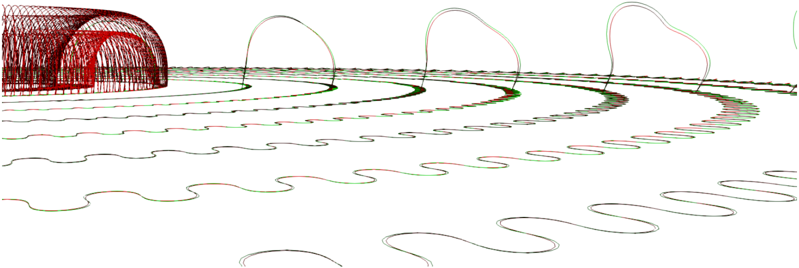
C^1 Isometric Embeddings of $\mathbb{H}^2$

Mélanie, Roland et Vincent

Hevea Project



The Poincaré Disk

• Let C and $\tilde{C}$ be the open cylinder and its universal covering

$$C := \{(\rho,\theta)\,|\,0<\rho<1, \theta\in\mathbb{R}/(2\pi\mathbb{Z})\}, \quad \tilde{C} =]0,1[\,\times\mathbb{R}$$

and h be the metric on C and $\tilde{C}$ given by

$$h := 4\frac{d\rho^2+\rho^2 d\theta^2}{(1-\rho^2)^2}.$$

• It is readily checked that the Riemannian cylinder (C,h) is isometric to the punctured Poincaré disk $\mathbb{H}^2\setminus\{O\}$ via the map

$$(\rho,\theta)\mapsto(x=\rho\cos\theta, y=\rho\sin\theta).$$

The initial embedding.

- We take as initial embedding $\mathcal{F}_0 : C \to \mathbb{E}^3$ the inclusion :

$$\mathcal{F}_0(\rho, \theta) := (\rho \cos\theta, \rho \sin\theta, 0).$$

The initial embedding.

- We take as initial embedding $\mathcal{F}_0 : C \to \mathbb{E}^3$ the inclusion :

$$\mathcal{F}_0(\rho,\theta) := (\rho\cos\theta, \rho\sin\theta, 0).$$

- We have

$$\mathcal{F}_0^*\langle .,.\rangle_{\mathbb{E}^3} = d\rho^2 + \rho^2 d\theta^2$$

and the isometric default of $\mathcal{F}_0$ is

$$\Delta_0 = \left(\frac{4}{(1-\rho^2)^2} - 1\right)(d\rho^2 + \rho^2 d\theta^2).$$

The initial embedding.

- We take as initial embedding $\mathcal{F}_0 : C \to \mathbb{E}^3$ the inclusion :

$$\mathcal{F}_0(\rho, \theta) := (\rho \cos \theta, \rho \sin \theta, 0).$$

- We have

$$\mathcal{F}_0^* \langle ., . \rangle_{\mathbb{E}^3} = d\rho^2 + \rho^2 d\theta^2$$

and the isometric default of $\mathcal{F}_0$ is

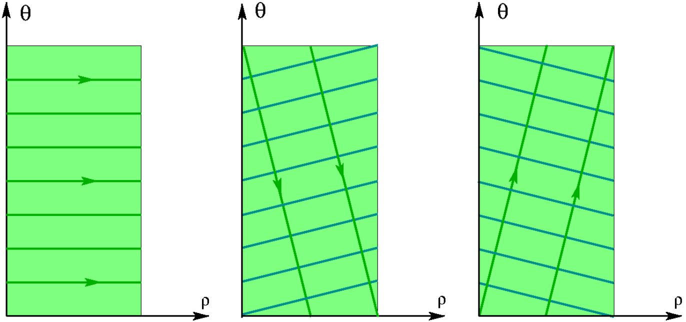
$$\Delta_0 = \left(\frac{4}{(1-\rho^2)^2} - 1 \right) (d\rho^2 + \rho^2 d\theta^2).$$

- The map $\mathcal{F}_0 : C \to \mathbb{E}^3$ is strictly short

Directions of corrugation

• Let $\mu > 1$. We choose the following three vector fields of C as directions of corrugation :

$$u_1 := \partial_\rho, \ \ u_2 := \frac{\partial_\rho - \mu\partial_\theta}{1+\mu^2}, \ \ u_3 := \frac{\partial_\rho + \mu\partial_\theta}{1+\mu^2}$$



Integral lines of u_1, u_2 and u_3 (green) together with the lines $\varpi_i(\rho,\theta) = 0$ (grey) for $\mu = \frac{4}{\pi}$.

Lines of "initial conditions".

- We also consider the three projections $\varpi_i : \tilde{C} \to \mathbb{R}$ defined by

$$\varpi_1(\rho,\theta) = \rho, \quad \varpi_2(\rho,\theta) = \rho - \mu\theta \quad \text{and} \quad \varpi_3(\rho,\theta) = \rho + \mu\theta$$

Lines of "initial conditions".

• We also consider the three projections $\varpi_i : \tilde{C} \to \mathbb{R}$ defined by

$$\varpi_1(\rho,\theta) = \rho, \quad \varpi_2(\rho,\theta) = \rho - \mu\theta \quad \text{and} \quad \varpi_3(\rho,\theta) = \rho + \mu\theta$$

• The maps $\varpi_i : \tilde{C} \to \mathbb{R}$, $i \in \{2,3\}$ do not descend to the quotient C in general since

$$\varpi_i(\rho,\theta + 2\pi) = \varpi_i(\rho,\theta) \mp 2\mu\pi$$

but their differentials do and are constant on C.

Lines of "initial conditions".

• We also consider the three projections $\varpi_i : \tilde{C} \to \mathbb{R}$ defined by

$$\varpi_1(\rho,\theta) = \rho, \quad \varpi_2(\rho,\theta) = \rho - \mu\theta \quad \text{and} \quad \varpi_3(\rho,\theta) = \rho + \mu\theta$$

• The maps $\varpi_i : \tilde{C} \to \mathbb{R}$, $i \in \{2,3\}$ do not descend to the quotient C in general since

$$\varpi_i(\rho,\theta + 2\pi) = \varpi_i(\rho,\theta) \mp 2\mu\pi$$

but their differentials do and are constant on C.

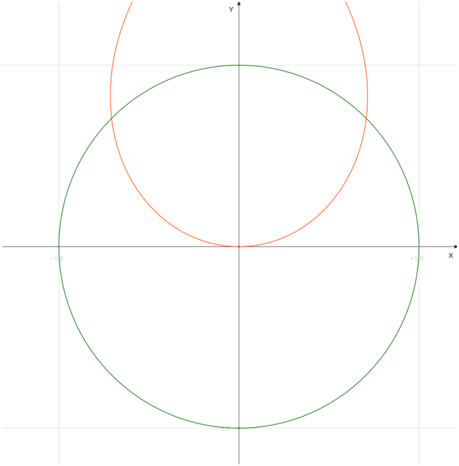
• We denote by ℓ_i the three linear forms $d\varpi_i$:

$$\ell_1 := d\rho,\ \ell_2 := d\rho - \mu d\theta,\ \ell_3 := d\rho + \mu d\theta.$$

• The vectors u_i are such that

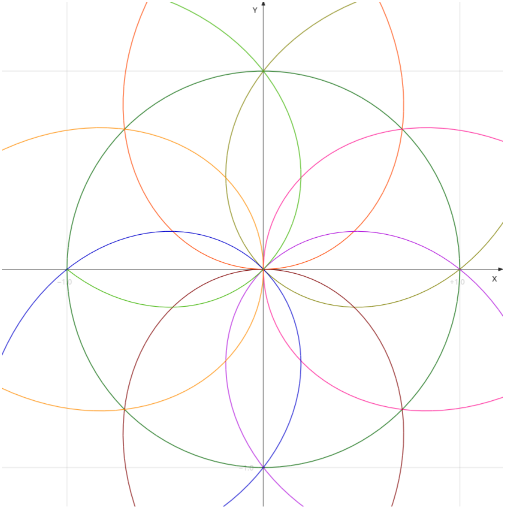
$$\ell_i(u_i) = 1$$

Lines of "initial conditions".



The curve $\varpi_2(p) = 0$ with $\mu = \frac{4}{\pi}$ (in red) and the boundary of the hyperbolic disk is (in green)

Lines of "initial conditions".



The curves $\varpi_2(p) = k$ with $k \in \mathbb{N}$. Along these curves the Corrugation Process (on the direction u_2) leaves the map unchanged.

The Corrugation Process

- We denote by

$$CP(f, g, u, N)$$

the Corrugation Process applied to $f : C \to \mathbb{E}^3$, in the direction $u \in \{u_1, u_2, u_3\}$ for the goal metric g and with N corrugations.

The Corrugation Process

- We denote by

$$CP(f, g, u, N)$$

the Corrugation Process applied to $f : C \to \mathbb{E}^3$, in the direction $u \in \{u_1, u_2, u_3\}$ for the goal metric g and with N corrugations.

- Precisely, for every $\tilde{p} = (\rho, \theta) \in \tilde{C}$, we put

$$CP(f, g, u, N)(\tilde{p}) := f(p) + \frac{r(p)}{N}\Big[\Gamma_1(p, N\varpi(\tilde{p}))\mathbf{t}(p) + \Gamma_2(p, N\varpi(\tilde{p}))\mathbf{n}(p)\Big]$$

where p is the image of $\tilde{p}$ by the covering map $\tilde{C} \to C$ and

$$\begin{aligned}
\Gamma_1(p, t) &:= \int_{s=0}^{t} \cos(\alpha(p)\cos(2\pi s)) - J_0(\alpha(p))ds \\
\Gamma_2(p, t) &:= \int_{s=0}^{t} \sin(\alpha(p)\cos(2\pi s))ds
\end{aligned}$$

The Corrugation Process

• We denote by $[X]^P$ the orthogonal projection of $X \in \mathbb{R}^3$ on the linear subspace $df_p(\ker(d\varpi)_p)$

The Corrugation Process

- We denote by $[X]^P$ the orthogonal projection of $X \in \mathbb{R}^3$ on the linear subspace $df_p(\ker(d\varpi)_p)$

- And we define

$$
\begin{aligned}
\mathbf{t}(p) &:= \frac{df_p(u_p) - [df_p(u_p)]^P}{\|df_p(u_p) - [df_p(u_p)]^P\|} \\
\mathbf{n}(p) &:= \text{a unit normal of } f \text{ at } p \\
r(p) &:= \sqrt{g_p(u_p, u_p) - \|[df_p(u_p)]^P\|^2} \\
\alpha(p) &:= J_0^{-1}\left(\frac{\|df_p(u_p) - [df_p(u_p)]^P\|}{r(p)}\right) \\
\varpi(\tilde{p}) &:= \varpi_i(\tilde{p}) \text{ if } u = u_i
\end{aligned}
$$

Descending to the quotient

Lemma.– *If* $\mu \in \frac{1}{2\pi}\mathbb{Z}$ *then map* $CP(f, g, u, N) : \tilde{C} \to \mathbb{E}^3$ *descends to the quotient* C.

• Indeed the only map which possibly does not descend to the quotient is $\varpi = \varpi_i$ when $i \in \{2, 3\}$ From the 1-periodicity of $t \mapsto \Gamma_1(p, t)$ and $t \mapsto \Gamma_2(p, t)$, we deduce that the condition

$$\varpi_i(\rho, \theta + 2\pi) - \varpi_i(\rho, \theta) \in \mathbb{Z}$$

ensures that $CP(f, g, u, N)$ descend to the quotient □

Choice of μ

- We have two conditions on μ :

$$\mu > 1 \quad \text{and} \quad \varpi_i(\rho, \theta + 2\pi) - \varpi_i(\rho, \theta) = 2\pi\mu \in \mathbb{Z}$$

- Any choice of the form

$$\mu = \frac{k}{2\pi} \quad \text{with} \quad k \in \mathbb{N}_{\geq 7}$$

is convenient.

- We have chosen

$$\mu = \frac{4}{\pi}$$

Choice of the metric sequence

The Nash's choice.– Let $\tilde{\delta}_k = 1 - 2^{-k}$ and define $\tilde{g}_k$ to be

$$\tilde{g}_k := \mathcal{F}_0^* \langle .,. \rangle_{\mathbb{E}^3} + \tilde{\delta}_k (h - \mathcal{F}_0^* \langle .,. \rangle_{\mathbb{E}^3})$$

where h is the Hyperbolic metric

$$h := 4 \frac{d\rho^2 + \rho^2 d\theta^2}{(1-\rho^2)^2}.$$

Choice of the metric sequence

The Nash's choice.– Let $\tilde{\delta}_k = 1 - 2^{-k}$ and define $\tilde{g}_k$ to be

$$\tilde{g}_k := \mathcal{F}_0^* \langle ., . \rangle_{\mathbb{E}^3} + \tilde{\delta}_k (h - \mathcal{F}_0^* \langle ., . \rangle_{\mathbb{E}^3})$$

where h is the Hyperbolic metric

$$h := 4 \frac{d\rho^2 + \rho^2 d\theta^2}{(1 - \rho^2)^2}.$$

- Recall that the isometric default $\Delta_0 := h - \mathcal{F}_0^* \langle ., . \rangle_{\mathbb{E}^3}$ is

$$\Delta_0 = \left(\frac{4}{(1 - \rho^2)^2} - 1 \right) (d\rho^2 + \rho^2 d\theta^2).$$

Choice of the metric sequence

- From the Taylor expansion

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots + (n+1)x^n + \dots$$

we deduce

$$\begin{aligned}
\Delta_0 &:= \left(\frac{4}{(1-\rho^2)^2} - 1\right)(d\rho^2 + \rho^2 d\theta^2) \\
&= \left(4\sum_{n=0}^{+\infty}(n+1)\rho^{2n} - 1\right)(d\rho^2 + \rho^2 d\theta^2) \\
&= \left(3 + 4\sum_{n=1}^{+\infty}(n+1)\rho^{2n}\right)(d\rho^2 + \rho^2 d\theta^2)
\end{aligned}$$

Choice of the metric sequence

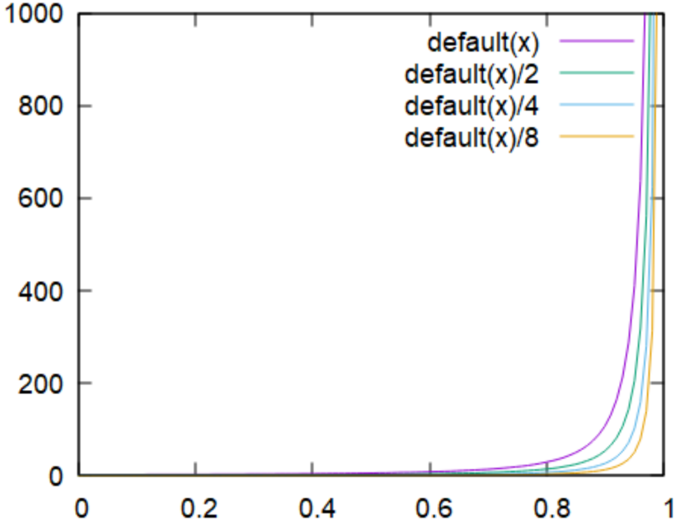
Our choice of the metric sequence.– For every $k \in \mathbb{N}^*$ we put

$$\delta_k(\rho) := 3 + 4\sum_{n=1}^{k}(n+1)\rho^{2n}$$

and we define g_k to be the metric

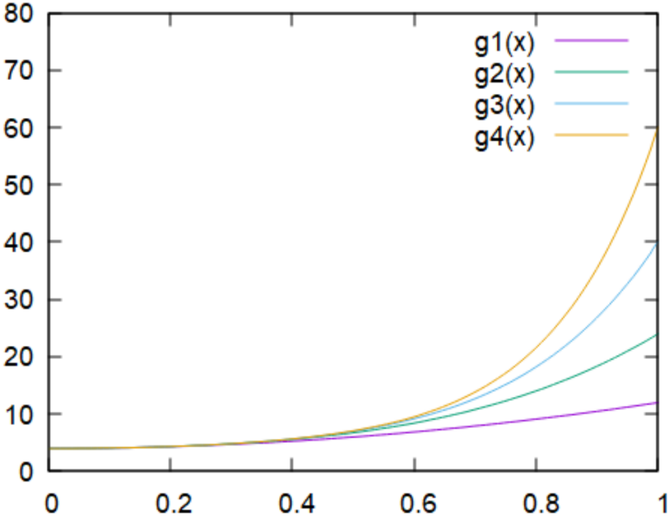
$$g_k := \mathcal{F}_0^*\langle .,.\rangle_{\mathbb{E}^3} + \delta_k(d\rho^2 + \rho^2 d\theta^2).$$

Comparison between the two metric sequences



Nash's choice : The metrics $\tilde{g}_k$ are unbounded at $\rho = 1$.

Comparison between the two metric sequences



Hevea's choice : The metrics g_k are bounded at $\rho = 1$.

The sequence $\mathcal{F}_{k,i}$

• Let $0 < b < 1$, we set

$$C_b := \{(\rho, \theta) \in C \mid \rho \leq b\}$$

and we denote by $\|g\|_{\infty, C_b}$ the supremum of $\|g_{(\rho,\theta)}\|$ on C_b.

Lemma.– *For every* $0 < b < 1$*, we have*

$$\sum_{k=1}^{+\infty} \|g_k - g_{k-1}\|_{\infty, C_b}^{\frac{1}{2}} < +\infty.$$

The sequence $\mathcal{F}_{k,i}$

- Let $0 < b < 1$, we set

$$C_b := \{(\rho,\theta) \in C \mid \rho \leq b\}$$

and we denote by $\|g\|_{\infty,C_b}$ the supremum of $\|g_{(\rho,\theta)}\|$ on C_b.

Lemma.– *For every* $0 < b < 1$*, we have*

$$\sum_{k=1}^{+\infty} \|g_k - g_{k-1}\|_{\infty,C_b}^{\frac{1}{2}} < +\infty.$$

- The Stage Theorem holds for $\mathcal{F}_{k,i}$

The sequence $\mathcal{F}_{k,i}$

• Let $0 < b < 1$, we set

$$C_b := \{(\rho, \theta) \in C \mid \rho \leq b\}$$

and we denote by $\|g\|_{\infty, C_b}$ the supremum of $\|g_{(\rho,\theta)}\|$ on C_b.

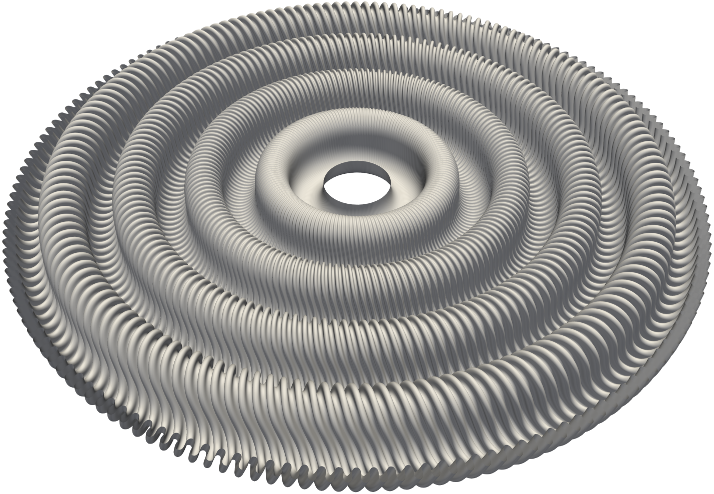
Lemma.– *For every* $0 < b < 1$*, we have*

$$\sum_{k=1}^{+\infty} \|g_k - g_{k-1}\|_{\infty, C_b}^{\frac{1}{2}} < +\infty.$$

• The Stage Theorem holds for $\mathcal{F}_{k,i}$

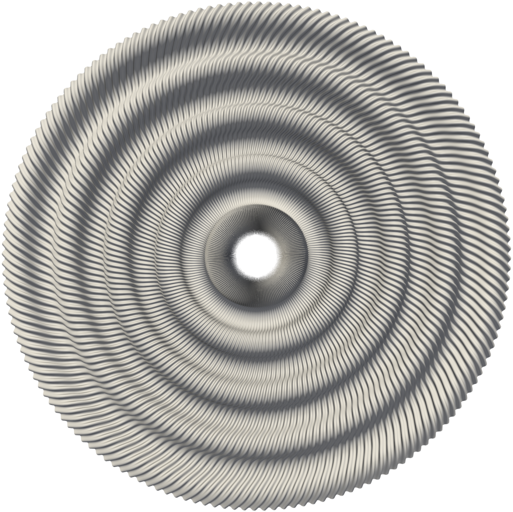
• La suite $(\mathcal{F}_{k,i})_{k,i}$ converge C^0 sur $\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$ et elle converge C^1 sur $\mathbb{H}^2$.

The map $\mathcal{F}_{1,2}$



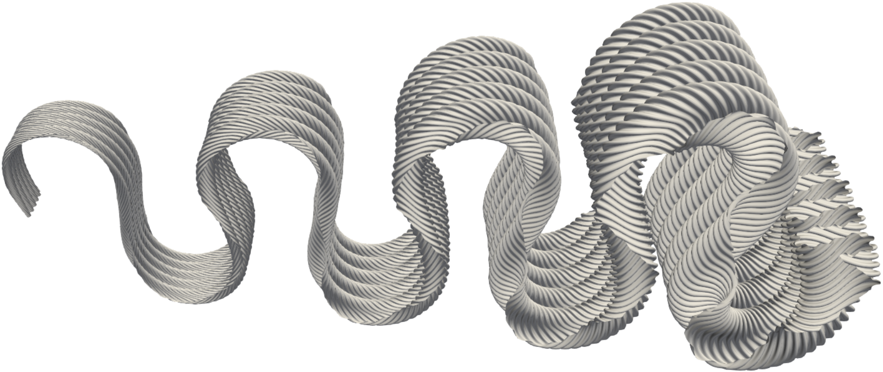
$N_{1,1} = 5$ *and* $N_{1,2} = 20$

The map $\mathcal{F}_{1,2}$



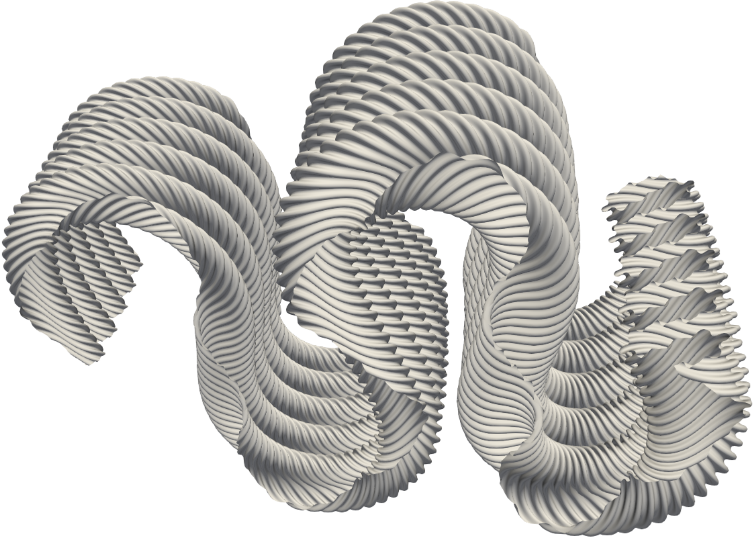
$N_{1,1} = 5$ *and* $N_{1,2} = 20$

The map $\mathcal{F}_{1,3}$



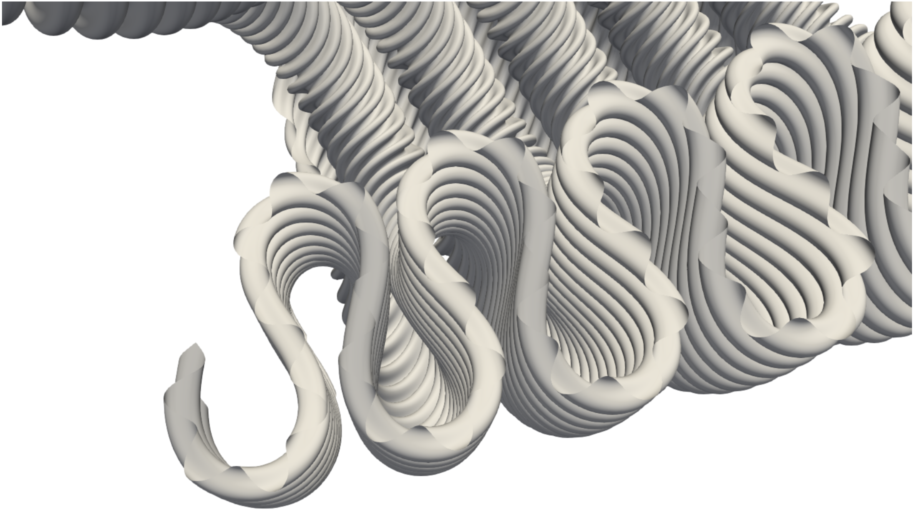
$N_{1,1} = 5$, $N_{1,2} = 20$ *and* $N_{1,3} = 200$.

The map $\mathcal{F}_{1,3}$



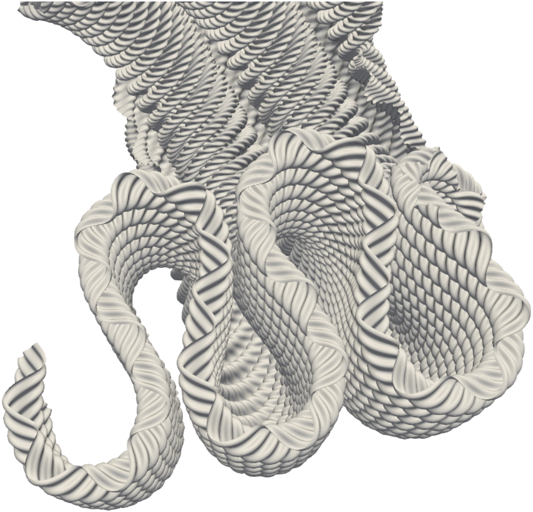
$N_{1,1} = 5$, $N_{1,2} = 20$ *and* $N_{1,3} = 200$.

The map $\mathcal{F}_{1,3}$



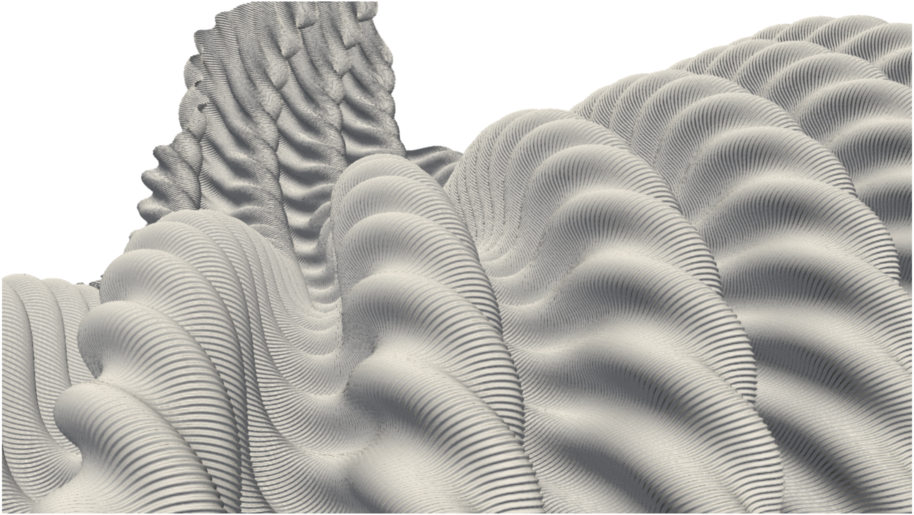
$N_{1,1} = 5$, $N_{1,2} = 20$ *and* $N_{1,3} = 200$.

The map $\mathcal{F}_{2,1}$



$N_{1,1} = 5$, $N_{1,2} = 20$, $N_{1,3} = 200$ *and* $N_{2,1} = 2000$.

The map $\mathcal{F}_{2,2}$



$N_{1,1} = 5$, $N_{1,2} = 20$, $N_{1,3} = 200$, $N_{2,1} = 2000$ *and* $N_{2,2} = 20000$.

Comportement au bord

• La suite $(\mathcal{F}_{k,i})_{k,i}$ converge C^0 sur $\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$ et elle converge C^1 sur $\mathbb{H}^2$.

Comportement au bord

• La suite $(\mathcal{F}_{k,i})_{k,i}$ converge C^0 sur $\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$ et elle converge C^1 sur $\mathbb{H}^2$.

Conjecture 1.– *L'application* $\theta \longmapsto \mathcal{F}_\infty(1, \theta)$ *n'est pas Lipschitzienne.*

Comportement au bord

• La suite $(\mathcal{F}_{k,i})_{k,i}$ converge C^0 sur $\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$ et elle converge C^1 sur $\mathbb{H}^2$.

Conjecture 1.– *L'application $\theta \longmapsto \mathcal{F}_\infty(1,\theta)$ n'est pas Lipschitzienne.*

Conjecture 2.– *L'application $\theta \longmapsto \mathcal{F}_\infty(1,\theta)$ est β-Hölder pour tout $\beta \in [0,1[$. En particulier la dimension fractale de la courbe*

$$\theta \longmapsto (1,\theta,\mathcal{F}_\infty(1,\theta))$$

est un.

Résultats en faveur de la conjecture 1

- Il est possible de montrer (Lemme 20) que sur le bord

$$\alpha_{k,i}(1,\theta) \geq \frac{Cte1}{\sqrt{k + Cte2}}$$

et donc que

$$\sum_{k}^{\infty} \sum_{i=1}^{3} \alpha_{k,i}(1,\theta) = +\infty$$

Résultats en faveur de la conjecture 1

• Pour simplifier, supposons que l'application de Gauss G_∞ de $\mathcal{F}_\infty$ soit donnée formellement par une série lacunaire de la forme

$$G_\infty(1,\theta) = \sum_{k=1}^{\infty} \alpha_k \cos 2\pi N_k \theta$$

Lacunaire : il existe $K > 0$ et $b > 1$ tels que

$$\forall k \geq K, \quad \frac{N_{k+1}}{N_k} \geq b.$$

Résultats en faveur de la conjecture 1

- It is well-known (I have to find a reference here...) that if

$$\sum_{k=1}^{\infty} \alpha_k^2 < +\infty$$

then G_∞ is convergent almost everywhere and if

$$\sum_{k=1}^{\infty} \alpha_k^2 = +\infty$$

then G_∞ is divergent almost everywhere.

Résultats en faveur de la conjecture 1

- On est donc dans le cas de la divergence presque partout. Si

$$\theta \longmapsto \mathcal{F}_\infty(1, \theta)$$

était Lipschitzienne, alors par le théorème de Rademacher elle devrait être dérivable presque partout, en "contradiction" avec la divergence presque partout de G_∞.

Résultats en faveur de la conjecture 2

• On peut démontrer la conjecture 1 sur le *Toy Model* que l'on décrit maintenant. On part des d.l. suivants

$$\begin{aligned}\Gamma_1(\alpha, t) &= -\frac{\alpha^2}{16\pi}\sin(4\pi t) + o(\alpha^2)\\ \Gamma_2(\alpha, t) &= \frac{\alpha}{2\pi}\sin(2\pi t) + o(\alpha^2).\end{aligned}$$

Définition.– Soit $\alpha > 0$. On définit la *courbe en huit* de paramètre α en posant

$$8_\alpha(t) = -\frac{\alpha}{8}\sin(4\pi t) + i\sin(2\pi t).$$

Ainsi

$$\Gamma_1(\alpha, t) + i\Gamma_2(\alpha, t) = \frac{\alpha}{2\pi} 8_\alpha(t) + o(\alpha^2).$$

Résultats en faveur de la conjecture 2

Parameter for the toy model.– We set $b > 1$ and

$$\alpha_k := \frac{1}{\sqrt{k}}, \quad N_k := b^k, \quad r_k := 4\sum_{n=0}^{k}(n+1) = 2(k+1)(k+2).$$

For $k \in \mathbb{N}$, we also define

$$A_k(t) := \sum_{n=1}^{k} \alpha_n \cos(2\pi N_n t) = \sum_{n=1}^{k} \frac{1}{\sqrt{n}} \cos(2\pi b^n t)$$

if $k \geq 1$ and $A_0 = 0$ if $k = 0$.

Résultats en faveur de la conjecture 2

The toy model.– Given an immersion $f_0 : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$, we put $\mathbf{t}_0 = \frac{f_0'}{\|f_0'\|}$ and we define recursively

$$\mathbf{t}_k(t) := e^{i\alpha_k \cos(2\pi N_k t)} \mathbf{t}_{k-1}(t)$$

and

$$f_k(t) := f_{k-1}(t) + \frac{r_k \alpha_k}{2\pi N_k} 8_k(N_k t) \mathbf{t}_{k-1}(t)$$

where 8_k stands for 8_α with $\alpha = \alpha_k$.

Résultats en faveur de la conjecture 2

- We thus have

$$\mathbf{t}_k(t) = e^{iA_k(t)}\mathbf{t}_0(t) \quad \text{and} \quad f_k(t) = f_{k-1}(t) + \frac{r_k \alpha_k}{2\pi N_k} \delta_k(N_k t) e^{iA_{k-1}(t)}\mathbf{t}_0(t)$$

Résultats en faveur de la conjecture 2

- We thus have

$$\mathbf{t}_k(t) = e^{iA_k(t)}\mathbf{t}_0(t) \quad \text{and} \quad f_k(t) = f_{k-1}(t) + \frac{r_k\alpha_k}{2\pi N_k}8_k(N_k t)e^{iA_{k-1}(t)}\mathbf{t}_0(t)$$

- Therefore

$$f_k(t) = f_0(t) + \left(\sum_{n=1}^{k} \frac{r_n\alpha_n}{2\pi N_n}8_n(N_n t)e^{iA_{n-1}(t)}\right)\mathbf{t}_0(t).$$

Résultats en faveur de la conjecture 2

- We thus have

$$\mathbf{t}_k(t) = e^{iA_k(t)}\mathbf{t}_0(t) \quad \text{and} \quad f_k(t) = f_{k-1}(t) + \frac{r_k\alpha_k}{2\pi N_k} 8_k(N_k t) e^{iA_{k-1}(t)}\mathbf{t}_0(t)$$

- Therefore

$$f_k(t) = f_0(t) + \left(\sum_{n=1}^{k} \frac{r_n\alpha_n}{2\pi N_n} 8_n(N_n t) e^{iA_{n-1}(t)} \right) \mathbf{t}_0(t).$$

- Observe that the serie is normally converging. We have

$$f_\infty(t) = f_0(t) + \left(\sum_{n=1}^{\infty} \frac{r_n\alpha_n}{2\pi N_n} 8_n(N_n t) e^{iA_{n-1}(t)} \right) \mathbf{t}_0(t).$$

Résultats en faveur de la conjecture 2

- We thus have

$$\mathbf{t}_k(t) = e^{iA_k(t)}\mathbf{t}_0(t) \quad \text{and} \quad f_k(t) = f_{k-1}(t) + \frac{r_k\alpha_k}{2\pi N_k}\delta_k(N_k t)e^{iA_{k-1}(t)}\mathbf{t}_0(t)$$

- Therefore

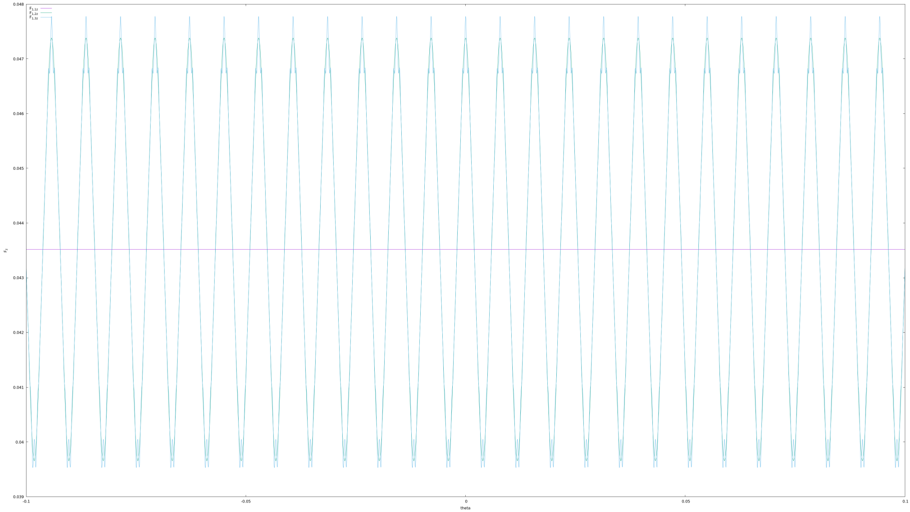
$$f_k(t) = f_0(t) + \left(\sum_{n=1}^{k} \frac{r_n\alpha_n}{2\pi N_n}\delta_n(N_n t)e^{iA_{n-1}(t)}\right)\mathbf{t}_0(t).$$

- Observe that the serie is normally converging. We have

$$f_\infty(t) = f_0(t) + \left(\sum_{n=1}^{\infty} \frac{r_n\alpha_n}{2\pi N_n}\delta_n(N_n t)e^{iA_{n-1}(t)}\right)\mathbf{t}_0(t).$$

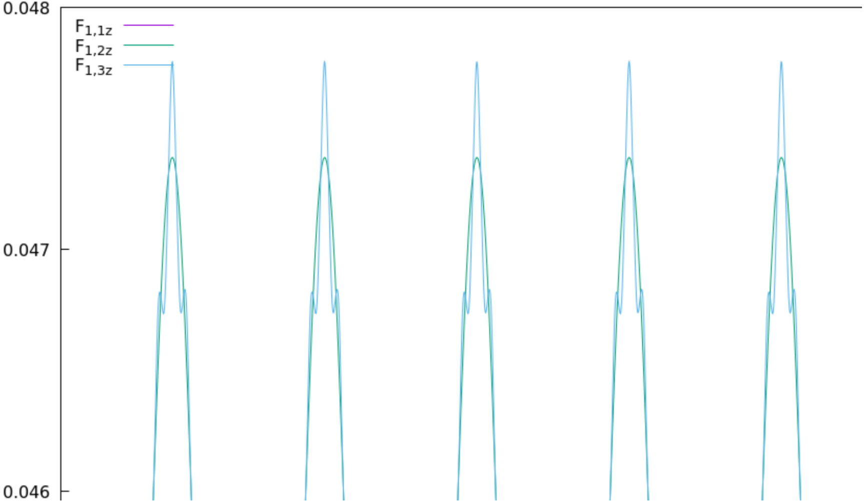
- We show (Proposition 30 and 31) that f_∞ is β-Hölder for every $\beta \in [0,1[$ and thus $\dim_H \Gamma(f_\infty) = 1$.

The maps $\mathcal{F}^z_{1,1}(1,\theta)$, $\mathcal{F}^z_{1,1}(1,\theta)$ and $\mathcal{F}^z_{1,3}(1,\theta)$



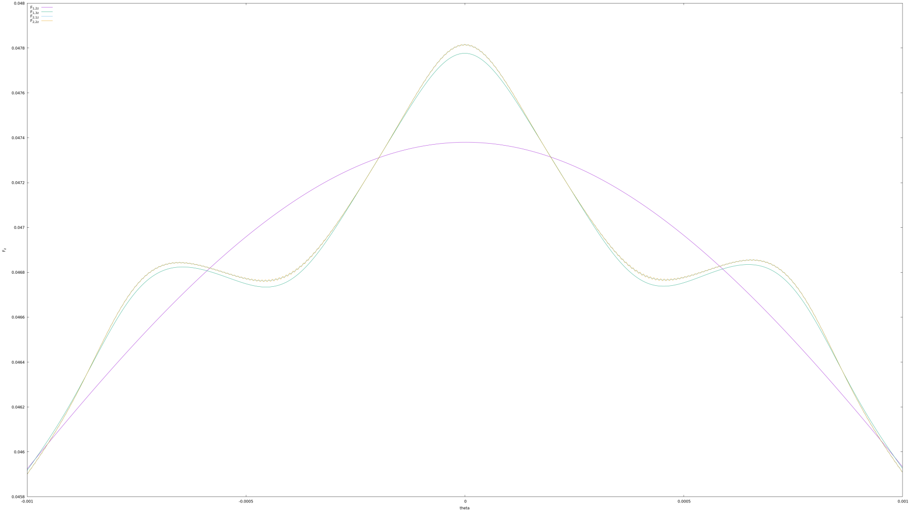
The maps $\mathcal{F}^z_{1,1}$ (purple) $\mathcal{F}^z_{1,2}$ (green) and $\mathcal{F}^z_{1,3}$ (blue) $N_{1,1} = 10$, $N_{1,2} = 100$, $N_{1,3} = 1000$.

The maps $\mathcal{F}^z_{1,1}(1,\theta)$, $\mathcal{F}^z_{1,1}(1,\theta)$ and $\mathcal{F}^z_{1,3}(1,\theta)$



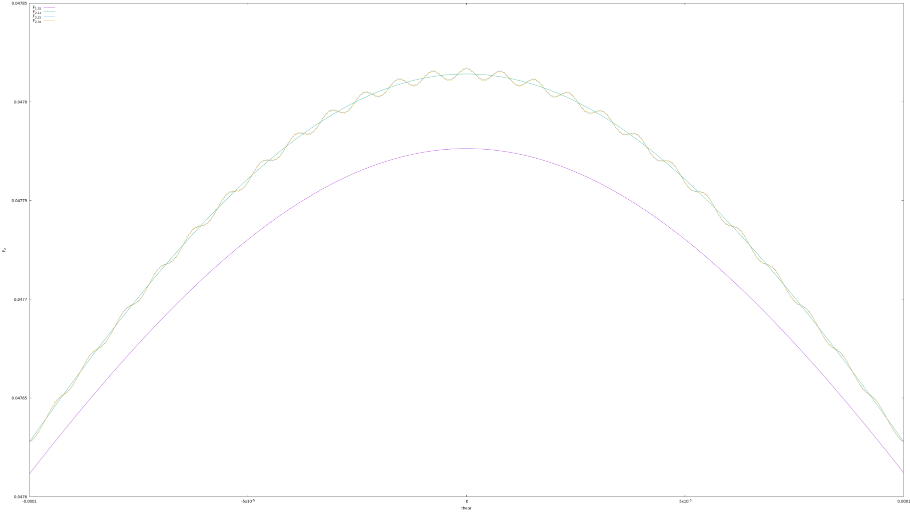
The maps $\mathcal{F}^z_{1,1}$ (purple) $\mathcal{F}^z_{1,2}$ (green) and $\mathcal{F}^z_{1,3}$ (blue) $N_{1,1} = 10$, $N_{1,2} = 100$, $N_{1,3} = 1000$.

The maps $\mathcal{F}^z_{1,2}(1,\theta)$, $\mathcal{F}^z_{1,3}(1,\theta)$ and $\mathcal{F}^z_{2,1}(1,\theta)$



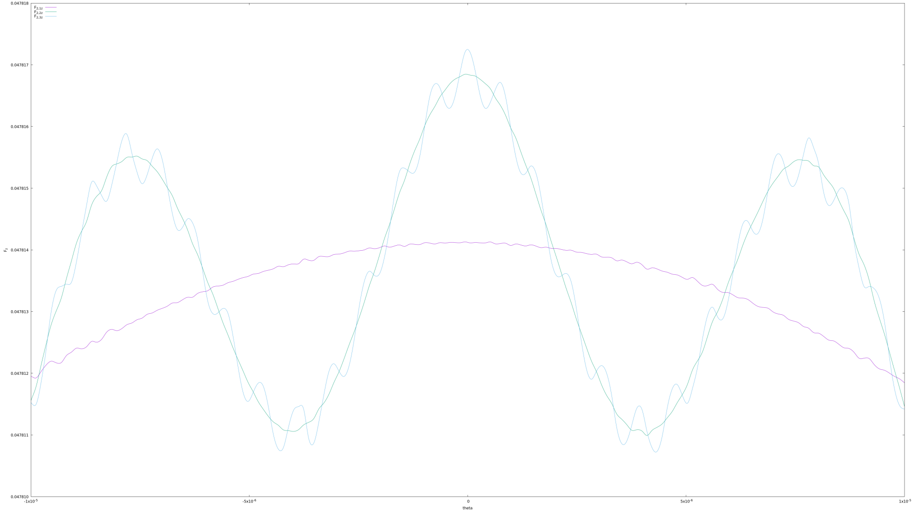
The maps $\mathcal{F}^z_{1,2}$ (purple) $\mathcal{F}^z_{1,3}$ (green) and $\mathcal{F}^z_{2,1}$ (yellow) $N_{1,1} = 10$, $N_{1,2} = 10^2$, $N_{1,3} = 10^3$, $N_{2,1} = 10^4$.

The maps $\mathcal{F}^z_{1,3}(1,\theta)$, $\mathcal{F}^z_{2,1}(1,\theta)$ and $\mathcal{F}^z_{2,2}(1,\theta)$



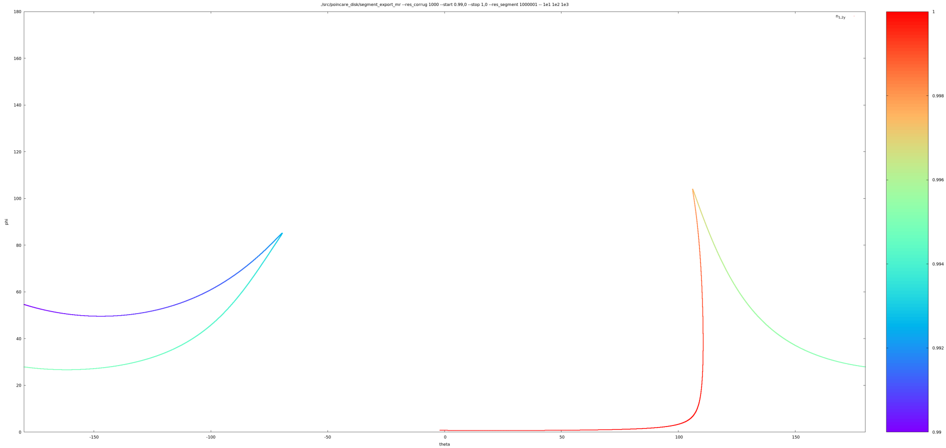
The maps $\mathcal{F}^z_{1,3}$ (purple) $\mathcal{F}^z_{2,1}$ (green) and $\mathcal{F}^z_{2,2}$ (yellow) $N_{1,1} = 10$, $N_{1,2} = 10^2$, $N_{1,3} = 10^3$, $N_{2,1} = 10^4$, $N_{2,2} = 10^5$.

The maps $\mathcal{F}^z_{2,1}(1,\theta)$, $\mathcal{F}^z_{2,2}(1,\theta)$ and $\mathcal{F}^z_{2,3}(1,\theta)$



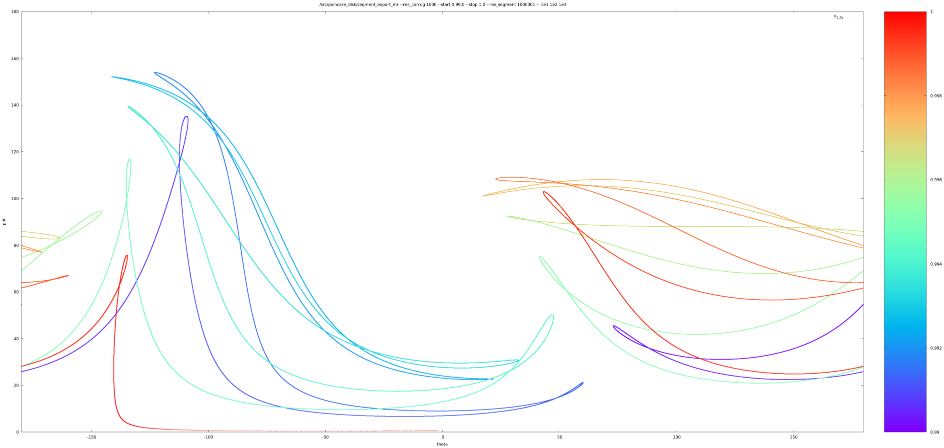
The maps $\mathcal{F}^z_{2,1}$ (purple) $\mathcal{F}^z_{2,2}$ (green) and $\mathcal{F}^z_{2,3}$ (yellow) $N_{1,1} = 10$, $N_{1,2} = 10^2$, $N_{1,3} = 10^3$, $N_{2,1} = 10^4$, $N_{2,2} = 10^5$, $N_{2,3} = 10^6$.

Comportement le long d'un rayon



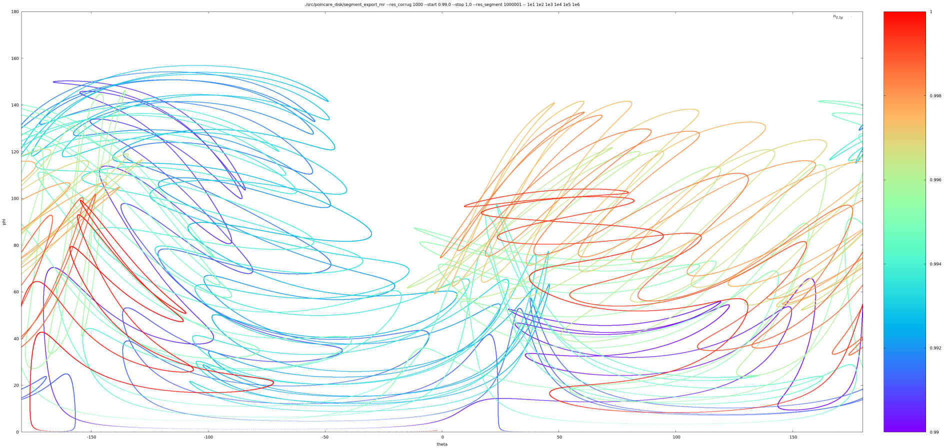
Application de Gauss de $\mathcal{F}_{12}$, $0.99 \leq \rho \leq 1$, $\theta = 0$

Comportement le long d’un rayon



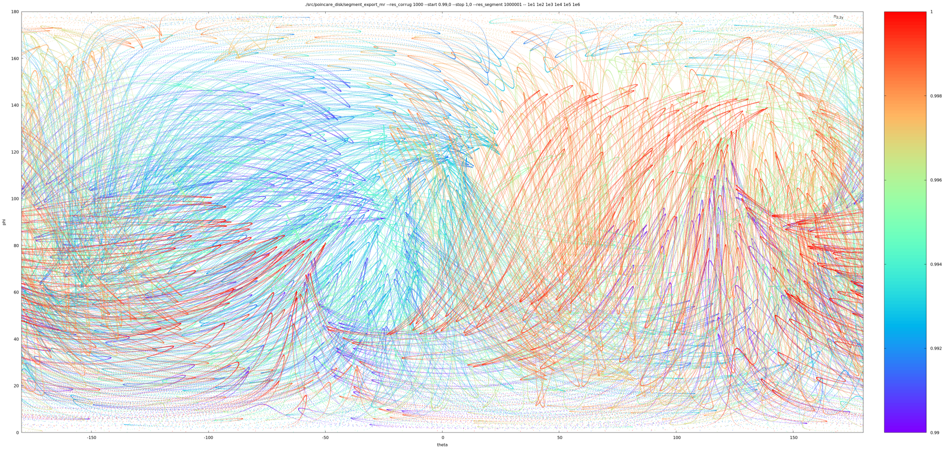
Application de Gauss de $\mathcal{F}_{13}$, $0.99 \leq \rho \leq 1$, $\theta = 0$

Comportement le long d'un rayon



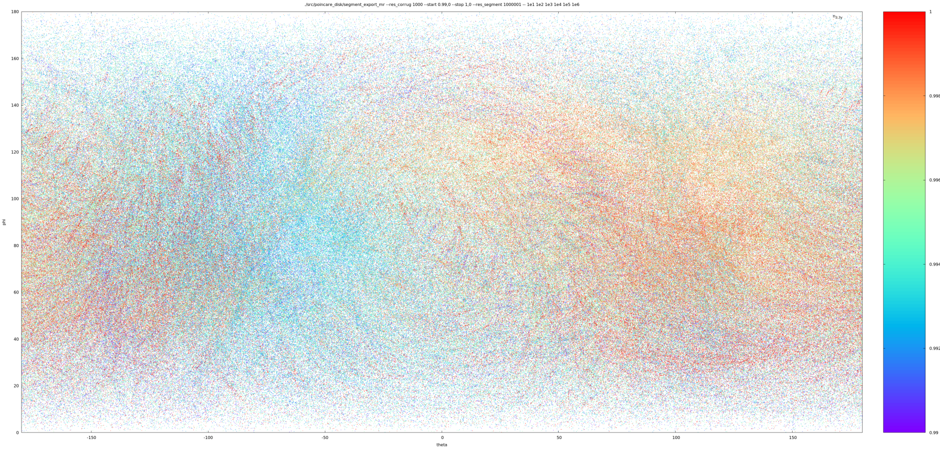
Application de Gauss de $\mathcal{F}_{21}$, $0.99 \leq \rho \leq 1$, $\theta = 0$

Comportement le long d’un rayon



Application de Gauss de $\mathcal{F}_{22}$, $0.99 \leq \rho \leq 1$, $\theta = 0$

Comportement le long d'un rayon



Application de Gauss de $\mathcal{F}_{23}$, $0.99 \leq \rho \leq 1$, $\theta = 0$