Regularity of limit sets of embedded Poincaré Disks

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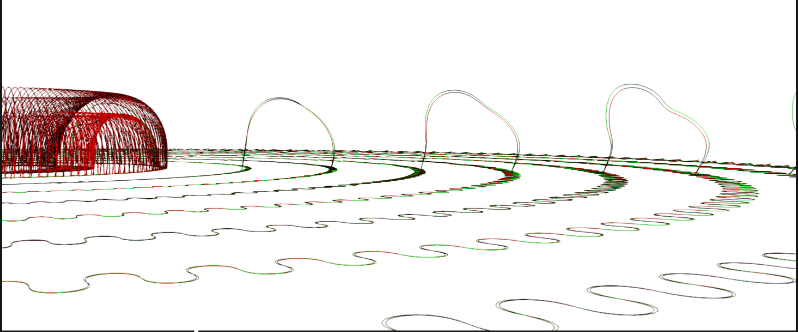
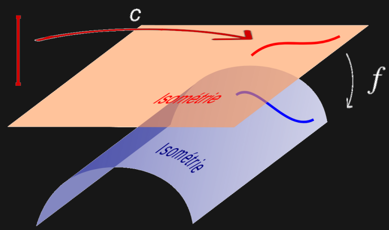


Isometric embeddings



- A C^1 map

$$f : (M^n, g) \xrightarrow{C^1} \mathbb{E}^q$$

between a riemannian manifold (M^n, g) and an Euclidean space $\mathbb{E}^q = (\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ is **isometric** it preserves the length of curves *i. e.*

$$Length(f \circ c) = Length(c)$$

for every C^1 piecewise parametrized curve $c : [0, 1] \longrightarrow M^n$.

Isometric embeddings

• In a coordinate system, the isometric condition amounts to solve a **non linear PDE** system :

$$\text{For all} \quad 1 \leq i \leq j \leq n, \qquad \left\langle \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j} \right\rangle = g_{ij}$$

of $s_n = \frac{n(n+1)}{2}$ equations. The number s_n is called the *Janet's dimension*.

Janet-Cartan Theorem (1926-27).– *Let (M^n, g) be a real-analytic Riemannian manifold. Every point of M has a neighborhood which has a real-analytic isometric embedding into $\mathbb{E}^q$ with $q = s_n$.*

Nash-Kuiper C^1 Embedding Theorem

John Forbes Nash and Nicolaas Kuiper

Définition.– A map $f : (M^n, g) \xrightarrow{C^1} \mathbb{E}^q$ is said to be *strictly short* if

$$f^*\langle .,. \rangle \leq \lambda g \quad \text{for some} \quad 0 < \lambda < 1.$$

Nash-Kuiper C^1 Embedding Theorem

Theorem (1954-55-59)– *Let* $f_0 : (M^n, g) \xrightarrow{C^1} \mathbb{E}^q$ *be a strictly short embedding of a Riemannian manifold. Then, for every* $\epsilon > 0$, *there exists a* C^1 *isometric embedding* $f : (M^n, g) \longrightarrow \mathbb{E}^q$ *such that* $\|f - f_0\|_{C^0} \leq \epsilon$.

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• Nash-Kuiper C^1 embedding theorem shows that the barrier of the Janet's dimension can be broken provided that the regularity is (drastically) lowered.

Nash C^∞ Embedding Theorem

- The 1954 result was the first step to prove the C^∞ *Nash embedding theorem* :

Theorem (Nash 1956, Gromov 1986, Gunther 1989).– *Every Riemannian manifold* (M^n, g) *can be* C^∞ *isometrically embedded into some Euclidean space* $\mathbb{E}^q$*. If* M^n *is compact then one can choose* $q = \max\{s_n + 2n, s_n + n + 5\}$.

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• The Janet dimension barrier is observed.

Conjectural Threshold

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• Recently, Cao, De Lellis and Inauen have shown the criticality of the Hölder space $C^{1,\frac{1}{2}}$ for polar caps and isometric extensions.

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• In the 1955 and 1959 papers, Kuiper focused on the construction of closed isometric embeddings.

Isometric embeddings with void limit set

Theorem (Kuiper, 1959).– *If there exists a closed strictly short embedding* $f_0 : (M^n, g) \to \mathbb{E}^q$ *then there exists a closed* C^1*-isometric embbeding* $f : (M^n, g) \to \mathbb{E}^q$.

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Theorem (Hilbert, 1901- Efimov, 1964).– *No surface can be* C^2 *immersed in Euclidean 3-space so as to be complete in the induced Riemannian metric, with Gauss curvature* $K \leq const < 0$*.*

• In particular, there is no C^2 isometric immersion of a hyperbolic plane into $\mathbb{E}^3$.

Gromov Boundary of Hyperbolic n-space

- The *Gromov boundary* $\partial_\infty \mathbb{H}^n$ of $\mathbb{H}^n$ is the set of all the geodesic rays $c : [0, \infty) \to \mathbb{H}^n$, where we regard two geodesic rays c and c' as the same if the Hausdorff distance between then is finite. The equivalence class of a geodesic ray c is denoted by $c(\infty)$.

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• A topology is defined on $\partial_\infty \mathbb{H}^n$ by saying that x_n converges to x if there is a sequence of rays (c_n) with $c_n(\infty) = x_n$ which converges uniformly on compact sets to a ray c satisfying $c(\infty) = x$.

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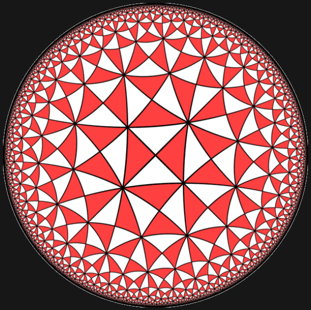
- The notion of Gromov boundary is in fact far more general and applies to δ-hyperbolic spaces (especially hyperbolic groups).

Our Result



- Let $D^2 = \{(x_1, x_2) \in \mathbb{R}^2, x_1^2 + x_2^2 \leq 1\}$ and $h = \dfrac{dx_1^2 + dx_2^2}{(1 - x_1^2 - x_2^2)^2}$. We denote by (*Int* D^2, h) the Poincaré Disk model of the Hyperbolic plane $\mathbb{H}^2$.

Our Result

Theorem (Hevea Team).– *There exists an embedding* $f : D^2 \to \mathbb{E}^3$ *of the closed unit disk into the Euclidean 3-space which is* β*-Hölder for any* $0 < \beta < 1$ *and whose restriction to the interior of the disk* $\mathit{Int}\, D^2$ *is a* C^1*-isometric embedding of the Hyperbolic plane* $\mathbb{H}^2 = (\mathit{Int}\, D^2, h)$.

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• Let L_∞ be the limit set of the restriction $f_{|Int\ D^2}$ to the interior of the disk. Since f is a β-Hölder embedding for any $0 < \beta < 1$, the limit set L_∞ is an embedded circle of Hausdorff dimension one.

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• Although any point of L_∞ lies in the closure of $f(Int\ D)$, each of them is at infinite distance of every other point of $f(Int\ D)$ for the induced distance of $\mathbb{E}^3$.

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- Since any isometry of $\mathbb{H}^2$ extends continuously to a homeomorphism of the boundary $\partial_\infty \mathbb{H}^2$, the same is true for the image by f of this isometry.

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- The proof of the theorem relies on an explicit construction based on the Nash-Kuiper approach.

The Hevea Team (current membership)

Roland Denis

Francis Lazarus

Mélanie Theillière

Boris Thibert

HEVEA : H-principlE, Visualization & Applications

A quotation from Gromov (2015)

"The above may seem shamefully easy ; you may smile at the geometers who, for years, tried to prove that isometric C^1-immersions, say of surfaces with C^∞- metrics g with positive curvatures to the 3-space, must be C^∞-smooth ; hence, convex. Well, mathematics teaches us humility ; another Nash may come up with something equally "obvious" that you have believed all your life to be impossible."

The Nash-Kuiper strategy

- Let $\mathcal{F}_0 : (Int\ D^2, h) \to \mathbb{E}^3$ be a (strictly) short embedding and let

$$\Delta_0 := h - f_0^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3} > 0$$

be its isometric default.

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• Let $(g_k)_{k\geq 1}$ be the sequence of increasing intermediary metrics given by

$$g_k := f_0^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3} + \left(1 - \frac{1}{2^k}\right)\Delta_0.$$

This sequence obviously converges toward h.

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• The Nash-Kuiper method builds iteratively a sequence of C^1 converging embeddings

$$\mathcal{F}_0, \quad \mathcal{F}_{1,1}, \mathcal{F}_{1,2}, \mathcal{F}_{1,3}, \quad \mathcal{F}_{2,1}, \mathcal{F}_{2,2}, \mathcal{F}_{2,3}, \quad ...$$

such that

$$\mathcal{F}_{k,3}^*\langle \cdot, \cdot \rangle_{\mathbb{R}^3} \approx g_k \quad \text{and} \quad \Delta_k := g_{k+1} - \mathcal{F}_{k,3}^*\langle \cdot, \cdot \rangle_{\mathbb{R}^3} > 0.$$

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- We choose three (constant) linear forms ℓ_1, ℓ_2 and ℓ_3 to write Δ_1 as a sum of three squares

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- The existence of such a decomposition follows from the fact that the vector space of symmetric bilinear forms over $\mathbb{R}^2$ is of dimension three.

The Nash-Kuiper strategy

- The map $\mathcal{F}_{1,1}$ is built so that to kill the term η_1 in the decomposition

$$\Delta_1 = \eta_1\, \ell_1 \otimes \ell_1 + \eta_2\, \ell_2 \otimes \ell_2 + \eta_3\, \ell_3 \otimes \ell_3.$$

Precisely

$$g_1 - \mathcal{F}_{1,1}^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3} \approx \eta_2\, \ell_2 \otimes \ell_2 + \eta_3\, \ell_3 \otimes \ell_3.$$

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• The map $\mathcal{F}_{1,2}$ is then built from $\mathcal{F}_{1,1}$ to kill the term η_2 :

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• And similarly, the map $\mathcal{F}_{1,3}$ is built from $\mathcal{F}_{1,2}$ to kill the last term so

that :

$$g_1 \approx \mathcal{F}_{1,3}^{*}\langle\cdot,\cdot\rangle_{\mathbb{R}^3}.$$

The Nash-Kuiper strategy

- Since $g_2 > g_1$, if the approximation $g_1 \approx \mathcal{F}_{1,3}^{*}\langle\cdot,\cdot\rangle_{\mathbb{R}^3}$ is good enough, we can deduce

$$g_2 - \mathcal{F}_{1,3}^{*}\langle\cdot,\cdot\rangle_{\mathbb{R}^3} > 0$$

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• We then can continue the process and build $\mathcal{F}_{2,1}, \mathcal{F}_{2,2}, \mathcal{F}_{2,3}$, and so on.

• It turns out that the parameters of this construction can be chosen so that

$$\sum \|d\mathcal{F}_{k+1,3} - d\mathcal{F}_{k,3}\| \leq Cte. \sum \|g_{k+1} - g_k\|^{\frac{1}{2}}$$

Thus the convergence of the right-hand side implies the C^1 convergence of the $\mathcal{F}_{k,3}$'s towards a C^1-isometric map $\mathcal{F}_\infty$.

The Nash-Kuiper strategy

• To sum up, all the process relies on the following Elementary Step which is used iteratively :

Elementary Step.– Let f be an immersion, η be and a positive function, ℓ be a linear form and μ the metric defined by

$$\mu = f^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3} + \eta\,\ell\otimes\ell.$$

For each $\epsilon > 0$, build an immersion F such that

i) $\|F - f\|_{C^0} \leq \epsilon$

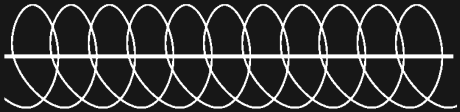
ii) $\|\mu - F^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3}\| \leq \epsilon$

• Observe that the first condition is needed to ensure the C^0 convergence of the sequence $(\mathcal{F}_{k,i})$.

• The second condition requires to elongate f in the "direction" ℓ.

The Nash-Kuiper strategy

- Nash uses a spiralling process to achieve the Elementary Step along a curve in the "direction" ℓ (needs to be in codimension at least 2).



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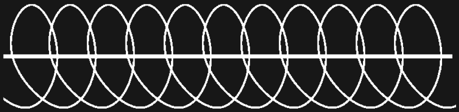
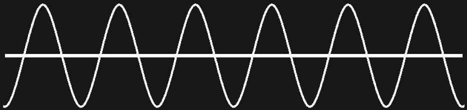
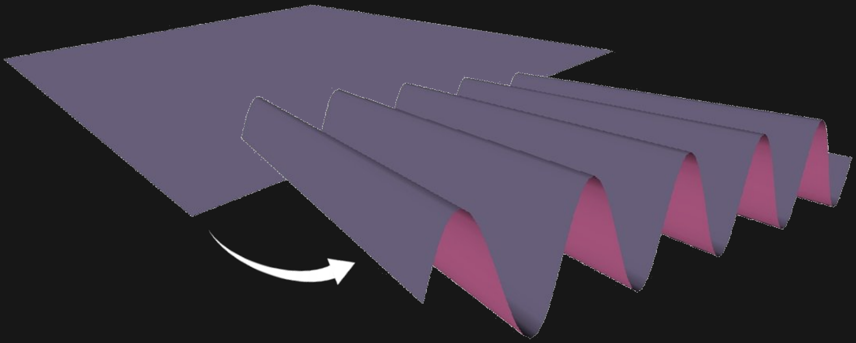


• The Kuiper uses oscillations to achieve the same task in codimension 1.

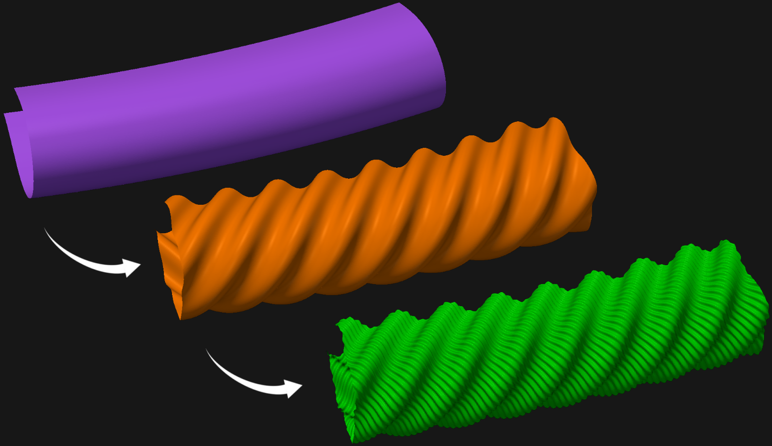


The Nash-Kuiper strategy



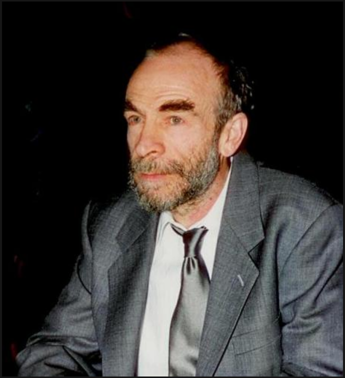
- Kuiper oscillations : the initial surface is foliated by curves in the "direction" ℓ and each curve is deformed through oscillations to create a corrugated surface.

The Nash-Kuiper strategy



- When the process is iterated in different directions, the oscillations are piling up while the number of corrugations increases rapidly.

Convex Integration Theory



Mikhaïl Gromov

- In the late 60's, Gromov turns the Nash and Kuiper method into a general procedure to solve a large class of differential relations : the Convex Integration Theory

Convex Integration

- The theory relies on an Elementary Step similar to the one of Nash and Kuiper.
- It uses an *corrugation process* to solve this Elementary Step.
- The theory has different variations depending on the corrugation process that is taken into account (the original one of Gromov, the one of Eliashberg and Mishachev, the Anzast of De Lellis and Székelyhidi, etc.)
- Here we shall use a recent variation discovered by Theillière.
- One of the advantages of this version is that it allows at coordinate-free expression.

Back to the Elementary Step

Elementary Step.– Let f be an immersion, η be and a positive function, ℓ be a linear form and μ the metric defined by

$$\mu = f^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3} + \eta\,\ell\otimes\ell.$$

For each $\epsilon > 0$, build an immersion F such that

i) $\|F - f\|_{C^0} \leq \epsilon$

ii) $\|\mu - F^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3}\|_{C^0} \leq \epsilon$

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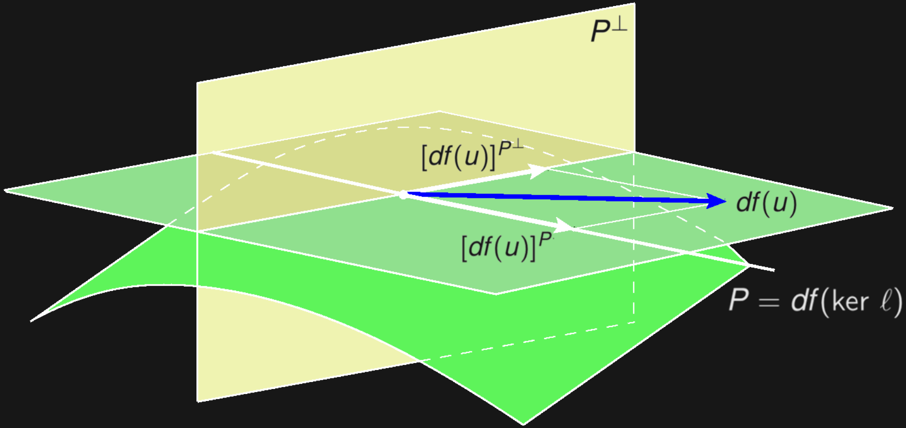
• We are going to reformulate this Elementary Step in a more tractable form.

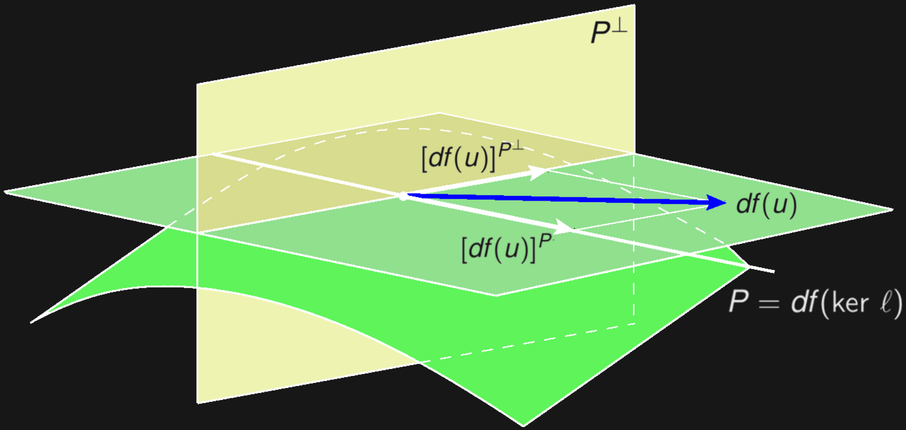
Back to the Elementary Step



- We set $P = df(\ker \ell)$ and we decompose df over $P \oplus P^\perp = \mathbb{E}^3$:

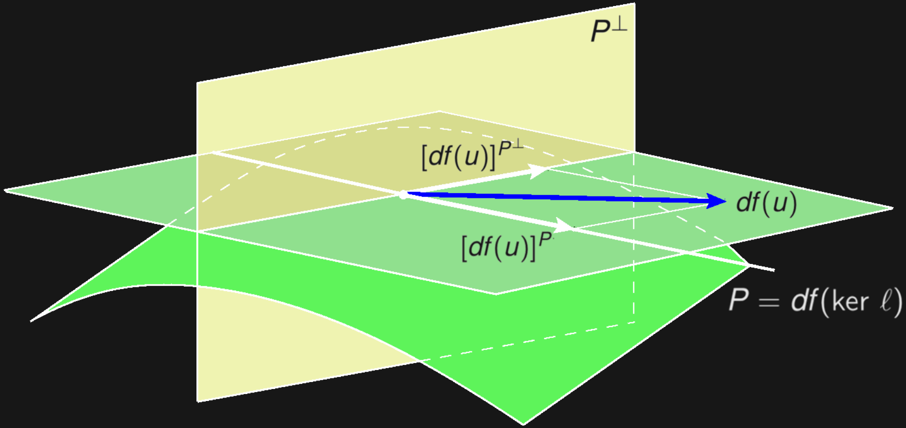
$$df = [df]^P + [df]^{P^\perp}.$$

Back to the Elementary Step



- Since $Im\, df \cap P^\perp \simeq \mathbb{R}$, we can see $[df]^{P^\perp}$ as a linear form.

Back to the Elementary Step



- This form vanishes on $\ker \ell$. It is thus proportional to ℓ :

$$[df]^{P^{\perp}} = [df(u)]^{P^{\perp}} \otimes \ell \quad \text{where } u \text{ is such that } \ell(u) = 1.$$

Back to the Elementary Step

- From $df = [df]^P \oplus^\perp [df]^{P^\perp}$, we obviously have

$$f^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3} = \left([df]^P\right)^* \langle\cdot,\cdot\rangle_{\mathbb{R}^3} + \left([df]^{P^\perp}\right)^* \langle\cdot,\cdot\rangle_{\mathbb{R}^3}$$

and from $[df]^{P^\perp} = [df(u)]^{P^\perp} \otimes \ell$, we deduce

$$f^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3} = \left([df]^P\right)^* \langle\cdot,\cdot\rangle_{\mathbb{R}^3} + \|[df(u)]^{P^\perp}\|^2 \ell \otimes \ell.$$

Back to the Elementary Step

- From $df = [df]^P \oplus^\perp [df]^{P\perp}$, we obviously have

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$$f^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3} = \left([df]^P\right)^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3} + \|[df(u)]^{P\perp}\|^2\ell\otimes\ell.$$

- The target metric

$$\mu = f^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3} + \eta\,\ell\otimes\ell$$

thus can be re-written as

$$\mu = \left([df]^P\right)^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3} + (\|[df(u)]^{P\perp}\|^2 + \eta)\ell\otimes\ell.$$

Back to the Elementary Step

- The target metric

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thus can be re-written as

$$\mu = \left([df]^P \right)^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3} + (\|[df(u)]^{P^\perp}\|^2 + \eta) \ell \otimes \ell.$$

- Under that form, μ appears to be a pullback

$$\mu = \Phi^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3} \quad \text{with} \quad \Phi = [df]^P + \left(\|[df(u)]^{P^\perp}\|^2 + \eta \right)^{\frac{1}{2}} \boldsymbol{\nu} \otimes \ell$$

and where $\boldsymbol{\nu}$ is any map onto the unit circle $U = \mathbb{S}^2(1) \cap P^\perp$.

Back to the Elementary Step

Reformulation of the Elementary Step.– Build a new map F such that

i) F and f are C^0 close :

$$\|F - f\|_{C^0} \leq \epsilon$$

ii) dF and $[df]^P + \gamma \otimes \ell$ are C^0 close :

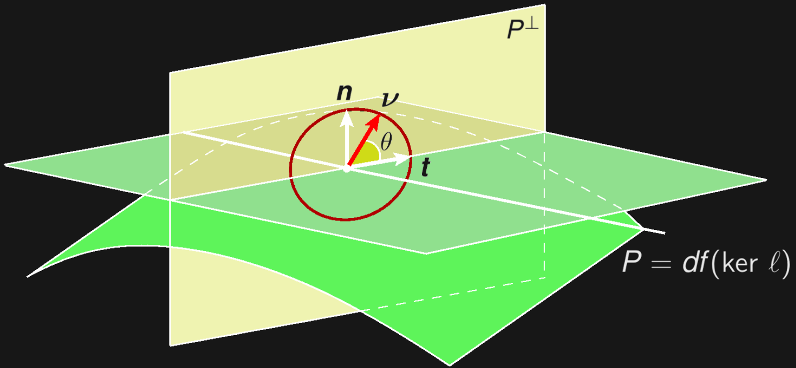
$$\|dF - ([df]^P + \gamma \otimes \ell)\|_{C^0} \leq \epsilon$$

with

$$\gamma := r\nu \quad \text{and} \quad r := \left(\|[df(u)]^{P^\perp}\|^2 + \eta\right)^{\frac{1}{2}}$$

and ν is any map onto $U = \mathbb{S}^2(1) \cap P^\perp$.

Back to the Elementary Step



- Let $\boldsymbol{n}$ = unit normal and $\boldsymbol{t} = \dfrac{[df(u)]^{P^{\perp}}}{\|[df(u)]^{P^{\perp}}\|}$.
- We have $P^{\perp} = Span(\boldsymbol{n}, \boldsymbol{t})$ and thus, for some θ :

$$\boldsymbol{\nu} = \cos\theta\, \boldsymbol{t} + \sin\theta\, \boldsymbol{n}.$$

Theillière's Corrugation Process

Theillière's corrugation process : Given a linear form ℓ, a number $N \in \mathbb{N}_{\geq 1}$ and $x \mapsto \gamma(x, \cdot)$ be any family of loops :

$$\gamma(x, \cdot) : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{E}^3$$

define F to be

$$F(x) := f(x) + \frac{1}{N} \int_0^{N\ell(x)} \gamma(x, s) - \overline{\gamma}(x) \, \mathrm{d}s$$

where

$$\overline{\gamma}(x) := \int_0^1 \gamma(x, s) \, \mathrm{d}s$$

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- Since $s \mapsto \gamma(x,s) - \overline{\gamma}(x)$ is 1-periodic and with vanishing average, we have over compact sets

$$F = f + O\left(\frac{1}{N}\right)$$

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where

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- Differentiating, we have over compact sets

$$dF_x = df_x + (\gamma(x, N\ell(x)) - \overline{\gamma}(x)) \otimes \ell + O\left(\frac{1}{N}\right)$$

Theillière's Corrugation Process

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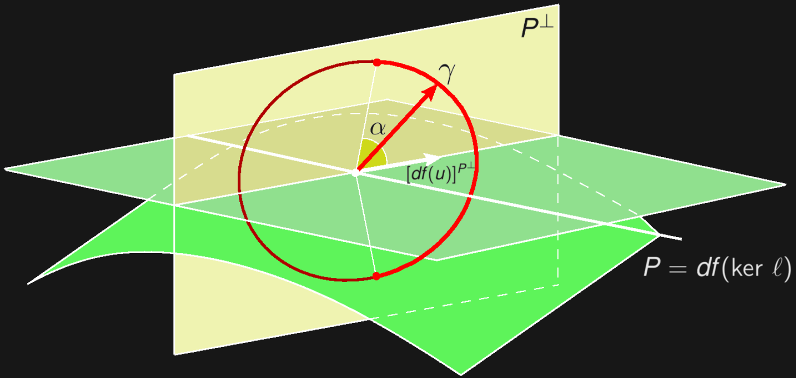
$$F(x) := f(x) + \frac{1}{N}\int_0^{N\ell(x)} \gamma(x,s) - \overline{\gamma}(x)\, ds.$$

- If moreover γ is chosen such that
$$\overline{\gamma}(x) = [df_x(u)]^{P^\perp}$$
then
$$\overline{\gamma} \otimes \ell = [df]^{P^\perp}$$
and over compact sets
$$dF_x = [df_x]^P + \gamma(x, N\ell(x)) \otimes \ell + O\left(\frac{1}{N}\right)$$

Theillière's Corrugation Process



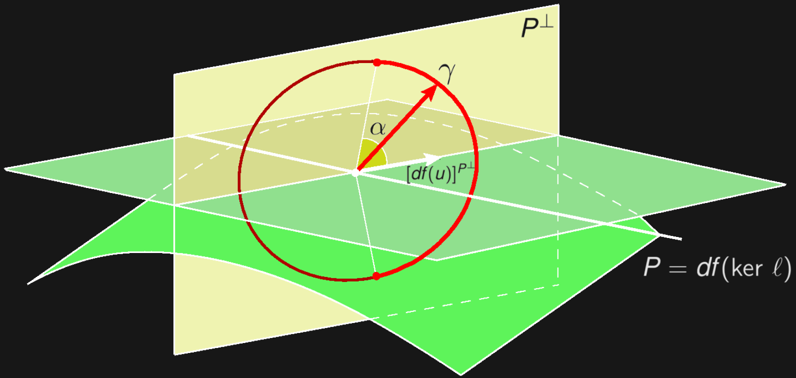
- We choose

$$\gamma(x,s) := r(x)\nu(x,s)$$

with

$$\nu(x,s) = \cos\theta(x,s)\,\boldsymbol{t}(x) + \sin\theta(x,s)\,\boldsymbol{n}(x).$$

Theillière's Corrugation Process



- and $\theta(x,s) = \alpha(x)\cos 2\pi s$.

$$\gamma(x,s) := r(x)\left(\cos\theta(x,s)\,\boldsymbol{t}(x) + \sin\theta(x,s)\,\boldsymbol{n}(x)\right).$$

Theillière's Corrugation Process

- Observe that

$$\overline{\gamma}(x) = \int_0^1 \gamma(x,s)ds = r(x)\left(\int_0^1 \cos(\alpha(x)\cos 2\pi s)ds\right)\boldsymbol{t}(x)$$

By choosing $\alpha(x)$ such that

$$\int_0^1 \cos(\alpha(x)\cos 2\pi s)ds = \frac{\|[df(u)]^{P^\perp}\|}{r(x)}$$

we obtain

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• With such a choice, the map F obtained by the Theillière's Integration Process solves the Elementary Step.

Theillière's Corrugation Process

- For short, we write

$$F = CP(f, \ell, \eta, N)$$

Theillière's Corrugation Process

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$$F = CP(f, \ell, \eta, N)$$
- The map F has the following expression

$$F(x) = f(x) + \frac{r(x)}{N}\left(\Gamma_1(\alpha(x), N\ell(x))\boldsymbol{t}(x) + \Gamma_2(\alpha(x), N\ell(x))\boldsymbol{n}(x)\right)$$

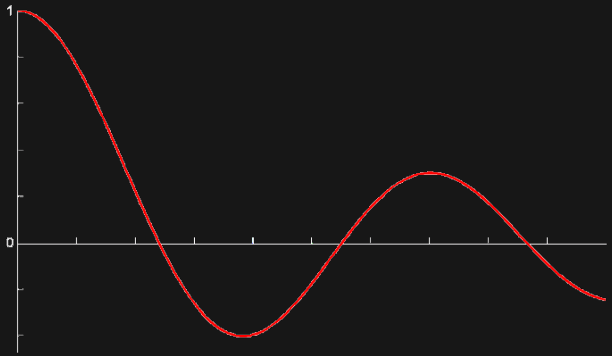
where

$$\begin{aligned}\Gamma_1(\alpha, t) &= \int_0^t \left(\cos(\alpha\cos 2\pi s) - J_0(\alpha)\right) ds\\ \Gamma_2(\alpha, t) &= \int_0^t \sin(\alpha\cos 2\pi s) ds\end{aligned}$$

and J_0 is the 0-th Bessel function of the first kind

$$J_0(\alpha) = \int_0^1 \cos(\alpha\cos 2\pi s)\, ds.$$

Theillière's Corrugation Process

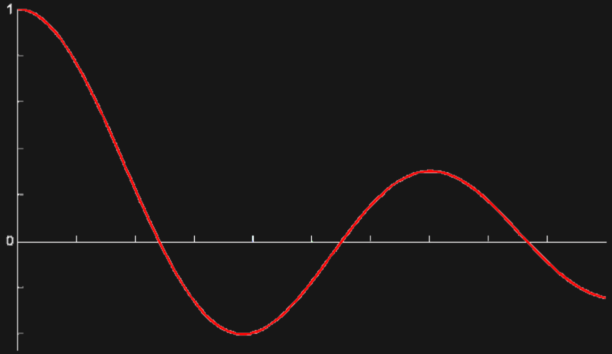


Graph of the Bessel function J_0

- The condition for the choice of α

$$\int_0^1 \cos(\alpha(x)\cos 2\pi s)ds = \frac{\|[df(u)]^{P^\perp}\|}{r(x)} \iff J_0(\alpha(x)) = \frac{\|[df(u)]^{P^\perp}\|}{r(x)}$$

Theillière's Corrugation Process

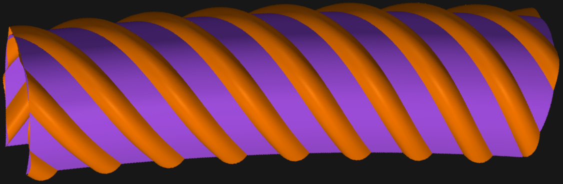


Graph of the Bessel function J_0

- In particular

$$0 < \alpha(x) < \kappa_0 = 2.40\text{...the first positive root of } J_0.$$

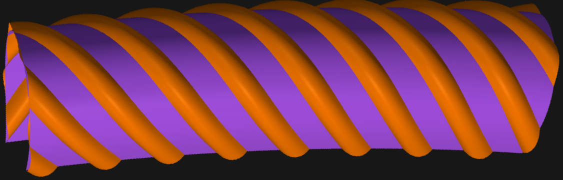
Theillière's Corrugation Process



The maps f (purple) and F (orange)

- Let $V := \boldsymbol{n} \wedge \boldsymbol{t}$. The basis $(\boldsymbol{t}, V, \boldsymbol{n})$ of $\mathbb{E}^3$ is direct and orthonormal.

Theillière's Corrugation Process



The maps f (purple) and F (orange)

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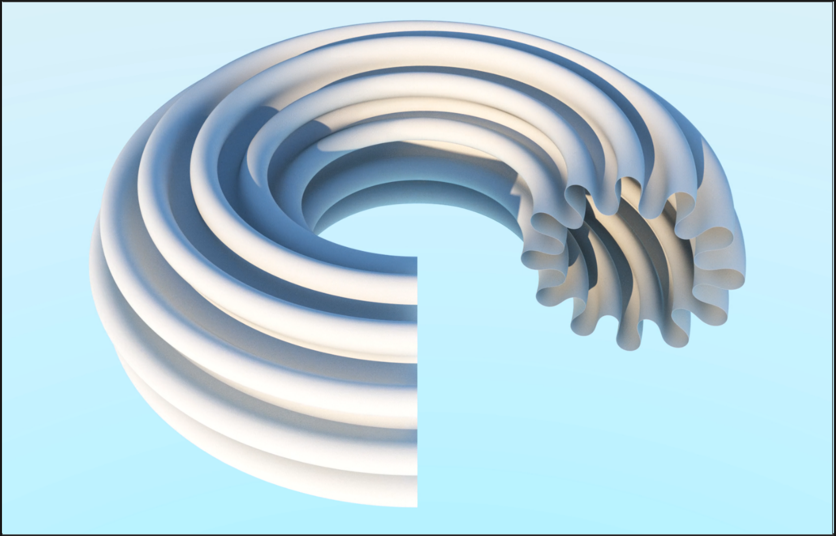
- By denoting $\Gamma := (\Gamma_1, 0, \Gamma_2)$, we can shorten the expression of F :

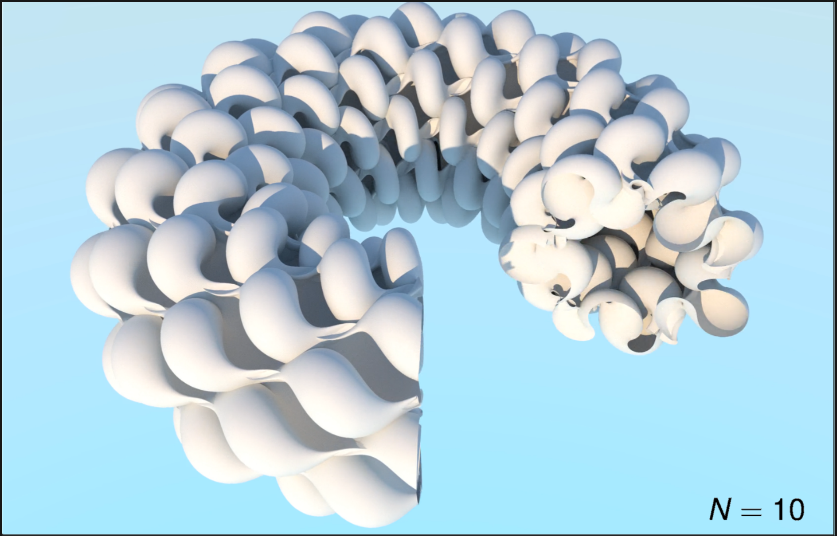
$$\begin{aligned} F &= f + \frac{r}{N}\Gamma_1(\alpha, N\ell)\,\boldsymbol{t} + \frac{r}{N}\Gamma_2(\alpha, N\ell)\,\boldsymbol{n} \\ &= f + \frac{r}{N}\Gamma(\alpha, N\ell) \cdot \begin{pmatrix} \mathbf{t} \\ V \\ \mathbf{n} \end{pmatrix} \end{aligned}$$

Theillière’s Corrugation Process



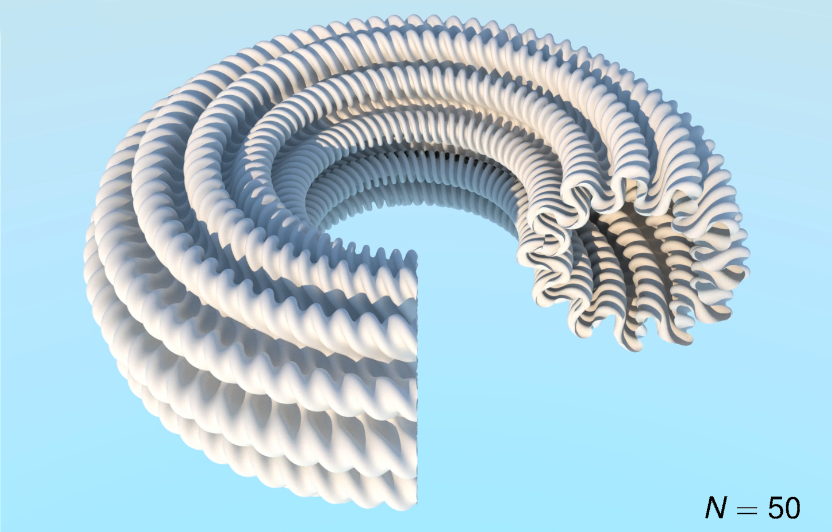
An initial (corrugated) map f

Theillière's Corrugation Process



The map F

Theillière's Corrugation Process



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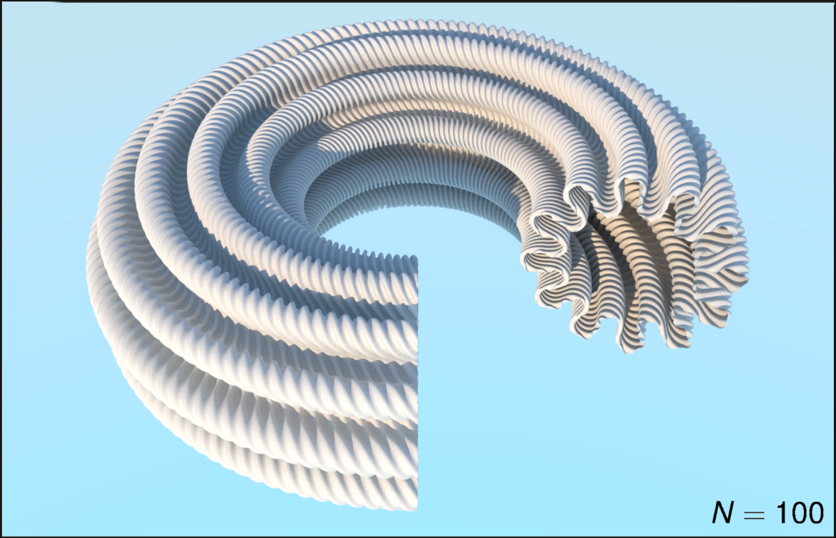

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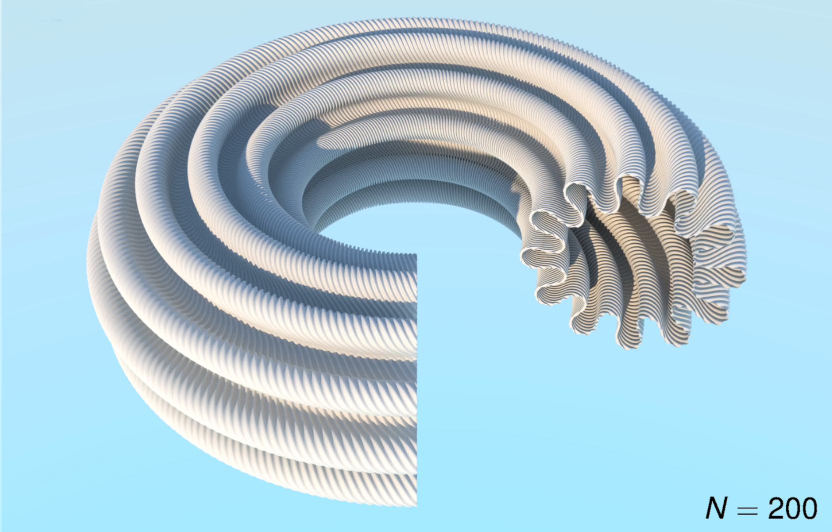

The map F

Kuiper construction of embeddings of $\mathbb{H}^2 \hookrightarrow \mathbb{E}^3$

- The non compactness of the hyperbolic space $\mathbb{H}^2$ is an issue since we need to work over compact sets to solve the Elementary Step.

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• Here, we are going to avoid the use of an exhaustion to obtain a direct control of L_∞.

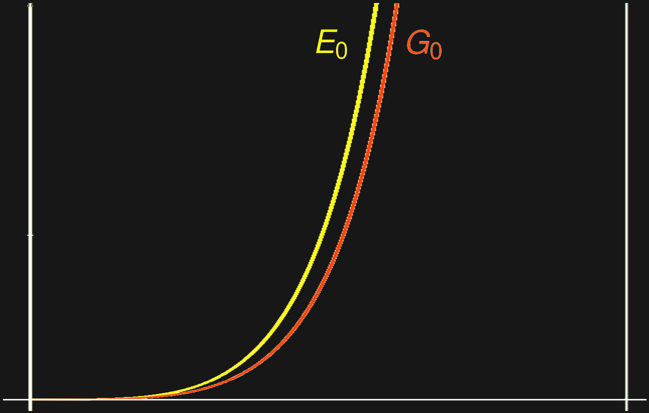
Construction of our embedding of $\mathbb{H}^2 \hookrightarrow \mathbb{E}^3$

Image

- We use polar coordinates (ρ, φ) on the closed unit disk D^2 and choose the following initial map

$$\mathcal{F}_0(\rho, \varphi) = 2\left(\rho \cos\varphi, \rho \sin\varphi, \frac{\sqrt{2}}{2}\rho^2\right).$$

Construction of our embedding of $\mathbb{H}^2 \hookrightarrow \mathbb{E}^3$

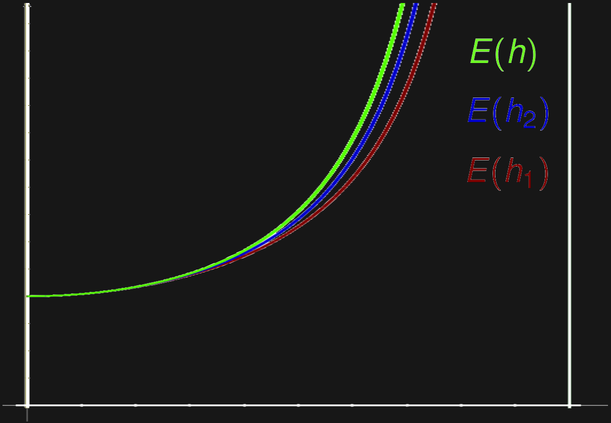


- The isometric default

$$\Delta_0 = h - \mathcal{F}_0^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3} = E_0(\rho) d\rho^2 + G_0(\rho) d\varphi^2$$

explodes at $\rho = 1$.

Construction of our embedding of $\mathbb{H}^2 \hookrightarrow \mathbb{E}^3$



- And so do the Nash-Kuiper intermediary metrics $h_k \uparrow h$:

$$h_k := \mathcal{F}_0^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3} + \left(1 - \frac{1}{2^k}\right) \Delta_0 = E(h_k) d\rho^2 + G(h_k) d\varphi^2.$$

Construction of our embedding of $\mathbb{H}^2 \hookrightarrow \mathbb{E}^3$

- We define our intermediary metrics (g_k) by considering the Taylor series of $\Delta_0 = E_0 d\rho^2 + G_0 d\varphi^2$:

$$E_0 = 4\sum_{n=1}^{\infty}(n+2)\rho^{2(n+1)}, \quad G_0 = 4\sum_{n=1}^{\infty}(n+1)\rho^{2(n+1)}$$

and by defining intermediary isometric defaults

$$\Delta_k = E_k d\rho^2 + G_k d\varphi^2$$

through the sucessive Taylor polynomials of E_0 and G_0 :

$$E_k = 4\sum_{n=1}^{k}(n+2)\rho^{2(n+1)}, \quad G_k = 4\sum_{n=1}^{k}(n+1)\rho^{2(n+1)}$$

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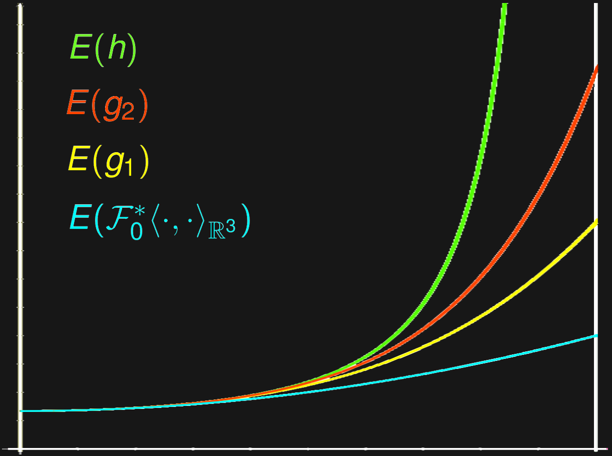
- We then set

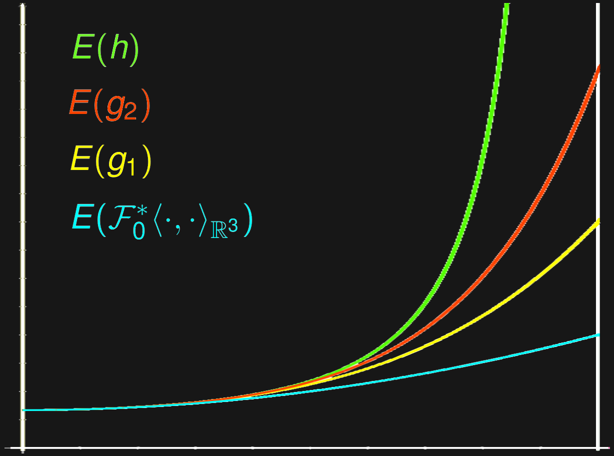
$$g_k := \mathcal{F}_0^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3} + \Delta_k.$$

Construction of our embedding of $\mathbb{H}^2 \hookrightarrow \mathbb{E}^3$



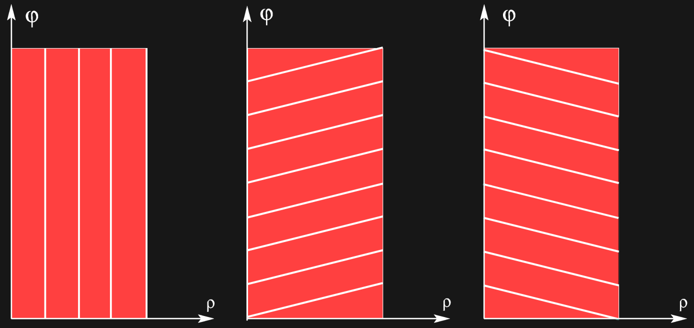
- The crucial point is that each g_k is defined over the **closed** unit disk D^2. Obviously, $g_k \uparrow h$.

Construction of our embedding of $\mathbb{H}^2 \hookrightarrow \mathbb{E}^3$



- We thus can solve the Elementary Step over the compact set D^2 and avoid the use of an exhaustion.

Construction of our embedding of $\mathbb{H}^2 \hookrightarrow \mathbb{E}^3$



Level curves of ℓ_1 (left), ℓ_2 (middle) and ℓ_3 (right) in the (ρ, φ)-parameter space.

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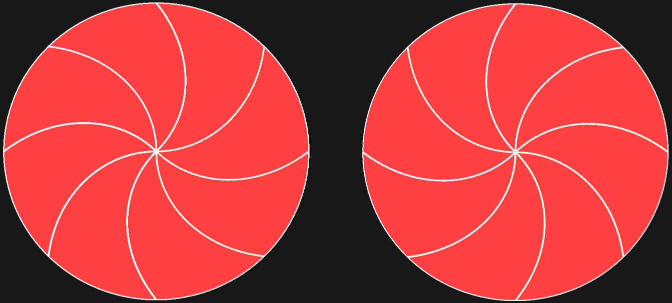
$$\ell_1 := d\rho,\ \ \ell_2 := d\rho - a\, d\varphi,\ \ \ell_3 := d\rho + a\, d\varphi \ \text{ with } \ a = \frac{4}{\pi}$$

Construction of our embedding of $\mathbb{H}^2 \hookrightarrow \mathbb{E}^3$



Level curves of ℓ_2 (left) and ℓ_3 (right) in the disk D^2.

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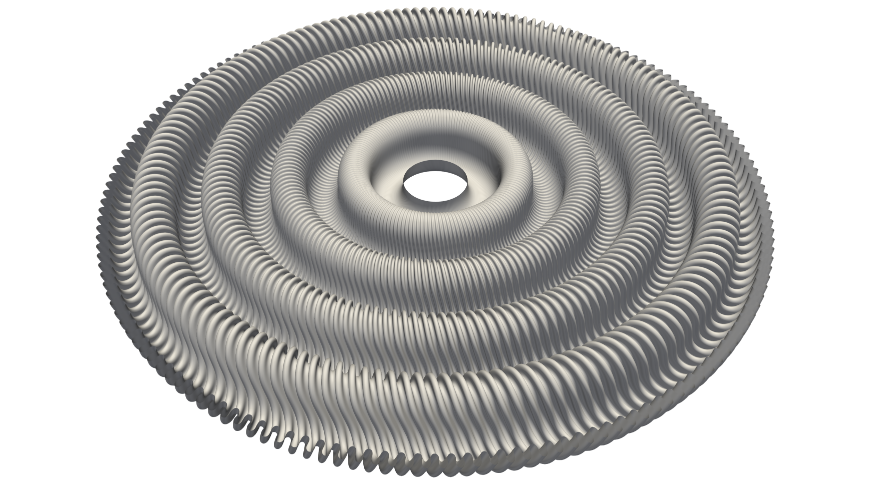
The map $\mathcal{F}_{1,2}$



$N_{1,1} = 5$ *and* $N_{1,2} = 20$

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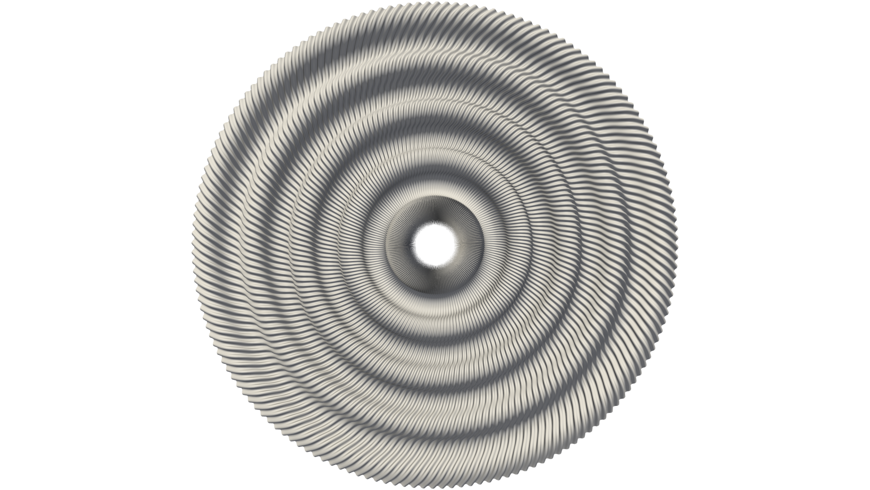

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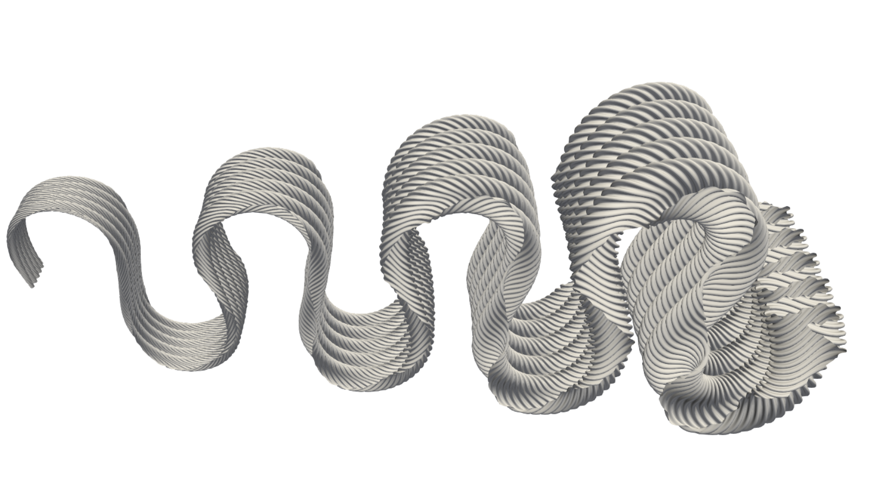

$N_{1,1} = 5$, $N_{1,2} = 20$ *and* $N_{1,3} = 200$.

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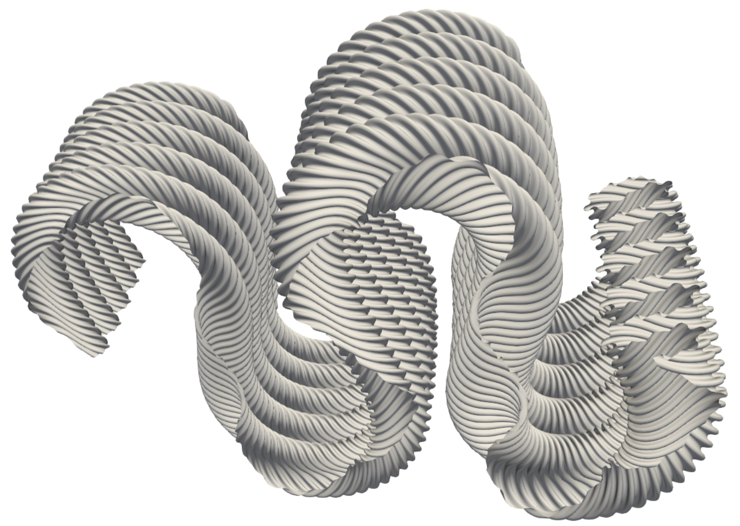

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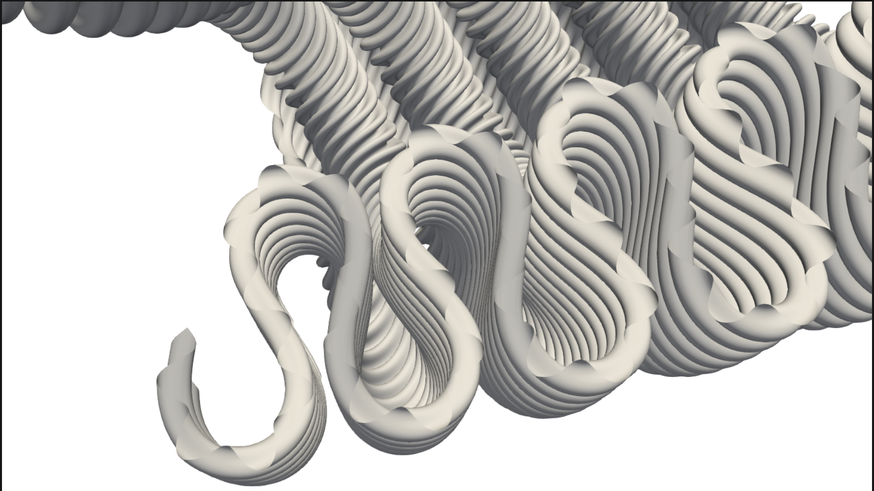

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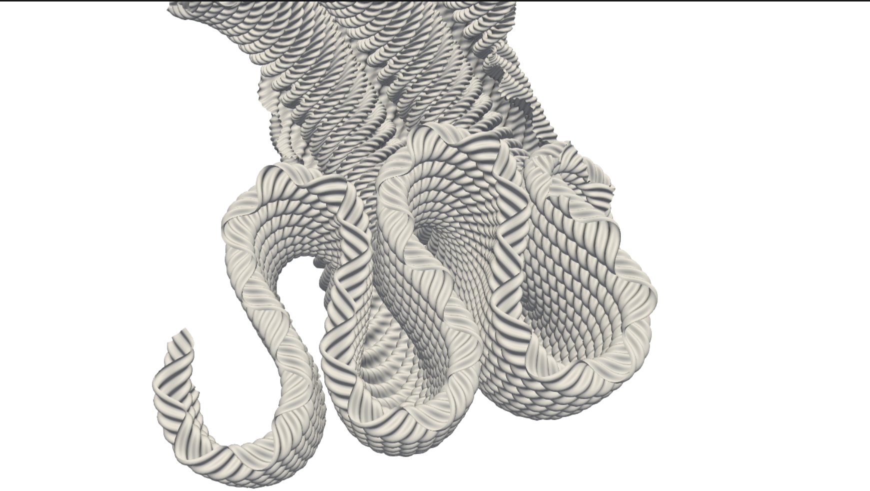

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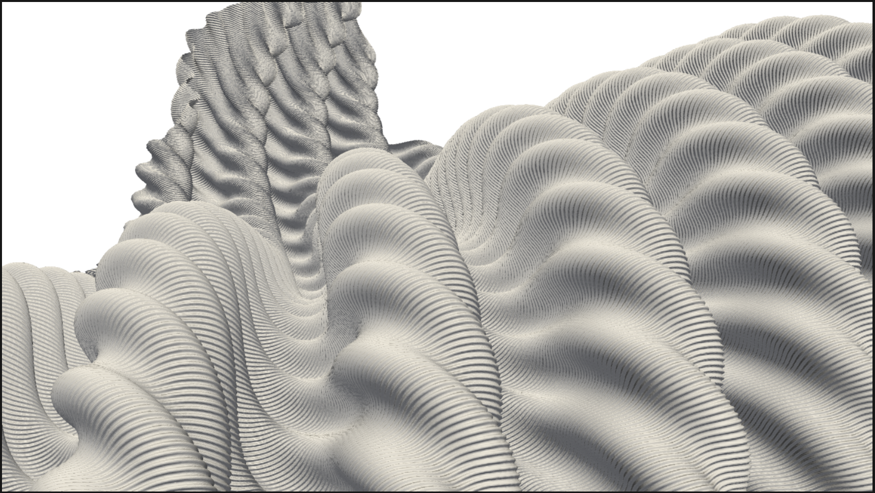

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C^0-convergence of $(\mathcal{F}_n)$

C^0 **convergence.–** *If the corrugation numbers are chosen large enough then sequence* $(\mathcal{F}_{k,3})$ *is* C^0*-converging on* D^2*. We denote by* $\mathcal{F}_\infty$ *the limit map.*

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- Indeed, let (τ_k) be a sequence of positive numbers such that

$$\sum \tau_k < +\infty.$$

From the Point *i*) of Elementary Step, we know that

$$\|\mathcal{F}_{k,3} - \mathcal{F}_{k-1,3}\|_{C^0(D^2)} = O\left(\frac{1}{N_{k,1}}\right) + O\left(\frac{1}{N_{k,2}}\right) + O\left(\frac{1}{N_{k,3}}\right)$$

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• Since $\sum \tau_k < +\infty$, the sequence $(\mathcal{F}_{k,3})$ is a Cauchy for $\|\cdot\|_{C^0(D^2)}$. $\square$

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• If the corrugation numbers are chosen large enough then, points *i*) and *ii*) of the Elementary Step leads to the following inequality :

$$\|d\mathcal{F}_{k,3} - d\mathcal{F}_{k-1,3}\|_{C^0(D^2)} \quad \leq \quad \tau_k + Cte\,\|g_k - g_{k-1}\|^{1/2}_{C^0(D^2)}.$$

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• It is readily checked that

$$\sum \|g_k - g_{k-1}\|^{1/2}_{C^0(K)} < +\infty$$

on any compact set $K \subset$ *Int* D^2. The C^1 convergence of the $(\mathcal{F}_{k,3})$ follows. $\square$

An isometric embedding of $\mathbb{H}^2$

IMAGE DE $\mathcal{F}_\infty$

Asymptotic behaviour

IMAGE DU BORD

Hölder regularity.– *If the corrugation numbers are chosen large enough then* $\mathcal{F}_\infty$ *is* β*-Hölder for any* $0 < \beta < 1$.

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  1. The map $\mathcal{F}_\infty$ at the boundary looks like a Weierstrass function

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  with $a_n = \dfrac{r_n}{N_n}$ and $N_n = b^n$ for some $b > 1$.

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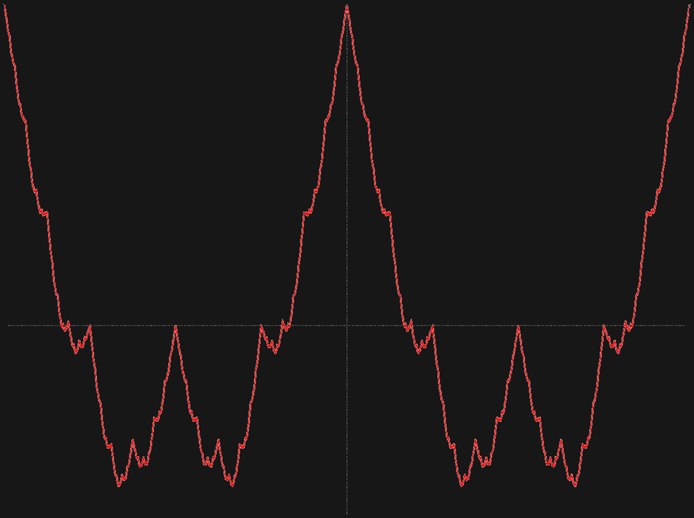
4. It is classical result about Weierstrass functions that $a_n = O(N_n^{-\beta})$ implies a β-Hölder regularity for W.

Asymptotic behaviour



Graph of a Weierstrass function of β-Hölder regularity for any $0 < \beta < 1$.

Asymptotic behaviour

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- We begin with a reindexing of the sequence $(\mathcal{F}_{k,i})$: we write $\mathcal{F}_n$ with $n = 3(k-1)+i$ for $\mathcal{F}_{k,i}$. We have

$$\mathcal{F}_n = \mathcal{F}_{n-1} + \frac{r_n}{N_n}\Gamma(\alpha_n, N_n\ell_n) \cdot \begin{pmatrix} \mathbf{t}_{n-1} \\ V_{n-1} \\ \mathbf{n}_{n-1} \end{pmatrix}.$$

- To iterate, we denote by $\mathcal{C}_j$ the *corrugation matrix map* defined by

$$\begin{pmatrix} \mathbf{t}_j \\ V_j \\ \mathbf{n}_j \end{pmatrix} = \mathcal{C}_j \cdot \begin{pmatrix} \mathbf{t}_{j-1} \\ V_{j-1} \\ \mathbf{n}_{j-1} \end{pmatrix}$$

where the dot $\cdot$ denotes the obvious action of $O(\mathbb{R}^3)$ on $(\mathbb{R}^3)^3$.

Asymptotic behaviour

- We thus have

$$\mathcal{F}_n \;=\; \mathcal{F}_{n-1} + \frac{r_n}{N_n}\Gamma(\alpha_n, N_n\ell_n) \cdot \left(\prod_{j=1}^{n-1} \mathcal{C}_j\right) \begin{pmatrix} \mathbf{t}_0 \\ V_0 \\ \mathbf{n}_0 \end{pmatrix}$$

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- We put

$$\mathcal{M}_n := r_{n+1} \prod_{j=1}^{n} \mathcal{C}_j$$

so that

$$\mathcal{F}_n = \mathcal{F}_{n-1} + \frac{1}{N_k}\Gamma(\alpha_k, N_n\varpi_n) \cdot \mathcal{M}_{n-1} \begin{pmatrix} \mathbf{t}_0 \\ V_0 \\ \mathbf{n}_0 \end{pmatrix}$$

Asymptotic behaviour

- Iterating the relation, we obtain

$$\mathcal{F}_n \quad = \quad \mathcal{F}_0 + \left(\sum_{q=1}^{n} \frac{1}{N_q} \Gamma(\alpha_q, N_q \ell_q) \cdot \mathcal{M}_{q-1} \right) \begin{pmatrix} \mathbf{t}_0 \\ V_0 \\ \mathbf{n}_0 \end{pmatrix}.$$

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- This explicit expression will help to show the β-Hölder regularity of the map $\mathcal{F}_\infty$.

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- In order to lighten the notations, and since $\rho = 1$, we write $F(\varphi)$ instead of $F(1, \varphi)$ for each map F appearing in the proof.

Hölder regularity

- We denote by Υ_n the map

$$\Upsilon_n = \frac{1}{N_n}\Gamma(\alpha_n, N_n\ell_n) \cdot \mathcal{M}_{n-1}.$$

Since $\mathcal{F}_0$ is C^∞, the proof reduces to show that

$$\Upsilon_\infty := \sum_{n=1}^{\infty} \Upsilon_n$$

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- To do so, we consider the difference

$$\begin{aligned}\Upsilon_n(\varphi) - \Upsilon_n(\varphi_0) &= \frac{1}{N_n}\Gamma(\alpha_n(\varphi), N_n\ell_n(\varphi)) \cdot \mathcal{M}_{n-1}(\varphi) \\ &\quad -\frac{1}{N_n}\Gamma(\alpha_n(\varphi_0), N_n\ell_n(\varphi_0)) \cdot \mathcal{M}_{n-1}(\varphi_0)\end{aligned}$$

Hölder regularity

- We write this difference as

$$\Upsilon_n(\varphi) - \Upsilon_n(\varphi_0) \quad = \quad A_n + B_n + C_n$$

with

$$\begin{aligned}
A_n &= \frac{1}{N_n}\Big(\Gamma(\alpha_n(\varphi), N_n\ell_n(\varphi)) - \Gamma(\alpha_n(\varphi_0), N_n\ell_n(\varphi))\Big) \cdot \mathcal{M}_{n-1}(\varphi) \\
B_n &= \frac{1}{N_n}\Big(\Gamma(\alpha_n(\varphi_0), N_n\ell_n(\varphi)) - \Gamma(\alpha_n(\varphi_0), N_n\ell_n(\varphi_0))\Big) \cdot \mathcal{M}_{n-1}(\varphi) \\
C_n &= \frac{1}{N_n}\Gamma(\alpha_n(\varphi_0), N_n\ell_n(\varphi_0)) \cdot \Big(\mathcal{M}_{n-1}(\varphi) - \mathcal{M}_{n-1}(\varphi_0)\Big).
\end{aligned}$$

Hölder regularity

- We first focus on the last term

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We have

$$\|\mathcal{M}_{n-1}(\varphi) - \mathcal{M}_{n-1}(\varphi_0)\| \le \|\mathcal{M}_{n-1}\|_{C^1(D^2)}|\varphi - \varphi_0|$$

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- We choose N_n large enough so that

$$\frac{1}{N_n}\|\mathcal{M}_{n-1}\|_{C^1(D^2)} \le \tau_n.$$

We then have

$$\|\sum_{n=1}^{\infty} C_n\| \le \left(\sum_{n=1}^{\infty}\tau_n\right)|\varphi - \varphi_0|.$$

Hölder regularity

- Next, we move to the first term

$$A_n \quad = \quad \frac{1}{N_n}\Big(\Gamma(\alpha_n(\varphi), N_n\ell_n(\varphi)) - \Gamma(\alpha_n(\varphi_0), N_n\ell_n(\varphi))\Big) \cdot \mathcal{M}_{n-1}(\varphi)$$

We have

$$\|\mathcal{M}_{n-1}(\varphi)\| = \left\| r_n(\varphi) \prod_{j=1}^{n} \mathcal{C}_j(\varphi) \right\| \leq r_n(\varphi)$$

and

$$\|A_n\| \leq r_n(\varphi) \, \|\Gamma\|_{C^1([0,\kappa_0]\times[0,1])} \, |\alpha_n(\varphi) - \alpha_n(\varphi_0)|$$

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- We also choose N_n such that

$$\frac{1}{N_n}\|r_n\|_{C^0(D^2)}\|\alpha_n\|_{C^1(D^2)} \leq \tau_n.$$

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- With such a choice

$$\|A_n\| \leq \tau_n \, \|\Gamma\|_{C^1([0,\kappa_0]\times[0,1])} \, |\varphi - \varphi_0|$$

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- It remains to deal with

$$\sum_{n=1}^{\infty} B_n.$$

This term is the non-Lipschitzian one and requires a finer processing.

Hölder regularity

Lemma.– *For $i \in \{1,2\}$ we have*

$$|\Gamma_i(\alpha, t) - \Gamma_i(\alpha, t_0)| \leq 4\alpha \cosh\alpha \left|\sin\left(\frac{t - t_0}{2}\right)\right|.$$

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Hint for a proof.– Start with the definition of Γ_1 and Γ_2 :

$$\left\{\begin{array}{rcl} \Gamma_1(\alpha,t) & = & \displaystyle\int_{s=0}^{t} \cos(\alpha\cos(2\pi s)) - J_0(\alpha)ds \\ \Gamma_2(\alpha,t) & = & \displaystyle\int_{s=0}^{t} \sin(\alpha\cos(2\pi s))ds, \end{array}\right.$$

use the Jacobi-Anger identity

$$e^{i\alpha\cos t} = J_0(t) + 2\sum_{n=1}^{\infty} i^n J_n(\alpha)\cos(nt)$$

and then the Chebyshev polynomials and the fact that the Taylor expansion of J_n is alternating. $\square$

Hölder regularity

- We use this Lemma to handle the term

$$B_n = \frac{1}{N_n}\Big(\Gamma(\alpha_n(\varphi_0), N_n\ell_n(\varphi)) - \Gamma(\alpha_n(\varphi_0), N_n\ell_n(\varphi_0))\Big) \cdot \mathcal{M}_{n-1}(\varphi)$$

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- Since $0 < \alpha(\varphi) \leq \kappa_0 = 2.40...$ we have

$$\|B_n\| \leq \frac{1}{N_n} 4\kappa_0 \cosh \kappa_0 \left|\sin\left(\frac{N_n\ell_n(\varphi) - N_n\ell_n(\varphi_0)}{2}\right)\right| \|\mathcal{M}_{n-1}(\varphi)\|$$

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- Adding that

$$\|\mathcal{M}_{n-1}(\varphi)\| \le r_n(\varphi)$$

we obtain

$$\|B_n\| \;\le\; \frac{\|r_n\|_{C^0(D^2)}}{N_n} 4\kappa_0 \cosh\kappa_0 \left|\sin\left(\frac{N_n\ell_n(\varphi) - N_n\ell_n(\varphi_0)}{2}\right)\right|$$

Hölder regularity

- We thus have

$$\|\sum_{n=1}^{\infty} B_n\| \leq Cte \sum_{n=1}^{\infty} \frac{\|r_n\|_{C^0(D^2)}}{N_n} \left|\sin\left(\frac{N_n\ell_n(\varphi) - N_n\ell_n(\varphi_0)}{2}\right)\right|.$$

where

$$\ell_n(\varphi) - \ell_n(\varphi_0) = \begin{cases} 0 & \text{for the step } i = 1 \\ -a(\varphi - \varphi_0) & \text{for the step } i = 2 \\ a(\varphi - \varphi_0) & \text{for the step } i = 3. \end{cases}$$

Hölder regularity

• We thus have

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• In that summation, the norm $\|r_n\|_{C^0(D^2)}$ diverges when n tends toward infinity.

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Hölder regularity

- By definition

$$g_{k+1} = \mathcal{F}_0^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3} + E_{k+1}d\rho^2 + G_{k+1}d\varphi^2$$

with

$$E_{k+1} = 4\sum_{q=1}^{k+1}(q+2)\rho^{2(q+1)} \quad \text{and} \quad G_{k+1} = 4\sum_{q=1}^{k+1}(q+1)\rho^{2(q+1)}.$$

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- We have

$$\begin{cases} \|E_{k+1}\|_{C^0(D^2)} &= 4\sum_{q=1}^{k+1}(q+2) = O(k^2) \\ \|G_{k+1}\|_{C^0(D^2)} &= 4\sum_{q=1}^{k+1}(q+1) = O(k^2) \end{cases}$$

thus $\|r_n\|_{C^0(D^2)} = O(n)$. $\square$

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- Recall that we have

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- If we assume that the corrugations numbers N_n are chosen large enough to satisfy the *Hadamard's lacunary condition* :

$$\frac{N_{n+1}}{N_n} \geq b \quad \text{for some} \quad b > 1$$

then

$$O\left(\frac{\|r_n\|_{C^0(D^2)}}{N_n}\right) = O\left(N_n^{-\beta}\right)$$

for any $0 < \beta < 1$.

Hölder regularity

• A classical result in lacunary Fourier series then shows the desired Hölder regularity.

Theorem (Hardy, 1916).– *Let* (N_n) *be a sequence of natural numbers satisfying the Hadamard's lacunary condition. Let* $f : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{C}$ *and* $t_0 \in \mathbb{R}/2\pi\mathbb{Z}$ *such that*

$$\forall t, \quad |f(t) - f(t_0)| \leq Cte \sum_{k=0}^{\infty} N_n^{-\beta} |e^{iN_n t} - e^{iN_n t_0}|$$

for some $0 < \beta < 1$. *Then* f *is* β*-Hölder at* t_0.

Thank you for your attention !

The Hevea Team