SOME PROBLEMS IN SCALING UP THE SOURCE TERMS IN AN UNDERGROUND WASTE REPOSITORY

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Oberwolfach Conference - October 31, November 4; 2005 - Reactive Flow and Transport in Complex Systems

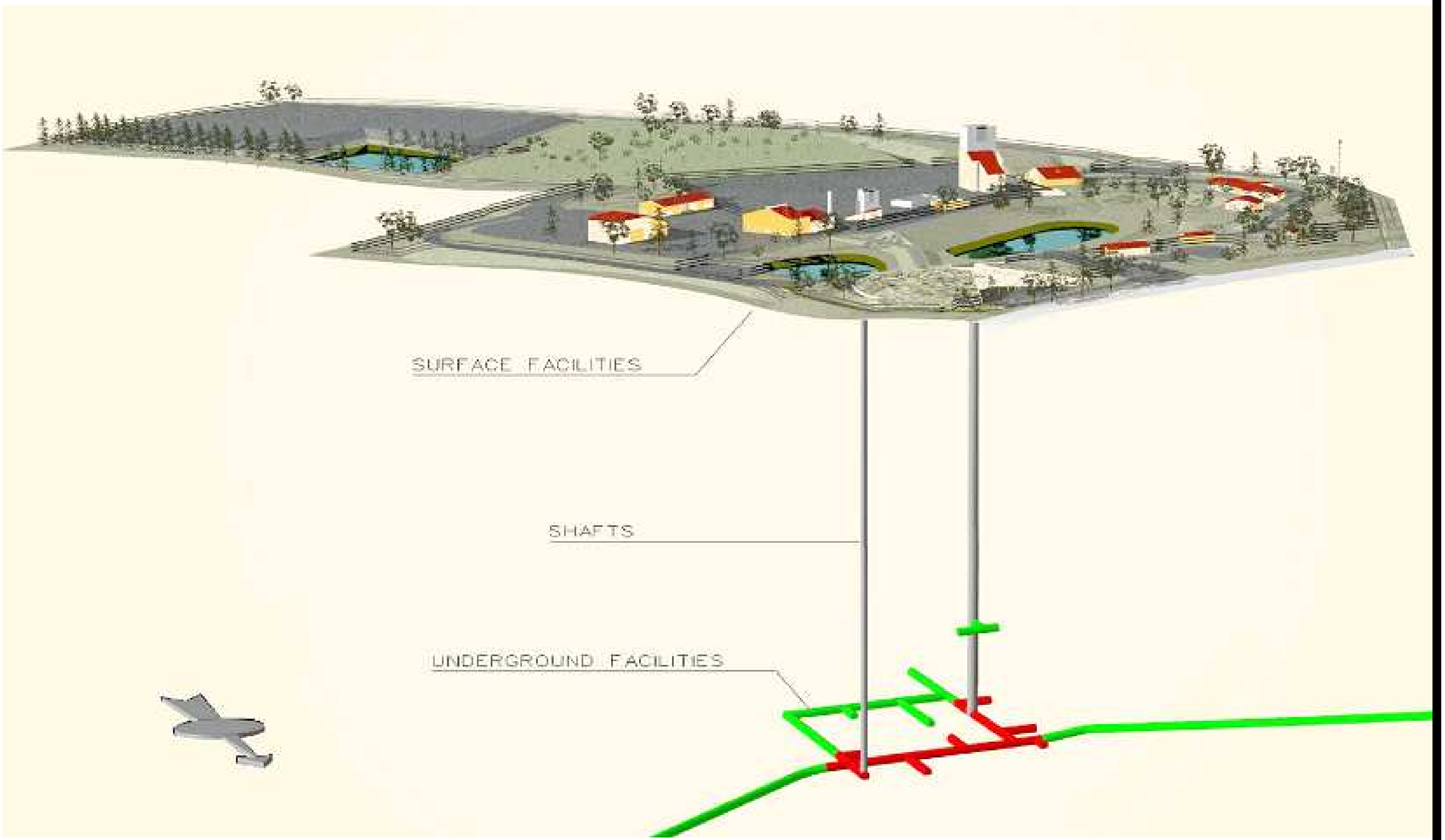


Figure 1: Situation of the underground laboratory site

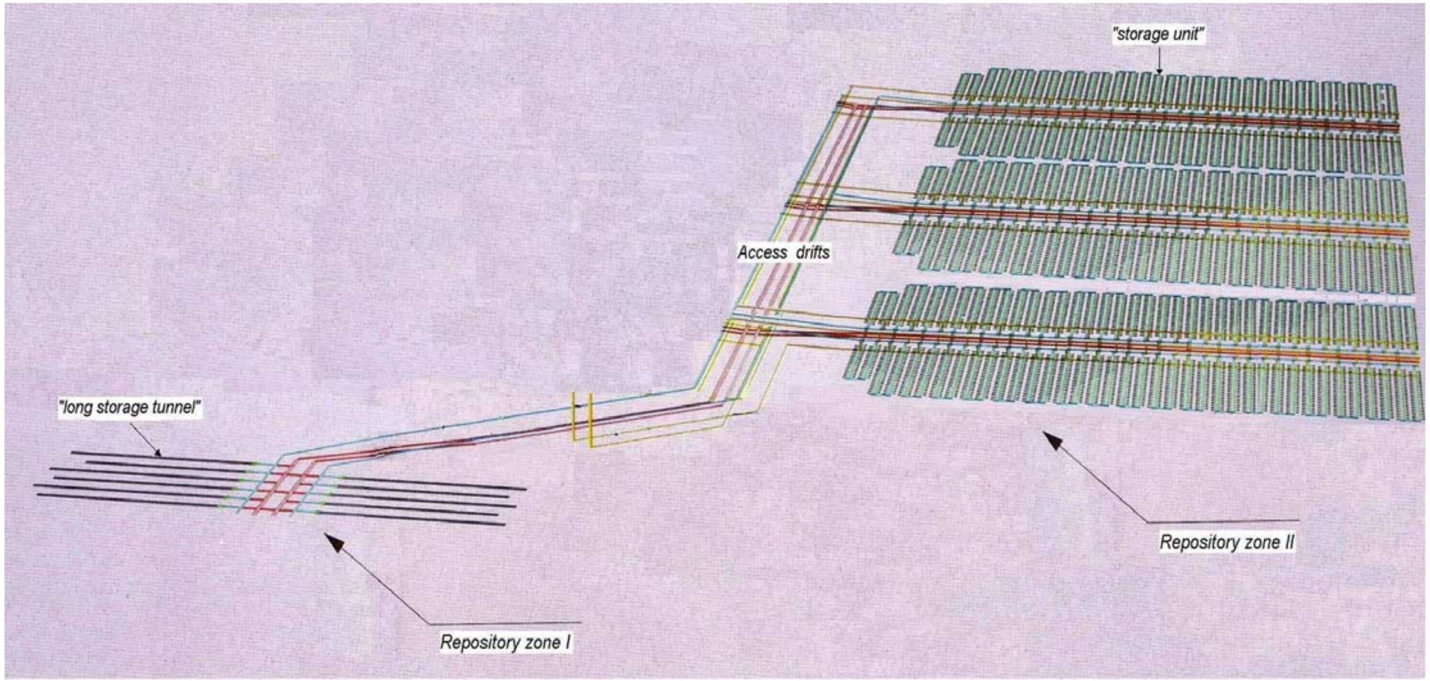


Figure 2: A part of a Waste Repository Site, with two different zones; a zone I, with 11 long storage tunnels; a part of the zone II, with 3 storage units

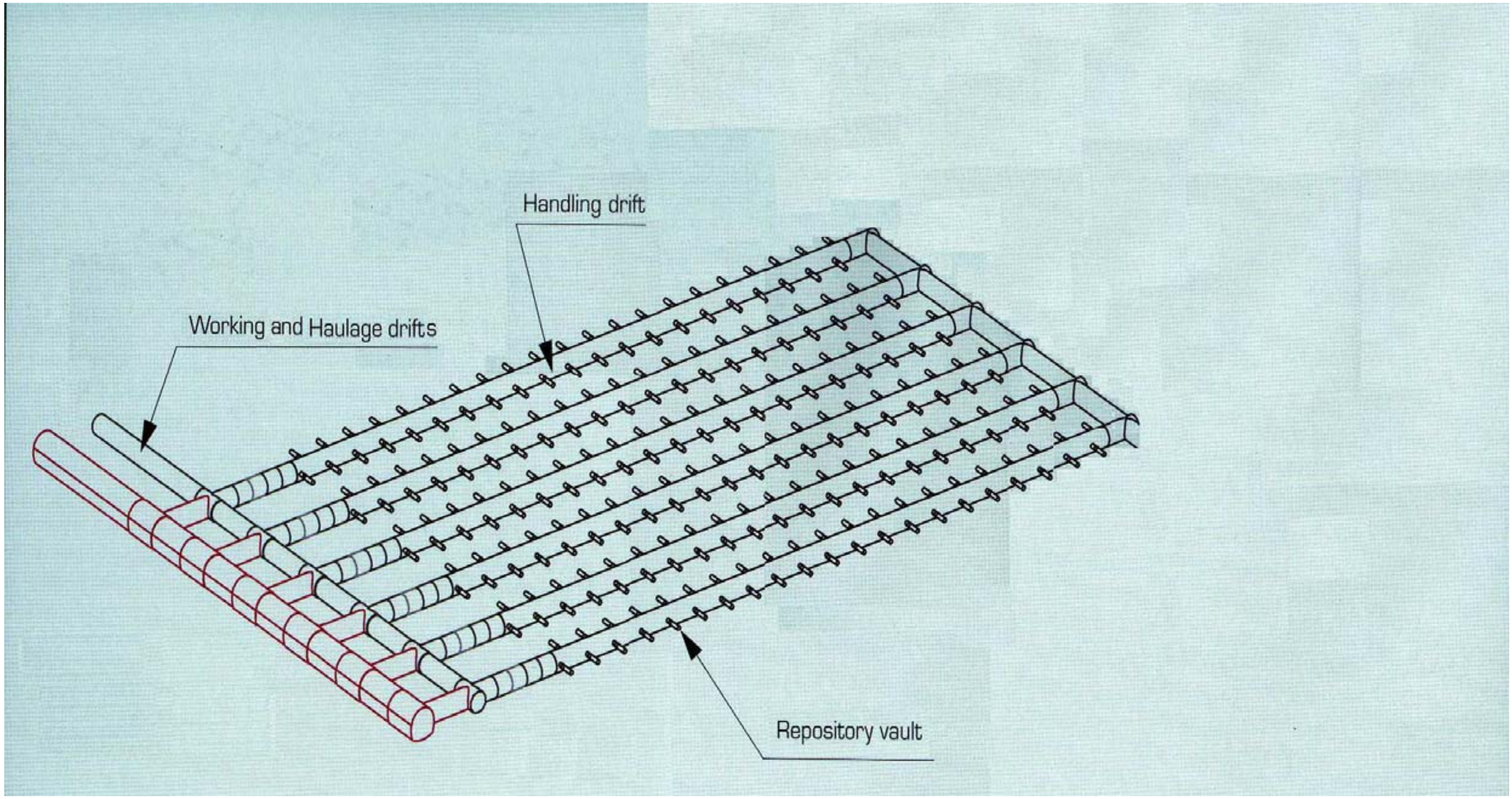


Figure 3: A Storage Unit (or Repository Module)

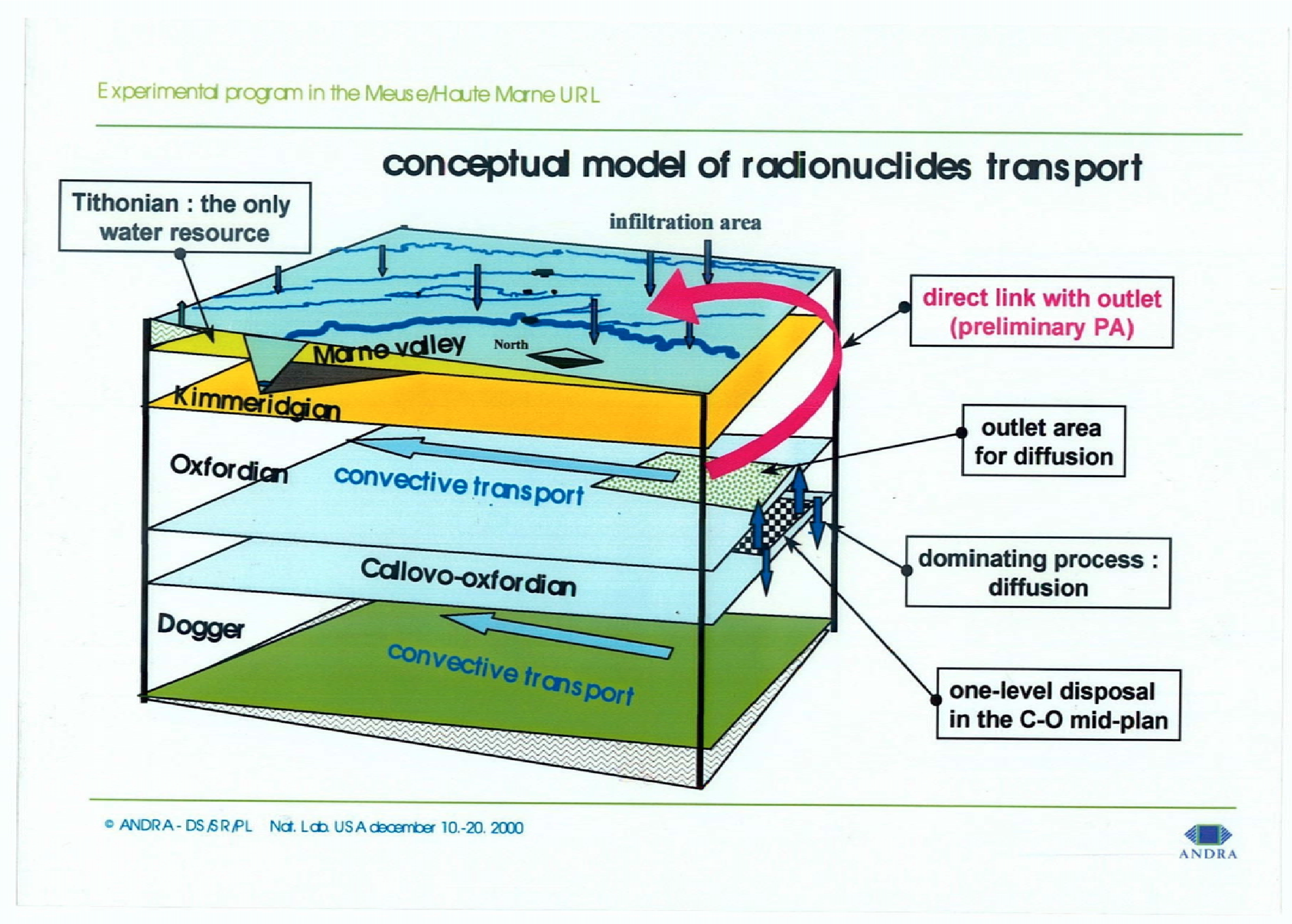


Figure 4: The **Far Field**, with geological layers containing the units

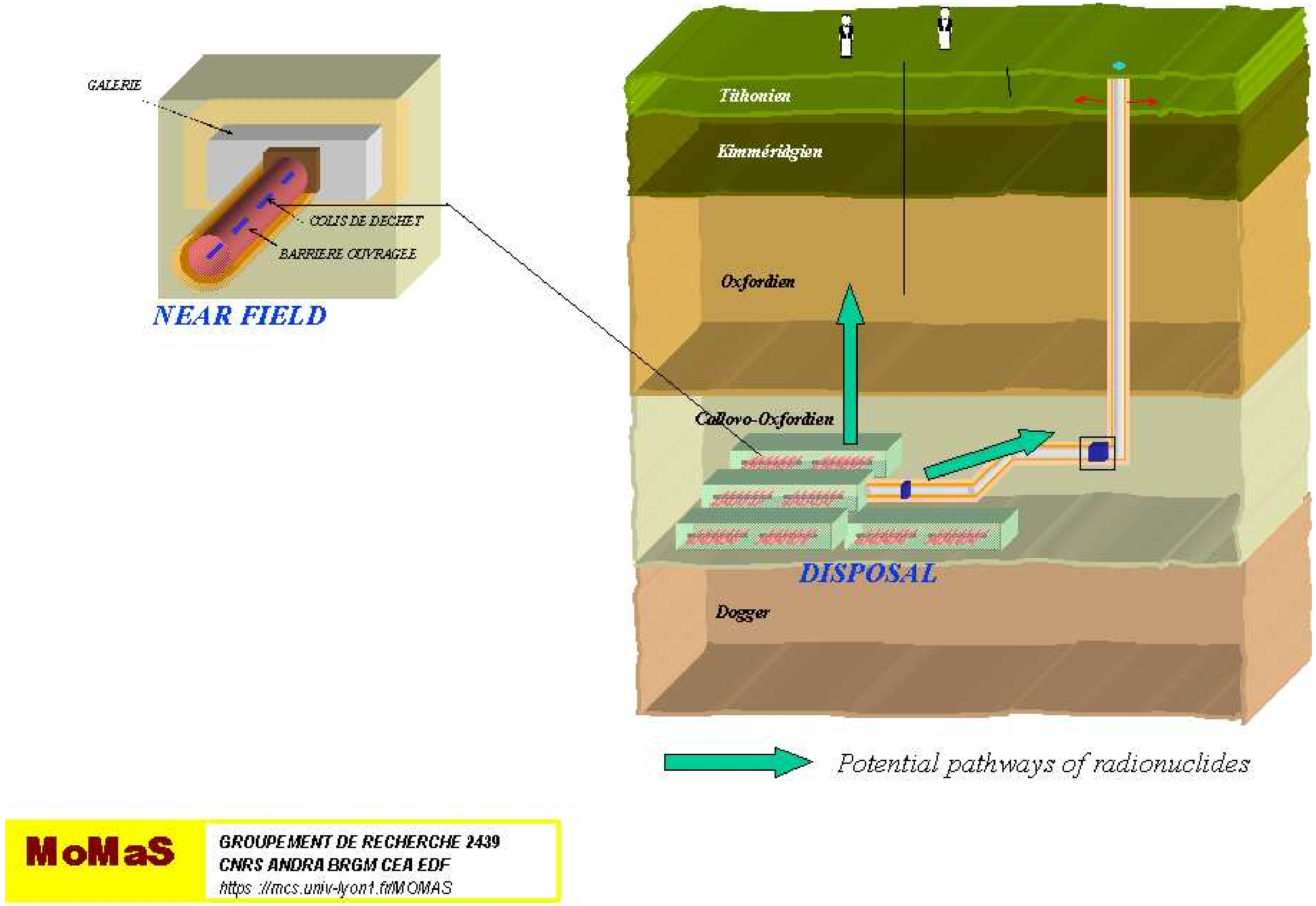


Figure 5: A **Repository Zone** : from **Waste Packages**(or containers sets) to **Storage Units**

There are several levels of upscaling

- from **waste packages** to a **storage unit** global model
- from **storage units** to a **zone** model
- from **similar zones** to the **repository site** global model

The use of Re-iterated Homogenization,could not be not straightforward !
(the phenomena to be taken in account at each level could be different leading to different equations parameters or boundary conditions)

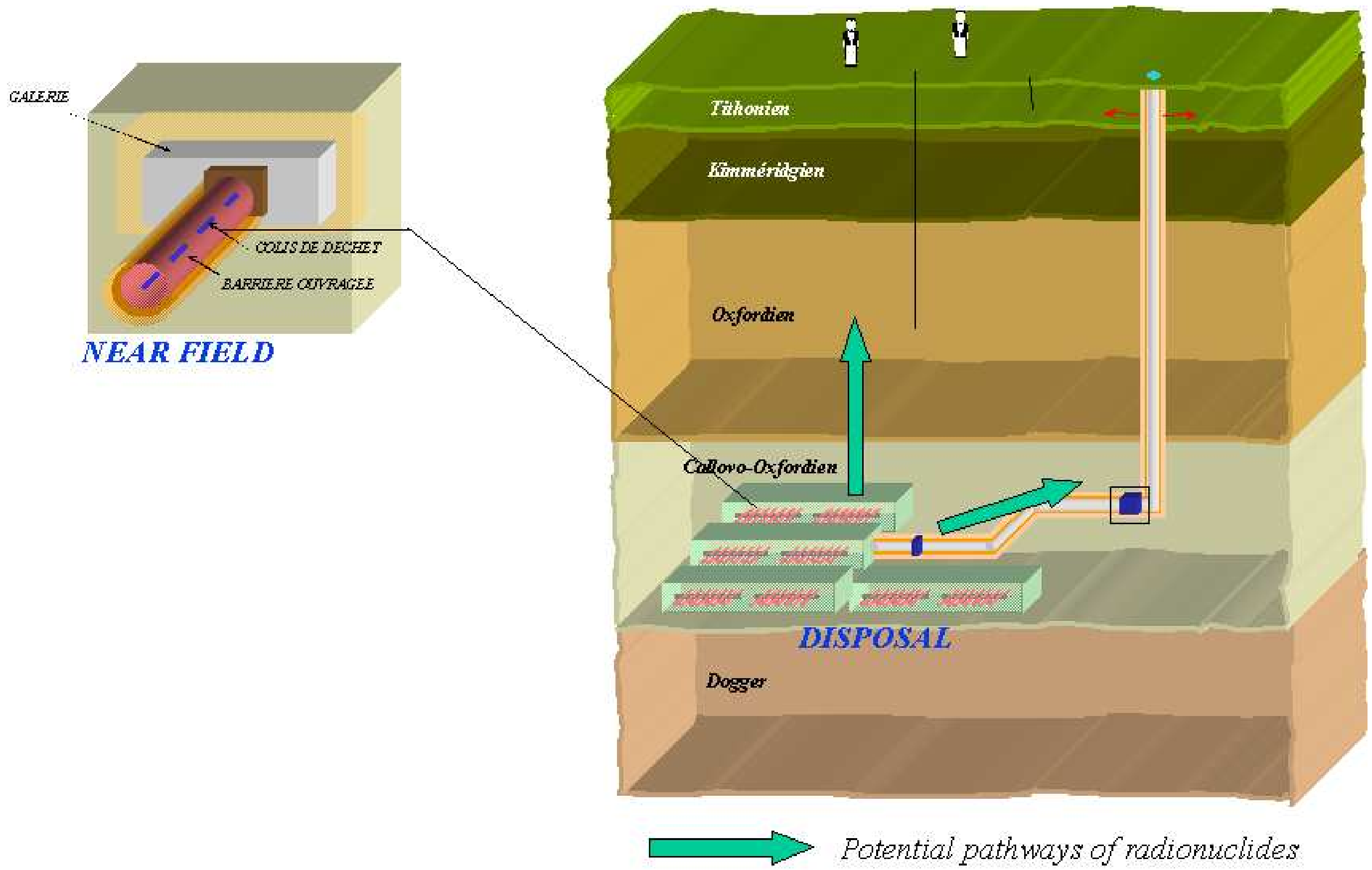
1 General Equations

$$R\omega\frac{\partial\rho}{\partial t} - \nabla\cdot(\mathbf{A}\nabla\rho) + (\mathbf{V}\cdot\nabla)\rho + \lambda R\omega\rho = 0 \tag{1}$$

- R the latency retardation factor,
- ω the porosity,
- $\mathbf{v}$ the Darcy's velocity
- $\lambda = \frac{\log 2}{\mathcal{T}}$; $\mathcal{T}$ the element radioactivity half life time
- according to the units width and their length we consider a storage $2D$ vertical section
- Iodine ^{129}I has half life time $\mathcal{T} = 1.57\ 10^7$ years and is releasing during a time $t'_m = 8\ 10^3$ years, with intensity $\Phi' = 10^{-1}$.

(I) From "STORAGE UNITS " to "a ZONE model"

(OR) From "Similar ZONES" to "the REPOSITORY SITE model"



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https ://mcs.univ-lyon1.fr/MOMAS

Figure 6: A **Repository Zone** : from **Waste Packages** to **Storage Units**

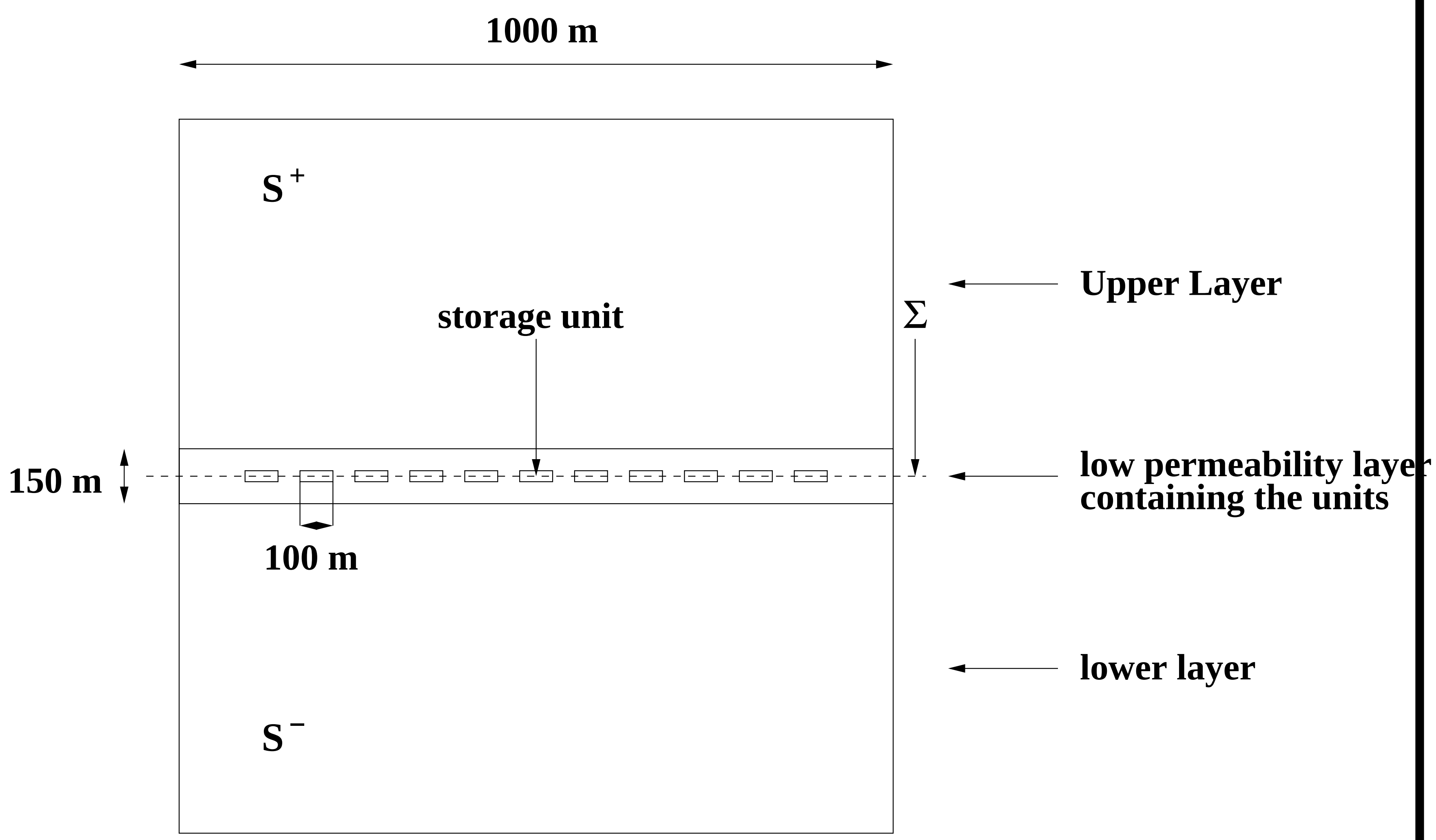


Figure 7: According to the units width and their length we consider a Repository Zone $2D$ vertical section

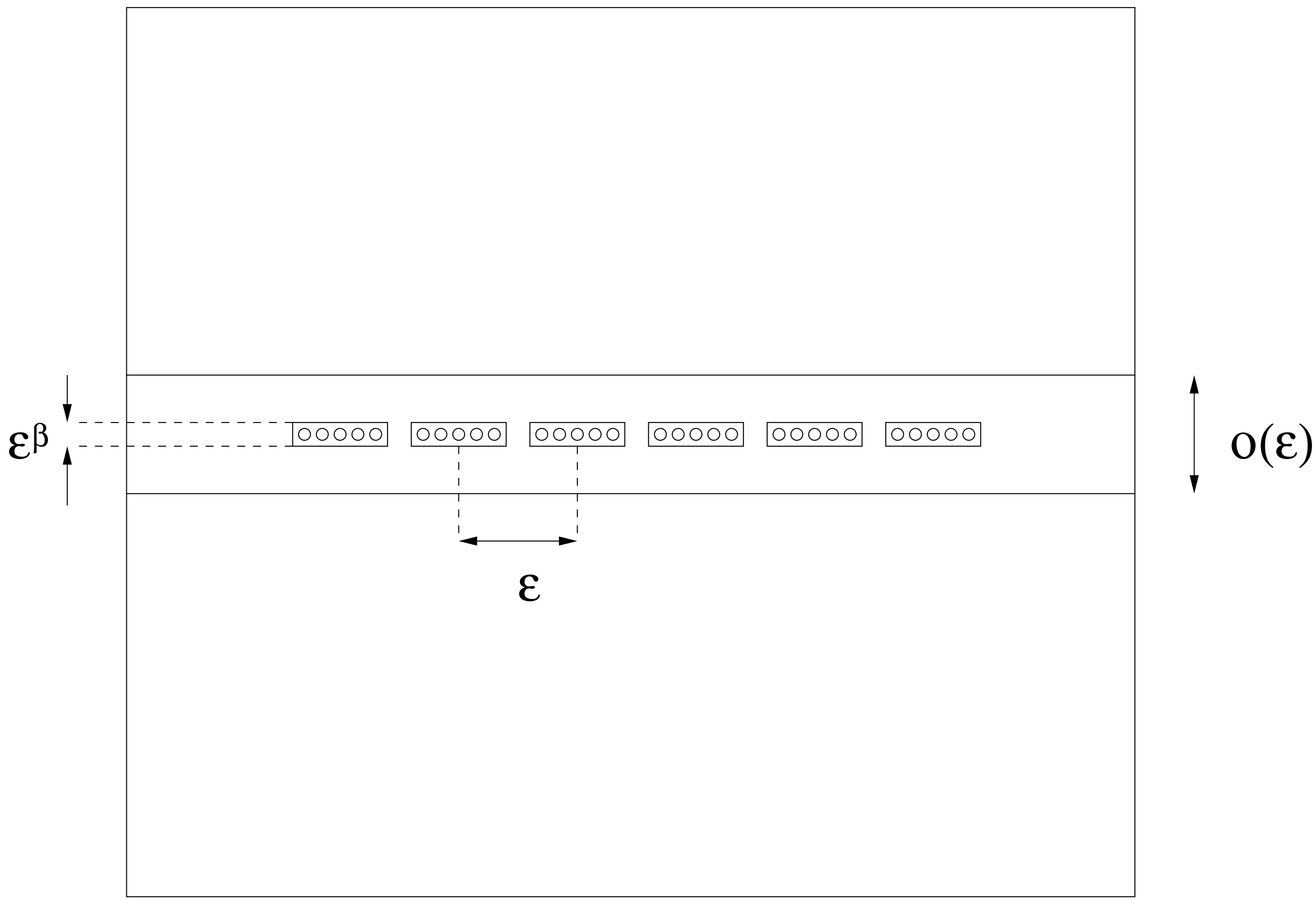


Figure 8: Storage Units (of Waste Packages), after renormalization $x - \frac{x'}{L}$

2 The Equations

$$\omega^\varepsilon \frac{\partial \varphi_\varepsilon}{\partial t} - \mathrm{div}\,(\mathbf{A}^\varepsilon \nabla \varphi_\varepsilon) + (\mathbf{v}^\varepsilon \cdot \nabla)\varphi_\varepsilon + \lambda \omega^\varepsilon \varphi_\varepsilon = 0 \quad \text{in} \ \Omega_\varepsilon^T \tag{2}$$

$$\varphi_\varepsilon(0,x) = \varphi_0(x) \quad x \in \Omega_\varepsilon \tag{3}$$

$$\mathbf{n} \cdot \sigma = \mathbf{n} \cdot (\mathbf{A}^\varepsilon \nabla \varphi_\varepsilon - \mathbf{v}^\varepsilon \varphi_\varepsilon) = \Phi(t) \quad \text{on } \Gamma_\varepsilon^T \tag{4}$$

$$\varphi_\varepsilon = 0 \quad \text{on} \ S_1, \tag{5}$$

$$\mathbf{n} \cdot (\mathbf{A}^\varepsilon \nabla \varphi_\varepsilon - \mathbf{v}^\varepsilon \varphi_\varepsilon) = 0 \quad \text{on} \ S_2 \tag{6}$$

with

$$\mathbf{A}^\varepsilon(x_2) = \mathbf{A}(\frac{x_2}{\varepsilon}); \ \mathbf{v}^\varepsilon(x,t) = \mathbf{v}(x, \frac{x_2}{\varepsilon}, t); \ \omega^\varepsilon(x_2) = \omega(x_2/\varepsilon). \tag{7}$$

3 A priori Energy estimates

give

$$\varphi_\varepsilon \rightharpoonup \varphi \qquad \text{weak* in} \quad L^\infty(0,T;L^2(\Omega)) \tag{8}$$

$$\nabla\varphi_\varepsilon \rightharpoonup \nabla\varphi \qquad \text{weakly in} \quad L^2(0,T;L^{\beta*}(\Omega)) \tag{9}$$

$$\beta* = \frac{2\beta}{3\beta - 2}.$$

with $\varphi \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;\ L^2(\Omega))$;

$$\omega^2 \frac{\partial \varphi}{\partial t} - \mathrm{div}\,(\mathbf{A}^2 \nabla \varphi) + (\mathbf{v}^2 \cdot \nabla)\varphi + \lambda \omega^2 \varphi = 0 \text{ in } \tilde{\Omega}^T \tag{10}$$

$$\varphi(x,0) = \varphi_0(x) \quad x \in \tilde{\Omega} = \Omega \backslash \Sigma \tag{11}$$

$$\varphi = 0 \quad \text{on} \quad S_1 \tag{12}$$

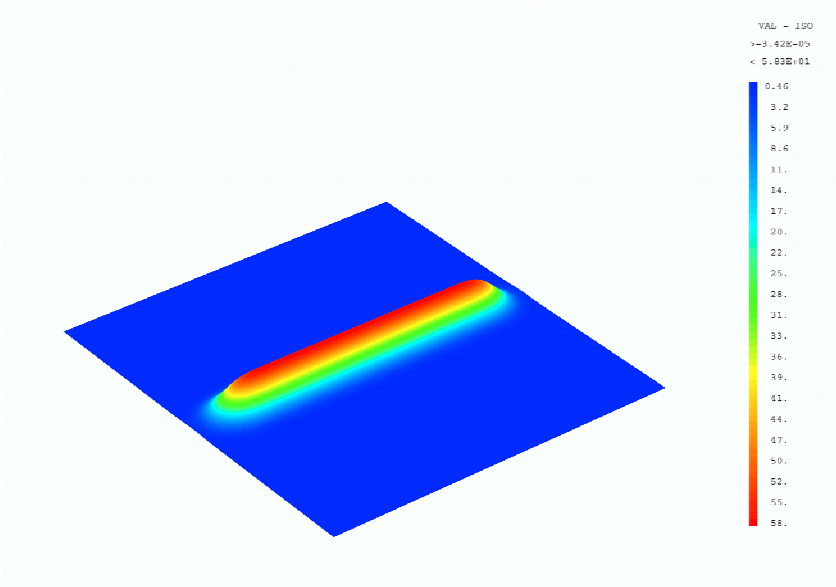
$$\mathbf{n} \cdot (\mathbf{A}^2 \nabla \varphi - \mathbf{v}^2 \varphi) = 0 \quad \text{on} \quad S_2 \tag{13}$$

$$[\varphi] = 0 \quad , \quad \left[\mathbf{e}_2 \cdot (\mathbf{A}^2 \nabla \varphi - \mathbf{v}^2 \varphi)\right] = -|\tilde{M}|\boldsymbol{\Phi} \quad \text{on} \quad \Sigma \quad , \tag{14}$$

where $[\cdot]$ denotes the jump over Σ, and $|\widetilde{M}|$ stands for the limit of a storage unit area; $(\mathcal{M}_\varepsilon)$ *area* $= |\widetilde{M}| + O(\varepsilon^{\beta-1})$

Remark 1 *We do not need exact periodicity in space, of the units. The same proof holds whenever each unit is randomly placed in a mesh of an $\varepsilon-$ net.*
The units do not even need to have the same shape as long as their thickness is small enough ($\ll \varepsilon$).
We may extend to a general case where the flux Φ depends also on the space $\Phi(x,t)$ and the units have different shapes $\mathcal{M}_\varepsilon(x)$, then the right hand side of (14) has to be replaced by $\lim_{\varepsilon \longrightarrow 0} |\mathcal{M}_\varepsilon(x)| \Phi(x', t)$.

The **long time behavior**:



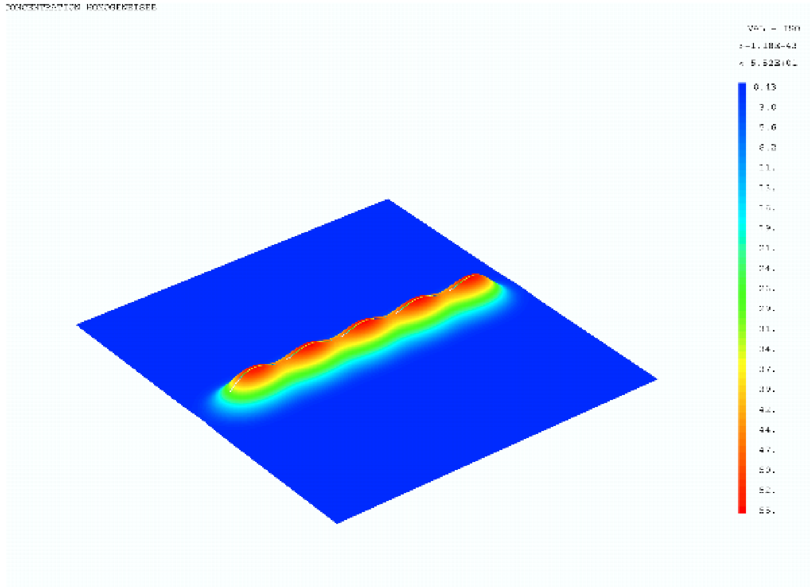


Figure 9: Global 'homogenized' solution φ vs. 'real' solution φ_ε at 200 Kyears.

4 Asymptotic expansion and Matching for the Short time

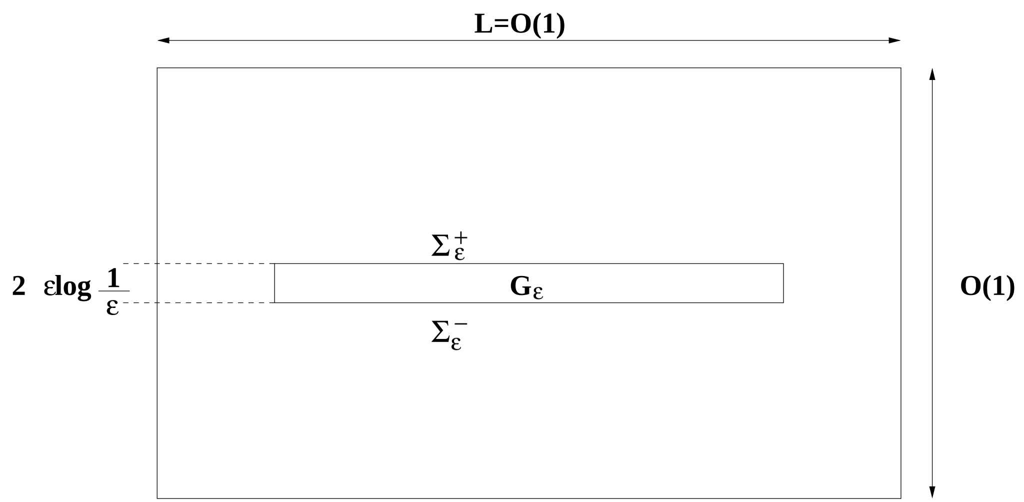


Figure 10: G_ε The inner layer; and $\Omega\backslash\overline{G_\varepsilon}$ the outer domain

In G_ε, the inner domain, we look for an asymptotic expansion of φ_ε:

$$\varphi_\varepsilon \simeq \varphi_\varepsilon^0 + \varepsilon \left(\chi_\varepsilon^k(\frac{x}{\varepsilon}) \frac{\partial \varphi_\varepsilon^0}{\partial x_k} + w_\varepsilon(\frac{x}{\varepsilon})\Phi - \varphi_\varepsilon^0 \rho_\varepsilon^k(\frac{x}{\varepsilon}) v_k^1 \right) \equiv \varphi_\varepsilon^1 \quad , \qquad (15)$$

where φ_ε^0 mimics the behaviour of φ but has two jumps respectively on $\Sigma_\varepsilon^+ = \{\varepsilon \log(1/\varepsilon)\} \times \,] - \delta/2, \delta/2\,[$ and on $\Sigma_\varepsilon^- = \{-\varepsilon \log(1/\varepsilon)\} \times \,] - \delta/2, \delta/2\,[$, instead of only one on Σ.
The functions $\chi_\varepsilon^k, \rho_\varepsilon^k$ and w_ε are 1-periodic solutions in y_1 of three auxiliary stationary diffusion type problems posed in an infinite strip

$$\mathcal{G}_\varepsilon = (\,] - 1/2, 1/2[\times \mathbf{R}\,) \backslash \mathcal{M}_\varepsilon \quad .$$

4.1 Error estimates for the Matched expansion

With the approximation:

$$F_\varepsilon = \begin{cases} \varphi_\varepsilon^0 \quad \text{in } \Omega\backslash\overline{G_\varepsilon}\ ; (\text{ outer expansion}) \\ \varphi_\varepsilon^0 + \varepsilon\left(\chi_\varepsilon^k(\frac{x}{\varepsilon})\frac{\partial\varphi_\varepsilon^0}{\partial x_k} + w_\varepsilon(\frac{x}{\varepsilon})\Phi - \varphi_\varepsilon^0\rho_\varepsilon^k(\frac{x}{\varepsilon})v_k^1\right) \quad \text{in } G_\varepsilon. \end{cases} \tag{16}$$

Theorem 1 *For any $0 < \tau < 1$ there exists a constant $C_\tau > 0$ non dependent on ε, such that*

$$|\varphi_\varepsilon - F_\varepsilon|_{L^2(0,T;H^1(\mathcal{B}_\varepsilon))} \leq C_\tau\, \varepsilon^\tau \quad , \tag{17}$$

where $\mathcal{B}_\varepsilon = \Omega\backslash\partial G_\varepsilon$.
The same estimate holds in $L^\infty(0,T;L^2(\Omega_\varepsilon))$ norm.

5 Conclusion of Part One

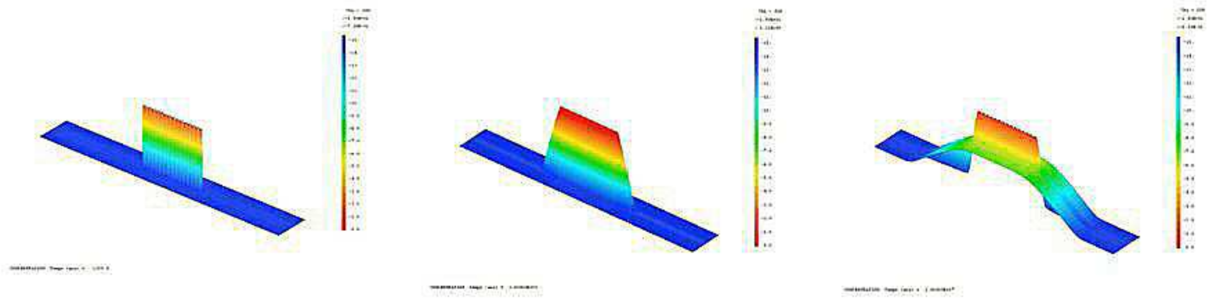
The expansion (16) clearly points out two important terms:

- the zero order term φ_ε^0
- and the first order term $\varepsilon\, w_\varepsilon(\frac{x}{\varepsilon})\Phi$.

On one hand the diffusion in the low permeable layer around the units is small and on the other hand the leaking is intensive during a short time; then: during that short time the first order term $\varepsilon\, w_\varepsilon(\frac{x}{\varepsilon})\Phi$ will dominate in φ_ε;
and after this short time the diffusion will become dominant, i.e. φ_ε^0 is now the most important term in the expansion.

Remark: The effects of the boundary layer caused by the non periodicity of the geometry on G_ε could be neglected for a $\varepsilon-$ order approximation .

6 Numerical Simulations



Niveaux de concentration aprés1209, 300 000 et 1 000 000 d'années; obtenus par simulations à partir du modèle détaillé (en haut) et du modèle « homogénéisé » (en bas)

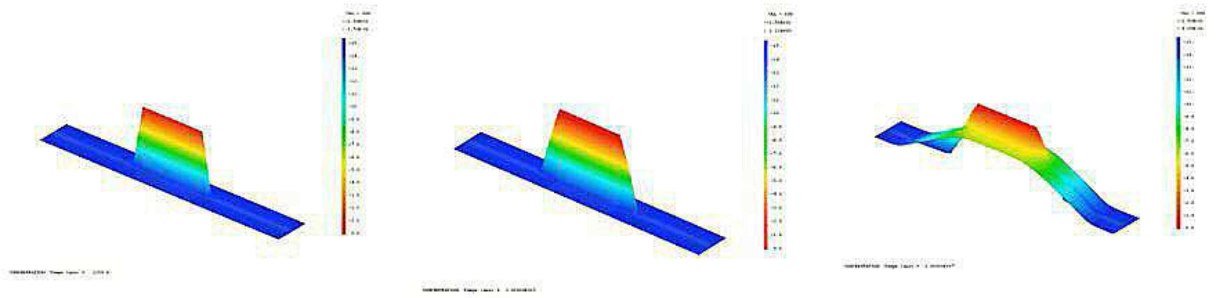
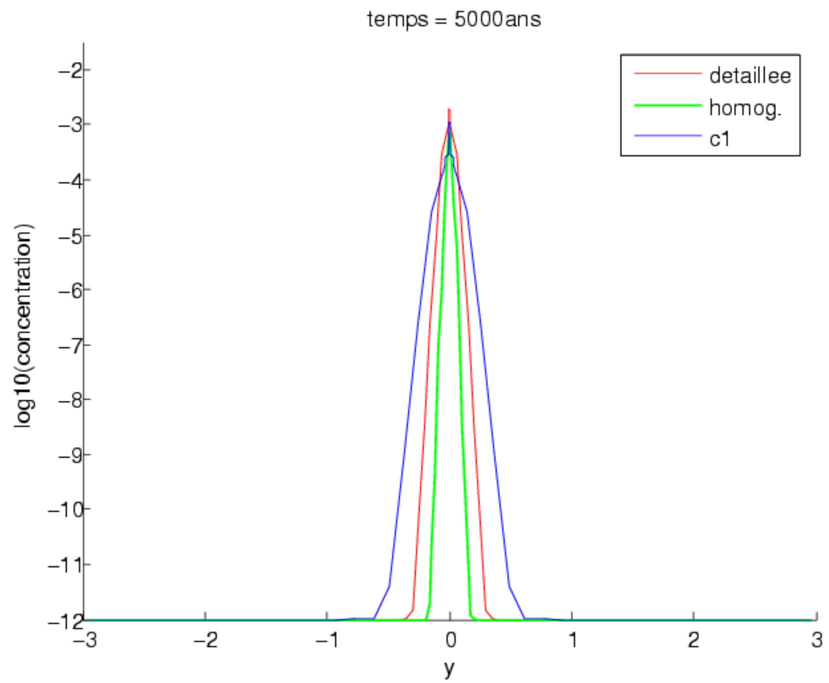
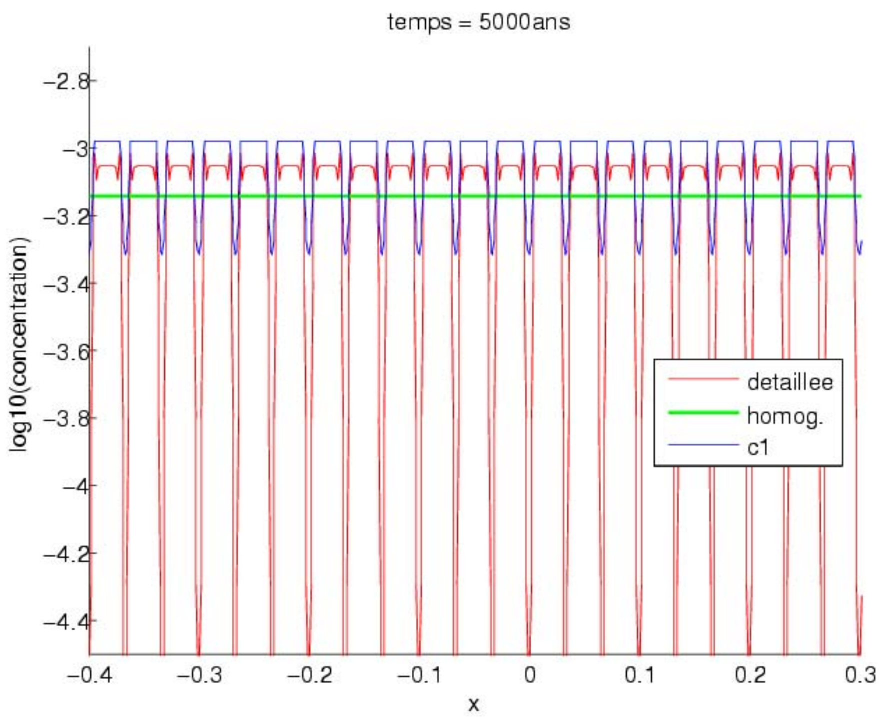
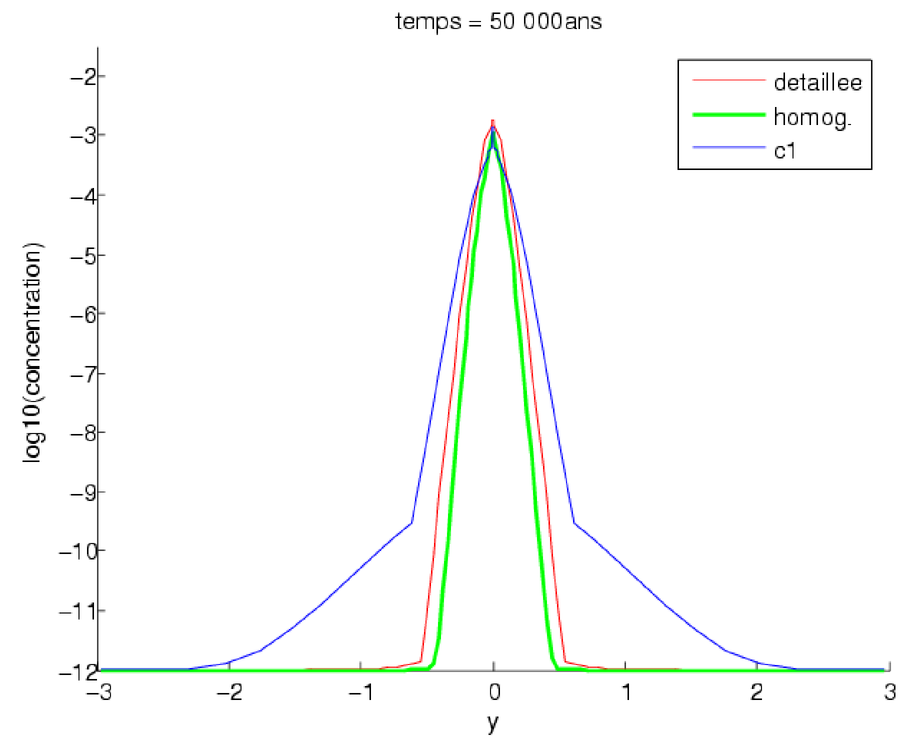
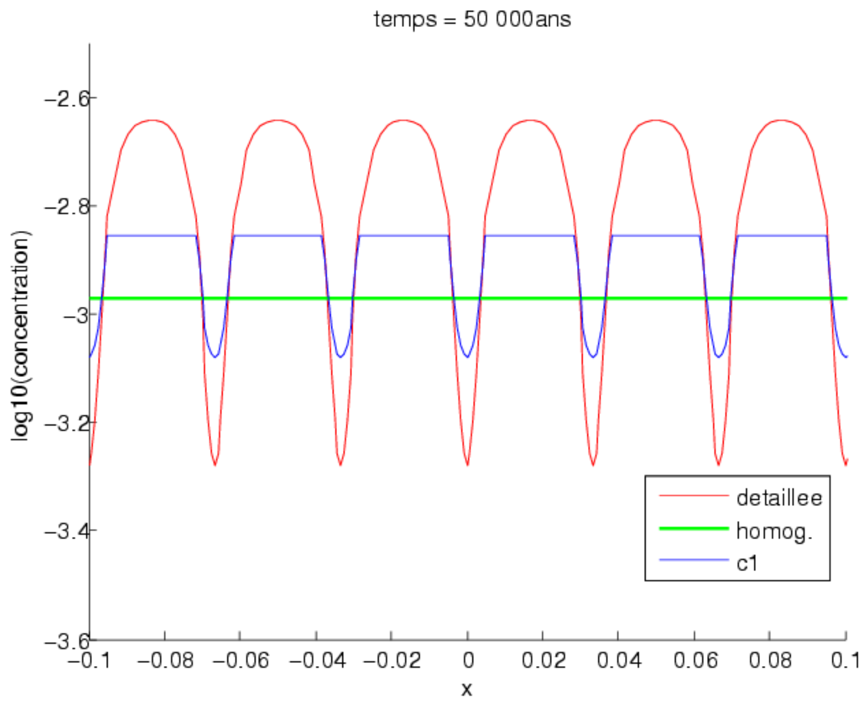


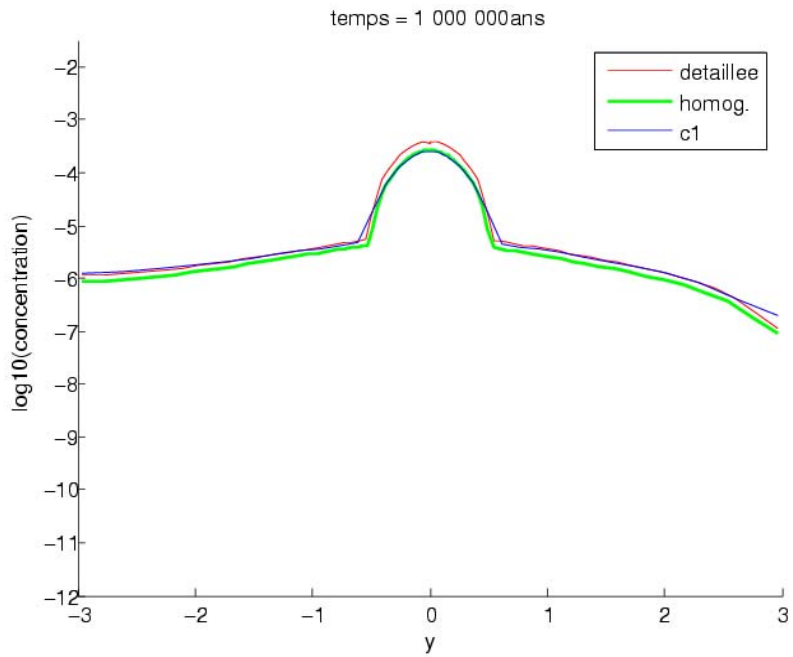
Fig.10 : ***Comparaison des niveaux de concentration en Iode129,*** *obtenus par une simulation détaillée à une échelle fine et ceux obtenus par une simulation basée sur le modèle « homogénéisé » correspondant. Malgré son caractère « global », cette dernière simulation, moins détaillée, rend cependant bien compte des pics de concentration, au voisinage des conteneurs.*

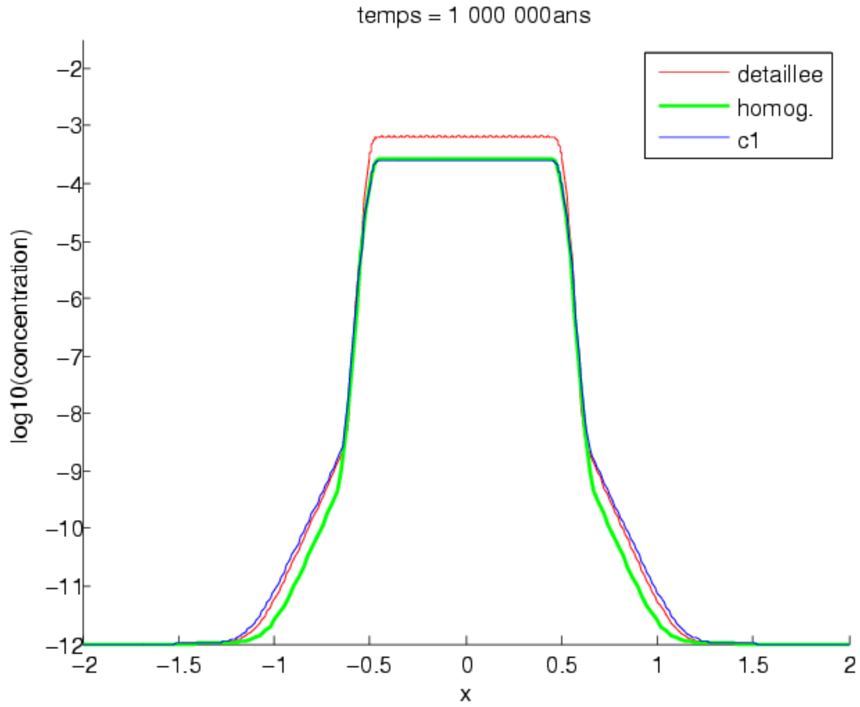

temps = 5000ans
detaillee
homog.
c1
log10(concentration)
y
−2
−3
−4
−5
−6
−7
−8
−9
−10
−11
−12
−3
−2
−1
0
1
2
3


temps = 5000ans
log10(concentration)
−2.8
−3
−3.2
−3.4
−3.6
−3.8
−4
−4.2
−4.4
−0.4
−0.3
−0.2
−0.1
0
0.1
0.2
0.3
x
detaillee
homog.
c1


temps = 50 000ans
detaillee
homog.
c1
log10(concentration)
y
−2
−3
−4
−5
−6
−7
−8
−9
−10
−11
−12
−3
−2
−1
0
1
2
3


temps = 50 000ans
log10(concentration)
−2.6
−2.8
−3
−3.2
−3.4
−3.6
−0.1
−0.08
−0.06
−0.04
−0.02
0
0.02
0.04
0.06
0.08
0.1
x
detaillee
homog.
c1


temps = 1 000 000ans
detaillee
homog.
c1
log10(concentration)
y
−2
−3
−4
−5
−6
−7
−8
−9
−10
−11
−12
−3
−2
−1
0
1
2
3


temps = 1 000 000ans
detaillee
homog.
c1
log10(concentration)
−2
−3
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−5
−6
−7
−8
−9
−10
−11
−12
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
x

(II) From ” WASTE PACKAGES” to a ”STORAGE UNIT”

Global model, with a possibly damaged zone

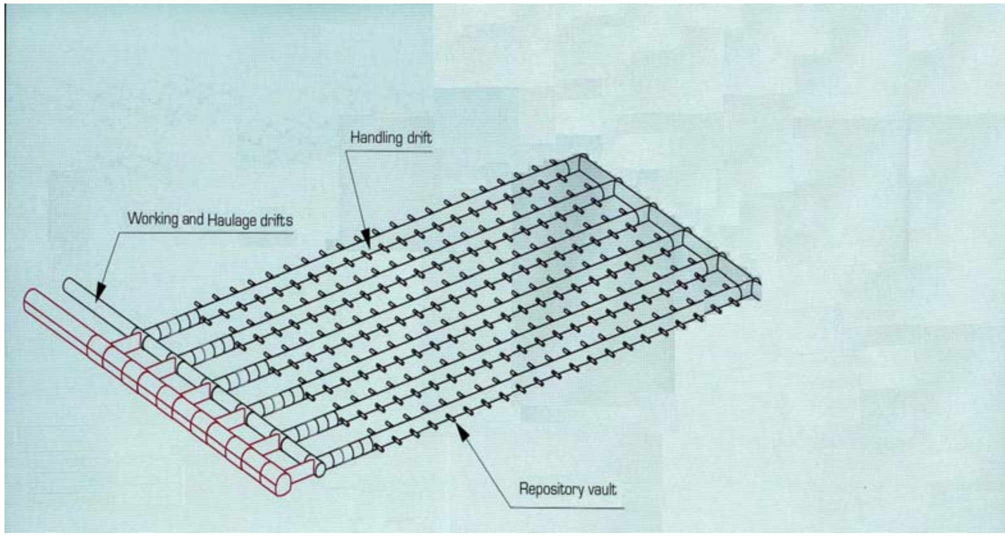


Figure 11: A Storage Unit (or Repository Module)

- Seeking a mathematical model describing the global behavior of one **Storage Unit** of an underground waste Repository Zone,
- Assuming it is made of a high number of Waste Packages (or containers sets) , located inside a low permeable rock, lying on a hypersurface Σ and linked by parallel filled shafts; all the parallel shafts being connected at the top to a main shaft, also filled.
- All the repository is embedded in a thin (100 m.) layer, called host layer, which is included between two higher permeability layers,
- The convection field (Hydrology regime) is given.

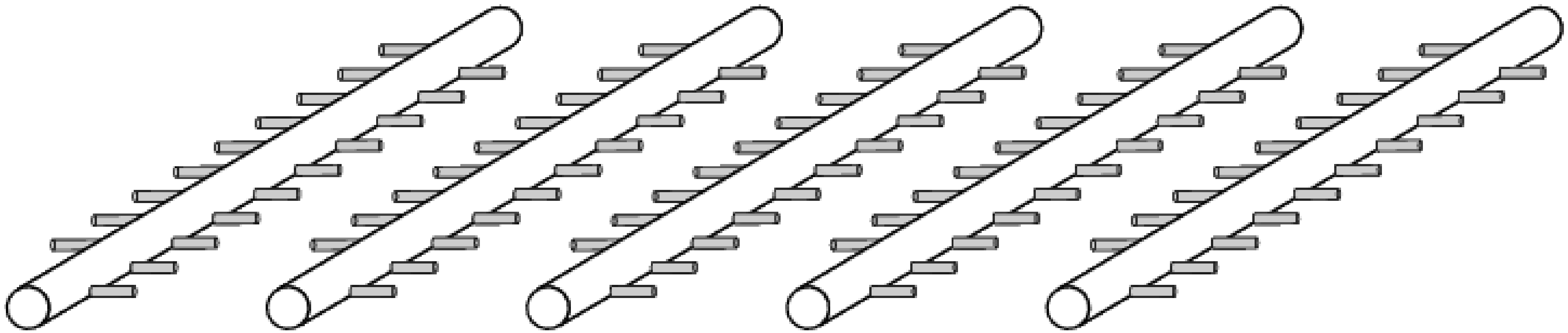


Figure 12: A part of a **Storage Unit**, with 5 rows of waste packages (or containers sets) along shafts

Denoting ε the ratio between the width of a unit (500 m.) and distance (50 m.)between two shafts

- $\Rightarrow$ The containers set have a diameter, of order ε^γ, γ close to three.
- $\Rightarrow$ In the renormalized model there are three scales: 1 for a disposal unit scale, ε for both the scale of a containers row and the shafts period, and ε^γ for the containers diameter.

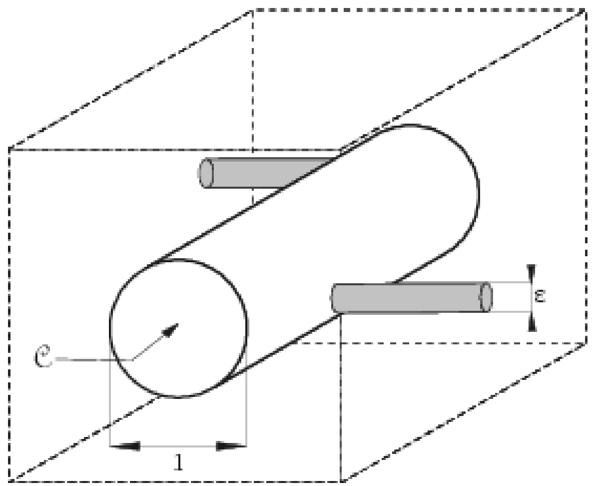


Figure 13: Cell of periodicity Y containing a cylindrical shaft $\mathcal{S} =]-1/2, 1/2[\times \mathcal{C}$ and a waste package P_ε; $\varepsilon^{\gamma-1}$ =diameter of P_ε.

7 The model and equations

The Darcy's velocity:

$$\mathbf{v}^{\varepsilon}(x) = \begin{cases} \mathbf{v}^{h}(x) \ \text{ in the host rock } \ \Omega_{\varepsilon} \backslash \mathcal{S}_{\varepsilon} \\ \varepsilon^{-\beta}\, \mathbf{v}^{d}(x', x_2/\varepsilon; x_3/\varepsilon) \ \text{ in the shafts } \ \mathcal{S}_{\varepsilon} \end{cases} .$$

The Diffusion/Dispersion

$$\mathbf{A}^{\varepsilon}(x) = \begin{cases} \mathbf{A}^{h}(x) \ \text{ in the host rock } \ \Omega_{\varepsilon} \backslash \mathcal{S}_{\varepsilon} \\ d(x)\, \mathbf{I} + \varepsilon^{-\beta}\, \mathbf{A}^{d}(x_2, x_2/\varepsilon, x_3/\varepsilon) \ \text{ in the shafts } \ \mathcal{S}_{\varepsilon} \end{cases} .$$

Because the convection in a storage unit goes mainly in the direction of the shafts: $\Rightarrow$

$$\mathbf{A}^{d}(x_2, y_2, y_3) = a(x_2, y_2, y_3) \ \left(\mathbf{e}_1 \otimes \mathbf{e}_1 \right)$$

"Microspic" model of a storage unit

$$\omega^{\varepsilon} \frac{\partial \varphi_{\varepsilon}}{\partial t} - \mathrm{div}\,(\mathbf{A}^{\varepsilon} \nabla \varphi_{\varepsilon}) + (\mathbf{v}^{\varepsilon} \cdot \nabla\,)\varphi_{\varepsilon} + \lambda\,\omega^{\varepsilon}\,\varphi_{\varepsilon} = 0 \quad \text{in} \quad \Omega_{\varepsilon}^{T} \tag{18}$$

$$\varphi_{\varepsilon}(0,x) = \varphi_0(x) \quad x \in \Omega_{\varepsilon} \tag{19}$$

$$\mathbf{n} \cdot (\mathbf{A}^{\varepsilon} \nabla \varphi_{\varepsilon} - \mathbf{v}^{\varepsilon}\,\varphi_{\varepsilon}) = \Phi_{\varepsilon}(t) \quad \text{on } \Gamma_{\varepsilon}^{T} \tag{20}$$

$$\mathbf{n} \cdot (\mathbf{A}^{\varepsilon} \nabla \varphi_{\varepsilon} - \mathbf{v}^{\varepsilon}\,\varphi_{\varepsilon}) = \kappa\,(\varphi_{\varepsilon} - g_{\varepsilon}) \quad \text{on} \quad \mathcal{K}_{\varepsilon}^{T} \cup \mathcal{H}_{\varepsilon}^{T} \tag{21}$$

$$\varphi_{\varepsilon} = 0 \quad \text{on} \quad \mathcal{Z}_{\varepsilon}^{T}\;. \tag{22}$$

with $\mathcal{H}_{\varepsilon}^{T}$ the shafts tops surface, $\mathcal{Z}_{\varepsilon}^{T}$ the Shafts Bottoms (sealed), $\mathcal{K}_{\varepsilon}^{T}$ the rest of the exterior boundary of Ω, and Γ_{ε} the Waste Packages boundary $\times(0,T)$.

g_{ε} will measure the **concentration entering at the shafts tops**; and $\varepsilon^{-\beta}$ **the Darcy's velocity range** inside the shafts.

8 Results

Depending on β*(the Darcy's velocity range) we have three different cases :*

- $0 \leq \beta < 1$

With a sufficiently strong source :

$$\lim_{\varepsilon \to 0} \varepsilon^{\gamma - 1} \, \Phi_\varepsilon(t) = \Phi(t) \quad \text{uniformly in } \; t \; . \tag{23}$$

and

$$g_\varepsilon = g = \begin{cases} g^h & \text{on the shafts cylindrical surfaces } \; \mathcal{K}_\varepsilon \\ g & \text{on the shafts tops } \; \mathcal{H}_\varepsilon \end{cases} \; .$$

The shafts do not make any contribution, i.e. the repository behaves as if they were not there. $\varphi_\varepsilon \to \varphi$ the unique solution of a problem, of same type as the microscopic problem:

$$\omega^h \frac{\partial \varphi}{\partial t} - \operatorname{div}(\mathbf{A}^h \nabla \varphi) + (\mathbf{v}^h \cdot \nabla)\varphi + \lambda \omega^h \varphi = 0 \quad \text{in} \quad \tilde{\Omega}^T$$

$$\varphi(x,0) = f_0(x) \quad x \in \tilde{\Omega} = \Omega \backslash \Sigma \ , \tag{24}$$

$$\mathbf{n} \cdot (\mathbf{A}^h \nabla \varphi - \mathbf{v}^h \varphi) = \kappa (\varphi - g) \quad \text{on} \quad S^T \tag{25}$$

$$[\varphi] = 0, \quad \left[\mathbf{e}_3 \cdot \mathbf{A}^h \nabla \varphi - (\mathbf{v}^h \cdot \mathbf{e}_3)\varphi\right] = -\Phi \mathcal{M} \text{ on } \Sigma \ . \tag{26}$$

$\tilde{\Omega}^T = (\Omega \backslash \Sigma) \times]0,T[; S^T = \partial\Omega \times]0,T[$

$[w](x') = w(x',0+) - w(x',0-)$, denotes the jump over Σ and $\mathcal{M}$ denotes the limit of the rescaled containers surface area , i.e.

$$\mathcal{M} = \lim_{\varepsilon \to 0} \varepsilon^{1-\gamma} |\partial P_\varepsilon| \quad . \tag{27}$$

- $\beta = 1$

With a source term,

$$\lim_{\varepsilon \to 0} \Phi_\varepsilon(t) = \Phi(t) \quad \text{uniformly in } \ t \ , \tag{28}$$

and some concentration entering the shafts tops g_ε

$$g_\varepsilon = \begin{cases} g^h & \text{on the shafts cylindrical surfaces } \mathcal{K}_\varepsilon \\ \varepsilon^{-1}\, g^d & \text{on the shafts tops } \ \mathcal{H}_\varepsilon \end{cases} \quad . \tag{29}$$

$\varphi_\varepsilon \to \varphi$ weakly in $L^2(0,T; W^{1,\gamma^*}(\Omega))$ and $\varphi_\varepsilon \longrightarrow \varphi^0 = \varphi(x_1, x_2, 0)$, $d\mu^\varepsilon(x) 2-scale$, where φ is the unique solution of a coupled problem. *The transport processes, inside and outside the "damaged" shafts are comparable and there are interactions between them.*

$\beta = 1$; The model could be seen as representing connected shafts, galleries and drifts with damaged sealings.

$$\omega^h \frac{\partial \varphi}{\partial t} - \operatorname{div}(\mathbf{A}^h \nabla \varphi) + (\mathbf{v}^h \cdot \nabla)\varphi + \lambda \omega^h \varphi = 0 \text{ in } \tilde{\Omega}^T; \quad (30)$$

$$\varphi(0, x) = \varphi_0(x) \text{in } \tilde{\Omega}; \quad (31)$$

$$\mathbf{n} \cdot (\mathbf{A}^h \nabla \varphi - \mathbf{v}^h \varphi) = \kappa(\varphi - g^h) \text{ on } S^T \quad (32)$$

$$[\mathbf{e}_3 \cdot (\mathbf{A}^h \nabla \varphi - \mathbf{v}^h \varphi)] = -\mathcal{M}\Phi - \frac{\partial}{\partial x_1}(\langle a \rangle \frac{\partial \varphi^0}{\partial x_1}) + \langle v_1^d \rangle \frac{\partial \varphi^0}{\partial x_1} \text{ on } \Sigma^T \quad (33)$$

$$\langle a \rangle \frac{\partial \varphi^0}{\partial x_1}(t, L, x_2, 0) + \langle v_1^d \rangle \varphi^0(t, L, x_2, 0) = \kappa g^d. \quad (34)$$

- $2 > \beta > 1$

With a sufficiently strong source:

$$\lim_{\varepsilon \to 0} \varepsilon^{\frac{\beta}{2}+\gamma-\frac{3}{2}} \Phi_\varepsilon(t) = \Phi(t) \quad \text{uniformly in} \;\; t \;, \tag{35}$$

and some concentration entering the shafts tops g_ε:

$$g_\varepsilon = \begin{cases} g^h & \text{everywhere on the boundary except on the shafts tops} \\ \varepsilon^{-\frac{\beta+1}{2}} g^d & \text{on the shafts tops} \;\; \mathcal{H}_\varepsilon \end{cases} . \tag{36}$$

The Transport process in the shafts is dominant and we do not see anything else in the corresponding global model . Then $\varepsilon^{(1-\beta)/2}\,\varphi_\varepsilon \longrightarrow \phi,\ d\mu^\varepsilon(x) 2-\mathit{scale}$,

$$\varepsilon^{\frac{1-\beta}{2}}\ \varphi_\varepsilon \longrightarrow \varphi^0 \tag{37}$$

$$\varepsilon^{\frac{1-\beta}{2}}\ \frac{\partial \varphi_\varepsilon}{\partial x_1} \longrightarrow \frac{\partial \varphi^0}{\partial x_1} \quad , \tag{38}$$

The global concentration φ^0 is the unique solution of a 1-dimensional problem defined for any $x \in]0, L[$ (assuming $g^d = \mathrm{Cte}$).

$$-\frac{\partial}{\partial x_1}\left(\mathbf{A}^d_{11}\ \frac{\partial \varphi^0}{\partial x_1}\right) + \mathbf{v}^d_1 \frac{\partial \varphi^0}{\partial x_1} = 0 \ \text{ in } \ ]0, L[\tag{39}$$

$$\varphi^0(0) = 0 \quad , \ \mathbf{A}^d_{11}\ \frac{\partial \varphi^0}{\partial x_1}(L) + (\mathbf{v}^d_1 + \kappa)\,\varphi^0(L) = \kappa g^d \quad .$$

9 Proofs

- The starting point is *a priori estimates* obtained from (18)-(22):

$$|\nabla \varphi_\varepsilon|_{L^2(0,T;L^2(\Omega_\varepsilon))} \leq C \tag{40}$$

$$|\varphi_\varepsilon|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} \leq C \tag{41}$$

$$|\varphi_\varepsilon|_{L^2(0,T;L^2(\mathcal{C}_\varepsilon))} \leq C\, \varepsilon^{\beta/2} \tag{42}$$

$$|\frac{\partial \varphi_\varepsilon}{\partial x_1}|_{L^2(0,T;L^2(\mathcal{C}_\varepsilon))} \leq C\, \varepsilon^{\beta/2} \quad . \tag{43}$$

- The global models obtained, at the limit, are defined on the hypersurface Σ and the general two-scale convergence has to be adapted to this situation

  We use the *two-scale convergence with respect to the rescaled measure* $d\mu^\varepsilon(x) = \varepsilon^{-1}\,\mathbf{1}_{\mathcal{C}_\varepsilon}\,d\,x$, where $d\,x$ is the Lebesgue measure.

  Definition 1 *A sequence* $\{\varphi_\varepsilon\}_{\varepsilon>0}$, $\varphi_\varepsilon \in L^p(\Omega_\varepsilon)$ *is said to converge two-scale, with respect to the singular measure* $d\mu(x)$, *to* $\varphi^0 \in L^p(\Sigma \times \mathcal{C})$ *if for any* $\psi \in C(\Omega; L^{p'}(\mathcal{C})\,)$

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\mathcal{S}_\varepsilon} \varphi_\varepsilon(x)\psi(x, \frac{x'}{\varepsilon})\,dx =$$
$$\int_\Sigma dx_1\,dx_2\ \int_{\mathcal{C}} \varphi^0(x_1, x_2, y_2, y_3)\,\psi(x_1, x_2, 0, y_2, y_3)\,dy_2\,dy_3$$

- Additional technical difficulty comes from the Dirichlet condition on $\mathcal{Z}_\varepsilon$, the sealed Shafts Bottoms, for the test functions, z_ε, used in the proofs of the above convergence theorems. For this,we start from a $z \in C^1(\overline{\Omega})$ and we construct the test functions z_ε satisfying the Dirichlet condition on $\mathcal{Z}_\varepsilon$, as needed in the proofs. We use the existence for any $z \in H^1(\Omega)$ of a sequence of functions $\{z_m\}_{m\in\mathbf{N}}$, $z_m \in C^1(\overline{\Omega})$ such that $z_m(0, x_2, 0) = 0$ and $z_m \to z$ in $H^1(\Omega)$ since on a 1-dimensional line $c = \{x \in \mathbf{R}^3 \ ; \ x_1 = 0, x_3 = 0\}$ the trace of a function from $H^1(\Omega)$ cannot be specified. $\square$

- We need also sharp estimate of the flux through all the containers sets boundaries Γ_ε for the a priori estimates.

- Mainly, for $\mathcal{L}^\varepsilon \in [H^1(\Omega)]'$ defined for any $\psi \in H^1(\Omega)$, by:

$$\mathcal{L}^\varepsilon \psi = \varepsilon^{1-\gamma} \int_{\Gamma_\varepsilon} \psi$$

,
we should prove

$$\mathcal{L}^\varepsilon \to \mathcal{M}\, \delta_\Sigma \text{ strongly in } [H^1(\Omega)]' \ .$$

(III) From a " LONG STORAGE UNITS model" to a Global "REPOSITORY ZONE model" with a possibly damaged zone

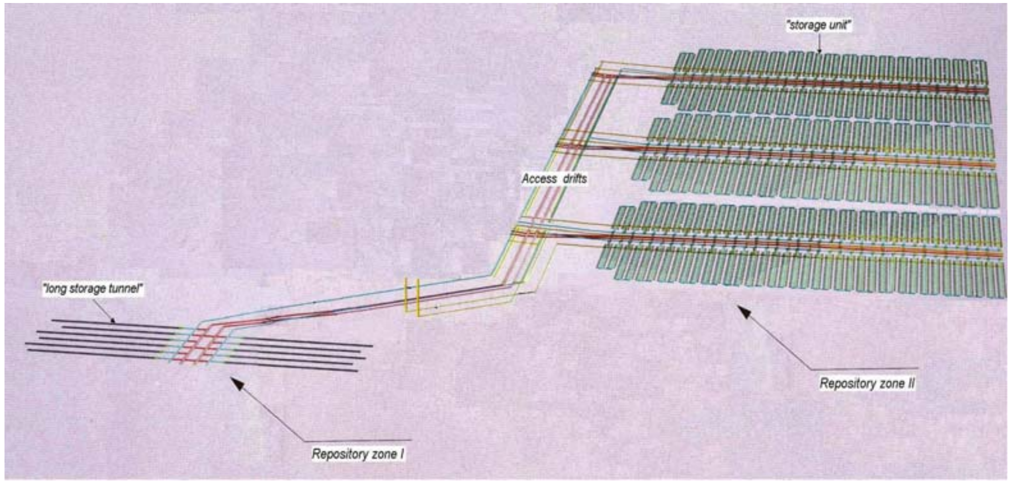
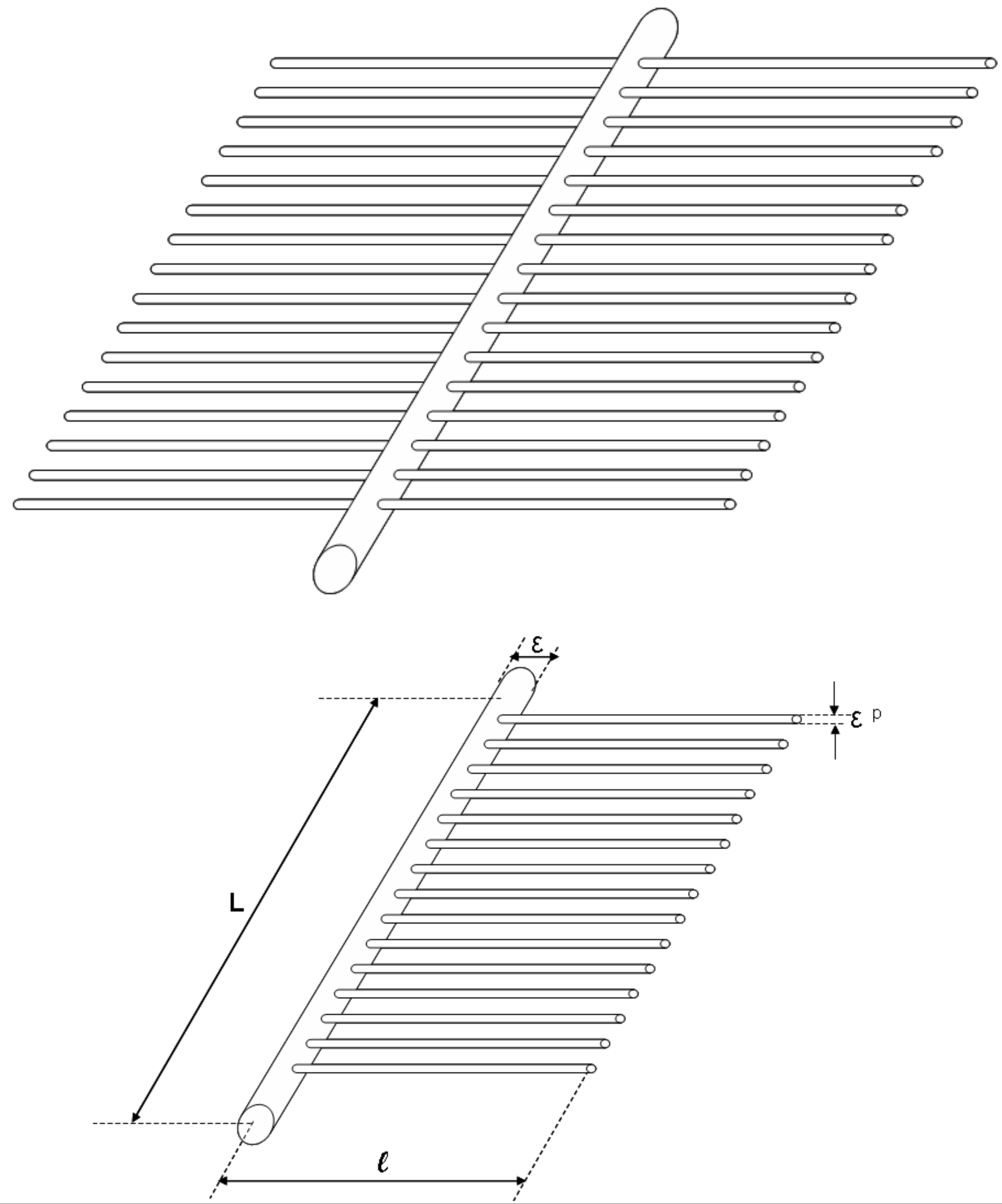


Figure 14: A part of a Waste Repository Site, with two different zones; a zone I, with 11 long storage tunnels; a part of the zone II, with 3 storage units

we study a simplified but typical repository zone, assuming it is made of a high number of similar long waste filled storage units, lying on a hypersurface Σ and linked by backfilled working and haulage drifts .
The units are periodically distributed on both sides of a backfilled drift (haulage and working drift) with period ε
The working and haulage drifts are represented by a single circular cylinder $\mathcal{C}_\varepsilon =]0, L[\times \varepsilon \mathcal{S}$, where the cross section of this cylinder is $\mathcal{S} = \{(y_2, y_3) \in \mathbf{R}^2 \ ; \ y_2^2 + y_3^2 < s^2\}$
the set of all units is denoted U_ε:

$$U_\varepsilon = \bigcup_{j=1}^{N(\varepsilon)} U_\varepsilon^j \ , \ N(\varepsilon) = O(\varepsilon^{-1}) \quad .$$


ε
ε p
L
ℓ

Like previously, the parameter β will characterize de degree of damaging (Darcy's velocity and consequently dispersion will be scaled by means of $\varepsilon^{-\beta}$). The main difference and difficulties compared to the two situations we studied previously, in [**?**] or in [**?**] are coming from the singular behavior of the drift. In [**?**] there vas no damaged zone at all, while in our second paper [**?**] the damaged drifts were periodically repeating, allowing to use the technique of singular measures.

But, the global models will only slightly differ; depending on β.

The first approximation (the weak limit) is independent of the choice of β and only further order correctors will differ, depending on β.

10 Definition of the problem:

We assume the convection, i.e. the Darcy's velocity, to be given by the hydrology and to have the form:

$$\mathbf{v} = \mathbf{v}^h + \boldsymbol{\chi}_{\mathcal{C}_\varepsilon} \, \varepsilon^{-\beta} \, |v^d| \, \mathbf{e}_1 \quad , \tag{44}$$

with $\boldsymbol{\chi}_{\mathcal{C}_\varepsilon}$ standing for the characteristic function of the drift, $\mathcal{C}_\varepsilon$, and where $|v^d|$, the absolute value of the velocity inside the drift, depends only on $r = |x'| = \sqrt{x_2^2 + x_3^2}$.

The evolution of the pollutant's concentration φ_ε in the Repository Zone Ω is governed by the equation:

$$\frac{\partial \varphi_\varepsilon}{\partial t} - \mathrm{div}\,(\mathbf{A}^\varepsilon \nabla \varphi_\varepsilon) + (\mathbf{v}^\varepsilon \cdot \nabla)\varphi_\varepsilon + \lambda\, \varphi_\varepsilon = f^\varepsilon \quad \text{in } \Omega^T = \Omega \times]0, T[, \tag{45}$$

$$\varphi_\varepsilon(0, x) = \Phi_0(x) \;\; x \in \Omega \;\; ; \varphi_\varepsilon = 0 \;\; \text{on } \Gamma^T; \;\; , \tag{46}$$

(47)

where f^ε the sources density is: $f^\varepsilon = \varepsilon^{-1} f\left(t, x, \frac{x_1}{\varepsilon}, \frac{x_3}{\varepsilon}\right)$

11 Zero order approximation:

$$\varphi_\varepsilon \rightharpoonup \varphi_0 \quad \text{weakly in} \ \ L^2(0,T;H^1(\Omega)) \ \ . \tag{48}$$

The limit φ_0 is the unique solution of the problem

$$\frac{\partial \varphi_0}{\partial t} - d\,\Delta\varphi_0 + (\mathbf{v}^h \cdot \nabla)\varphi_0 + \lambda\,\varphi_0 = \langle f\rangle\,\delta_\Sigma \quad \text{in} \ \ \Omega^T \tag{49}$$

$$\varphi_0(0,x) = \Phi_0(x) \ \ x \in \Omega \tag{50}$$

$$\varphi_\varepsilon = 0 \ \ \text{on}\ \Gamma^T \tag{51}$$

$$\langle f\rangle(t,x) = \int_{\varepsilon\mathcal{S}} f(t,x,y_1,y_3)\,dy_1\,dy_3 \ \ . \tag{52}$$

12 asymptotic expansion,with matching, of the solution

Outside the drift we cut-off the limit and we pose:

$$\varphi_\varepsilon \approx \left(1 - \frac{\log r}{\log(s\,\varepsilon)}\right)\varphi_0(t,x) + \frac{1}{\log(s\,\varepsilon)}\,\varphi_1(t,x_1) \quad . \tag{53}$$

with: $r = \sqrt{x_2^2 + x_3^2} \le s\varepsilon$

Inside the drift $\mathcal{C}_\varepsilon$ we seek an expansion:

$$\varphi_\varepsilon \;\approx \frac{1}{\log(s\,\varepsilon)}\;\Psi_1^\varepsilon(t,x_1,\frac{x_2}{\varepsilon},\frac{x_3}{\varepsilon}) \quad . \tag{54}$$

Matching the **concentration**'s values and the **flux** on the drift surface, we conclude:

$$\varphi_1(t,x_1) = \Psi_1^\varepsilon(t,x_1,s)$$
$$s\ \frac{\partial \Psi_1^\varepsilon}{\partial \rho}(t,x_1,s) = -\varphi_0(r = s\,\varepsilon) \approx -\varphi_0(t,x_1,0,0)\ \ ;$$

with,the **interior approximation** in the drift Ψ_1^ε :

$$-|v^d|\ \frac{\partial^2 \Psi_1^\varepsilon}{\partial x_1^2} - \varepsilon^{\beta-2}\, d\left(\frac{\partial^2}{\partial \rho} + \frac{1}{\rho}\ \frac{\partial}{\partial \rho}\right)\ \Psi_1^\varepsilon + v^d\,\frac{\partial \Psi_1^\varepsilon}{\partial x_1} = 0 \ \text{ in }\ \mathcal{S}$$
$$s\ \frac{\partial \Psi_1^\varepsilon}{\partial \rho} = -\varphi_0(t,x_1,0,0)\ \text{ for }\ \rho = s\ \ ,\quad \Psi_1^\varepsilon = 0\ \text{ for }\ x_1 = 0,1 \quad (5.5)$$

Finally, with the above matching (by computing "exactly" the solution Ψ_1^ε)we obtain the strong convergence :

denoting : $R^\varepsilon = \varphi_\varepsilon - \varphi(\varepsilon)$;

then $\lim_{\varepsilon \to 0} |R^\varepsilon|_{L^2(0,T;H^1(\Omega))} = 0$.

and, for any $1 < q < 2$

$\lim_{\varepsilon \to 0} |\varphi_\varepsilon - \varphi_0|_{L^2(0,T;W^{1,q})(\Omega))} = 0$.

13 More about convergence

From the results above we conclude that the contribution of our corrector inside the damaged drift is not very significant. Indeed, what we gain is the $L^2(0,T;H^1(\Omega))$ estimate, while the convergence without corrector is in $L^2(0,T;W^{1,q}(\Omega))$ for $q<2$. That is because the norm of the corrector is negligible in $W^{1,q}$ for $q<2$ but not in H^1.

All this is due to the fact that integral norms in L^p and $W^{1,p}$ spaces are not suitable for a precise asymptotic analysis of tiny objects like the present drift.

Although uniform estimates are not expected for the part of the error that comes from the sources, since the limit of the source function f^ε is only a measure, we want to prove the uniform convergence to zero for the other part of the error coming from the matching.
And finally we obtain the result:

Theorem 2 $R_\varepsilon = \varphi_\varepsilon - \varphi(\varepsilon)$, *the error of the approximation can be decomposed as*
$R_\varepsilon = r_\varepsilon + W_\varepsilon$, *where*
$|r_\varepsilon|_{L^2(0,T;H^1(\Omega)} \leq C\sqrt{\varepsilon}$
and W_ε *tends uniformly to zero on* Ω^T.

14 Conclusion of Part III

It appears at the end that whatever was the magnitude of the convection (i.e. how big is the power β), it does not make an important difference.

As we can see, on the macroscopic scale, there is barely a mild logarithmic singularity around the drift; this is mainly due to the fact that the units are long comparing to the drift diameter and, in the limit, there is a uniform density of the source everywhere on Σ, which is relatively important compared to the effect of the strong convection, which was localized only in the vicinity of the drift itself, i.e. localized on a very thin cylinder.

15 bibliography

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(IV) Positions and contents of the Waste Packages are Random

work in progress, with A. Piatnitski

The "local sources" f^ε are periodically repeated, lying on a plan Σ; the ***emission starting time*** and the ***emission time evolution***, of each local source, are both random :
$f^\varepsilon(x,t) = \mathbb{1}_{B_\varepsilon} \frac{1}{\varepsilon^\gamma} f(T_{\mathbf{x}'/\varepsilon}\omega, t)$.

Theorem 1

$$\lim_{\varepsilon \to 0} \|u^\varepsilon - u^0\|_{L^2(0,\infty;H^1(G))} = 0 \quad a.s.;$$

with :

$$\partial_t u^0 - \mathrm{div}(a(x)\nabla u^0) + \mathrm{div}(b(x)u^0) = F(t)\delta_\Sigma(x); \tag{56}$$

$$F(t) = s_1 s_2 \mathbf{E}\{f(\cdot,t)\}. \tag{57}$$

THE END

Thank you