be map
- $N > 0$

We define a new map $f : [0,1]^m \to \mathbb{R}^n$ by setting

$$\forall \in [0,1]^n, \qquad f(x) := f_0(x) + \frac{1}{N}\Gamma(x, Nx_1)$$

where

$$\Gamma(x,t) = \int_{s=0}^{t} \gamma(x,s) - \overline{\gamma}(x) ds \quad \text{and} \quad \overline{\gamma}(x) = \int_0^1 \gamma(x,s) ds$$

We say that f is obtained from f_0 by a **Corrugation Process.** We denote it by $f = CP_\gamma(f_0, \partial_1, N)$.

Theillière's Corrugation Formula

Definition (recall).– We say that a family of loops $\gamma : [0,1]^m \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$ *fulfills the average condition with respect to* f_0 *and in the direction* ∂_1 if

$$\forall x \in [0,1]^m, \quad \int_0^1 \gamma(x,t)dt = \partial_1 f_0(x).$$

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$$\forall x \in [0,1]^m, \quad \int_0^1 \gamma(x,t)dt = \partial_1 f_0(x).$$

Proposition (recall).– *If γ fulfills the average condition with respect to f_0 and in the direction ∂_1 then the map $F = CI_\gamma(f_0, \partial_1, N)$ satisfies*

(P_1) $\|F - f_0\|_{C^0} = O(1/N)$,

(P_2) $\|\partial_i F - \partial_i f_0\|_{C^0} = O(1/N)$ *for all* $i \neq 1$,

(P_3) $\forall x \in [0,1]^m, \quad \partial_1 F(x) = \gamma(x, Nx_1)$.

Theillière's Corrugation Formula

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Theillière's Corrugation Formula

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Conclusion.– We can equally choose CI_γ or CP_γ to run the Convex Integration Theory.

Theillière's Corrugation Formula

Proof of the proposition.– Since $f(x) = f_0(x) + \frac{1}{N}\Gamma(x, Nx_1)$ we have

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- We have $d\Gamma = \sum_{i=1}^{m} \partial_i \Gamma dx_i + \partial_t \Gamma dt$. Thus, if $i \neq 1$,

$$\partial_i f(x) - \partial_i f_0(x) = \frac{1}{N}(\partial_i \Gamma)(x, Nx_1)$$

and

$$\|\partial_i f - \partial_i f_0\|_{C^0} \leq \frac{1}{N}\|\partial_i \Gamma(.,.)\|_{C^0} = O(\frac{1}{N})$$

Theillière's Corrugation Formula

- From

$$f(x) = f_0(x) + \frac{1}{N}\int_{s=0}^{Nx_1} \gamma(x,s) - \overline{\gamma}(x)ds$$

we deduce

$$\partial_1 f(x) = \partial_1 f_0(x) + \gamma(x, Nx_1) - \overline{\gamma}(x) + \frac{1}{N}\int_{s=0}^{Nx_1} (\partial_1\gamma)(x,s) - (\partial_1\overline{\gamma})(x)ds$$

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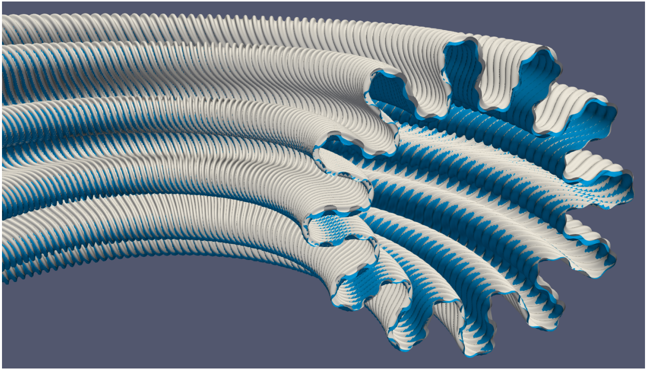
- We have

$$\left|\int_{s=0}^{Nx_1} (\partial_1\gamma)(x,s) - (\partial_1\overline{\gamma})(x)ds\right| \leq \sup_{t\in[0,1], x\in[0,1]^m} \|(\partial_1\gamma)(x,t) - (\partial_1\overline{\gamma})(x)\|$$

☐

Theillière's Corrugation Formula

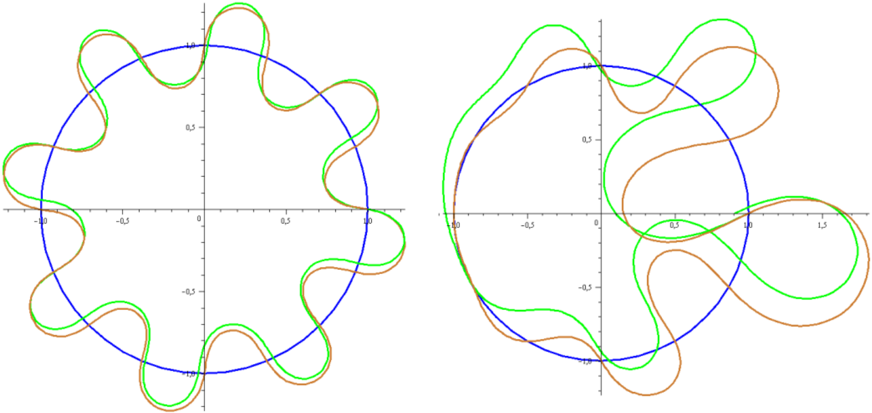


Comparison between CI_γ and CP_γ

Proposition.– *Let* $f = CP_\gamma(f_0, \partial_1, N)$ *and* $F = CI_\gamma(f_0, \partial_1, N)$*, we have*

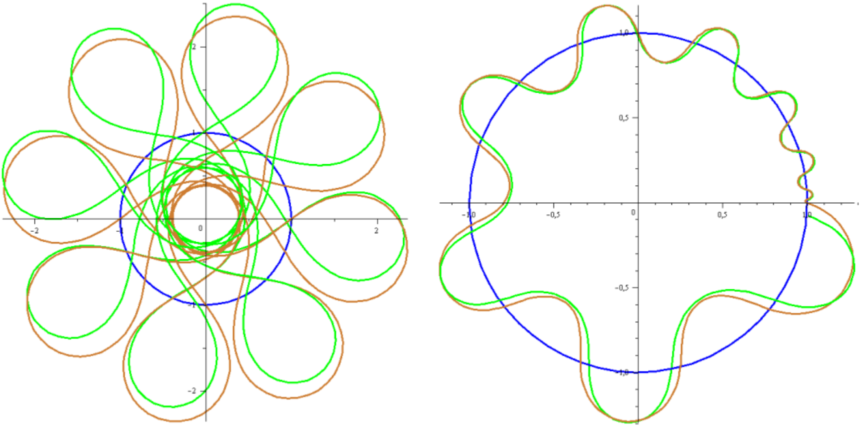
$$\|F - f\|_{C^0} = O(\frac{1}{N}).$$

Theillière's Corrugation Formula



Comparison between CI_γ (green) and CP_γ (orange)

Theillière's Corrugation Formula



Comparison between CI_γ (green) and CP_γ (orange)

Theillière's Corrugation Formula

- To prove the proposition, we need two preparatory lemmas.

Lemma.– Let $g : [a, b] \times \mathbb{R} \to \mathbb{R}^n$. We have

$$\int_{s=a}^{b} g(s, Ns)ds = \frac{G(b, Nb) - G(a, Na)}{N} - \frac{1}{N} \int_{s=a}^{b} (\partial_1 G)(s, Ns)ds.$$

where $G(x, t) = \int_{s=0}^{t} g(x, s)ds$ and ∂_1 denotes the partial derivative with respect to the first variable.

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where $G(x,t) = \int_{s=0}^{t} g(x,s)ds$ and ∂_1 denotes the partial derivative with respect to the first variable.

Proof of the lemma.– Let Φ denotes the map $s \mapsto \Phi(s) = (s, Ns)$. We have

$$\begin{array}{rcl} (G \circ \Phi)'(s) & = & (\partial_1 G) \circ \Phi(s) + N\,(\partial_2 G) \circ \Phi(s) \\ & = & (\partial_1 G) \circ \Phi(s) + N\, g \circ \Phi(s). \end{array}$$

Theillière's Corrugation Formula

Thus

$$g\circ\Phi(s)=\frac{1}{N}(G\circ\Phi)'(s)-\frac{1}{N}(\partial_1 G)\circ\Phi(s)$$

and

$$\int_{s=a}^{b} g\circ\Phi(s)ds=\frac{G\circ\Phi(b)-G\circ\Phi(a)}{N}-\frac{1}{N}\int_{s=a}^{b}(\partial_1 G)\circ\Phi(s)ds.$$

□

Theillière's Corrugation Formula

Lemma.– *We further assume that, for all $x \in [a,b]$, the average $\overline{g}(x) = \int_{t=0}^{1} g(x,t)dt$ vanishes. Then*

$$\int_{s=a}^{b} g(s,Ns)ds = \frac{G(b,Nb) - G(a,Na)}{N} - \frac{\overline{G}(b) - \overline{G}(a)}{N} + O\left(\frac{1}{N^2}\right)$$

where $\overline{G}(x) = \int_{t=0}^{1} G(x,t)dt$.

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where $\overline{G}(x) = \int_{t=0}^{1} G(x,t)dt$*.*

Proof of the lemma.– The proof reduces to apply the previous lemma twice. We first apply it to the map g to obtain

$$\int_{s=a}^{b} g(s,Ns)ds = \frac{G(b,Nb) - G(a,Na)}{N} - \frac{1}{N}\int_{s=a}^{b} (\partial_1 G)(s,Ns)ds. \quad (1)$$

As we have assumed that $\overline{g}(x) = 0$ for all $x \in [a,b]$, the map $t \mapsto G(x,t)$ is 1-periodic (however, note that its average does not vanish in general).

Theillière's Corrugation Formula

- Let h be the map defined by

$$h : (x,t) \mapsto (\partial_1 G)(x,t) - (\partial_1 \overline{G})(x)$$

and $H(x,t) = \int_{s=0}^{t} h(x,s)ds$. Applying the previous lemma to the map h we obtain:

$$\int_{s=a}^{b} h(s,Ns)ds = \frac{H(b,Nb) - H(a,Na)}{N} - \frac{1}{N}\int_{s=a}^{b} (\partial_1 H)(s,Ns)ds.$$

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- Since $\partial_1 \overline{G} = \overline{\partial_1 G}$, we have $\overline{h} = 0$ and thus $t \mapsto H(x,t)$ is 1-periodic. In particular, the maps $t \mapsto H(x,t)$ and $t \mapsto \partial_1 H(x,t)$ are bounded. We thus deduce

$$\int_{s=a}^{b} h(s,Ns)ds = O\left(\frac{1}{N}\right).$$

Theillière's Corrugation Formula

- Since

$$\int_{s=a}^{b}(\partial_1 G)(s,Ns)ds = \int_{s=a}^{b}(\partial_1 \overline{G})(s)ds + \int_{s=a}^{b} h(s,Ns)ds$$

we also deduce that

$$\int_{s=a}^{b}(\partial_1 G)(s,Ns)ds \quad = \quad \overline{G}(b) - \overline{G}(a) + O\left(\frac{1}{N}\right).$$

It remains to report this last relation in the equation (1) to obtain the desired result. ☐

Theillière's Corrugation Formula

Proof of the proposition.– The proof relies on the evaluation of the difference $F - f_0$ where $F = CI_\gamma(f_0, \partial_1, N)$ and $f_0 : [0,1]^m \to \mathbb{R}^n$. To avoid the writing of useless coordinates, we do the proof for $m = 1$. We are going to prove that

$$F(x) - f_0(x) = \frac{\Gamma(x, Nx)}{N} - \frac{\overline{\Gamma}(x) - \overline{\Gamma}(0)}{N} + O\left(\frac{1}{N^2}\right)$$

for all $x \in [0,1]$. Since $f(x) = f_0(x) + \frac{\Gamma(x,Nx)}{N}$, this formula will prove the theorem.

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- We have

$$F(x) - f_0(x) = \int_{s=0}^{x} F'(s) - f_0'(s) ds.$$

Theillière's Corrugation Formula

The Average Condition implies that

$$F(x) - f_0(x) = \int_{s=0}^{x} F'(s) - \overline{\gamma}(s)ds = \int_{s=0}^{x} \gamma(s, Ns) - \overline{\gamma}(s)ds.$$

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• Let $g(s,t) := \gamma(s,t) - \overline{\gamma}(s)$. Since g is 1-periodic with respect to the second variable and since the average $\overline{g}(s)$ vanishes for every $s \in [0,1]$, we can use the above Lemma to write:

$$\begin{aligned} F(x) - f_0(x) &= \frac{\Gamma(x, Nx) - \Gamma(0,0)}{N} - \frac{\overline{\Gamma}(x) - \overline{\Gamma}(0)}{N} + O\left(\frac{1}{N^2}\right) \\ &= \frac{\Gamma(x, Nx)}{N} - \frac{\overline{\Gamma}(x) - \overline{\Gamma}(0)}{N} + O\left(\frac{1}{N^2}\right). \end{aligned}$$

with $\Gamma(s,t) = \int_{w=0}^{t} g(s,w)dw$. ☐

Theillière's Corrugation Formula

What do we gain by using $f = CP_\gamma(f_0, \partial_1, N)$ **instead of** $F = CI_\gamma(f_0, \partial_1, N)$ **?**

$$\begin{aligned} f(x) &= f_0(x) + \frac{1}{N}\int_{t=0}^{Nx_1} \gamma(x,t) - \overline{\gamma}(x)dt \\ F(x) &= f_0(0, x_2, ..., x_n) + \int_0^{x_1} \gamma(u, x_2, ..., x_n; Nu)du \end{aligned}$$

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• **Pointwise property of** CP_γ**:** only the value of f_0 at x and the loop $t \mapsto \gamma(x,t)$ are needed to know the value of f at x.

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- **Relative property of** CP_γ**:** if $t \mapsto \gamma(x,t)$ is constant then $f(x) = f_0(x)$.

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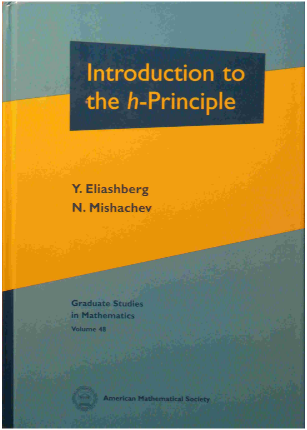
- **Relative property of** CP_γ**:** if $t \mapsto \gamma(x,t)$ is constant then $f(x) = f_0(x)$.

- **Periodicity property of** CP_γ**:** If $N \in \mathbb{Z}$ and $f_0(x + \partial_1) = f_0(x)$ then $f(x + \partial_1) = f(x)$.

Theillière's Corrugation Formula



Remark.– In 2002, Y. Eliashberg and N. Mishachev also proposed a new convex integration formula with similar properties but their construction is much less straightforward than the Theillière's one.

Theillière's Corrugation Formula

- **Coordinate free expression of CP_γ:** Let

1) $f_0 : U \to (W, h)$ where $U \subset M$ and h is a metric on W,
2) $\exp_y : T_y W \to W$ be the exponential map induced by h,
3) $\pi : U \to \mathbb{R}$ be a submersion,
4) $\gamma : U \times \mathbb{R}/\mathbb{Z} \to f_0^* TW$ be a family of loops such that $\gamma(x, .) : \mathbb{R}/\mathbb{Z} \to (f_0^* TW)_x$ for every $x \in U$.

We define

$$CP_\gamma(f_0, \pi, N) : x \mapsto \exp_{f_0(x)} \frac{1}{N} \int_0^{N\pi(x)} \gamma(x, t) - \overline{\gamma}(x) dt.$$

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This expression reduces to the previous one if $M = [0, 1]^m$, $W = \mathbb{E}^n$ and $\pi(x) = x_1$. In that case $d\pi = dx_1 = \langle \partial_1, \cdot \rangle$ and $\exp_{f_0(x)} y = f_0(x) + y$.

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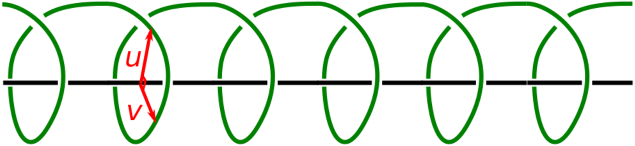
- This property was observed by Patrick Massot and has no counterpart in the classical version of the Convex Integration Theory.

Theillière's Corrugation Formula



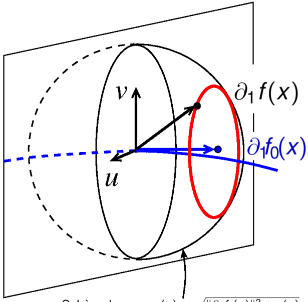
The corrugation process and the Nash strain

- Recall that the Nash's strain (in the direction ∂_1) is the helical deformation given by

$$f(x) = f_0(x) + \frac{\sqrt{\rho(x)}}{2\pi N}\big(\cos(2\pi N x_1)\,\mathbf{u}(x) + \sin(2\pi N x_1)\,\mathbf{v}(x)\big)$$

where **u** and **v** are orthonormal vectors of the normal bundle of f_0.

Theillière's Corrugation Formula



Sphère de rayon $r(x) = \sqrt{\|\partial_1 f_0(x)\|^2 + \rho(x)}$

- The Convex Integration version of the Nash's proof replaces the Nash's strain by the following deformation

$$CI_\gamma(f_0, \partial_1, N) = f_0(x) + \int_0^{x_1} \sqrt{\rho(u, x_2, ..., x_n)}\mathbf{e}^{i2\pi Nu}\,\mathrm{d}u$$

where $\mathbf{e}^{i\theta} = \cos\theta\,\mathbf{u} + \sin\theta\,\mathbf{v}$.

Theillière's Corrugation Formula

- The above expression of $CI_\gamma(f_0, \partial_1, N)$ comes from the choice

$$\gamma(x,t) = \sqrt{\rho(x)}\big(\cos(2\pi t)\,\mathbf{u}(x) + \sin(2\pi t)\,\mathbf{v}(x)\big) + \partial_1 f_0(x).$$

Since

$$\overline{\gamma}(x) = \int_0^1 \gamma(x,t)dt = \partial_1 f_0(x)$$

we have

$$\begin{array}{rcl} \Gamma(x,t) & = & \displaystyle\int_0^t \gamma(x,s) - \overline{\gamma}(x)ds \\ & = & \displaystyle\sqrt{\rho(x)}\int_0^t \cos(2\pi s)\,\mathbf{u}(x) + \sin(2\pi s)\,\mathbf{v}(x)ds \\ & = & \displaystyle\frac{\sqrt{\rho(x)}}{2\pi}\left(\sin(2\pi t)\,\mathbf{u}(x) - (\cos(2\pi t) - 1)\,\mathbf{v}(x)\right) \end{array}$$

Theillière's Corrugation Formula

- The corresponding expression of CP_γ is

$$\begin{aligned} CP_\gamma(f_0, \partial_1, N)(x) &= f_0(x) + \frac{1}{N}\Gamma(x, Nx_1) \\ &= f_0(x) + \frac{\sqrt{\rho(x)}}{2\pi N}\left(\sin(2\pi Nx_1)\,\mathbf{u}(x) - \cos(2\pi Nx_1)\,\mathbf{v}(x)\right) \\ &\quad + \frac{\sqrt{\rho(x)}}{2\pi N}\mathbf{v}(x) \end{aligned}$$

This analytical expression is equal to the Nash strain up to an angle shift of $\frac{\pi}{2}$ and an extra translation.

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- The angle shift could be corrected by choosing

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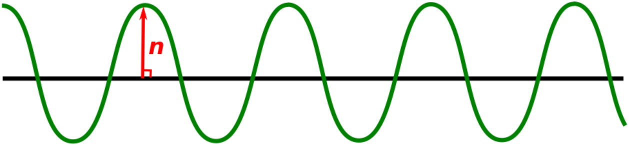
- Observe that, unlike $CI_\gamma(f_0,\partial_1,N)$, the analytical expression of $CP_\gamma(f_0,\partial_1,N)$ involves no integral.

Theillière's Corrugation Formula



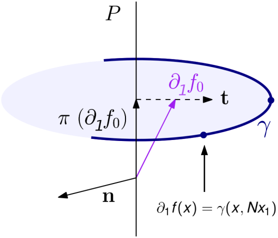
The corrugation process and the Kuiper strain

- Recall that the Kuiper's strain (in the direction ∂_1) is the corrugation given by

$$f(x) = f_0(x) - \frac{\rho(x)}{4N}\sin(2Nx_1)\mathbf{t}(x) + \frac{\sqrt{\rho(x)}}{N}\sin\left(Nx_1 - \frac{\rho(x)}{4}\sin(2Nx_1)\right)\mathbf{n}(x)$$

where $\mathbf{t} = \frac{\partial f_0}{\partial x_1} / \| \frac{\partial f_0}{\partial x_1} \|$ is the normalized derivative of f_0 in the direction ∂_1 and $\mathbf{n}$ is a unit normal vector.

Theillière's Corrugation Formula

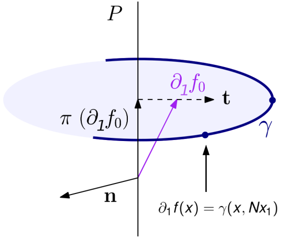


$$P = Span(\partial_2 f_0, ..., \partial_m f_0).$$

- The Convex Integration version of the Nash-Kuiper's proof replaces the Kuiper's strain by

$$CI_\gamma(f_0, \partial_1, N) = f_0(0, x_2, ..., x_m) + \int_0^{x_1} \gamma(u, x_2, ..., x_m; Nu)du$$

Theillière's Corrugation Formula



$$P = Span(\partial_2 f_0, ..., \partial_m f_0).$$

- In the above expression, γ is chosen to be

$$\gamma(x,t) = r(x)\left(\cos(\alpha(x)\cos 2\pi t) - J_0(\alpha(x))\right)\ \mathbf{t}(x) \\ + r(x)\sin(\alpha(x)\cos 2\pi t)\ \mathbf{n}(x) + \partial_1 f_0(x)$$

where

$$r = \sqrt{\|\partial_1 f_0\|^2 + \rho - \|\pi(\partial_1 f_0)\|^2}, \qquad \alpha = J_0^{-1}\left(\frac{\|\partial_1 f_0 - \pi(\partial_1 f_0)\|}{r}\right).$$

Theillière's Corrugation Formula

- We have

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$$\begin{aligned}
\Gamma(x,t) &= \int_0^t \gamma(x,s) - \overline{\gamma}(x)ds \\
&= r(x)\int_0^t (\cos(\alpha(x)\cos 2\pi s) - J_0(\alpha(x)))\,\mathbf{t}(x) \\
&\qquad\qquad + \sin(\alpha(x)\cos 2\pi s)\,\mathbf{n}(x)ds \\
&= r(x)\left(\int_0^t \cos(\alpha(x)\cos 2\pi s) - J_0(\alpha(x))ds\right)\,\mathbf{t}(x) \\
&\quad + r(x)\left(\int_0^t \sin(\alpha(x)\cos 2\pi s)ds\right)\,\mathbf{n}(x)
\end{aligned}$$

Theillière's Corrugation Formula

- If we set

$$K_c(a,t) := \int_{u=0}^{t} \Big(\cos(a\cos 2\pi u) - J_0(a) \Big) du$$

$$K_s(a,t) := \int_{u=0}^{t} \sin(a\cos 2\pi u) du$$

we then can write

$$\Gamma(x,t) = r(x)K_c(\alpha(x),t)\,\mathbf{t}(x) + r(x)K_s(\alpha(x),t)\,\mathbf{n}(x).$$

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- Note that K_c is 1/2-periodic in t and K_s 1-periodic in t.

Theillière's Corrugation Formula

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- This is a "without integration" formula in the following sense: the two integrals that appear do not involve any data of our specific problem, K_c and K_s are functions of two variables that are independent of f_0, r, α, **t** and **n**.

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- Once K_c and K_s have been tabulated, no numerical integration is needed to determine $CP_\gamma(f_0, \partial_1, N)$.

Theillière's Corrugation Formula

Sergio Conti, Camillo de Lellis and László Székélyhidi

Observation and cultural note.– The above formula is analogous to the *ansatz* used by S. Conti, C. De Lellis and L. Székelyhidi to study the $C^{1,\alpha}$ regularity of Nash-Kuiper embeddings.

Theorem (S. Conti, C. De Lellis, L. Székelyhidi, 2012).– *Let* M^n *be a compact Riemannian manifold,* $f_0 : (M^n, g) \to \mathbb{E}^{n+1}$ *be a strictly short map and* $\alpha \in \left]0, \frac{1}{2(n+1)s_n}\right[$*, then there exists an isometric map* $f_\infty : (M^n, g) \to \mathbb{E}^{n+1}$ *of classe* $C^{1,\alpha}$.

Convex Integration without Integration

Question.– Can we expect a "without integration formula" for other differential relations ?

Ricky Stern, Wild animals in the urban landscape

Convex Integration without Integration

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Ricky Stern, Wild animals in the urban landscape

Answer.– Yes, provided that the differential relation allows a uniform construction of the family of loops.

Convex Integration without Integration

Our approach so far: Given $\mathscr{R} \subset J^1([0,1]^m, \mathbb{R}^n)$ and

$$x \mapsto \mathfrak{S}_0(x) = (x, f_0(x), v_1(x), ..., v_m(x)) \in \mathscr{R}$$

we build on an *ad hoc* basis a family of loops $\gamma : [0,1]^m \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$ such that, for all $x \in [0,1]^m$,

$$(i) \quad t \mapsto (x, f_0(x), \gamma(x,t), v_2(x), ..., v_m(x)) \in \mathscr{R}$$

$$(ii) \quad \int_0^1 \gamma(x,t)dt = \partial_1 f_0(x)$$

$$(iii) \quad \gamma(x,0) = \gamma(x,1) = v_1(x)$$

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We then set $F_1 = CI_\gamma(f_0, \partial_1, N)$ and, for N large enough, we obtain

$$x \mapsto \mathfrak{S}_1(x) = (x, F_1(x), \partial_1 F_1(x), v_2(x), ..., v_m(x)) \in \mathscr{R}$$

Convex Integration without Integration

A new approach: Assume we are given a map

$$\begin{array}{cccc} \gamma : & "\mathscr{R} \times \mathbb{R}^n" & \longrightarrow & C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \\ & (\sigma, w) & \longmapsto & \gamma(\sigma, w) \end{array}$$

such that

$$(1) \quad t \mapsto (x, y, \gamma(\sigma, w)(t), v_2, ..., v_m) \in \mathscr{R}$$

$$(2) \quad \int_0^1 \gamma(\sigma, w)(t)dt = w$$

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such that

$$(1) \quad t \mapsto (x, y, \curlyvee(\sigma, w)(t), v_2, ..., v_m) \in \mathscr{R}$$

$$(2) \quad \int_0^1 \curlyvee(\sigma, w)(t) dt = w$$

$$(3) \quad \curlyvee(\sigma, w)(0) = \curlyvee(\sigma, w)(1) = v_1$$

- Then, for any section

$$x \mapsto \mathfrak{S}_0(x) = (x, f_0(x), v_1(x), ..., v_m(x)) \in \mathscr{R}$$

the following family of loops γ will satisfy points (*i*) to (*iii*):

$$\gamma(x, t) := \curlyvee(\mathfrak{S}_0(x), \partial_1 f_0(x))(t).$$

Convex Integration without Integration

- Recall that, in Lecture 4, we solved an open and ample differential relation $\mathscr{R} \subset J^1([0,1]^m, \mathbb{R}^n)$ by considering the projection

$$p^{\perp_m} : (x, y, v_1, ..., v_m) \longmapsto (x, y, v_1, ..., v_{m-1}) = z$$

and the slices $\mathscr{R}_z^{\perp_m}$ of the differential relation $\mathscr{R}$ over $(p^{\perp_m})^{-1}(z)$:

$$\begin{aligned} \mathscr{R}_z^{\perp_m} &:= \mathscr{R} \cap (p^{\perp_m})^{-1}(z) \\ &= \{u \in \mathbb{R}^n \,|\, (x, y, v_1, ..., v_{m-1}, u) \in \mathscr{R}\}. \end{aligned}$$

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- Then by solving iteratively each $\mathscr{R}^{\perp_k}$, $k \in \{m, m-1, ..., 1\}$, we have obtain a holonomic section $j^1 f$ from a section $\mathfrak{S}_0$ of $\mathscr{R}$

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• Here, we do not assume that $\mathscr{R} \subset J^1(M, N)$ is ample. We thus have to introduce new notations to take into account the convex hull of each slice of $\mathscr{R}$.

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• Let $\sigma = (x, y, v_1, ..., v_m) \in \mathscr{R}$. We denote by

$$\mathscr{R}(\sigma, \partial_m)$$

the connected component of $\mathscr{R}_z^{\perp m}$, $z = (x, y, v_1, ..., v_{m-1})$, that contains v_m.

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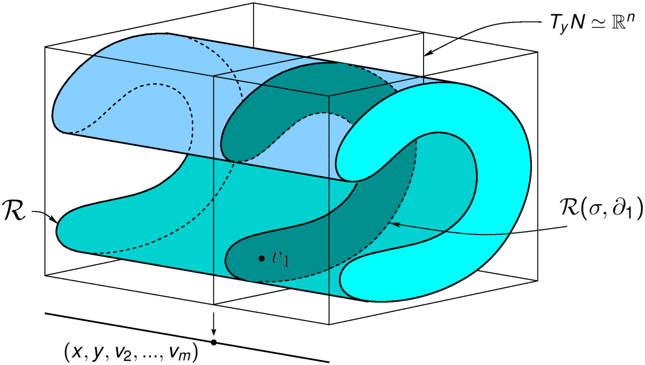
• We consider the bundle $p_y^* TN$ over $\mathscr{R}$ induced by the projection $p_y : \mathscr{R} \to N$, $\sigma = (x, y, L) \mapsto y$, and we define

$$IntConv(\mathscr{R}, \partial_k) := \{(\sigma, w) \in p_y^* TN \mid w \in IntConv\ \mathscr{R}(\sigma, \partial_k)\}.$$

Convex Integration without Integration

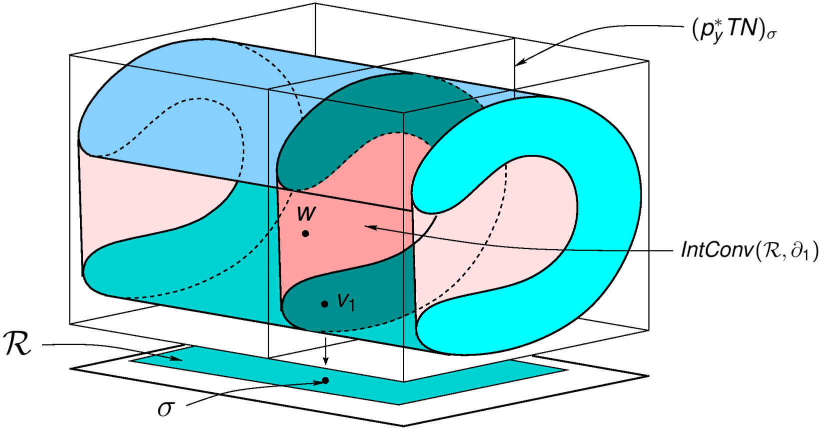


- An illustration showing

$$\mathscr{R}(\sigma, \partial_1) = \{w \in T_yN \mid (x, y, w, v_2, ..., v_m) \in \mathscr{R}\}$$

with $\sigma = (x, y, v_1, ..., v_m) \in \mathscr{R}$...

Convex Integration without Integration



- ... and

$$IntConv(\mathscr{R}, \partial_1) = \{(\sigma, w) \in p_y^* TN \mid w \in IntConv\, \mathscr{R}(\sigma, \partial_1)\}$$

Convex Integration without Integration

Definition.– Let U be a chart of M and $\mathscr{R}$ be a differential relation of $J^1(U,W)$. We say that a loop family

$$\begin{array}{rccc} \gamma: & IntConv(\mathscr{R},\partial_k) & \longrightarrow & C^1(\mathbb{R}/\mathbb{Z},TN) \\ & (\sigma,w) & \longmapsto & \gamma(\sigma,w)(\cdot) \end{array}$$

is *surrounding in the direction* ∂_k if for every (σ,w) we have

(1) $t \mapsto \gamma(\sigma,w)(t)$ is a loop in $\mathscr{R}(\sigma,\partial_k)$,

(2) the average of $t \mapsto \gamma(\sigma,w)(t)$ is w,

(3) there exists a continuous homotopy

$$H : IntConv(\mathscr{R},\partial_k)\times[0,1]\to TN$$

such that for all $t \in [0,1]$:

$$H(\sigma,w,0)=\gamma(\sigma,w)(0), \quad H(\sigma,w,1)=v_k \quad \text{and} \quad H(\sigma,w,t)\in\mathscr{R}(\sigma,\partial_k).$$

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- If such a $\mathfrak{\gamma}$ exists, we denote the induced Corrugation Process by

$$CP_{\mathfrak{\gamma}}(\mathfrak{S}_0, \partial_1, N)$$

rather than $CP_{\gamma}(f_0, \partial_1, N)$. The relation between $\mathfrak{\gamma}$ and γ is given by

$$\gamma(x,t) := \mathfrak{\gamma}(\mathfrak{S}_0(x), \partial_1 f_0(x))(t).$$

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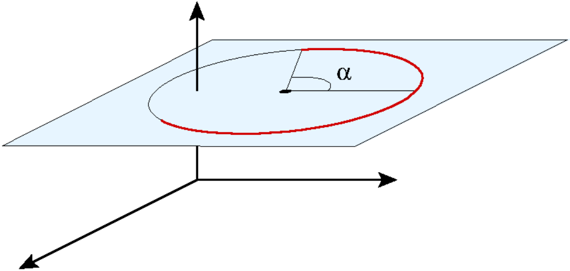
• The third point in the definition ensures the homotopic properties needed to glue local solutions and to state a potential *h*-principle for $\mathscr{R}$.

• We are going to focus on maps $\mathfrak{\gamma}$ whose loops are shaped on a same pattern.

Convex Integration without Integration

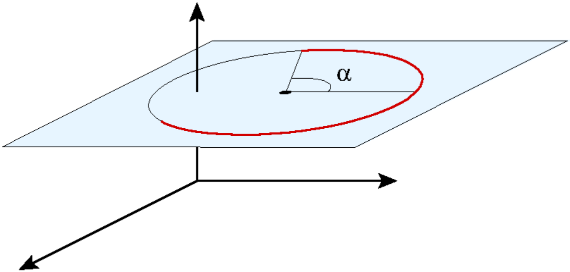


Définition.– A *loop pattern* is map

$$c : A \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^p$$

where $A \subset \mathbb{R}^q$ is a parameter space.

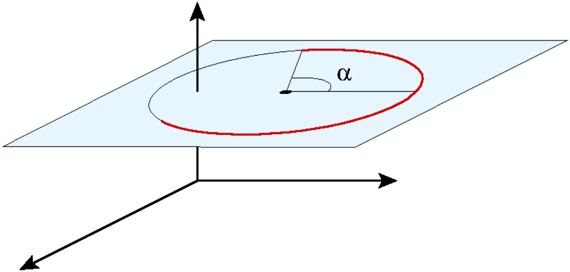
Convex Integration without Integration



Example.– Let $\alpha_0 \simeq 2.4$ be the first positif zero of J_0. The pattern $c = (c_1, c_2, c_3) : [0, \alpha_0] \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ behind the loops used to solve the ϵ-isometric relation in codimension one is given by

$$c(\alpha, t) = \Big(\cos(\alpha \cos 2\pi t) - J_0(\alpha), \sin(\alpha \cos 2\pi t), 1 \Big).$$

Convex Integration without Integration



Example.– Indeed, recall that γ was chosen to be

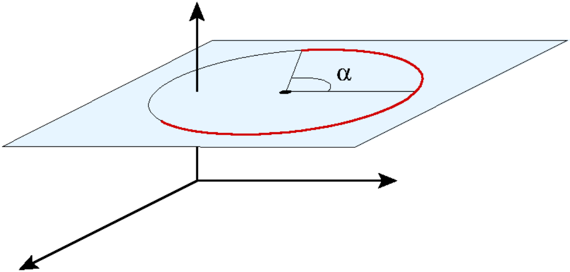
$$\gamma(x,t) = r(x)\left(\cos(\alpha(x)\cos 2\pi t) - J_0(\alpha(x))\right)\ \mathbf{t}(x)$$

$$+r(x)\sin(\alpha(x)\cos 2\pi t)\ \mathbf{n}(x) + \partial_1 f_0(x)$$

that is

$$\gamma(x,t) = r(x)c_1(\alpha(x),t)\ \mathbf{t}(x) + r(x)c_2(\alpha(x),t)\ \mathbf{n}(x) + c_3(\alpha(x),t)\partial_1 f_0(x)$$

Convex Integration without Integration

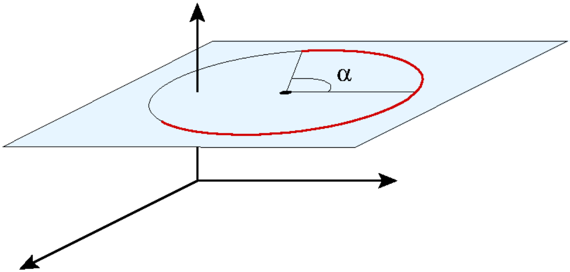


Example.– Thus, for every point x, the loop $t \mapsto \gamma(x,t)$ is the linear image of the loop $t \mapsto c(\alpha(x),t)$ by the linear map

$$\mathscr{e}(x) : \mathbb{R}^3 \longrightarrow T_{f_0(x)}N$$

defined by $\mathscr{e}(x)(\epsilon_1) = r(x)\mathbf{t}(x)$, $\mathscr{e}(x)(\epsilon_2) = r(x)\mathbf{n}(x)$ and $\mathscr{e}(x)(\epsilon_3) = \partial_1 f_0(x)$ where $(\epsilon_1, \epsilon_2, \epsilon_3)$ is the standard basis of $\mathbb{R}^3$.

Convex Integration without Integration



Example.– Summing up, we thus have

$$\gamma(x, t) = \mathscr{e}(x) \circ c(\alpha(x), t)$$

for all x and t.

Convex Integration without Integration

- To state the definition of a *c*-shaped surrounding family, we need a formal framework to define what it means to be the linear image of a loop of *c*.

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- We consider its pull back by the projection

$$\begin{array}{rccc} q: & IntConv(\mathscr{R}, \partial_k) & \longrightarrow & N \\ & (\sigma, w) & \longmapsto & y \end{array}$$

A section $\mathscr{e}$ of q^*E defines a family of linear maps $\mathscr{e}(\sigma, w) : \mathbb{R}^p \to T_yN$.

Convex Integration without Integration

Definition.– We say that a surrounding family in the direction ∂_k

$$\gamma : IntConv(\mathscr{R}, \partial_k) \to C^1(\mathbb{R}/\mathbb{Z}, TN)$$

is *c-shaped* if there exist a section $\mathscr{e}$ of $q^*E \to IntConv(\mathscr{R}, \partial_k)$ and a map $\mathbf{a} : IntConv(\mathscr{R}, \partial_k) \to A$ such that

$$\gamma(\sigma, w)(t) = \mathscr{e}(\sigma, w) \circ c(\mathbf{a}(\sigma, w), t)$$

for all $((\sigma, w), t) \in IntConv(\mathscr{R}, \partial_k) \times \mathbb{R}/\mathbb{Z}$.

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- If $(c_1, \dots, c_p)$ denote the components of c in the standard basis of $\mathbb{R}^p$ and if $\mathbf{e}_1, \dots, \mathbf{e}_p$ denote the image of this basis by $\mathscr{e}$, we thus have

$$\gamma(\sigma, w)(t) = \sum_{i=1}^p c_i(\mathbf{a}(\sigma, w), t)\, \mathbf{e}_i(\sigma, w).$$

Convex Integration without Integration

Notation.– We write: $\boldsymbol{\gamma}(\sigma, w)(t) = c(\mathbf{a}(\sigma, w), t) \cdot \mathbf{e}(\sigma, w)$.

Convex Integration without Integration

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Definition.– Let $c : A \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^p$ and $\mathscr{R} \subset J^1(U, N)$ be a differential relation. If there exists a c-shaped family $\widetilde{\gamma}$ in the direction ∂_k, we say that $\mathscr{R}$ is a *Kuiper relation with respect* (c, ∂_k)*.*

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• If $\mathscr{R} \subset J^1(M, N)$, we say that $\mathscr{R}$ is a *Kuiper relation with respect c* if $\mathscr{R}_{|U}$ it is a Kuiper relation with respect (c, ∂_k) for every chart $U \subset M$ and for every direction ∂_k.

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- The interest of Kuiper relations lies in the fact that the analytical expression of the Corrugation Process for such relations involves no integrals. This is stated in the following proposition.

Convex Integration without Integration

Proposition.– *Let* $\mathscr{R} \subset J^1(U, \mathbb{R}^n)$ *be a Kuiper relation with respect to* (c, ∂_k) *and* $x \mapsto \mathfrak{S}_0(x) = (x, f_0(x), L(x))$ *be a section of* $\mathscr{R}$, *then*

$$CP_{\widetilde{\gamma}}(\mathfrak{S}_0, \partial_k, N)(x) = f_0(x) + \frac{1}{N}\sum_{i=1}^{p} C_i(a(x), Nx_k)e_i(x)$$

where

- $C_i(a,t) = \displaystyle\int_0^t \big(c_i(a,u) - \overline{c}(a)\big)du$
- $a(x) = \boldsymbol{a}(\mathfrak{S}_0(x), \partial_1 f_0(x)) \in \mathbb{R}^q$
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Convex Integration without Integration

Proposition.– *Let $\mathscr{R} \subset J^1(U, \mathbb{R}^n)$ be a Kuiper relation with respect to (c, ∂_k) and $x \mapsto \mathfrak{S}_0(x) = (x, f_0(x), L(x))$ be a section of $\mathscr{R}$, then*

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where

- $C_i(a, t) = \int_0^t \big(c_i(a, u) - \overline{c}(a)\big) du$
- $a(x) = \boldsymbol{a}(\mathfrak{S}_0(x), \partial_1 f_0(x)) \in \mathbb{R}^q$
- $e_i(x) = \boldsymbol{e}_i(\mathfrak{S}_0(x), \partial_1 f_0(x)) \in \mathbb{R}^n$

- If $\mathscr{R} \subset J^1(U, N)$ then $e_i(x) = \mathbf{e}_i(\mathfrak{S}_0(x), \partial_1 f_0(x)) \in T_{f_0(x)} N$ and

$$CP_{\widetilde{\gamma}}(\mathfrak{S}_0, \partial_k, N)(x) = \exp_{f_0(x)} \left(\frac{1}{N} \sum_{i=1}^{p} C_i(a(x), Nx_k) e_i(x) \right)$$

Convex Integration without Integration

Proof.– We have

$$CP_{\tilde{\gamma}}(\mathfrak{S}_0, \partial_k, N)(x) = f_0(x) + \frac{1}{N}\int_{t=0}^{x_k} \gamma(x,t) - \overline{\gamma}(x)dt$$

where $\gamma(x,t) := \tilde{\gamma}(\mathfrak{S}_0(x), \partial_k f_0(x))(t)$ and $\overline{\gamma}(x) = \partial_k f_0(x)$.

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$$\gamma(x,t) := c(a(x),t)\cdot e(x)$$

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- Since $\mathfrak{\gamma}$ is c-shaped and surrounding, we have

$$\gamma(x,t) := c(a(x),t)\cdot e(x)$$

- Therefore

$$\begin{aligned}\int_{s=0}^{t} \gamma(x,s) - \overline{\gamma}(x)ds &= \left(\int_{s=0}^{t} c(a(x),s) - \overline{c}(a(x))ds\right)\cdot e(x)\\ &= C(a(x),t)\cdot e(x)\end{aligned}$$

☐

Convex Integration without Integration

- Here is an example of a Kuiper relation:

Theorem (Theillière).– *The differential relation of codimension one immersions is a Kuiper relation with respect to the loop pattern* $c : [0, \alpha_0] \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ *defined by*

$$c(\alpha, t) = \Big(\cos(\alpha \cos 2\pi t) - J_0(\alpha), \sin(\alpha \cos 2\pi t), 1\Big).$$

Convex Integration without Integration

Proof.– We do the proof for $N = \mathbb{R}^{m+1}$. Let

$$\mathscr{R} = \{\sigma = (x, y, v_1, \ldots, v_m) \mid (v_1, \ldots, v_m) \text{ are l.i. in } \mathbb{R}^{m+1}\}$$

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• We consider the direction ∂_1. Let $\sigma \in \mathscr{R}$ et $P = Span(v_2, ..., v_m)$. We have: $\dim\ P = m - 1$.

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- $\mathscr{R}(\sigma, \partial_1) = \{w \in \mathbb{R}^{m+1} \mid (x, y, w, v_2, ..., v_m) \in \mathscr{R}\} = \mathbb{R}^{m+1} \setminus P$

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- P is of codimension 2 in $\mathbb{R}^{m+1}$ therefore $\mathscr{R}(\sigma, \partial_1)$ is connected and its convex hull is $\mathbb{R}^{m+1}$

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- P is of codimension 2 in $\mathbb{R}^{m+1}$ therefore $\mathscr{R}(\sigma, \partial_1)$ is connected and its convex hull is $\mathbb{R}^{m+1}$
- $IntConv(\mathscr{R}, \partial_1) = \{(\sigma, w) \mid w \in IntConv\ \mathscr{R}(\sigma, \partial_1)\} = \mathscr{R} \times \mathbb{R}^{m+1}$

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Proof.– We do the proof for $N=\mathbb{R}^{m+1}$. Let

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- We consider the direction ∂_1. Let $\sigma\in\mathscr{R}$ et $P=Span(v_2,...,v_m)$. We have: $\dim\ P=m-1$.

- $\mathscr{R}(\sigma,\partial_1)=\{w\in\mathbb{R}^{m+1}\mid(x,y,w,v_2,...,v_m)\in\mathscr{R}\}=\mathbb{R}^{m+1}\setminus P$

- P is of codimension 2 in $\mathbb{R}^{m+1}$ therefore $\mathscr{R}(\sigma,\partial_1)$ is connected and its convex hull is $\mathbb{R}^{m+1}$

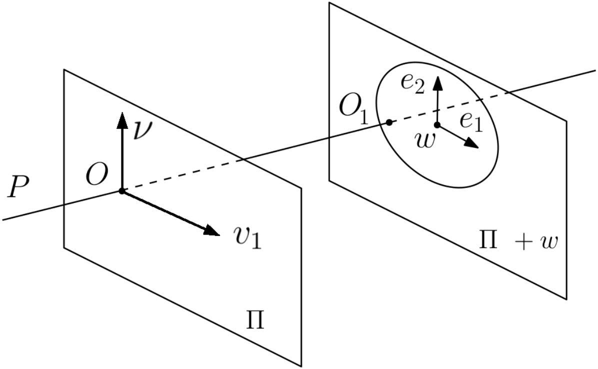
- $IntConv(\mathscr{R},\partial_1)=\{(\sigma,w)\mid w\in IntConv\ \mathscr{R}(\sigma,\partial_1)\}=\mathscr{R}\times\mathbb{R}^{m+1}$

- For every $(\sigma,w)\in\mathscr{R}\times\mathbb{R}^{m+1}$, we have to build (in a continuous way) a loop

$$\gamma(\sigma,w):\mathbb{R}/\mathbb{Z}\to\mathbb{R}^{m+1}\setminus P$$

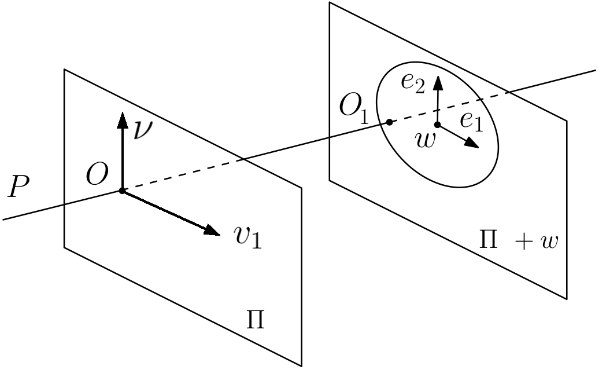
whose average is w.

Convex Integration without Integration



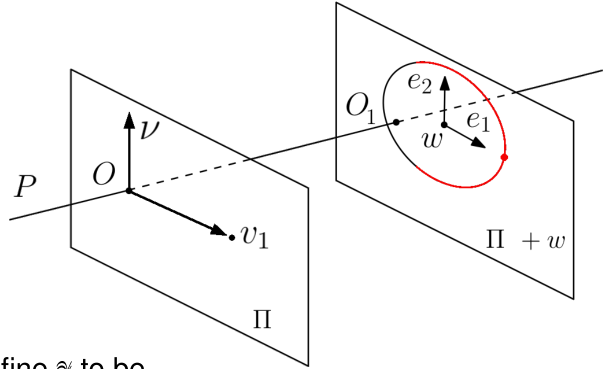
- Let $\nu = v_1 \wedge ... \wedge v_m$, $\prod = span(v_1, \nu)$ and $O_1 = P \cap (\prod + w)$.

Convex Integration without Integration



- Let $\nu = v_1 \wedge ... \wedge v_m$, $\prod = span(v_1, \nu)$ and $O_1 = P \cap (\prod + w)$.
- We choose $r = \sqrt{dist^2(w, O_1) + 1} = \sqrt{dist^2(w, P) + 1}$ and we put $\mathbf{e}_1 := r\frac{v_1}{\|v_1\|}$, $\mathbf{e}_2 := r\frac{\nu}{\|\nu\|}$ and $\mathbf{e}_3 := w$

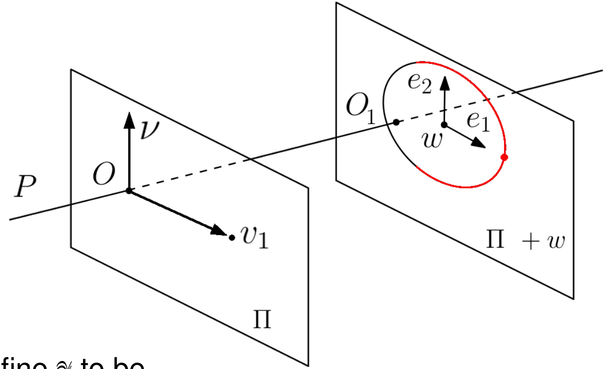
Convex Integration without Integration



We then define γ to be

$$\gamma(\sigma, w)(t) = (\cos(\alpha \cos 2\pi t) - J_0(\alpha))\, \mathbf{e}_1 + \sin(\alpha \cos 2\pi t)\, \mathbf{e}_2 + w$$

Convex Integration without Integration



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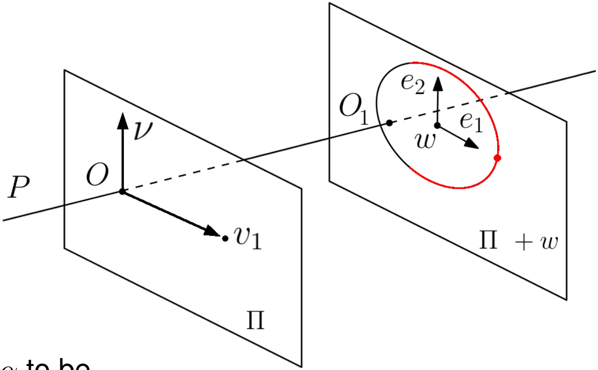
We could choose the function α to be $\alpha \equiv \alpha_0$. But, to ensure an (extra) relative property, we are going to do a different choice...

Convex Integration without Integration

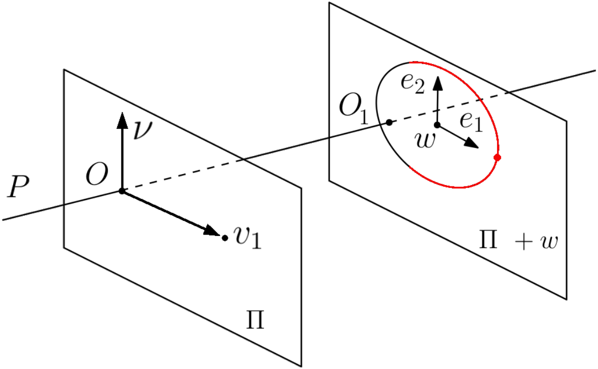


We choose α to be

- equal to α_0 if $dist(v_1, w) > \delta$,

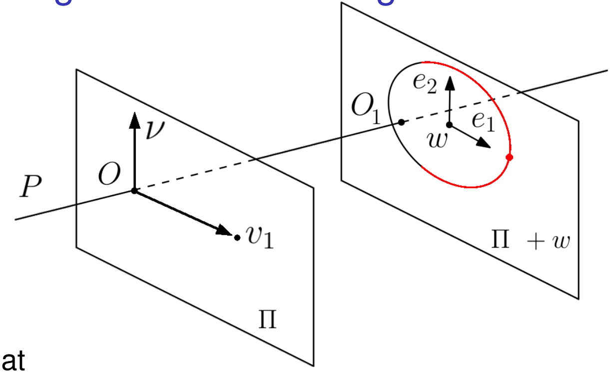
- and if not, $\alpha \in [0, \alpha_0[$ is chosen in such a way that $\tilde{\gamma}(\sigma, w) \equiv w$ if $v_1 = w$ and $dist(w, P) > \delta$.

Convex Integration without Integration



- The reason for a such choice is the following: if the component v_1 of the formal solution is already a solution, i. e. $v_1 = w$, we choose $\tilde{\gamma}(\sigma, w)$ to be constant equals to w except for the case where w is too close to the singular locus P.

Convex Integration without Integration



Observe that

(1) $t \mapsto (x, y, \gamma(\sigma, w)(t), v_2, ..., v_m) \in \mathscr{R}$

(2) $\int_0^1 \gamma(\sigma, w)(t)dt = w$

(3) An obvious homotopy joins v_1 to $\gamma(\sigma, w)(0)$

$\square$

Convex Integration without Integration

Expression of $f = CP_{\widetilde{\gamma}}(\mathfrak{S}_0, \partial_1, N)$: This expression is easily deduced from the proof of the theorem. If

$$\mathfrak{S}_0(x) = (x, f_0(x), v_1(x), ..., v_m(x))$$

then

$$f(x) = f_0(x) + \frac{r(x)}{N} K_c(\alpha(x), Nx_1)\, \mathbf{t}_0(x) + \frac{r(x)}{N} K_s(\alpha(x), Nx_1)\, \mathbf{n}(x)$$

with

$$K_c(\alpha, t) := \int_{u=0}^{t} \Big(\cos(\alpha \cos 2\pi u) - J_0(\alpha) \Big) du$$

$$K_s(\alpha, t) := \int_{u=0}^{t} \sin(\alpha \cos 2\pi u) du$$

and $\mathbf{t}_0 := \frac{v_1}{\|v_1\|}$, $\mathbf{n} := \frac{v_1 \wedge \ldots \wedge v_m}{\|v_1 \wedge \ldots \wedge v_m\|}$, $r = \sqrt{dist^2(\partial_1 f_0, P) + 1}$, $P = Span(v_2, ..., v_m)$.

Circle shaped loops and codimension one immersions

Exercice.– We consider the relation $\mathscr{R}$ of immersions of M^m into $\mathbb{R}^{m+1}$.

1) Prove that this relation is Kuiper with respect to the constant loop pattern $c : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ defined by $c(t) = (\cos 2\pi t, \sin 2\pi t, 1)$ and $\mathbf{e}_1 = r\frac{v_1}{\|v_1\|}$, $\mathbf{e}_2 = r\frac{v_1\wedge\ldots\wedge v_m}{\|v_1\wedge\ldots\wedge v_m\|}$, $\mathbf{e}_3 = w$ with r any C^1 function greater than $dist(P, w)$ (the $\mathbf{e}_i$'s are given for the direction ∂_1).

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2) We consider $M = \mathbb{R}/\mathbb{Z} \times [-1, 1]$ and the map $f_0 : M \to \mathbb{R}^3$ given by

$$f_0(x_1, x_2) = (x_2\cos(2\pi x_1), x_2\sin(2\pi x_1), x_2).$$

Check that f_0 fails to be an immersion along $\mathbb{S}^1 \times \{0\}$ and describe its image.

Circle shaped loops and codimension one immersions

3) Let $\mathfrak{S}_0$ be the section of $J^1(M,\mathbb{R}^3)$ defined by

$$x=(x_1,x_2)\longmapsto\mathfrak{S}_0(x)=(x,f_0(x),v_1(x),\partial_2 f_0(x))$$

where $v_1(x)=(-\sin(2\pi x_1),\cos(2\pi x_1),0)$. Show that $\mathfrak{S}_0$ is a formal solution of $\mathscr{R}$.

Circle shaped loops and codimension one immersions

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where $v_1(x)=(-\sin(2\pi x_1),\cos(2\pi x_1),0)$. Show that $\mathfrak{S}_0$ is a formal solution of $\mathscr{R}$.

4) Let $\mathbf{r}(\sigma,w)=\sqrt{1+\|w\|^2}$. Show that $\mathbf{r}(\sigma,w)>dist(w,P)$ where $P=Span(v_2)$ (we recall that $\sigma=(x,y,v_1,v_2)$).

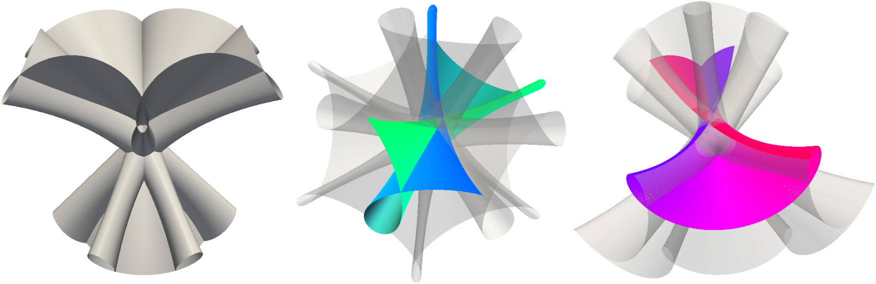
Circle shaped loops and codimension one immersions

5) Show that $f_1 = CP_{\widetilde{\gamma}}(\mathfrak{S}_0, \partial_1, N)$ has the following expression:

$$\begin{aligned} f_1(x) &= x_2 \begin{pmatrix} \cos(2\pi x_1) \\ \sin(2\pi x_1) \\ 1 \end{pmatrix} + \frac{\sqrt{4\pi^2 x_2^2 + 1}}{2\pi N} \sin(2\pi N x_1) \begin{pmatrix} -\sin(2\pi x_1) \\ \cos(2\pi x_1) \\ 0 \end{pmatrix} \\ &+ \frac{\sqrt{4\pi^2 x_2^2 + 1}}{2\sqrt{2}\pi N} (1 - \cos(2\pi N x_1)) \begin{pmatrix} \cos(2\pi x_1) \\ \sin(2\pi x_1) \\ -1 \end{pmatrix} \end{aligned}$$

for every $x = (x_1, x_2) \in \mathbb{R}/\mathbb{Z} \times [-1, 1]$.

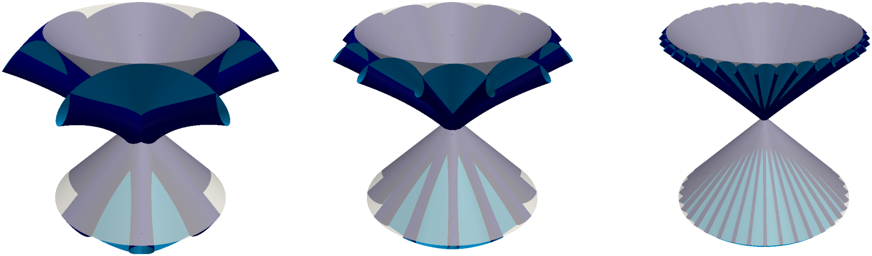
Circle shaped loops and codimension one immersions



Desingularisation of a cone obtained by $f_1 = CP_{\tilde{\gamma}}(\mathfrak{S}_0, \partial_1, N = 6)$ and

with $\mathbf{r}(\sigma, w) = 2\pi + \dfrac{\|w\|^2}{2\pi}$.

Circle shaped loops and codimension one immersions



The C^0-density phenomenon: Corrugation Process with
$N = 6,\ 12,\ 36.$

Mélanie Theillière