A Numerical Simulation of an Underground Waste Repository using a Homogenized Model: Advantages and Comparison to the Detailed Model

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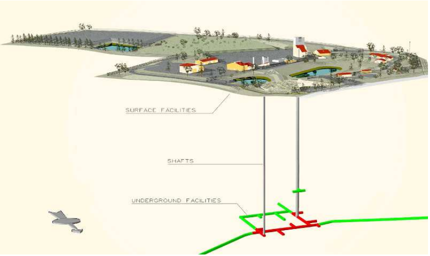
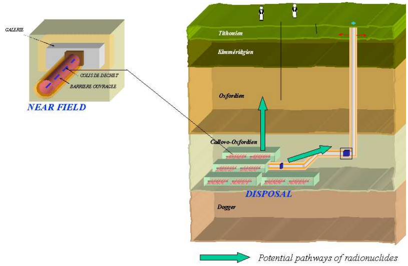
Physical Frame



Physical Frame



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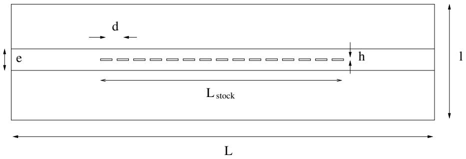
General Model

$$R\omega\frac{\partial\rho}{\partial t} - \nabla\cdot(\mathbf{A}\nabla\rho) + (\mathbf{V}\cdot\nabla)\rho + \lambda R\omega\rho = 0 \qquad (1)$$

R Latency retardation factor,
ω Porosity,
$\mathbf{v}$ Darcy Velocity
$\lambda = \frac{\log 2}{\mathcal{T}}$; $\mathcal{T}$ half-life time of the element radioactivity
According to the units width and their length, only 2D vertical section is considered.
Iodine ^{129}I has half life time $\mathcal{T} = 1.57\ 10^7$ years

Renormalization

- Study time : 10^7 ans.
- Characteristic length : $L = 6km$, $l = 800m$, $e = 135m$, $d = 100m$, $h = 5m$, $L_{stock} = 1500m$.



Normalization : : $\varepsilon = \frac{L_{stock}}{\text{Number of units}}$, $L = 4L_{stock}$, $e \simeq \varepsilon$, $d \simeq \varepsilon$, $h \simeq \varepsilon^2$

Numerical simulation of the Global model

$$\omega^\varepsilon \frac{\partial \varphi_\varepsilon}{\partial t} - \operatorname{div}(\mathbf{A}^\varepsilon \nabla \varphi_\varepsilon) + (\mathbf{v}^\varepsilon \cdot \nabla)\varphi_\varepsilon + \lambda\, \omega^\varepsilon \varphi_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon^T \tag{2}$$

$$\varphi_\varepsilon(0,x) = \varphi_0(x) \;\; x \in \Omega_\varepsilon \tag{3}$$

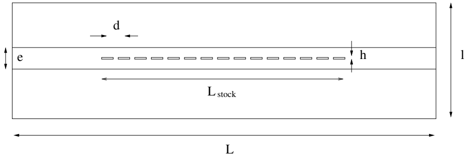
$$\mathbf{n} \cdot \sigma = \mathbf{n} \cdot (\mathbf{A}^\varepsilon \nabla \varphi_\varepsilon - \mathbf{v}^\varepsilon \varphi_\varepsilon) = \Phi(t) \;\; \text{on } \Gamma_\varepsilon^T \tag{4}$$

$$\varphi_\varepsilon = 0 \quad \text{on} \;\; S_1, \tag{5}$$

$$\mathbf{n} \cdot (\mathbf{A}^\varepsilon \nabla \varphi_\varepsilon - \mathbf{v}^\varepsilon \varphi_\varepsilon) = 0 \quad \text{on} \;\; S_2 \tag{6}$$

avec

$$\mathbf{A}^\varepsilon(x_2) = \mathbf{A}(\frac{x_2}{\varepsilon});\;\; \mathbf{v}^\varepsilon(x,t) = \mathbf{v}(x, \frac{x_2}{\varepsilon}, t);\;\; \omega^\varepsilon(x_2) = \omega(x_2/\varepsilon). \tag{7}$$



Results

Homogenized Model

$\varphi \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;\ L^2(\Omega))$;

$$\omega^2 \frac{\partial \varphi}{\partial t} - \operatorname{div}(\mathbf{A}^2 \nabla \varphi) + (\mathbf{v}^2 \cdot \nabla)\varphi + \lambda \omega^2 \varphi = 0 \text{ in } \tilde{\Omega}^T \quad (8)$$

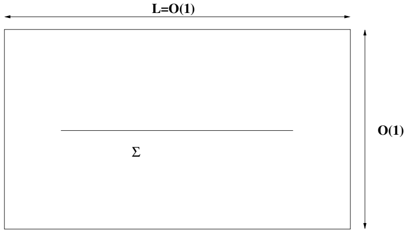
$$\varphi(x,0) = \varphi_0(x) \quad x \in \tilde{\Omega} = \Omega \backslash \Sigma \quad (9)$$

$$\varphi = 0 \quad \text{on} \quad S_1 \quad (10)$$

$$\mathbf{n} \cdot (\mathbf{A}^2 \nabla \varphi - \mathbf{v}^2 \varphi) = 0 \quad \text{on} \quad S_2 \quad (11)$$

$$[\varphi] = 0 \quad , \quad \left[\mathbf{e}_2 \cdot (\mathbf{A}^2 \nabla \varphi - \mathbf{v}^2 \varphi)\right] = -|\tilde{M}|\Phi \quad \text{on} \quad \Sigma \quad (12)$$

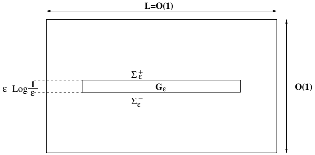
where $|\tilde{M}|$ is the limit of the area of one normalized unit $\mathcal{M}_\varepsilon$.



Results

Asymptotic Expansion

- We want more accurate information around the units



$$\omega^2 \frac{\partial \varphi_\varepsilon^0}{\partial t} - \operatorname{div}(\mathbf{A}^2 \nabla \varphi_\varepsilon^0) + (\mathbf{v}^2 \cdot \nabla)\varphi_\varepsilon^0 + \lambda \omega^2 \varphi_\varepsilon^0 = 0 \text{ in } \tilde{\Omega}^T \quad (13)$$

$$\varphi(x,0) = \varphi_{\varepsilon 0}^0(x) \;\; x \in \tilde{\Omega} = \Omega \backslash \Sigma \quad (14)$$

$$\varphi_\varepsilon^0 = 0 \quad \text{on} \;\; S_1 \quad (15)$$

$$\mathbf{n} \cdot (\mathbf{A}^2 \nabla \varphi_\varepsilon^0 - \mathbf{v}^2 \varphi_\varepsilon^0) = 0 \quad \text{on} \;\; S_2 \quad (16)$$

$$[\varphi_\varepsilon^0] = 0 \quad , \quad [\mathbf{e}_2 \cdot (\mathbf{A}^2 \nabla \varphi_\varepsilon^0 - \mathbf{v}^2 \varphi_\varepsilon^0)] = -|\partial \mathcal{M}_\varepsilon| \Phi \;\; \text{on} \;\; \frac{1}{2}\Sigma_\varepsilon^{\pm} \quad (17)$$

the inner asymptotic expansion

In G_ε, the inner domain, we seek an asymptotic expansion φ_ε :

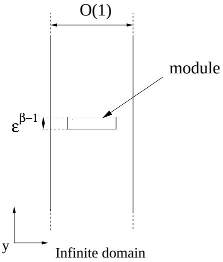
$$\varphi_\varepsilon \simeq \varphi_\varepsilon^0 + \varepsilon \left(\underbrace{\chi_\varepsilon^k(\frac{x}{\varepsilon})\frac{\partial \varphi_\varepsilon^0}{\partial x_k}}_{shape} + \underbrace{w_\varepsilon(\frac{x}{\varepsilon})\Phi}_{source} - \underbrace{\varphi_\varepsilon^0 \rho_\varepsilon^k(\frac{x}{\varepsilon}) v_k^1}_{convection} \right) \equiv \varphi_\varepsilon^1 \quad , \tag{18}$$

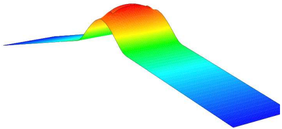
where φ_ε^0 mimics the behaviour of φ but with two jumps on $\Sigma_\varepsilon^+ = \{\varepsilon \log(1/\varepsilon)\} \times \,]-\delta/2, \delta/2\,[$ and on $\Sigma_\varepsilon^- = \varepsilon \log(1/\varepsilon)\} \times \,]-\delta/2, \delta/2\,[$, instead of one one Σ.
the functions $\chi_\varepsilon^k, \rho_\varepsilon^k$ and w_ε are 1 perisodic in y_1 solutions of three auxiliary stationary problems (diffusion type)posed in on infinit strip.

$$\mathcal{G}_\varepsilon = (\,]-1/2, 1/2[\times \mathbf{R}\,) \backslash \mathcal{M}_\varepsilon \quad .$$

Source Corrector

$$-\text{div}\,(\mathbf{A}\,\nabla w_\varepsilon) = 0 \quad \text{in} \;\; \mathcal{G}_\varepsilon$$
$$\mathbf{n}\cdot\,\mathbf{A}\,\nabla w_\varepsilon = 1 \quad \text{on} \;\; \partial\mathcal{M}_\varepsilon \;;$$
$$w_\varepsilon \quad \text{is} \;\; 1-\text{periodic in} \;\; y_1$$
$$\lim_{y_2\to\pm\infty} \mathbf{A}\,\nabla w_\varepsilon(y) = \mp\frac{1}{2}\,|\partial\mathcal{M}_\varepsilon|\,\mathbf{e}_2 \quad . \qquad (19)$$





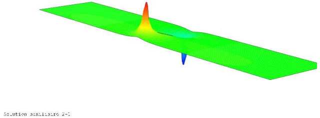
Shape correctors

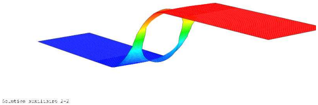
$$-\mathrm{div}\left(\mathbf{A}\,\nabla(\chi_\varepsilon^k + y_k)\right) = 0 \quad \text{in} \;\; \mathcal{G}_\varepsilon$$

$$\mathbf{n}\,\cdot\,\mathbf{A}\,\nabla(\chi_\varepsilon^k + y_k) = 0 \quad \text{on} \;\; \partial\mathcal{M}_\varepsilon \;; \qquad (20)$$

$$\chi_\varepsilon^k \quad \text{is} \;\; 1-\text{periodic in} \;\; y_1$$

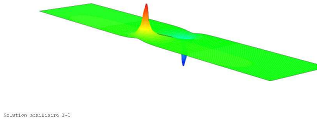
$$\lim_{y_2\to\infty} \nabla\chi_\varepsilon^k = 0 \quad , \qquad (21)$$

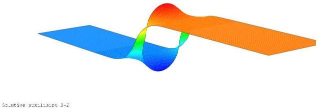




Convection Correctors

$$
\begin{aligned}
&-\mathrm{div}\,(\mathbf{A}\,\nabla\rho_\varepsilon^k) = 0 \quad \text{in} \;\; \mathcal{G}_\varepsilon \\
&\mathbf{n}\,\cdot\,(\mathbf{A}\,\nabla\rho_\varepsilon^k + \mathbf{e}_k) = 0 \quad \text{on} \;\; \partial\mathcal{M}_\varepsilon\;; \qquad (22) \\
&\rho_\varepsilon^k \quad \text{is} \;\; 1-\text{periodic in} \;\; y_1 \\
&\lim_{y_2\to\infty} \nabla\rho_\varepsilon^k = 0 \quad . \qquad (23)
\end{aligned}
$$

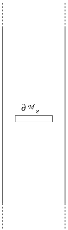


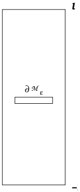


Stabilization at the infinite

TAB.: Dependance of the auxiliary solutions behaviour in function of the truncature length h

h	1	2	4
$c_+^1 - c_-^1$	$-3.09164\ 10^{-11}$	$2.25161\ 10^{-11}$	$-1.69610\ 10^{-11}$
$c_+^2 - c_-^2$	801.80	801.79	801.79
$d_+^1 - d_-^1$	$2.04024\ 10^{-17}$	$4.37095\ 10^{-14}$	$-2.29154\ 10^{-13}$
$d_+^2 - d_-^2$	0.65787	0.65786	0.65786





A priori estimates

with the approximation :

$$F_\varepsilon = \begin{cases} \varphi_\varepsilon^0 & \text{in } \Omega\backslash\overline{G_\varepsilon}\ ; (\text{ outer expansion}) \\ \varphi_\varepsilon^0 + \varepsilon\left(\chi_\varepsilon^k(\frac{x}{\varepsilon})\frac{\partial\varphi_\varepsilon^0}{\partial x_k} + w_\varepsilon(\frac{x}{\varepsilon})\Phi - \varphi_\varepsilon^0\rho_\varepsilon^k(\frac{x}{\varepsilon})v_k^1\right) & \text{in } G_\varepsilon. \end{cases} \tag{24}$$

Theorem

For any $0 < \tau < 1$, there exists a constant $C_\tau > 0$ indépendent of ε, such that

$$|\varphi_\varepsilon - F_\varepsilon|_{L^2(0,T;H^1(\mathcal{B}_\varepsilon))} \leq C_\tau\, \varepsilon^\tau \quad , \tag{25}$$

where $\mathcal{B}_\varepsilon = \Omega\backslash\partial G_\varepsilon$.

Remarks.....

$$|\varphi_\varepsilon - F_\varepsilon|_{L^2(0,T;H^1(\mathcal{B}_\varepsilon))} \leq C_\tau\, \varepsilon^\tau \quad , \tag{26}$$

- Estimates in gradient.... No a priori Continuity of the expansion....
- $0 < \tau < 1$ Which ε is small enough ?
- All the auxiliary problem are up to a constant

$$F_\varepsilon \text{is defined up to } \varepsilon \left(\alpha_1 \frac{\partial \varphi_\varepsilon^0}{\partial x_k} + \alpha_2 \Phi - \varphi_\varepsilon^0 \alpha_3 v_k^1 \right) \quad \text{in}\, G_\varepsilon. \tag{27}$$

Which constant $(\alpha_1, \alpha_2, \alpha_3)$ to choose .....

Results

Other terms....

in the inner layer $G_\varepsilon =]0, L[\times] - d\,\varepsilon, d\,\varepsilon\,[\cap \Omega_\varepsilon$, we will look for higher order terms :

$$\varphi_\varepsilon(x) \approx \Phi^0_\varepsilon(t,x) + \varepsilon\;\Phi^1_\varepsilon(t,x,y) + \varepsilon^2\;\Phi^2_\varepsilon(t,x,y)\cdots\;,\quad y = \frac{x}{\varepsilon}\quad .$$

we have :

$$\Phi^1_\varepsilon(t,x,y) = \chi^k_\varepsilon(y)\,\frac{\partial\phi^0_\varepsilon}{\partial x_k}(t,x)\;-\;\rho^j_\varepsilon(y)\,v_j(y_2)\phi^0_\varepsilon(t,x)\;+\;w_\varepsilon(y)\,\Phi(t)$$

we seek ϕ^0_ε as follows :

$$\phi^0_\varepsilon(t,x) = \varphi^0_\varepsilon(t,x) + \varepsilon\,\varphi^1_\varepsilon(t,x) + \varepsilon^2\,\varphi^2_\varepsilon(t,x) + \cdots\quad . \tag{28}$$

in $\Omega\backslash G_\varepsilon$ **:** we seek an expansion as follows :

$$\varphi_\varepsilon = \varphi^0_\varepsilon(x) + \varepsilon\,\varphi^1_\varepsilon(x) + \cdots\quad , \tag{29}$$

A second corrector

$$\omega^\varepsilon \frac{\partial \varphi_\varepsilon^1}{\partial t} - \mathrm{div}_x(\mathbf{A}^\varepsilon \nabla_x \varphi_\varepsilon^1) + (\mathbf{v}^\varepsilon \cdot \nabla)\varphi_\varepsilon^1 + \lambda\, \omega^\varepsilon\, \varphi_\varepsilon^1 = 0 \quad (30)$$

$$[\varphi_\varepsilon^1] = \chi_\varepsilon^k(\pm d)\, \frac{\partial \varphi_\varepsilon^0}{\partial x_k} - \rho_\varepsilon^j(\pm d)\, v_j(\pm d)\, \varphi^0 + w_\varepsilon\, \Phi(t) \quad \text{on} \quad \Sigma_\varepsilon^\pm \quad (31)$$

$$[-D\, \frac{\partial \varphi_\varepsilon^1}{\partial y_2} + v_2\, \varphi_\varepsilon^1] = \frac{1}{2}(\delta_{k2} b_\ell^\pm + \delta_{\ell 2} b_k^\pm)\, \frac{\partial^2 \varphi_\varepsilon^0}{\partial x_k \partial x_\ell} + \delta_{k2}\,(\, v_j\, d_j^\pm - \sigma)\, \frac{\partial \varphi_\varepsilon^0}{\partial x_k} \quad \text{on} \quad \Sigma_\varepsilon^\pm \quad (32)$$

$$\varphi_\varepsilon^1(0,x) = 0 \quad \text{and} \quad \varphi_\varepsilon^1 = 0 \quad \text{on} \quad S^\pm \; . \quad (33)$$

Remarks

- There is continuity of the solution : continuité.
- This approach fix the constant for the auxiliary problems solutions :

$$\int_{\mathcal{G}_\varepsilon} \chi_\varepsilon^k(y)dy = \int_{\mathcal{G}_\varepsilon} \rho_\varepsilon^j(y)dy = \int_{\mathcal{G}_\varepsilon} w_\varepsilon(y)dy = 0$$

- A new problem to solve : :
  - Two Jumps at the same place ($\Sigma_\varepsilon^\pm$)....
  - "very weak" solution

Decomposition into two more simple problems

$$\varphi_\varepsilon^1 = p + \pi$$

with p défined in $\Omega\backslash G_\varepsilon$ and π defined in $\Omega\backslash\Sigma^\pm$

$$\omega^\varepsilon \frac{\partial p}{\partial t} - \mathrm{div}_x(\mathbf{A}^\varepsilon \nabla_x p) + (\mathbf{v}^\varepsilon \cdot \nabla) p + \lambda \omega^\varepsilon p = 0 \ \text{ in } \ \Omega_\varepsilon^+ \cup \Omega_\varepsilon^- \tag{34}$$

$$p = A^\pm \quad \text{on} \quad \Sigma_\varepsilon^\pm \tag{35}$$

$$p(0,x) = 0 \quad \text{and} \quad p = 0 \quad \text{on} \quad S^\pm \quad . \tag{36}$$

$$\omega^\varepsilon \frac{\partial \pi}{\partial t} - \mathrm{div}_x(\mathbf{A}^\varepsilon \nabla_x \pi) + (\mathbf{v}^\varepsilon \cdot \nabla) \pi + \lambda \omega^\varepsilon \pi = 0 \ \text{ in } \ \Omega\backslash(\Sigma_\varepsilon^+ \cup \Sigma_\varepsilon^-) \tag{37}$$

$$[-D \, \frac{\partial \pi}{\partial y_2} + v_2 \, \pi] = B^\pm - [-D \, \frac{\partial p}{\partial y_2} + v_2 \, p] \ \text{ on } \ \Sigma_\varepsilon^\pm \tag{38}$$

$$u_\varepsilon(0,x) = 0 \quad \text{and} \quad \pi = 0 \quad \text{on} \quad S^\pm \quad . \tag{39}$$

Conclusions

- Global Model obtained by homogenization give very good results in "long times"
- The model with matched expansion take into account the local oscilation in the near field.
- The two approach describe precisely enough the Detailed behaviour without expensive computations.
  - For this situation ratio around 10
  - For more complex situation (increasing the number of units), the size of homogenized problems do not increase.