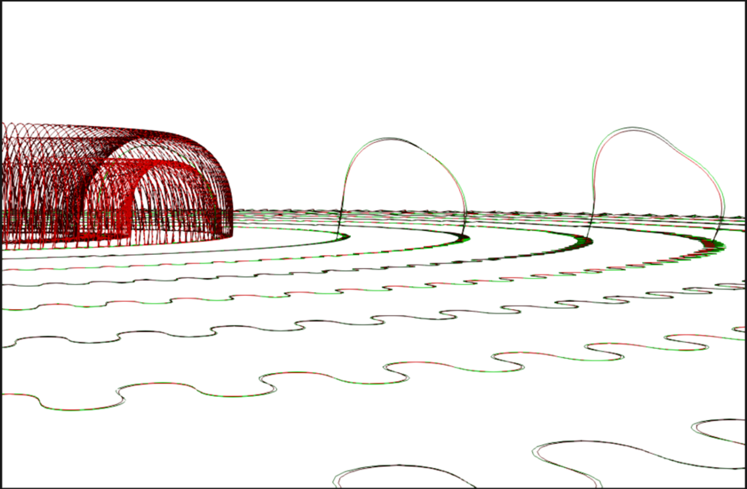
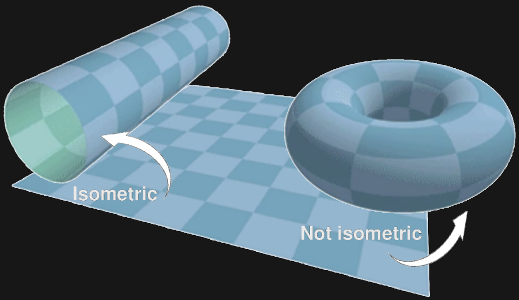
Limit sets of embedded Poincaré Disks



Vincent Borrelli - Univ. of Lyon - Hevea Project

Isometric embeddings



- A C^1 map

$$f : (M^n, g) \xrightarrow{C^1} \mathbb{E}^q$$

between a Riemannian manifold (M^n, g) and an Euclidean space $\mathbb{E}^q = (\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ is **isometric** it preserves the length of curves *i. e.*

$$\mathit{Length}(f \circ \gamma) = \mathit{Length}(\gamma)$$

for every C^1 piecewise parametrized curve $\gamma : [0, 1] \longrightarrow M^n$.

Janet's Dimension

- In a coordinate system, the isometric condition amounts to solve a **non linear PDE** system :

$$\text{For all} \quad 1 \leq i \leq j \leq n, \qquad \left\langle \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j} \right\rangle = g_{ij}$$

of $s_n = \frac{n(n+1)}{2}$ equations. The number s_n is called the *Janet's dimension*.

Janet-Cartan Theorem (1926)

Janet-Cartan Theorem (1926-27).– *Let* (M^n, g) *be a real-analytic Riemannian manifold. Every point of* M^n *has a neighborhood which has a real-analytic isometric embedding into* $\mathbb{E}^q$ *with* $q = s_n$.

Nash-Kuiper C^1 Embedding Theorem (1954)


John Forbes Nash and Nicolaas Kuiper

Définition.– A map $f : (M^n, g) \xrightarrow{C^1} \mathbb{E}^q$ is said to be *strictly short* if

$$f^*\langle ., . \rangle_{\mathbb{E}^q} \leq \Lambda g \quad \text{for some} \quad 0 < \Lambda < 1.$$

Nash-Kuiper C^1 Embedding Theorem (1954)

Theorem (1954-55-59)– *Let* $f_0 : (M^n, g) \xrightarrow{C^1} \mathbb{E}^q$*,* $q \geq n+1$*, be a strictly short embedding of a Riemannian manifold. Then, for every* $\epsilon > 0$*, there exists a* C^1 *isometric embedding* $f : (M^n, g) \longrightarrow \mathbb{E}^q$ *such that* $\|f - f_0\|_{C^0} \leq \epsilon$*.*

Limit set

Image : *Dimensions*, A. Alvarez, É. Ghys, J. Leys

- Let M^n be a non compact manifold, $f : M^n \to \mathbb{R}^q$ be a map and $(x_n)_{n\in\mathbb{N}}$ be a divergent sequence of points of M^n. If the sequence $(f(x_n))_{n\in\mathbb{N}}$ is convergent in $\mathbb{R}^q$, its limit is called a *limit point* of f.

Isometric embeddings with void limit set

Extension of the NK-Theorem (Kuiper, 1959).– *If there exists a closed strictly short embedding* $f_0 : (M^n, g) \to \mathbb{E}^q$ *then there exists a closed* C^1*-isometric embbeding* $f : (M^n, g) \to \mathbb{E}^q$.

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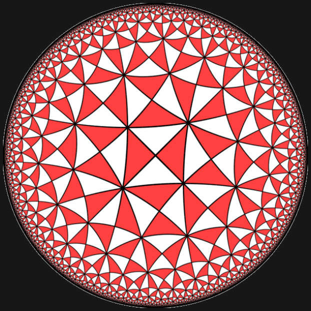
Corollary (Kuiper).– *The Hyperbolic Plane* $\mathbb{H}^2$ *has a* C^1*-isometric embedding in* $\mathbb{E}^3$ *which is closed.*

The Poincaré Disk Model



$$\mathbb{H}^2 = (Int\ D^2, h)$$

$$D^2 = \{(x_1, x_2) \in \mathbb{R}^2, x_1^2 + x_2^2 \leq 1\} \quad \text{and} \quad h = 4\frac{dx_1^2 + dx_2^2}{(1 - x_1^2 - x_2^2)^2}.$$

- The Gauss curvature of $\mathbb{H}^2$ is $K \equiv -1$.

Hilbert's Theorem & Efimov's Theorem

Theorem (Hilbert, 1901- Efimov, 1964).– *No surface can be C^2 immersed in Euclidean 3-space so as to be complete in the induced Riemannian metric, with Gauss curvature $K \leq const < 0$.*

• In particular, there is no C^2 isometric immersion of the Hyperbolic Plane $\mathbb{H}^2$ into $\mathbb{E}^3$.

Corollary (Recall).– *The Hyperbolic Plane $\mathbb{H}^2$ has a C^1-isometric embedding in $\mathbb{E}^3$ which is closed.*

Limit Set Extension of the Nash-Kuiper Theorem

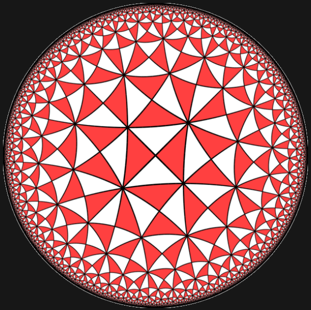
Limit Set Extension of the Nash-Kuiper Theorem (De Lellis 2017).– *Let* $f_0 : (M^n, g) \to \mathbb{E}^q$ *be a strictly short embedding then there exists a* C^1 *isometric embedding* $f : (M^n, g) \to \mathbb{E}^q$ *with same limit set* $L(f) = L(f_0)$*.*

A "Hyperbolic Sphere"

- It is easy to construct a strictly short embedding $f_0 : \mathbb{H}^2 \to \mathbb{E}^3$ such that $L(f_0)$ is a single point and $f_0(\mathbb{H}^2) \cup L(f_0)$ is homeomorphic to a 2-sphere.

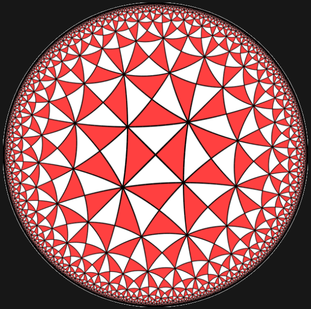
- As a consequence, there exists a "hyperbolic sphere" that is a C^1 isometric embedding $f : \mathbb{H}^2 \to \mathbb{E}^3$ such that $f(\mathbb{H}^2) \cup L(f)$ is homeomorphic to a 2-sphere.

Isometric embeddings with $L(f) = \mathbb{S}^1$



- The inclusion $f_0 : \mathit{Int}\ D^2 \subset \mathbb{E}^3$ is a strictly short embedding thus there exists a C^1 isometric embedding $f : \mathbb{H}^2 \to \mathbb{E}^3$ such that $L(f) = \mathbb{S}^1$.

Isometric immersions with $L(f) = \Gamma$



Here, we address the following question :

Given a Jordan curve $\Gamma \subset \mathbb{E}^3$, does it bound an C^1 immersed hyperbolic disk ?

The Hevea Team (current membership)

Roland Denis

Francis Lazarus

Mélanie Theillière

Boris Thibert

HEVEA : H-principlE, Visualization & Applications

Our Results

Result 1.– *Let Γ be any smooth immersed circle in Euclidean 3-space. There exists a map $f : D^2 \to \mathbb{E}^3$ such that*

1) *the restriction $f_{|Int\ D^2}$ to $Int\ D^2$ is a C^1 isometric immersion of $\mathbb{H}^2$*
2) $L(f_{|Int\ D^2}) = f(\partial D^2) = \Gamma$
3) *f is β-Hölder for any $0 < \beta < 1$*

Moreover, if Γ is the boundary of an embedded disk then f can be chosen to be an embedding.

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• Circles in Hyperbolic Plane can have arbitrarily large perimeter, however the length of Γ is finite : the length functional is lower semi-continuous.

Our Results

- Any point of Γ is at infinite distance of every other point of $f(\mathit{Int}\, D^2)$ for the induced distance of $\mathbb{E}^3$. Indeed any geodesic ray joining a point $p \in \mathit{Int}\, D^2$ to a point $p_\infty \in \partial D^2$ has infinite length as well as its image in $\mathbb{E}^3$.

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Result 1 bis.– *There exists a topological embedding of the sphere* $D^2/\partial D^2 \to \mathbb{E}^3$ *which is* β*-Hölder for any* $0 < \beta < 1$ *and whose restriction to* $(Int\ D^2, h)$ *is a* C^1 *isometric embedding.*

Our Results

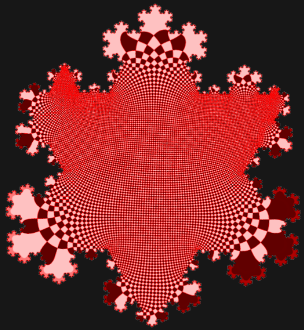

Image : N. Dym, Y. Lipman and R. Slutsky

Result 2.– *Let* $\Gamma \subset \mathbb{R}^2 \simeq \mathbb{C}$ *be a planar Jordan curve. Then there exists a topological embedding* $f : D^2 \to \mathbb{E}^3$ *such that*

1) *its restriction* $f_{|\,Int\ D^2}$ *to* $Int\ D^2$ *is a* C^1 *isometric embedding of* $\mathbb{H}^2$
2) $L(f_{|\,Int\ D^2}) = f(\partial D^2)$ *is homothetic to* Γ

Our Results

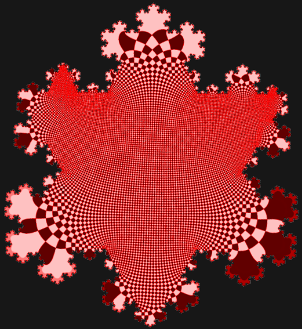

Image : N. Dym, Y. Lipman and R. Slutsky

In particular, there exists a C^1 isometric embedding of the Hyperbolic disk bounded by a Koch snowflake.

Our Results

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- This Riemannian structure on U does not depend on the biholomorphic map f_0. Indeed, if f_1 is another biholomorphic map then $f_1^{-1} \circ f_0$ is an holomorphic automorphism of $Int\, D^2$, and such a map is isometric for h.

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- Result 2 is about the realization of $U \subset \mathbb{C}$ with its hyperbolic metric as a C^1 surface of $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ with boundary $\partial U = \Gamma$.

Proof of Result 1

Result 1 (recall).– *Let Γ be any smooth immersed circle Γ in Euclidean 3-space. There exists a map $f : D^2 \to \mathbb{E}^3$ such that*

1) *the restriction $f_{|Int\ D^2}$ to Int D^2 is a C^1 isometric immersion of $\mathbb{H}^2$*
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• Except for Point 3, the proof of Result 1 amounts to construct an immersion $f_0 : D^2 \to \mathbb{E}^3$ such that

a) f_0 is stricly short : $f_0^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3} \leq \Lambda h$ for some $0 < \Lambda < 1$.
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• Indeed, once this short map is obtained, the Limit Set Extension of the Nash-Kuiper Theorem ensures that there is a map $f : D^2 \to \mathbb{E}^3$ such that

a) $f_{|Int\ D^2}$ is a C^1 isometric immersion of $\mathbb{H}^2$
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Short maps

- We now focus on the construction of an immersion $f_0 : D^2 \to \mathbb{E}^3$ such that
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- Let $\gamma : \mathbb{S}^1 \to \Gamma \subset \mathbb{R}^3$ be a smooth immersion of the circle. By standard arguments of the theory of immersions, the map γ extends to a smooth immersion $g : D^2 \to \mathbb{R}^3$.

Short maps

- Since D^2 is compact, there exists $C > 0$ such that

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- Recall that

$$h = \frac{4}{(1-\rho^2)^2}\langle\cdot,\cdot\rangle_{\mathbb{R}^2}$$

thus for $\rho = (x_1^2 + x_2^2)^{1/2}$ close enough to 1, the map g is short.

Short maps

• We build a strictly short map $f_0 = g \circ \Phi$ by composing g with a radial C^∞ diffeomorphism $\Phi : D^2 \to D^2$:

$$\begin{cases} \Phi(x,y) &= \dfrac{\alpha(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}(x,y) \quad \text{if } (x,y) \neq (0,0) \\ \Phi(0,0) &= (0,0) \quad \text{else.} \end{cases}$$

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- A computation shows that

$$f_0^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3} = (g \circ \Phi)^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3} \leq C \left(\alpha'^2(\rho) + \frac{\alpha^2(\rho)}{\rho^2} \right) \langle \cdot, \cdot \rangle_{\mathbb{R}^2}.$$

Short maps

- To get a short map, it is enough to find $\alpha : [0,1] \to [0,1]$ such that

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solution.

- Let $c > 1$. We consider the sum

$$s_k = \sum_{j=0}^{k} \frac{1}{2j+1}.$$

The series $\sum \frac{1}{2j+1}$ being divergent, there exists an index K such that

$$s_K < c \leq s_{K+1}.$$

Short maps

- We define a familly of increasing C^∞ diffeomorphisms $\alpha_c : [0,1] \to [0,1]$ by putting

$$\alpha_c(\rho) := \frac{1}{c}\left\{\left(\sum_{j=0}^{K} \frac{\rho^{2j+1}}{2j+1}\right) + (c - s_K)\rho^{2(K+1)+1}\right\}$$

for all $\rho \in [0,1]$.

Short maps

- It is then easy to check that if $c \geq \sqrt{\frac{C}{2\Lambda}}$ then α_c is a solution of

$$C\left(\alpha'^2(\rho) + \frac{\alpha^2(\rho)}{\rho^2}\right) \leq \frac{4\Lambda}{(1-\rho^2)^2}$$

and consequently the corresponding map f_0 is strictly short. $\square$

The Nash-Kuiper strategy in a nutshell

- We now focus on Point 3 of Result 1, that is the β-Hölder regularity of $f : D^2 \to \mathbb{E}^3$ for any $0 < \beta < 1$.

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• We first describe this method when the source space is a (small enough) compact disk $B^2 \subset \mathbb{R}^2$ endowed with a given metric g.

• Let $f_0 : (B^2, g) \to \mathbb{E}^3$ be a (strictly) short embedding and let

$$\Delta_0 := g - f_0^*\langle \cdot, \cdot \rangle_{\mathbb{R}^3} > 0$$

be its isometric default.

The Nash-Kuiper strategy in a nutshell

- Let $(g_k)_{k\geq 1}$ be the sequence of increasing intermediary metrics given by

$$g_k := f_0^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3} + \left(1 - \frac{1}{2^k}\right)\Delta_0.$$

This sequence obviously converges toward g.

The Nash-Kuiper strategy in a nutshell

- The Nash-Kuiper method builds iteratively a sequence of C^∞ embeddings/immersions

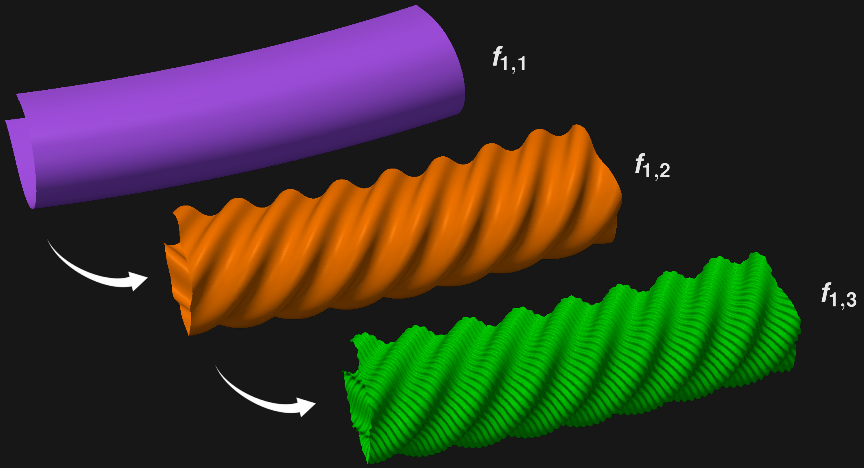
$$f_0, \quad f_{1,1}, f_{1,2}, f_{1,3}, \quad f_{2,1}, f_{2,2}, f_{2,3}, \quad \dots$$

such that

$$f_{k,3}^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3} \approx g_k \quad \text{and} \quad \Delta_k := g_{k+1} - f_{k,3}^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3} > 0.$$

The Nash-Kuiper strategy in a nutshell

- Given f_0, the maps $f_{1,1}, f_{1,2}, f_{1,3}$ are built iteratively by stacking layers of corrugations.



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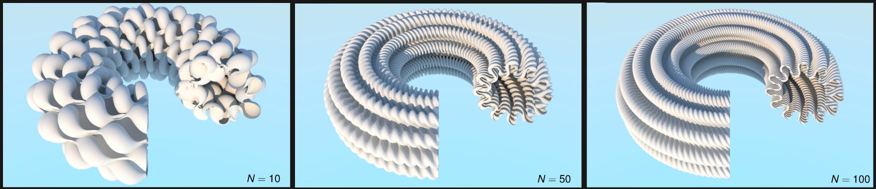


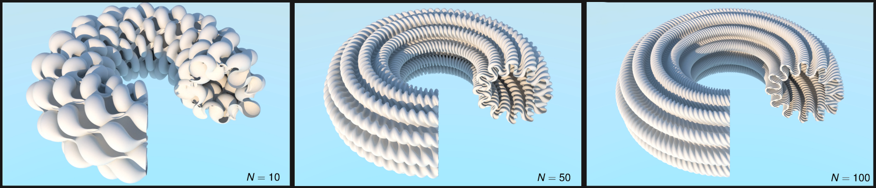
- C^0**-density.–** The numbers of corrugations

$$N_{1,1}, \quad N_{1,2}, \quad N_{1,3}$$

control the C^0 closeness of $f_{1,3}$ to f_0 :

$$\|f_{1,3} - f_0\|_{C^0(B^2)} = O\left(\frac{1}{N_{1,1}}\right) + O\left(\frac{1}{N_{1,2}}\right) + O\left(\frac{1}{N_{1,3}}\right).$$

The Nash-Kuiper strategy in a nutshell



- **Approximation of the metric.–** The numbers of corrugations

$$N_{1,1}, \quad N_{1,2}, \quad N_{1,3}$$

also control the C^0-closeness of $f_{1,3} : (B^2, g_1) \to \mathbb{E}^3$ to an isometric map :

$$\|f_{1,3}^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3} - g_1\|_{C^0(B^2)} = O\left(\frac{1}{N_{1,1}}\right) + O\left(\frac{1}{N_{1,2}}\right) + O\left(\frac{1}{N_{1,3}}\right).$$

The Nash-Kuiper strategy in a nutshell

- Since $g_2 > g_1$, if the approximation $g_1 \approx f_{1,3}^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3}$ is good enough, we can deduce

$$g_2 - f_{1,3}^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3} > 0$$

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- It turns out that the parameters of this construction can be chosen so that

$$\|df_{k+1} - df_k\|_{C^0(B^2)} \leq Cte. \|g_{k+1} - g_k\|_{C^0(B^2)}^{\frac{1}{2}}$$

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- Thus the convergence of

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- From the definition of $g_k = f_0^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3} + \left(1 - \frac{1}{2^k}\right) \Delta_0$, we deduce

$$\|g_{k+1} - g_k\|_{C^0(B^2)} \leq \frac{1}{2^{k+1}} \|\Delta_0\|_{C^0(B^2)}.$$

The Nash-Kuiper strategy in a nutshell

- Thus the convergence of

$$\sum \|g_{k+1} - g_k\|_{C^0(B^2)}^{\frac{1}{2}}$$

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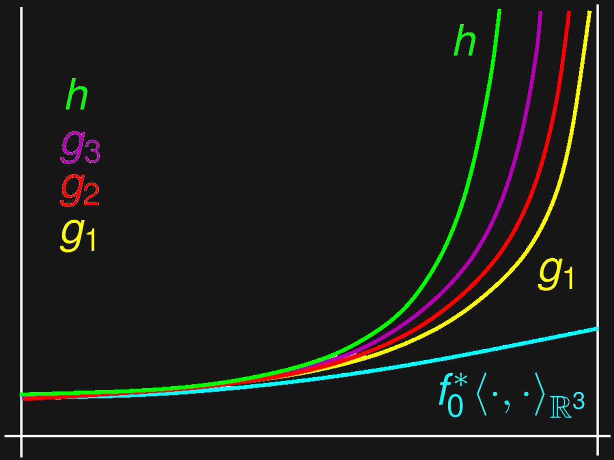
- It follows that

$$\sum \|g_{k+1} - g_k\|_{C^0(B^2)}^{\frac{1}{2}} < +\infty$$

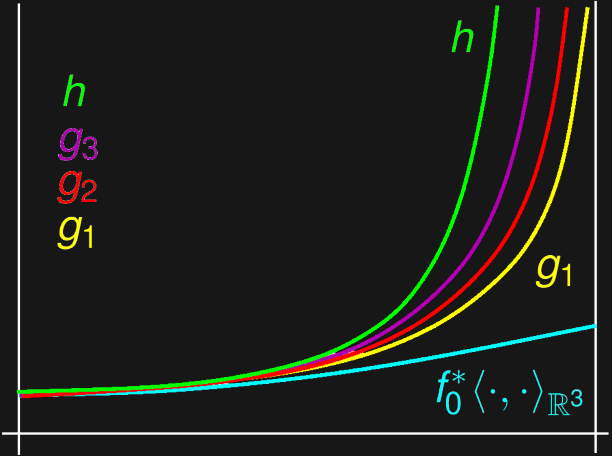
implying the C^1 convergence towards a C^1-isometric map f.

$\mathbb{H}^2 \hookrightarrow \mathbb{E}^3$: The Non Compactness Issue



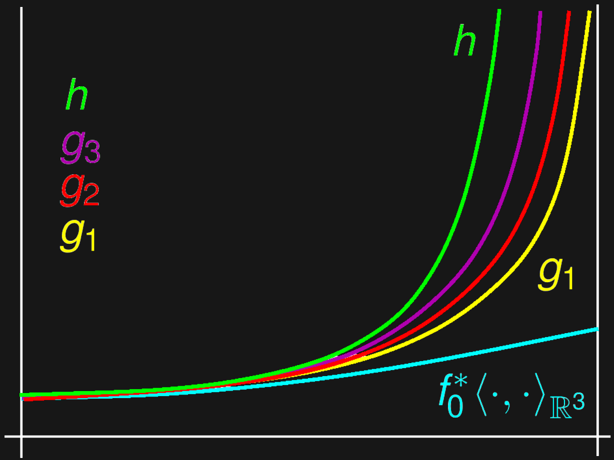
- The Hyperbolic metric $h = 4\dfrac{dx_1^2 + dx_2^2}{(1-\rho^2)^2}$ explodes as $\rho \to 1$.

$\mathbb{H}^2 \hookrightarrow \mathbb{E}^3$: The Non Compactness Issue



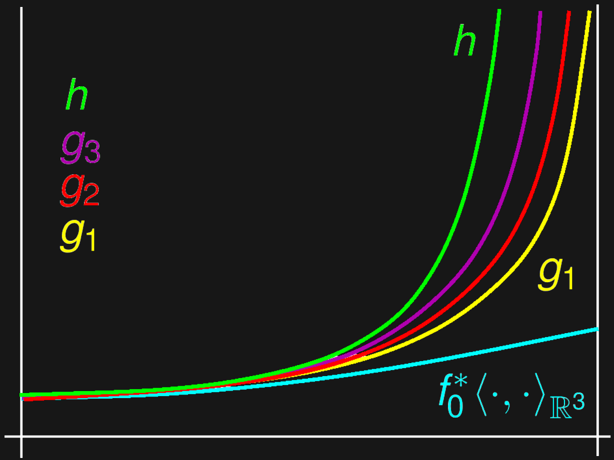
- This is not the case of the pullback metric $f_0^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3}$.

$\mathbb{H}^2 \hookrightarrow \mathbb{E}^3$: The Non Compactness Issue



- At the boundary ∂D^2, all the intermediary metrics are exploding.

$\mathbb{H}^2 \hookrightarrow \mathbb{E}^3$: The Non Compactness Issue



- In particular $\|\Delta_0\|_{C^0(\mathbb{H}^2)} = +\infty$ and $\|g_{k+1} - g_k\|_{C^0(\mathbb{H}^2)} = +\infty$.

Kuiper construction of embeddings of $\mathbb{H}^2 \hookrightarrow \mathbb{E}^3$

- To circumvent this difficulty, Nash and Kuiper use an infinite covering of $\mathbb{H}^2$ by neighborhoods so that each neighborhood intersects at most finitely many other elements of the cover (other conditions are also required but we will ignore them).

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- "The sequence of steps [...] would be infinite, but any compact portion would be effected by only a finite number of them. So this particular infiniteness does not raise any convergence problems."

- Here, we are going to avoid the use of a covering to obtain a direct control of the regularity of f on $D^2 = Int\, D^2 \cup \partial D^2$.

Our construction of $\mathbb{H}^2 \hookrightarrow \mathbb{E}^3$

- We consider an exhaustion of *Int* D^2 by a stricty increasing sequence

$$D_0 \subset D_1 \subset \cdots \subset D_n \subset \cdots \subset \mathit{Int}\ D^2$$

of concentric disks of radii

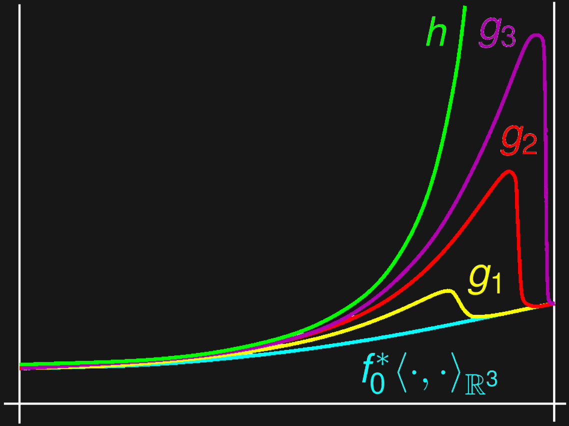
$$\rho_n = \sqrt{1 - \frac{1}{n+2}}.$$

Our construction of $\mathbb{H}^2 \hookrightarrow \mathbb{E}^3$

- Let $\Upsilon_n : D^2 \to [0, 1]$ be smooth cutoff functions such that

$$(\Upsilon_n)_{|D_n} \equiv 1 \quad \& \quad \text{support of } \Upsilon_n = \mathit{Int}\, D_{n+1}.$$

Our construction of $\mathbb{H}^2 \hookrightarrow \mathbb{E}^3$



- We define

$$g_n := f_0^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3} + (1 - \frac{1}{2^n})\Upsilon_n \Delta_0 \quad \text{with} \quad \Delta_0 = h - f_0^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3}.$$

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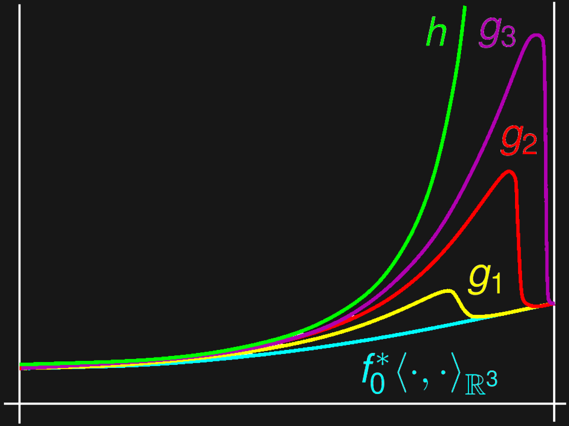
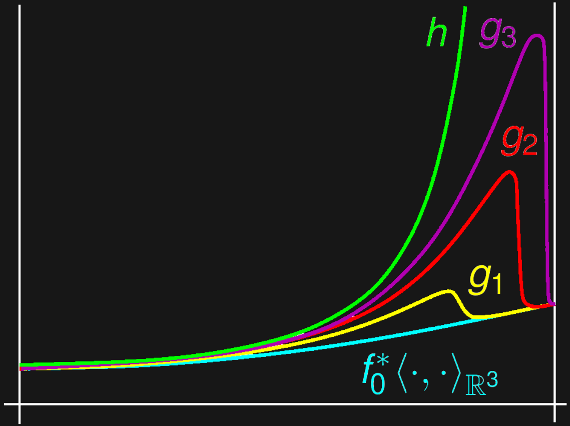


- Observe that each g_n is well defined on the **closed disk** D^2 and is equal to $f_0^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3}$ over the boundary ∂D^2.

Our construction of $\mathbb{H}^2 \hookrightarrow \mathbb{E}^3$



- This brings us back to a compact setting !

C^0 Convergence

- Let (τ_k) be a decreasing sequence of positive numbers such that

$$\sum_{k=1}^{\infty} \tau_k < +\infty$$

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- By the C^0-density property, we can choose the $N_{k,i}$ such that

$$\|f_k - f_{k-1}\|_{C^0(D^2)} \leq \tau_k.$$

It ensues that the sequence f_k is C^0 converging on $D^2 = Int\ D^2 \cup \partial D^2$.

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- Let f be the limit. Since for all k we have $g_k = f_0^*\langle \cdot, \cdot \rangle_{\mathbb{R}^3}$ over ∂D^2, at each iteration, we can left f_0 unchanged over ∂D^2

$$f(\partial D^2) = f_0(\partial D^2) = \Gamma.$$

C^1 Convergence

- A direct computation shows that for every disk D_n, we have

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- Since

$$\sum \|df_{k+1} - df_k\|_{C^0(D_n)} \leq Cte. \sum \|g_{k+1} - g_k\|_{C^0(D_n)}^{\frac{1}{2}}$$

it ensues that the maps f_k converge over $Int\ D^2 = \bigcup_{n\geq 1} D_n$ toward a C^1 isometric map.

Hölder Regularity

- We apply the following classical interpolation inequality :

$$\|F\|_{C^{0,\beta}} \leq \|F\|_{C^1}^{\beta} \|F\|_{C^0}^{1-\beta}$$

where

$$\|F\|_{C^{0,\beta}} = \sup_{x \neq y} \frac{|F(y) - F(x)|}{|y - x|^{\beta}} \quad \text{and} \quad \|F\|_{C^1} = \|F\|_{C^0} + \|dF\|_{C^0}.$$

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- We use it on D^2 with $F = f_k - f_{k-1}$. Observe that by our choice of the corrugation numbers

$$\|f_k - f_{k-1}\|_{C^0(D^2)} \leq \tau_k.$$

Hölder Regularity

- A direct computation shows that on the whole disk D^2 and for all $k \geq 1$ we have

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- From the fact that

$$\|f_k - f_{k-1}\|_{C^1(D^2)} \leq Cte.\|g_k - g_{k-1}\|^{\frac{1}{2}}_{C^0(D^2)}$$

we deduce

$$\|f_k - f_{k-1}\|_{C^1(D^2)} \leq Cte_1.(k+2)$$

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- So, the maps f_k are $C^{0,\beta}$ converging towards f for every $0 < \beta < 1$. $\square$

Proof of Result 2

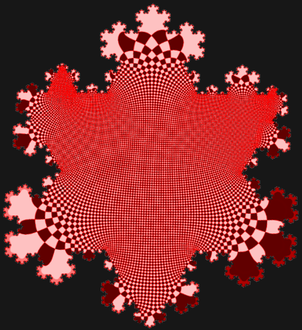
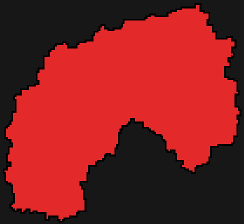


Image : N. Dym, Y. Lipman and R. Slutsky

Result 2 (recall).– *Let* $\Gamma \subset \mathbb{R}^2 \simeq \mathbb{C}$ *be a planar Jordan curve. Then there exists a topological embedding* $f : D^2 \to \mathbb{E}^3$ *such that*

1) *its restriction* $f_{|\,Int\,D^2}$ *to* $Int\,D^2$ *is a* C^1 *isometric immersion of* $\mathbb{H}^2$
2) $L(f_{|\,Int\,D^2}) = f(\partial D^2)$ *is homothetic to* Γ

Some Mega Famous Theorems



The Jordan-Schoenflies Theorem (1887 and 1914 ?).– Let $\Gamma \subset \mathbb{R}^2$ be a Jordan curve. Then $\mathbb{R}^2 \setminus \Gamma$, consists of exactly two connected components, one bounded (the interior U) and the other unbounded (the exterior V). The curve Γ is the boundary of each component. Moreover U and V are homeomorphic to the inside and outside of a standard circle.

Some Mega Famous Theorems

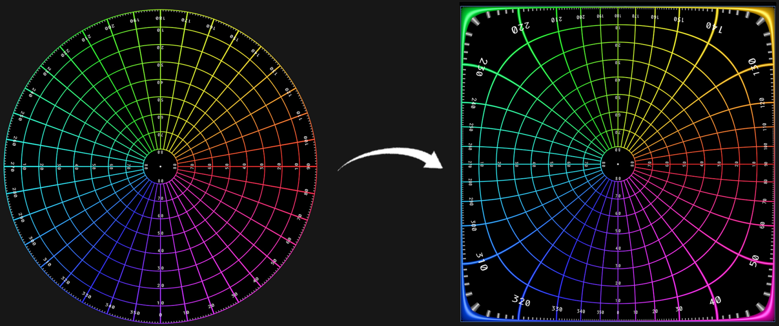


Image : Paul Bourke

The Riemann Mapping Theorem (1851 to 1900).– If $U \subset \mathbb{C}$ is a non-empty simply connected open subset which is not all of $\mathbb{C}$, then there exists a biholomorphic mapping $f_0 : Int\, D^2 \rightarrow U$.

Some Mega Famous Theorems

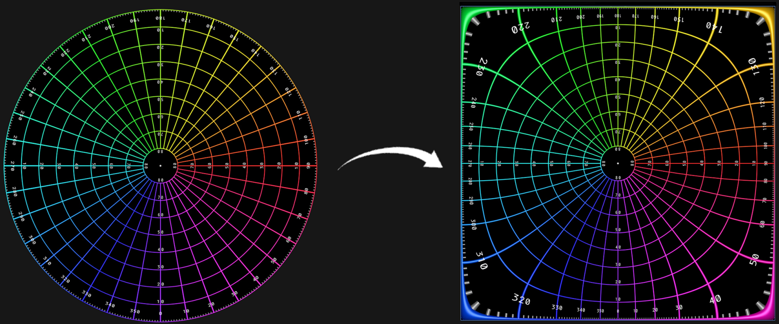


Image : Paul Bourke

Carathéodory Theorem (1907).– The function f_0 has a continuous and injective extension to $D^2 = \mathit{Int}\, D^2 \cup \partial D^2$ if and only if $\Gamma = \partial U$ is a Jordan curve.

Proof of Result 2

- The idea is to prove that the map $f_0 : Int\ D^2 \to U$ given by the RMT is (strictly) short and then to apply the Limit Set Extension of the Nash-Kuiper Theorem.

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• Since $h = \frac{4}{(1-|z|^2)^2}\langle \cdot, \cdot \rangle_{\mathbb{R}^2}$, the map f_0 is short if and only if for every $z \in \mathit{Int}\ D^2$

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for some $0 < \Lambda < 1$.

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- The fundamental space in the study of how the derivative $f_0'(z)$ explodes is the **little Bloch space.**

Proof of Result 2

The little Bloch space.– The space

$$\mathcal{B}_0 = \{f : \mathit{Int}\ D^2 \xrightarrow{\mathit{holo}} \mathbb{C} \mid \lim_{|z|\to 1^-} (1-|z|^2)|f'(z)| = 0\}$$

is called the *little Bloch space*. It is a Banach space with the norm

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in \mathit{Int}\ D^2} (1-|z|^2)|f'(z)|.$$

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A corollary of the Koebe Distorsion Lemma.– Let $f_0 : D^2 \to \mathbb{C}$ be analytic in $Int\ D^2$ and continuous in $Int\ D^2 \cup \partial D^2$ then $f_0 \in \mathcal{B}_0$ i. e.

$$\lim_{|z|\to 1^-} (1 - |z|^2)|f_0'(z)| = 0.$$

Proof of Result 2

- We can always assume $f_0(0) = 0$ so that

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- Thus, f_0 is strictly short if and only if

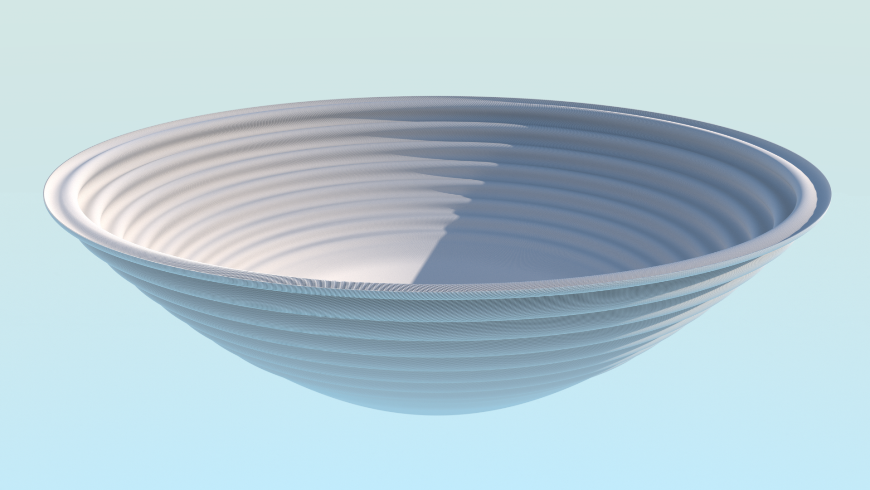
$$|f_0'(z)|^2 \leq \frac{\Lambda}{(1 - |z|^2)^2} \iff (1 - |z|^2)|f_0'(z)| \leq \sqrt{\Lambda}.$$

It ensues that if $\|f_0\|_{\mathcal{B}} < 1$ then f_0 is strictly short. Otherwise the map

$$\frac{f_0}{2\|f_0\|_{\mathcal{B}}}$$

is strictly short and has $\dfrac{1}{2\|f_0\|_{\mathcal{B}}}\Gamma$ for limit set. $\square$

Some Pictures



Result 3.– An explicit C^1 isometric embedding of $\mathbb{H}^2$ with a limit set of β-Hölder regularity for any $0 < \beta < 1$ (but not C^1). $N_{1,1} = 20$.

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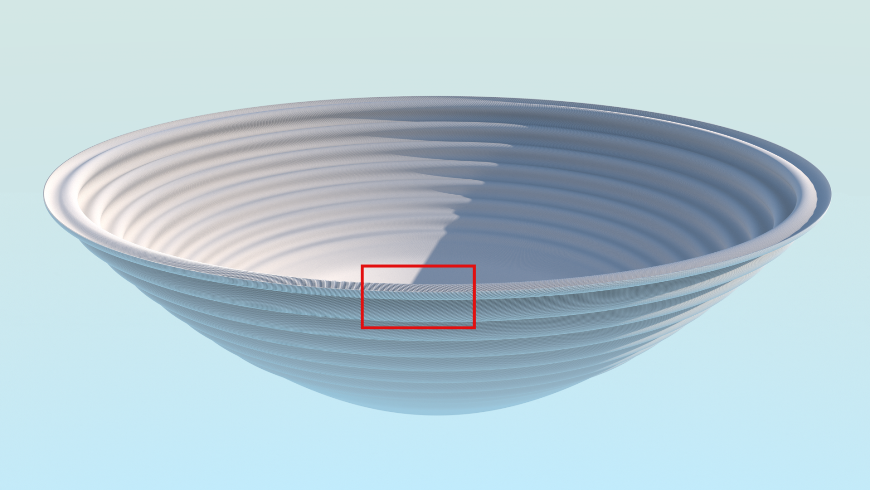

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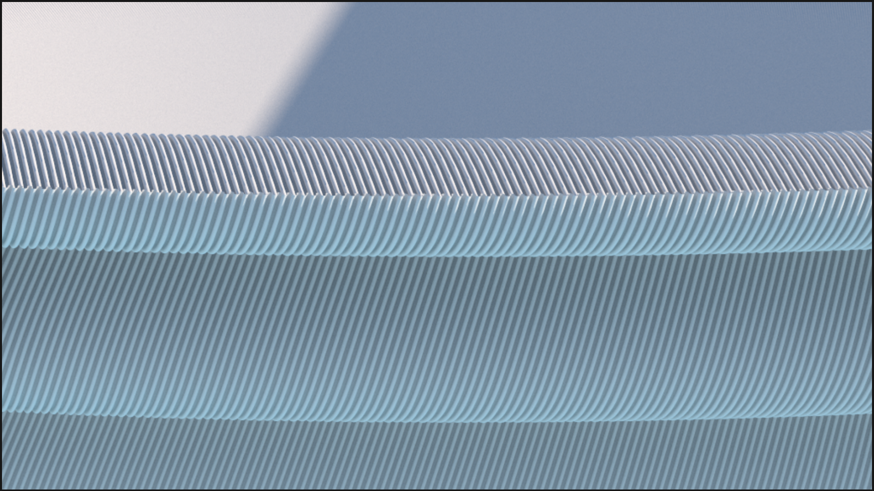

A first zoom to make visible the second layer of corrugations, $N_{1,2} = 400$.

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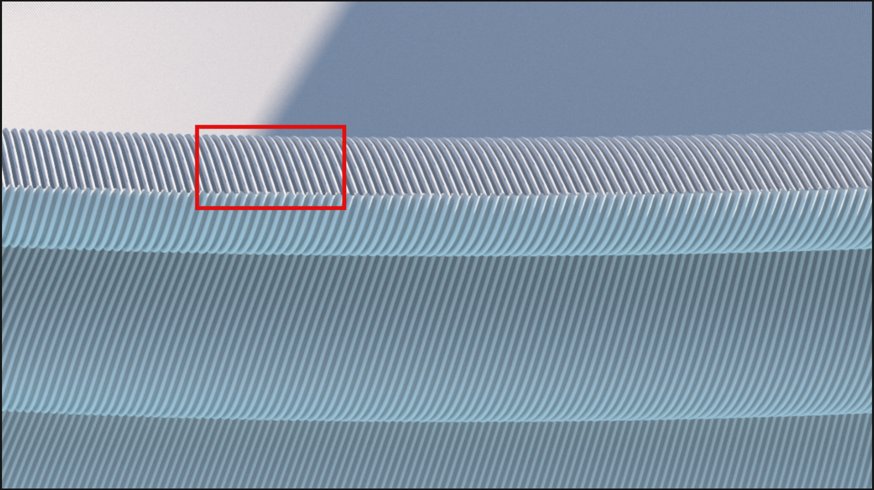

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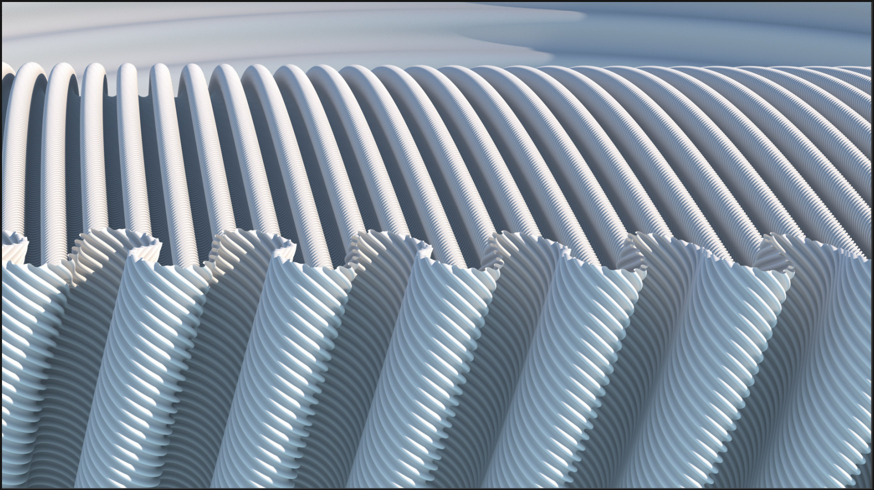

Another zoom makes visible that the limit set is not a planar curve, $N_{1,3} = 8000$.

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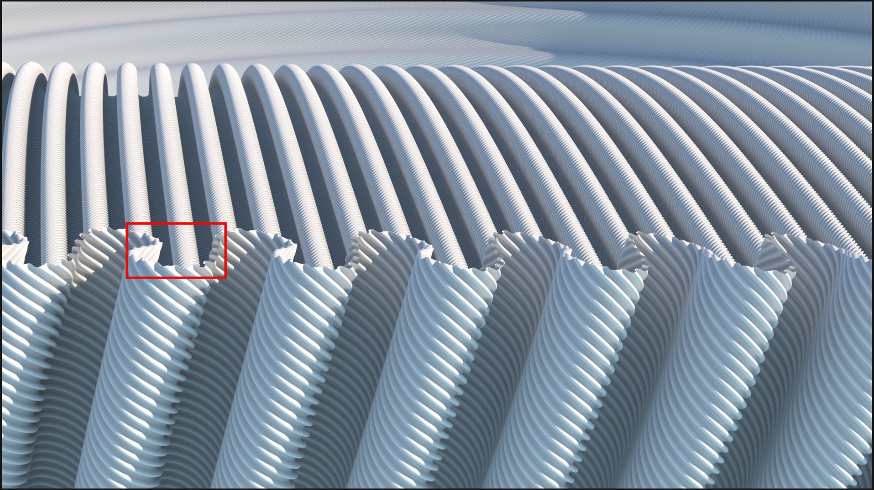

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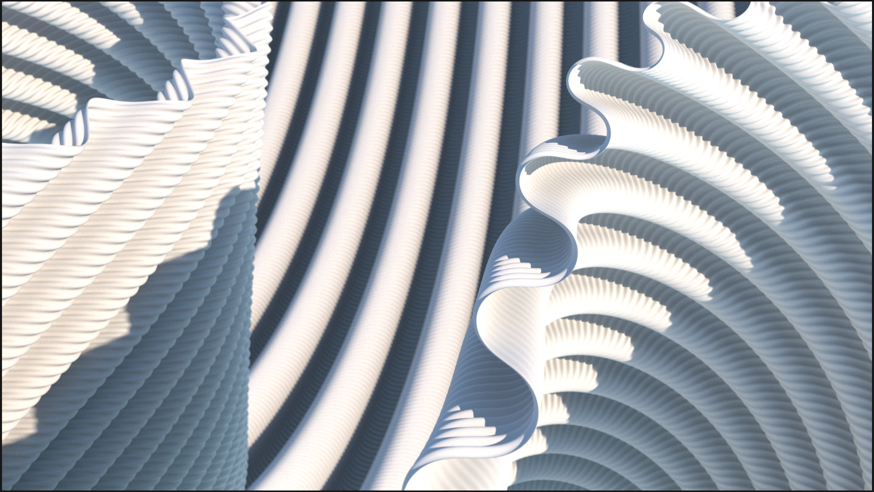

The analytic expression of the limit set curve is similar to a lacunary Fourier series
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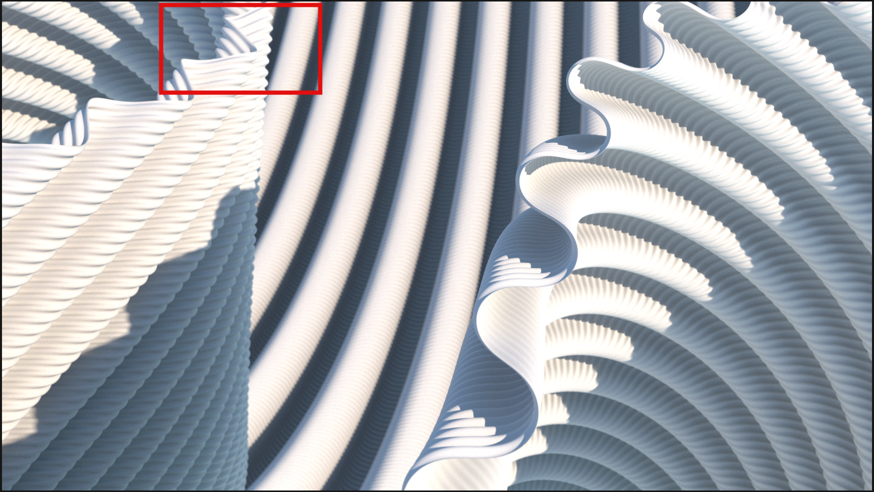

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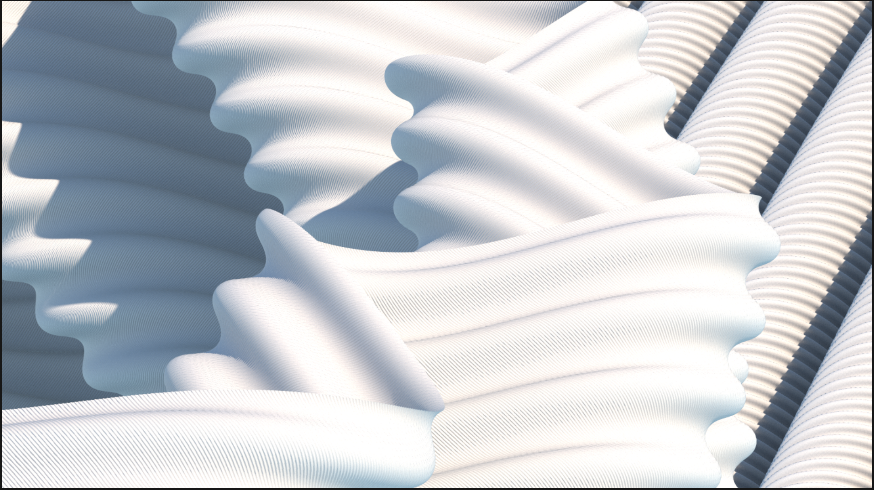
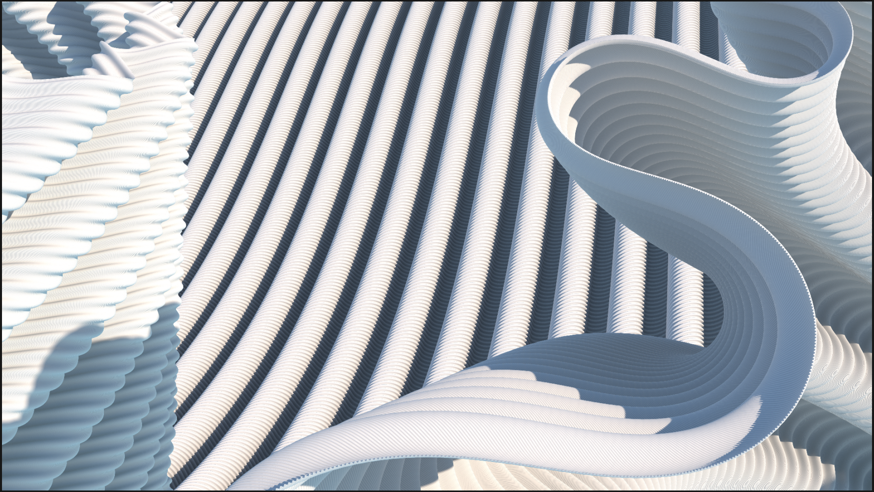


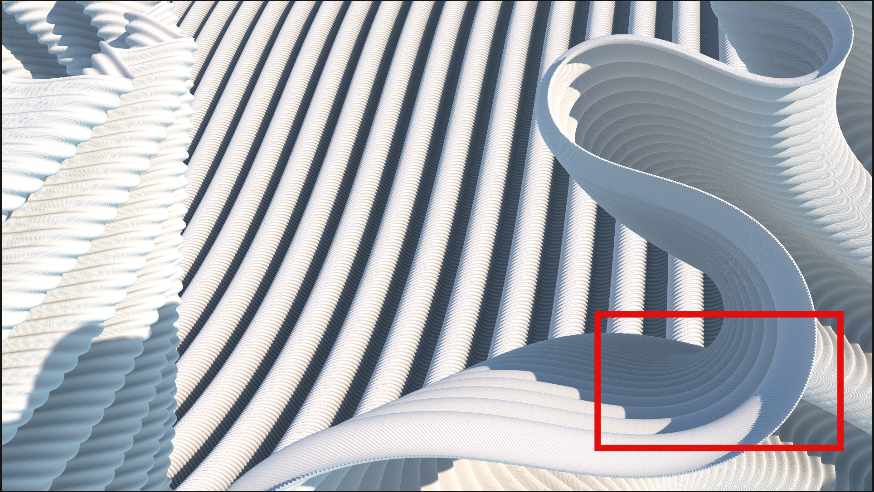
The analytic expression of the limit set curve is similar to a lacunary Fourier series $N_{2,2} = 3200000$.

Some Pictures



A zoom out to select another spot

Some Pictures



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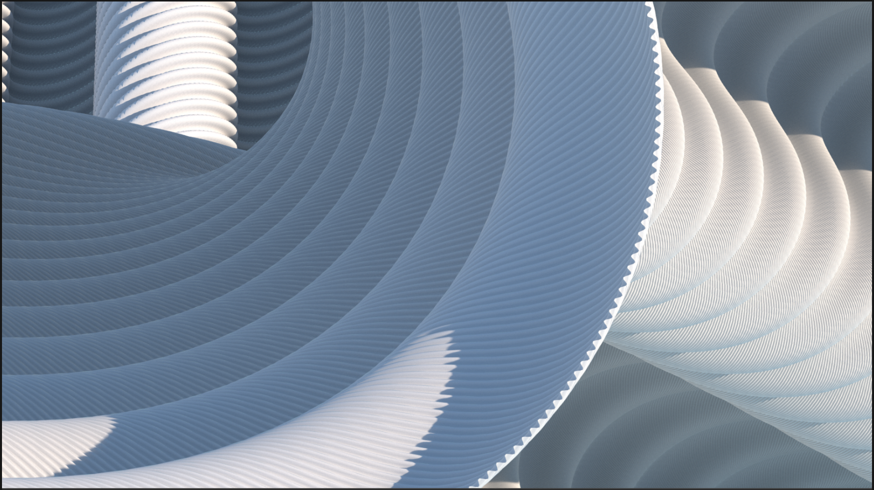

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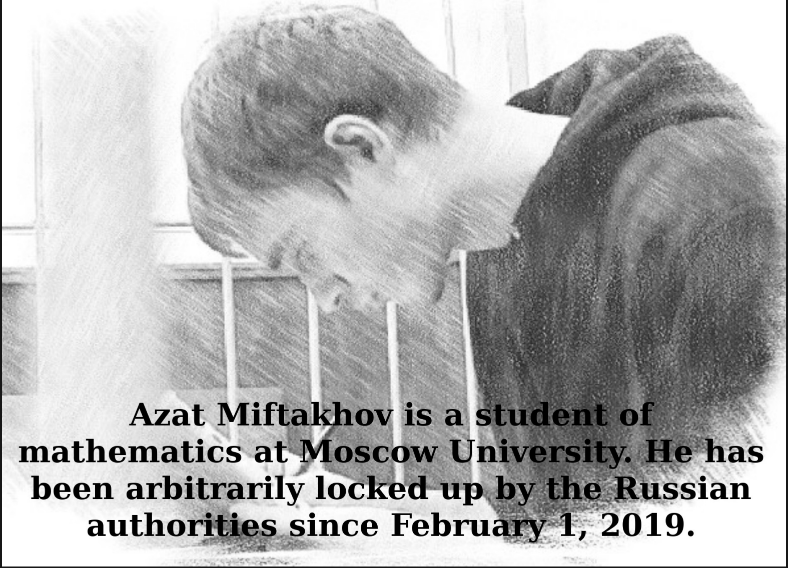
Thank you for your attention !

Azat Miftakhov



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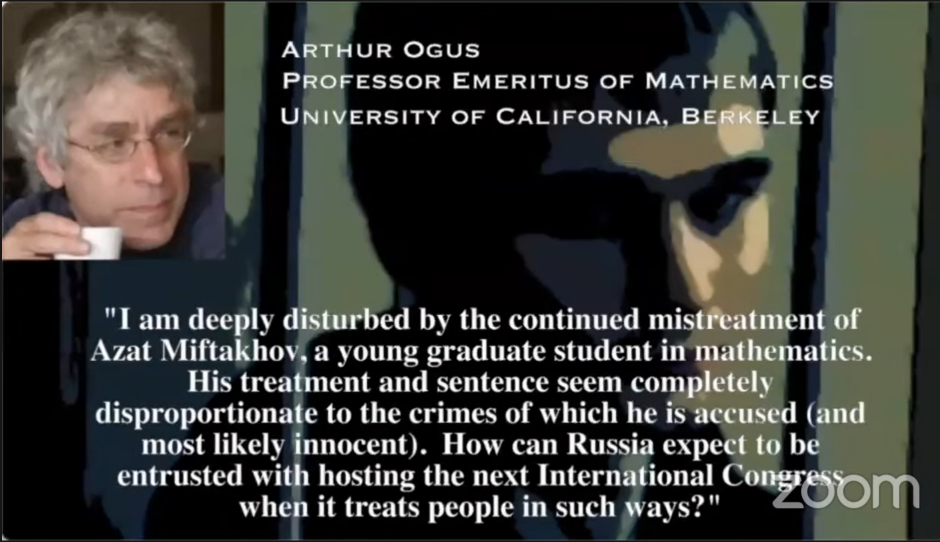

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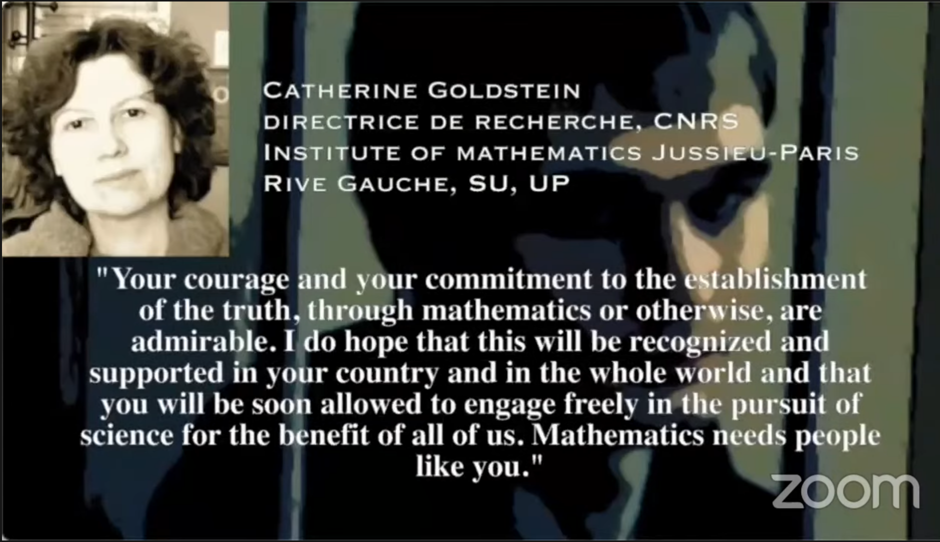

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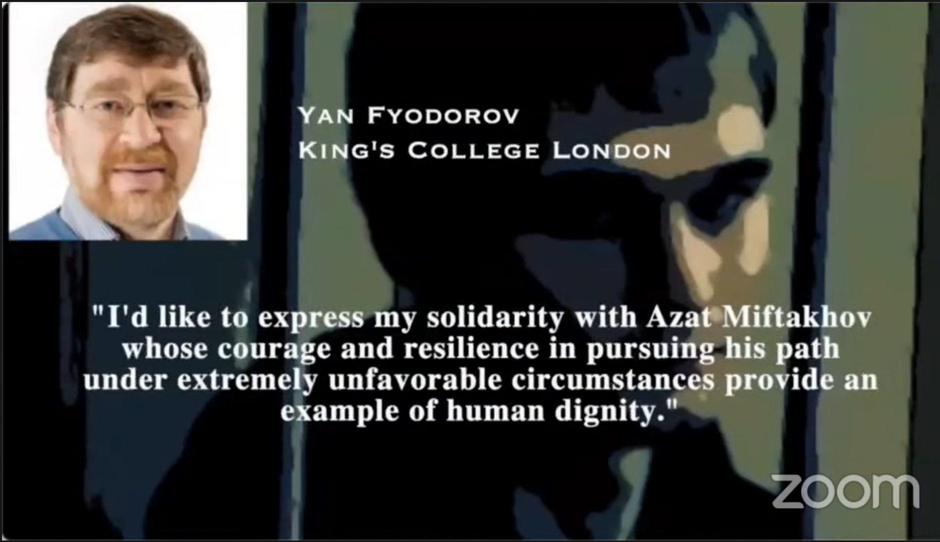

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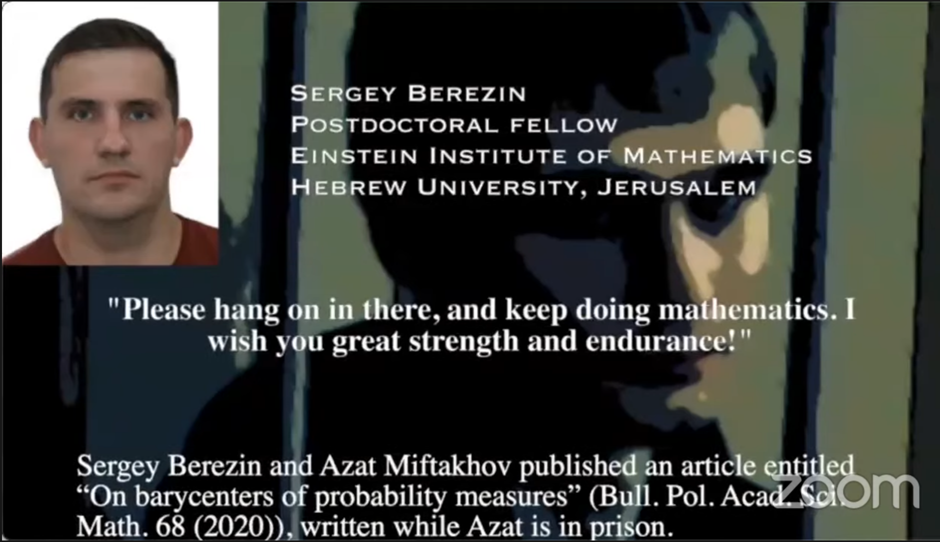

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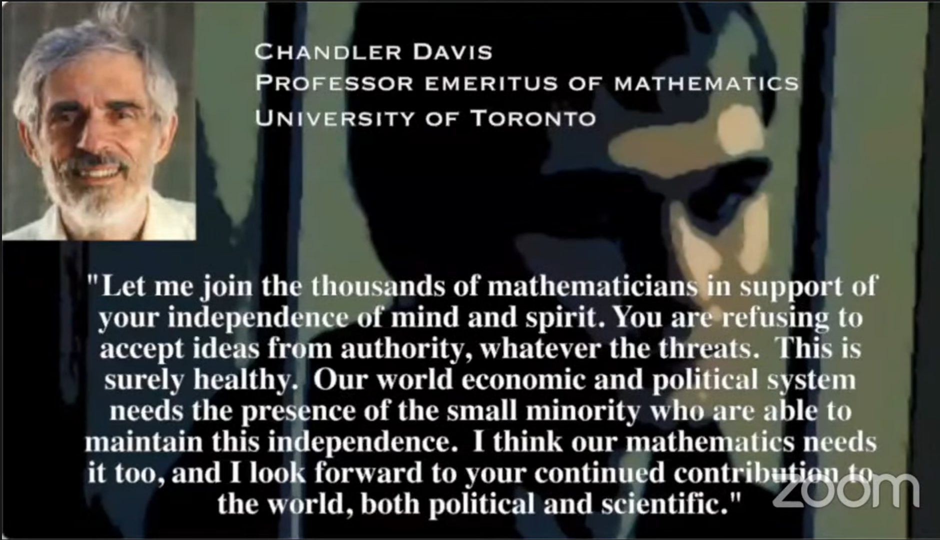

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