IHP, 23 septembre 2010

Schéma de volumes finis hybrides pour un écoulement diphasique, immiscible et incompressible

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Two-phase flow problem

- Purpose : Propose numerical methods for the simulation of two-phase flow problems in heterogeneous and anisotropic porous media on non-conforming meshes.

$$\begin{cases} -\nabla \cdot (\lambda(s)\mathbf{K}\nabla p) = q_w + q_n, \\ \mathbf{u} = -\lambda(s)\mathbf{K}\nabla p, \\ \omega \dfrac{\partial s}{\partial t} + \nabla \cdot (\mathbf{u} f(s)) - \nabla \cdot (\mathbf{K}\nabla \varphi(s)) = q_n. \end{cases}$$

- The first equation is uniformly elliptic in p;
- The second is parabolic degenerate in s.

Two-phase flow problem

Macroscopic description of immiscible two-phase flow:

$$\begin{cases} \omega \dfrac{\partial \rho_i s_i}{\partial t} + \nabla \cdot (\rho_i \mathbf{u}_i) = \bar{q}_i \\ \mathbf{u}_i = -\lambda_i(s_i)\mathbf{K}\nabla p_i \\ s_n + s_w = 1 \\ p_n - p_w = \pi(s_n) \end{cases}$$

ω - the porosity, $\mathbf{K}$ - the absolute permeability, ρ_i, $\mathbf{u}_i$, p_i, λ_i - the density, velocity, pressure and the relative permeability of the phase i, $\pi(s)$ - the capillary pressure, s_i - the saturation of the phase i.

Two-phase flow problem

If two phase are incompressible the system becomes:

$$
\begin{cases}
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\mathbf{u}_i = -\lambda_i(s_i)\mathbf{K}\nabla p_i \\
s_n + s_w = 1 \\
p_n - p_w = \pi(s_n)
\end{cases}
\Rightarrow
\begin{cases}
\omega \dfrac{\partial s}{\partial t} - \nabla \cdot (\lambda_n(s)\mathbf{K}\nabla p_n) = q_n \\
-\omega \dfrac{\partial s}{\partial t} - \nabla \cdot (\lambda_w(s)\mathbf{K}\nabla p_w) = q_w \\
p_n - p_w = \pi(s)
\end{cases}
$$

ω - the porosity, $\mathbf{K}$ - the absolute permeability, ρ_i, $\mathbf{u}_i$, p_i, λ_i - the density, velocity, pressure and the relative permeability of the phase i, $\pi(s)$ - the capillary pressure, s_i - the saturation of the phase i.

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Introducing the global pressure (Chavent, Jaffré, '86).

$$p = p_w + \int_0^s \frac{\lambda_n}{\lambda_n + \lambda_w}(\tau) \pi'(\tau) d\tau,$$

lead to the following system...

Two-phase flow: global pressure formulation

...lead to the following system:

$$\begin{cases} -\nabla \cdot (\lambda(s)\mathbf{K}\nabla p) = q_w + q_n, \\ \mathbf{u} = -\lambda(s)\mathbf{K}\nabla p, \\ \omega \dfrac{\partial s}{\partial t} + \nabla \cdot (\mathbf{u} f(s)) - \nabla \cdot (\mathbf{K}\nabla \varphi(s)) = q_n. \end{cases}$$

where $\lambda(s) = \lambda_n(s) + \lambda_w(s)$,

$$f(s) = \frac{\lambda_n(s)}{\lambda_n(s) + \lambda_w(s)} \quad \text{and} \quad \varphi(s) = \int_0^s \frac{\lambda_w \lambda_n}{\lambda_w + \lambda_n}(\tau)\pi'(\tau)d\tau.$$

Semi-discretization

Implicit time discretization

$$\begin{cases} -\nabla \cdot (\lambda(s^{n-1})\mathbf{K}\nabla p^n) = q_w^n + q_n^n, \\ \mathbf{u}^n = -\lambda(s^{n-1})\mathbf{K}\nabla p^n, \\ \omega \dfrac{s^n - s^{n-1}}{\delta t} + \nabla \cdot (\mathbf{u^n} f(s^n)) - \nabla \cdot (\mathbf{K}\nabla \varphi(s^n)) = q_n^n. \end{cases}$$

Two equations are not coupled anymore.

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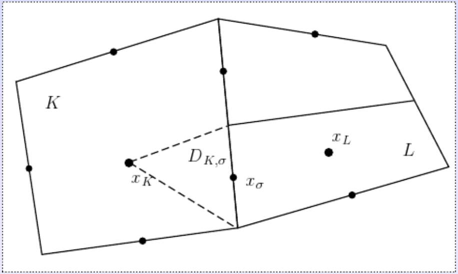
Two equations are not coupled anymore.

- SUSHI scheme for a diffusion term;
- Upwind scheme for a convection term.

Diffusion term: space discretization

At each time step one has to solve: $-\nabla \cdot (\mathbf{K}\nabla p) = q$, or in the discrete form

$$\left\{ \begin{array}{l} \sum F_{K,\sigma}(p) = q_K \\ F_{K,\sigma} + F_{L,\sigma} = 0, \end{array} \right. \quad \text{where} \quad F_{K,\sigma}(p) \approx \int_\sigma \nabla p \cdot \mathbf{n}_{K\sigma}$$

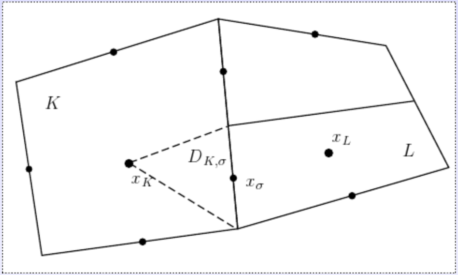


A discrete solution lives in the space
$X_\mathcal{D} = \{((v_K)_{K\in\mathcal{M}}, (v_\sigma)_{\sigma\in\mathcal{E}}), v_K \in \mathbb{R}, v_\sigma \in \mathbb{R}\}$.

Diffusion term: space discretization

For each sub-cell $D_{K\sigma}$ we define it's own discrete gradient

- $\nabla_{K,\sigma} p = \nabla_K p + R_{K,\sigma} p \cdot \mathbf{n}_{K,\sigma},$
- $\nabla_K p = \dfrac{1}{|K|} \displaystyle\sum_{\sigma \in \mathcal{E}_K} m(\sigma)(p_\sigma - p_K)\mathbf{n}_{K,\sigma}$
- $R_{K,\sigma} p = \dfrac{\alpha_K}{d_{K,\sigma}}(p_\sigma - p_K - \nabla_K p \cdot (\mathbf{x}_\sigma - \mathbf{x}_K))$



Diffusion term

We can define the discrete flux through

$$\sum_{\sigma \in \mathcal{E}_K} (v_{\sigma'} - v_K) F_{K,\sigma}(p) = \int_K \nabla_{\mathcal{D}} v \cdot \mathbf{K} \nabla_{\mathcal{D}} p \, d\mathbf{x} \text{ for all } v \in X_{\mathcal{D}}$$

where

$$\nabla_{\mathcal{D}} p|_K(\mathbf{x}) = \nabla_{K,\sigma} p \text{ for all } p \in X_{\mathcal{D}}$$

The discrete gradient $\nabla_{\mathcal{D}} \cdot$ satisfies

- $\nabla_{\mathcal{D}}(P_{\mathcal{D}}(\psi)) = \nabla \psi$ for all $\psi(\mathbf{x})$ piecewise linear on the elements of the mesh.

Discrete weak formulation

For each $n \in \{1, \dots, N\}$ *find* $s^n \in X_{\mathcal{D},0}$ *and* $p^n \in X_{\mathcal{D},0}$ *such that for all* $w^n, v^n \in X_{\mathcal{D},0}$:

$$
\begin{cases}
\displaystyle\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} (w_K^n - w_\sigma^n)\lambda_K^{n-1} F_{K,\sigma}(p^n) = \sum_{K\in\mathcal{M}} |K| w_K^n (q_{w,K}^n + q_{n,K}^n), \\
\displaystyle\sum_{K\in\mathcal{M}} |K| \omega_K v_K^n \frac{s_K^n - s_K^{n-1}}{\delta t} \\
\displaystyle+ \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} f_{K,\sigma}(s^n, p^n)(v_K^n - v_\sigma^n)\lambda_K^{n-1} F_{K,\sigma}(p^n) \\
\displaystyle+ \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} (v_K^n - v_\sigma^n) F_{K,\sigma}(\varphi^n) = \sum_{K\in\mathcal{M}} |K| v_K^n q_{n,K}^n.
\end{cases}
$$

Convection term

- The standard upwind value $f^n_{K,\sigma} = f_{K,\sigma}(s^n, p^n)$ is defined by

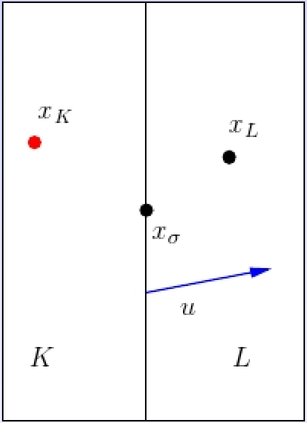
$$f^n_{K,\sigma} = \begin{cases} f^n_K, & \text{if } F_{K,\sigma}(p^n) \geq 0 \\ f^n_L, & \text{if } F_{K,\sigma}(p^n) < 0. \end{cases}$$

$$F_{K,\sigma}(p^n) \geq 0 \Rightarrow f^n_{K,\sigma} = f^n_{L,\sigma} = f^n_K$$

.

- This implies that

$$F_{K,\sigma}(p^n) f^n_{K,\sigma} + F_{L,\sigma}(p^n) f^n_{L,\sigma} = 0.$$



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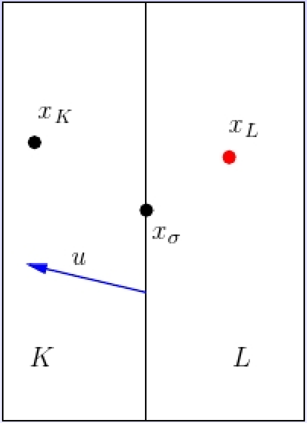

Convection term

- We define the partially upwind value $f^n_{K,\sigma} = f_{K,\sigma}(s^n, p^n)$ by

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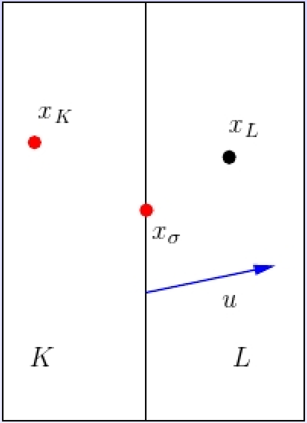
$$F_{K,\sigma}(p^n) \geq 0 \Rightarrow \quad \begin{array}{l} f^n_{K,\sigma} = f^n_K, \\ f^n_{L,\sigma} = f^n_\sigma. \end{array}$$

- The flux conservation is imposed by the scheme

$$\sum_{M=\{K,L\}} [F_{M,\sigma}(\varphi^n) + F_{M,\sigma}(p^n) f_{M,\sigma}(s^n)] = 0$$



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$$F_{K,\sigma}(p^n) \leq 0 \Rightarrow \quad \begin{array}{l} f^n_{K,\sigma} = f^n_\sigma, \\ f^n_{L,\sigma} = f^n_L. \end{array}$$

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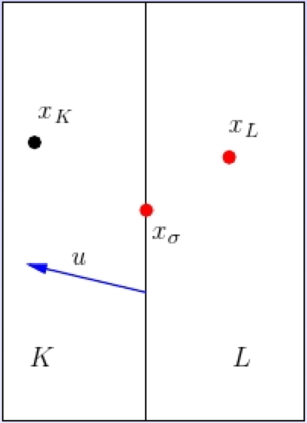

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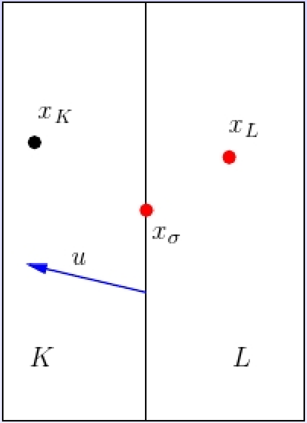
- The upwind value $f_{K,\sigma}(s^n, p^n)$ is in the space

$$\mathcal{F} = \{(q_{K,\sigma})_{K\in\mathcal{M},\sigma\in\mathcal{E}_K}, q_{K,\sigma} \in \mathbb{R}\}.$$



Convection term

- We can show that for all $v, w \in X_{\mathcal{D}}$ and $q \in \mathcal{F}$ it holds

$$|\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} q_{K,\sigma}(v_K - v_\sigma)F_{K,\sigma}(w)| \leq C\|q\|_\infty \|\nabla_{\mathcal{D}} v\|_{L^2(\Omega)} \|\nabla_{\mathcal{D}} w\|_{L^2(\Omega)},$$

where $\|q\|_\infty = \max_{K\in\mathcal{M},\sigma\in\mathcal{E}_K} |q_{K,\sigma}|$.

Convergence result

An approximate solution satisfies

$$\|\varphi_{\mathcal{D},\delta t}\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla_{\mathcal{D},\delta t}\varphi_{\mathcal{D},\delta t}\|_{L^2(Q_T)} + \|\nabla_{\mathcal{D},\delta t}p_{\mathcal{D},\delta t}\|_{L^\infty(0,T;L^2(\Omega))} \leq C;$$

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- There exists at least one solution;
- Uniqueness? Hard to prove because there is a lack of monotonicity: For φ increasing we do not have in general that

$$\nabla_{\mathcal{D},\delta t}\varphi(s) \cdot \nabla_{\mathcal{D},\delta t} s \geq 0$$

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Up to a subsequence:

- $\varphi_{\mathcal{D},\delta t} \to \phi \in L^2(0,T;H_0^1(\Omega))$ in $L^2(Q_T)$ as $h, \delta t \to 0$;

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Convergence result

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- $\nabla_{\mathcal{D},\delta t} p_{\mathcal{D},\delta t} \rightharpoonup \nabla\mathrm{p}$.

Convergence result

We introduce the function space

$$\psi = \{\psi \in C^{2,1}(\overline{\Omega} \times [0,T]), \quad \psi = 0 \text{ on } \partial\Omega \times [0,T], \quad \psi(\cdot, T) = 0\}.$$

We set $\psi_K^n = \psi(\mathbf{x}_K, t_n)$ and $\psi_\sigma^n = \psi(\mathbf{x}_\sigma, t_n)$

$$\begin{cases} \displaystyle\sum_{n=1}^{N} \sum_{K,\sigma} \lambda(s_K^{n-1})(\psi_K^n - \psi_\sigma^n) F_{K,\sigma}(p^n) = \sum_{n=1}^{N} \sum_{K,\sigma} |K| \psi_K^n (q_{w,K}^n + q_{n,K}^n), \\ \displaystyle\sum_{n=1}^{N} \sum_{K} |K| \omega_K \psi_K^n \frac{s_K^n - s_K^{n-1}}{\delta t} \\ \displaystyle+ \sum_{n=1}^{N} \sum_{K,\sigma} f_{K,\sigma}(s^n, p^n) \lambda(s_K^{n-1})(\psi_K^n - \psi_\sigma^n) F_{K,\sigma}(p^n) \\ \displaystyle+ \sum_{n=1}^{N} \sum_{K,\sigma} (\psi_K^n - \psi_\sigma^n) F_{K,\sigma}(\varphi^n) = \sum_{K \in \mathcal{M}} |K| \psi_K^n q_{n,K}^n. \end{cases}$$

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Diffusion term

The strong/weak consistency of the discrete gradient implies

$$\begin{aligned}\sum_{n,K}\sum_{\sigma\in\mathcal{E}_K}\delta t(\psi_K^n-\psi_\sigma^n)F_{K,\sigma}(\varphi^n) &= \sum_{n,K}\sum_{\sigma\in\mathcal{E}_K}\delta t|D_{K,\sigma}|\mathbf{K}\nabla_{K,\sigma}\psi^n\cdot\nabla_{K,\sigma}\varphi^n \\ &= \int_0^T\int_\Omega \mathbf{K}\nabla_{\mathcal{D},\delta t}\psi^n\cdot\nabla_{\mathcal{D},\delta t}\varphi^n\, d\mathbf{x}dt \\ &\to \int_0^T\int_\Omega \mathbf{K}\nabla\psi\cdot\nabla\phi\, d\mathbf{x}dt.\end{aligned}$$

Unlike the diffusion term we generally have

$$\begin{aligned}\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\lambda_K^{n-1} \quad & f_{K,\sigma}(s,p)(\psi_K^n-\psi_\sigma^n)F_{K,\sigma}(p^n) \\ &\neq \sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}|D_{K,s}|\lambda_K^{n-1}f_{K,\sigma}(s,p)\nabla_{K,s}\psi^n\cdot\mathbf{K}_K\nabla_{K,\sigma}p^n,\end{aligned}$$

Convergence of the convection term

We define

$$T_C = \sum_{n=1}^{N} \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \lambda_K^{n-1} f_{K,\sigma}(s,p)(\psi_K^n - \psi_\sigma^n) F_{K,\sigma}(p^n).$$

We set $T_C = T_C^1 - T_C^2$

$$T_C^1 = \sum_{n,K,\sigma} \delta t \lambda_K^{n-1} f_K^n (\psi_K^n - \psi_\sigma^n) F_{K,\sigma}(p^n) \to \int_0^T \int_\Omega \lambda(s) f(s) K \nabla\psi \nabla P \, d\mathbf{x} dt$$

$$T_C^2 = \sum_{n,K} \sum_{\sigma\in\mathcal{E}_K, F_{K,\sigma}\le 0} \delta t \lambda_K^{n-1} (\psi_K^n - \psi_\sigma^n)(f_K^n - f_\sigma^n) F_{K,\sigma}(p^n) \to 0.$$

We will now show the convergence to zero of the term T_C^2.

Convection term

$$
\begin{aligned}
|T_C^2| &= |\sum_{n,K} \sum_{\sigma \in \mathcal{E}_K, F_{K,\sigma} \leq 0} \delta t \lambda(s_K^{n-1})(f_K^n - f_\sigma^n)(\psi_K^n - \psi_\sigma^n) F_{K,\sigma}(p^n)| \\
&\leq \sum_{n,K} \sum_{\sigma \in \mathcal{E}_K} |\delta t \lambda(s_K^{n-1})(f_K^n - f_\sigma^n)(\psi_K^n - \psi_\sigma^n) F_{K,\sigma}(p^n)| \\
&\leq \sum_{n,K} \sum_{\sigma \in \mathcal{E}_K} \delta t \xi_{K,\sigma}^n (f_K^n - f_\sigma^n) F_{K,\sigma}(p^n).
\end{aligned}
$$

$\xi^n \in \mathcal{F}_\mathcal{D}$ is defined by

$$
\xi_{K,\sigma}^n = \lambda_K^{n-1} |\psi_K^n - \psi_\sigma^n| \mathrm{sgn}((f_K^n - f_\sigma^n) F_{K,\sigma}(p^n)).
$$

We obtain

$$
|T_C^2| \leq C \|\xi\|_\infty \|\nabla_{\mathcal{D},\delta t} f(s)\|_{L^2(Q_T)} \|\nabla_{\mathcal{D},\delta t} p\|_{L^2(Q_T)}.
$$

Convection term

$$
\begin{aligned}
|T_C^2| &= |\sum_{n,K} \sum_{\sigma\in\mathcal{E}_K, F_{K,\sigma}\le 0} \delta t \lambda(s_K^{n-1})(f_K^n - f_\sigma^n)(\psi_K^n - \psi_\sigma^n) F_{K,\sigma}(p^n)| \\
&\le \sum_{n,K} \sum_{\sigma\in\mathcal{E}_K} |\delta t \lambda(s_K^{n-1})(f_K^n - f_\sigma^n)(\psi_K^n - \psi_\sigma^n) F_{K,\sigma}(p^n)| \\
&\le \sum_{n,K} \sum_{\sigma\in\mathcal{E}_K} \delta t \xi_{K,\sigma}^n (f_K^n - f_\sigma^n) F_{K,\sigma}(p^n).
\end{aligned}
$$

- Let $|f(s_1) - f(s_2)| \le C|\varphi(s_1) - \varphi(s_2)|$

Convection term

$$
\begin{aligned}
|T_C^2| &= |\sum_{n,K} \sum_{\sigma\in\mathcal{E}_K, F_{K,\sigma}\leq 0} \delta t \lambda(s_K^{n-1})(f_K^n - f_\sigma^n)(\psi_K^n - \psi_\sigma^n) F_{K,\sigma}(p^n)| \\
&\leq \sum_{n,K} \sum_{\sigma\in\mathcal{E}_K} |\delta t \lambda(s_K^{n-1})(f_K^n - f_\sigma^n)(\psi_K^n - \psi_\sigma^n) F_{K,\sigma}(p^n)| \\
&\leq \sum_{n,K} \sum_{\sigma\in\mathcal{E}_K} \delta t \xi_{K,\sigma}^n (f_K^n - f_\sigma^n) F_{K,\sigma}(p^n).
\end{aligned}
$$

- Let $|f(s_1) - f(s_2)| \leq C|\varphi(s_1) - \varphi(s_2)|$
- Then $\|\nabla_{\mathcal{D},\delta t} f(s)\|_{L^2(Q_T)} \leq C$ so that $T_C^2 \to 0$.

Numerical tests

We set

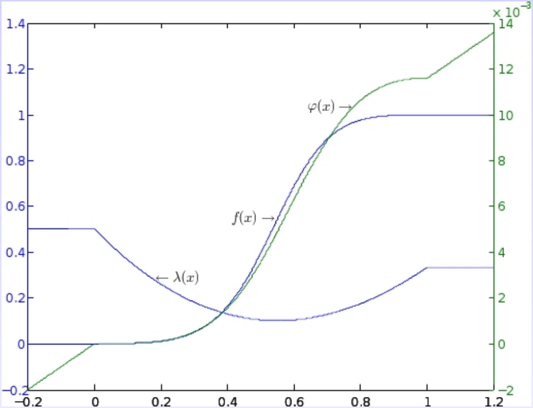
$$\lambda_n(s) = \frac{(1-s)^3}{3}, \lambda_w(s) = \frac{s^3}{2};$$

$$\pi(s) = \frac{1}{2}\sqrt{\frac{s}{1-s}};$$

$$\lambda(s) = \lambda_n(s) + \lambda_w(s);$$

$$f(s) = \frac{\lambda_n(s)}{\lambda(s)};$$

$$\varphi(s) = \int_0^s \lambda_w(\tau) f(\tau) \pi'(\tau) d\tau.$$



Since the maximum principle is not satisfied we prolong the curves outside of $[0, 1]$.

Implementation

At each time step one has to solve a system of nonlinear equations

$$\begin{cases} \sum_{K\in\mathcal{M}} \frac{\omega_K|K|}{\delta t} s_K^n + \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \Phi_{K,\sigma}^n = RHS \\ \\ \Phi_{K,\sigma}^n + \Phi_{L,\sigma}^n = 0, \end{cases}$$

where

$$\Phi_{K,\sigma}^n = \sum_{\sigma'\in\mathcal{E}_K} A_K^{\sigma\sigma'}(\varphi(s_K^n) - \varphi(s_\sigma^n)).$$

Implementation

- The linearized system is degenerate if $s_\sigma^{n,k} = 0, 1$ for some k (may be for $k = 0$)

$$\begin{cases} \dfrac{\omega_K |K|}{\delta t} \delta s_K^{n,m} + \displaystyle\sum_{\sigma \in \mathcal{E}_K} \delta \Phi_{K,\sigma}^{n,m} = RHS \\ \delta \Phi_{K,\sigma}^{n,m} + \delta \Phi_{L,\sigma}^{n,m} = RHS \\ s^{n,0} = s^{n-1}, \end{cases}$$

where

$$\delta \Phi_{K,\sigma}^{n,m} = \sum_{\sigma' \in \mathcal{E}_K} A_K^{\sigma\sigma'} (\varphi'(s_K^{n,m-1}) \delta s_K^{n,m} - \varphi'(s_\sigma^{n,m-1}) \delta s_\sigma^{n,m}).$$

Implementation

- The linearized system is degenerate if $s_\sigma^{n,k} = 0, 1$ for some k (may be for $k = 0$)
- A good choice of discrete unknowns is: $((s_K)_{K\in\mathcal{M}}, (\varphi_\sigma)_{K\in\mathcal{E}})$

$$\begin{cases} \dfrac{\omega_K|K|}{\delta t}\delta s_K^{n,m} + \displaystyle\sum_{\sigma\in\mathcal{E}_K} \delta\Phi_{K,\sigma}^{n,m} = RHS \\ \\ \delta\Phi_{K,\sigma}^{n,m} + \delta\Phi_{L,\sigma}^{n,m} = RHS \\ \\ s^{n,0} = s^{n-1}, \end{cases}$$

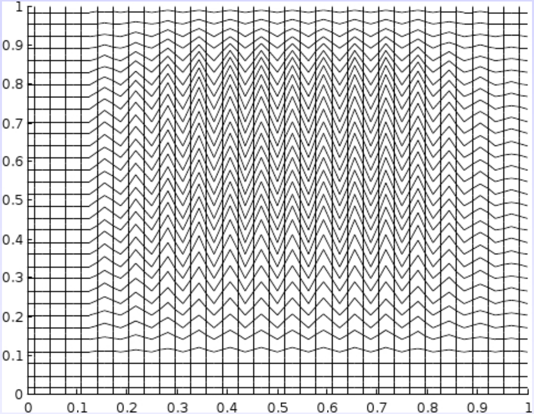
where

$$\delta\Phi_{K,\sigma}^{n,m} = \sum_{\sigma'\in\mathcal{E}_K} A_K^{\sigma\sigma'}(\varphi'(s_K^{n,m-1})\delta s_K^{n,m} - \delta\varphi_\sigma^{n,m}).$$

Numerical results

The 5-spot problem on a distorted mesh

- The square domain $\Omega = (0,1)^2$ is initially filled with oil $(s_0 \equiv 1)$;
- Injected in the left-down corner;
- Production in the right-upper corner;
- The absolute permeability is one in the whole domain.



Numerical results

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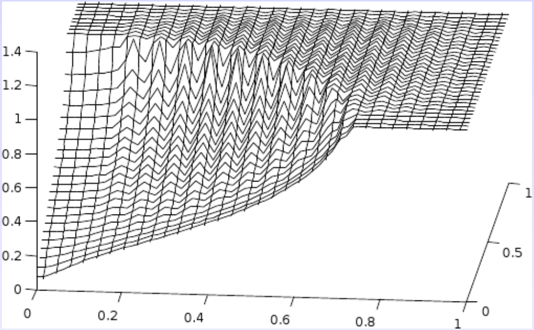
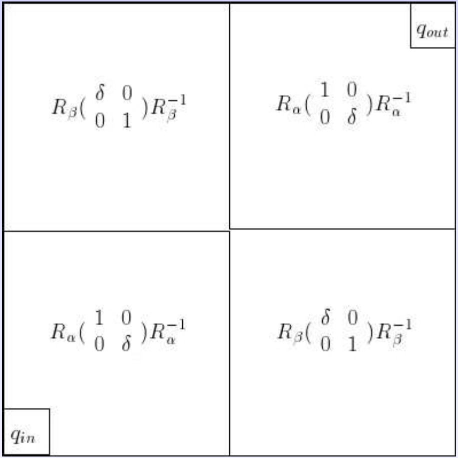


Numerical results

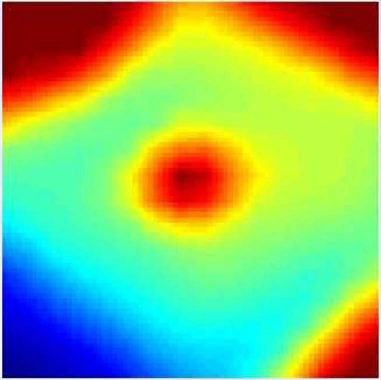
The 5-spot problem with anisotropy

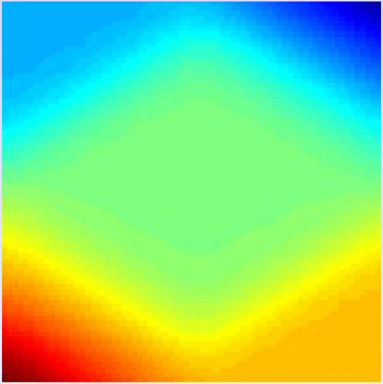
- The square domain $\Omega = (0,1)^2$ is initially filled with oil $(s_0 \equiv 1)$;
- Injected in the left-down corner;
- Production in the right-upper corner;
- The anisotropy: $\delta = 10^{-6}, \alpha = -\pi/6, \beta = -\pi/3$;



Numerical results

The 5-spot problem with anisotropy

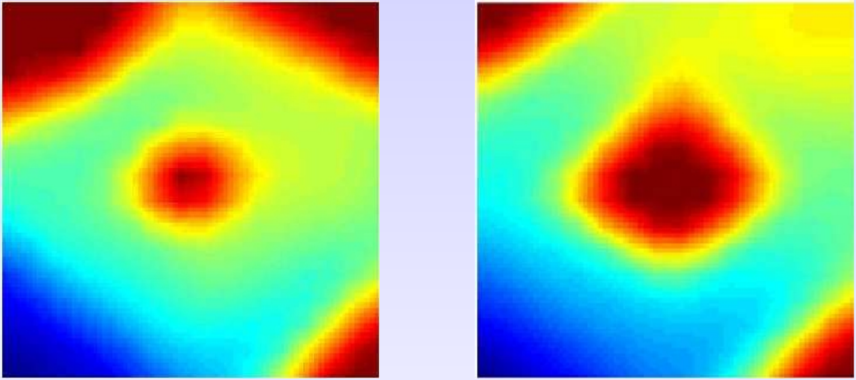




Saturation (left) and pressure (right) field at time t=1.

Numerical results

The 5-spot problem with anisotropy



Saturation field at time t=1 for two values of δt; $\delta t = 0.001$ (left) and $\delta t = 1$ (right).

Merci de votre attention!